STABILITY OF HERMITIAN–YANG–MILLS EQUATION

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Abstract. We show that on a smoothly indecomposable vector bundle over a complex surface with the trivial canonical line bundle, there are no critical points of the Hermitian-Yang-Mills functional other than the absolute minima.

0. Introduction. On a holomorphic hermitian vector bundle $(\mathcal{E}, h)$ over a compact complex hermitian manifold M, we consider the Hermitian-Yang-Mills functional (2.11)

$$\mathcal{Y}(B) = \frac{1}{2}\|F(B)\|^2, \quad B \in A^{0,1}(\mathfrak{g}_{\mathbf{C}}),$$

where $\|\cdot\|$ denotes the L^2-norm of the (0,2)-part of the traceless curvature tensor. Thus the zero set (or the absolute minima) of $\mathcal{Y}$ consists of possible other holomorphic structures on $E = |\mathcal{E}|$ fixing the determinant $\det \mathcal{E}$. We show

THEOREM (3.3). *On complex surfaces with the trivial canonical line bundle, there are no critical points of $\mathcal{Y}$ other than the absolute minima, when E is smoothly indecomposable.*

The complex surfaces satisfying the condition of the theorem are complex tori, K3 surfaces and Kodaira surfaces. Yang-Mills theory on these surfaces are considered in [5]. Donaldson's functional $\mathcal{L}$ [4, 6] have a similar property, namely h is a critical point of $\mathcal{L}$ if and only if it is an absolute minimum or an Einstein-Hermitian metric. But his functional is not bounded below by 0. This kind of phenomenon is not true in Yang-Mills theory [8, 2]. We expect from the above theorem that the space of Cauchy-Riemann operators (2.10) on such surfaces are path connected (cf. [7, p. 157]). A naive idea is the following. If $\gamma : [0,1] \to A^{0,1}(\mathfrak{g}_{\mathbf{C}})$ is a path joining two absolute minima, then the (negative) gradient flow of $\mathcal{Y}$ gives rise to a homotopy $\{\gamma_t\}$ of γ fixing the end points. The integral

$$E(\gamma_t) = \int_0^1 \mathcal{Y}(\gamma_t(s))\, ds$$

is a decreasing function of t. If the limit path γ_∞ exists, then $E(\gamma_\infty) = 0$ and hence γ_∞ lies in the zero set of $\mathcal{Y}$. So far, this is not carried out.

This paper is organized as follows. Although most notations are standard, e.g., as in [6], section 1 is introduced to fix notations. In section 2, we describe Hermitian-Yang-Mills functional. Main theorems appear only in section 3. Appendix explains the Serre duality for semi-connections, which is used in the proof of the theorem.

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1. Connections on a Lie algebra bundle. Let $\mathfrak{g}$ be a smooth bundle of real Lie algebras over a smooth manifold M of dimension m. The space of differential p-forms on M (resp. with values in $\mathfrak{g}$) is denoted by A^p (resp. $A^p(\mathfrak{g})$). Then the Lie braket

$$[\ ,\ ]: A^0(\mathfrak{g}) \otimes A^0(\mathfrak{g}) \to A^0(\mathfrak{g})$$

extends canonically to a map

$$[\ ,\ ]: A^p(\mathfrak{g}) \otimes A^q(\mathfrak{g}) \to A^{p+q}(\mathfrak{g})$$

and

$$[\xi_1, \xi_2] = -(-1)^{p_1 p_2}[\xi_2, \xi_1]$$

for $\xi_i \in A^{p_i}(\mathfrak{g})$. For $\phi \in A^\bullet(\mathfrak{g}) := \sum_{p=1}^m A^p(\mathfrak{g})$, let

$$ad(\phi): A^\bullet(\mathfrak{g}) \to A^\bullet(\mathfrak{g})$$

be the map $\xi \mapsto [\phi, \xi]$ for $\xi \in A^\bullet(\mathfrak{g})$. Then the Jacobi identity is

$$ad(\xi_1)[\xi_2, \xi_3] = [ad(\xi_1)\xi_2, \xi_3] + (-1)^{p_1 p_2}[\xi_2, ad(\xi_1)\xi_3]$$

for $\xi_i \in A^{p_i}(\mathfrak{g})$.

Now we assume a Riemannian structure h on $\mathfrak{g}$, which is *invariant* in the sense that

$$h([X,Y],Z) = h(X,[Y,Z]) \tag{1.1}$$

for all $X, Y, Z \in A^0(\mathfrak{g})$. We call such a pair $(\mathfrak{g}, h)$ a *metrized Lie algebra bundle.* The Riemannian structure h extends canonically to a map

$$h: A^p(\mathfrak{g}) \otimes A^q(\mathfrak{g}) \to A^{p+q} \tag{1.2}$$

and the equation (1.1) becomes

$$h([\xi_1, \xi_2], \xi_3) = h(\xi_1, [\xi_2, \xi_3]) \tag{1.3}$$

for $\xi_i \in A^{p_i}(\mathfrak{g})$, or equivalently

$$h(ad(\xi_1)\xi_2, \xi_3) + (-1)^{p_1 p_2} h(\xi_2, ad(\xi_1)\xi_3) = 0. \tag{1.4}$$

Now we assume that M is a compact, oriented Riemannian manifold of dimension m. Then the Hodge $\star$ extends to a map

$$\star: A^p(\mathfrak{g}) \to A^{m-p}(\mathfrak{g}).$$

We have a pointwise inner product

$$\langle\ ,\ \rangle : A^p(\mathfrak{g}) \otimes A^p(\mathfrak{g}) \to A^0 \tag{1.5}$$

satisfying

$$\langle \xi_1, \xi_2 \rangle = \star h(\xi_1, \star\xi_2)$$

and the global inner product

$$(\ ,\ ) : A^p(\mathfrak{g}) \otimes A^p(\mathfrak{g}) \to \mathbb{R} \tag{1.6}$$

defined by

$$(\xi_1, \xi_2) = \int_M h(\xi_1, \star\xi_2)$$

for $\xi_i \in A^p(\mathfrak{g})$. The induced norms of $\langle\ ,\ \rangle$ and $(\ ,\ )$ will be denoted by $|\cdot|$ and $\|\cdot\|$, i.e.,

$$|\xi| = \langle \xi, \xi \rangle^{1/2}, \quad \|\xi\| = (\xi, \xi)^{1/2}. \tag{1.7}$$

The adjoint of

$$ad(\xi) : A^\bullet(\mathfrak{g}) \to A^\bullet(\mathfrak{g}) \tag{1.8}$$

with respect to the inner product $(\ ,\ )$ is denoted by $ad(\xi)^*$. Then for $\xi \in A^p(\mathfrak{g})$,

$$ad(\xi)^* = -(-1)^{(m+1+p)q} \star ad(\xi) \star \qquad \text{on } A^q(\mathfrak{g}). \tag{1.9}$$

1.10. DEFINITION. A *connection* D on a metrized Lie algebra bundle $(\mathfrak{g}, h)$ is an $\mathbb{R}$-linear map

$$D : A^0(\mathfrak{g}) \to A^1(\mathfrak{g})$$

such that

(1) $D(fX) = df \cdot X + fD(X)$
(2) $d\,h(X,Y) = h(DX,Y) + h(X,DY)$
(3) $D[X,Y] = [DX,Y] + [X,DY]$

for any $f \in C^\infty(M)$ and $X,\ Y,\ Z \in A^0(\mathfrak{g})$.

A connection D on $(\mathfrak{g}, h)$ extends in an obvious way to a map

$$d_D : A^p(\mathfrak{g}) \to A^{p+1}(\mathfrak{g}).$$

Then $d_D \circ d_D$ is equal to the *curvature tensor* R of D, which is a 2-form on M with values in the bundle $Der(\mathfrak{g})$ of deriviations on $\mathfrak{g}$.

2. Holomorphic Lie algebra bundle. In this section we assume that M is a compact complex manifold with a hermitian metric. As in the previous section let $(\mathfrak{g}, h)$ be a metrized real Lie algebra bundle, where the metric h is invariant (1.1). The induced *hermitian metric* on the complexified Lie algebra bundle

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes \mathbf{C}$$

is also denoted by h. For $X = X_1 + \sqrt{-1}X_2$ ($X_i \in \mathfrak{g}$), we define the *conjugate transpose*

$$X^\dagger = -X_1 + \sqrt{-1}X_2. \tag{2.1}$$

Then $X \mapsto X^\dagger$ is an *involutive conjugate linear isomorphism* on $\mathfrak{g}_{\mathbf{C}}$ such that

(1) $X^\dagger = -X$ if and only if $X \in \mathfrak{g}$

(2) $h(X^\dagger, Y^\dagger) = h(Y, X) = \overline{h(X, Y)}$

(3) $h([X, Y], Z) = h(Y, [X^\dagger, Z]) = -h(X, [Y^\dagger, Z])$

for $X,\ Y,\ Z \in \mathfrak{g}_{\mathbf{C}}$.

The conjugate transpose map extends obviously to a conjugate linear isomorphism of $A^\bullet(\mathfrak{g}_{\mathbf{C}}) \to A^\bullet(\mathfrak{g}_{\mathbf{C}})$, which, in turn, defines an isomorphism

$$\mathfrak{g}_{\mathbf{C}} \simeq \mathfrak{g}_{\mathbf{C}}{}^\vee \tag{2.2}$$

of $\mathfrak{g}_{\mathbf{C}}$ and its dual $\mathfrak{g}_{\mathbf{C}}{}^\vee$. Thus for $\xi \in A^\bullet(\mathfrak{g}_{\mathbf{C}})$ the corresponding dual element $\xi^\vee$ is characterized by

$$\xi^\vee(\phi) = h(\phi, \xi^\dagger), \quad \phi \in A^\bullet(\mathfrak{g}_{\mathbf{C}}). \tag{2.3}$$

2.4. Lemma. *For $\xi_i \in A^{p_i}(\mathfrak{g}_{\mathbf{C}})$ and $\xi \in A^p(\mathfrak{g}_{\mathbf{C}})$,*

(1) $[\xi_1^\dagger, \xi_2^\dagger] = -[\xi_1, \xi_2]^\dagger$

(2) $\langle \xi_1^\dagger, \xi_2^\dagger \rangle = \langle \xi_2, \xi_1 \rangle = \overline{\langle \xi_1, \xi_2 \rangle}$

(3) $h(\xi_1^\dagger, \xi_2^\dagger) = (-1)^{p_1 p_2} h(\xi_2, \xi_1) = \overline{h(\xi_1, \xi_2)}$

(4) $h([\xi_1, \xi_2], \xi_3) = (-1)^{p_1 p_2} h(\xi_2, [\xi_1^\dagger, \xi_3]) = -h(\xi_1, [\xi_2^\dagger, \xi_3])$

(5) $ad(\xi)^* = (-1)^{q(p+1)} \star ad(\xi^\dagger) \star$ *on* $A^q(\mathfrak{g}_{\mathbf{C}})$.

Now we assume that $\mathfrak{g}_{\mathbf{C}}$ has a *holomorphic structure*

$$\overline{\partial} : A^{0,0}(\mathfrak{g}_{\mathbf{C}}) \to A^{0,1}(\mathfrak{g}_{\mathbf{C}}), \quad \overline{\partial}^2 = 0 \tag{2.5}$$

such that the Lie algebra structure on each fiber varies holomorphically. In other words,

$$\overline{\partial}[\xi_1, \xi_2] = [\overline{\partial}\xi_1, \xi_2] + (-1)^{p_1}[\xi_1, \overline{\partial}\xi_2] \tag{2.6}$$

for $\xi_i \in A^{p_i}(\mathfrak{g}_{\mathbf{C}})$.

2.7. PROPOSITION. *Let $D = \partial + \bar{\partial}$ be the connection on $\mathfrak{g}_{\mathbf{C}}$ compatible with h and the holomorphic structure. Then the isomorphism (2.2) is holomorphic if and only if*

$$\partial(\xi^{\dagger}) = (\bar{\partial}(\xi))^{\dagger}$$

for all $\xi \in A^{\bullet}(\mathfrak{g}_{\mathbf{C}})$. In this case,

$$\partial[\xi_1, \xi_2] = [\partial\xi_1, \xi_2] + (-1)^{p_1}[\xi_1, \partial\xi_2]$$

for any $\xi_i \in A^{p_i}(\mathfrak{g}_{\mathbf{C}})$ and hence D is a connection on the metrized Lie algebra bundle in the sense of (1.10).

PROOF: Note that (2.2) is holomorphic if and only if

$$\bar{\partial}(X^{\vee}) = (\bar{\partial}(X))^{\vee}, \quad \forall X \in A^0(\mathfrak{g}_{\mathbf{C}})$$

i.e.,

$$(\bar{\partial}(X^{\vee}))(Y) = (\bar{\partial}(X))^{\vee}(Y), \quad \forall X, Y \in A^0(\mathfrak{g}_{\mathbf{C}})$$

i.e.,

$$d''h(Y, X^{\dagger}) - h(\bar{\partial}Y, X^{\dagger}) = h(Y, (\bar{\partial}(X))^{\dagger})$$

i.e.,

$$h(Y, \partial(X^{\dagger})) = h(Y, \bar{\partial}(X)^{\dagger})$$

i.e.,

$$\partial(X^{\dagger}) = \bar{\partial}(X)^{\dagger}.$$

This shows the first assertion. Now

$$\begin{aligned} h(\partial[X_1, X_2], X_3) &= d'h([X_1, X_2], X_3) - h([X_1, X_2], \bar{\partial}X_3) \\ &= -d'h(X_1, [X_2^{\dagger}, X_3]) + h(X_1, [X_2, \bar{\partial}X_3]) \\ &= -h(\partial X_1, [X_2^{\dagger}, X_3]) - h(X_1, \bar{\partial}[X_2^{\dagger}, X_3] - [X_2, \bar{\partial}X_3]) \\ &= h([\partial X_1, X_2], X_3) - h(X_1, [\bar{\partial}(X_2^{\dagger}), X_3]) \\ &= h([\partial X_1, X_2] + [X_1, \bar{\partial}(X_2^{\dagger})^{\dagger}], X_3). \end{aligned}$$

This shows the second assertion. ∎

Note that each $B \in A^{0,1}(\mathfrak{g}_{\mathbf{C}})$ defines a *semi-connection* or the (0,1)-part of a connection (cf. [6])

$$\bar{\partial}_B := \bar{\partial} + ad(B) : A^{0,0}(\mathfrak{g}_{\mathbf{C}}) \to A^{0,1}(\mathfrak{g}_{\mathbf{C}}) \tag{2.8}$$

on $\mathfrak{g}_{\mathbf{C}}$ such that

$$\bar{\partial}_B[\xi_1, \xi_2] = [\bar{\partial}_B\xi_1, \xi_2] + (-1)^{p_1}[\xi_1, \bar{\partial}_B\xi_2]$$

for $\xi_i \in A^{p_i}(\mathfrak{g}_{\mathbf{C}})$. Put

$$F(B) = \overline{\partial}(B) + \frac{1}{2}[B, B] \in A^{0,2}(\mathfrak{g}_{\mathbf{C}}). \tag{2.9}$$

Then

$$\overline{\partial}_B \circ \overline{\partial}_B = ad(F(B))$$

for any $B \in A^{0,1}(\mathfrak{g}_{\mathbf{C}})$ and $\overline{\partial}_B(F(B)) = 0$ is the Bianchi identity.

Note that semi-connections $\overline{\partial}_B$, in general, do not define a holomorphic structure and $F(B)$ is the obstruction.

2.10. DEFINITION. A semi-connection $\overline{\partial}_B$ is called a *Cauchy-Riemann operator* if $F(B) = 0$.

2.11. DEFINITION. The functional

$$\mathcal{Y} : A^{0,1}(\mathfrak{g}_{\mathbf{C}}) \to \mathbf{R}$$

defined by

$$\mathcal{Y}(B) = \frac{1}{2}\|F(B)\|^2 = \int_M \star|F(B)|^2$$

is called the *Hermitian-Yang-Mills* functional.

This Hermitian-Yang-Mills functional measures the integrability of semi-connections and the zero set (or the absolute minima) consists of Cauchy-Riemann operators (2.10). Now the first and second variational formulae are easily obtained as in the Yang-Mills theory [3].

2.12. PROPOSITION (THE FIRST VARIATIONAL FORMULA). *Let $B \in A^{0,1}(\mathfrak{g}_{\mathbf{C}})$ and let $\{B_t\}$ be a 1-parameter family of elements in $A^{0,1}(\mathfrak{g}_{\mathbf{C}})$ with $B_0 = B$ and $\frac{d}{dt}|_0 B_t = V \in A^{0,1}(\mathfrak{g}_{\mathbf{C}})$. Then*

$$\frac{d}{dt}|_0\mathcal{Y}(B_t) = Re(\overline{\partial}_B(V), F(B)).$$

2.13. COROLLARY. *$B \in A^{0,1}(\mathfrak{g}_{\mathbf{C}})$ is a critical point of $\mathcal{Y}$ if and only if $(\overline{\partial}_B)^* F(B) = 0$.*

2.14. COROLLARY. *If $\overline{\partial}_B : A^{0,1}(\mathfrak{g}_{\mathbf{C}}) \to A^{0,2}(\mathfrak{g}_{\mathbf{C}})$ is surjective for all $B \in A^{0,1}(\mathfrak{g}_{\mathbf{C}})$, then there are no critical points of $\mathcal{Y}$ other than the absolute minima.*

2.15. PROPOSITION (THE SECOND VARIATIONAL FORMULA). *Let $B \in A^{0,1}(\mathfrak{g}_{\mathbf{C}})$ be a critical point of $\mathcal{Y}$ and let $\{B_t\}$ be a 1-parameter family of elements in $A^{0,1}(\mathfrak{g}_{\mathbf{C}})$ with $B_0 = B$ and $\frac{d}{dt}|_0 B_t = V \in A^{0,1}(\mathfrak{g}_{\mathbf{C}})$. Then*

$$(\frac{d}{dt})^2|_0\mathcal{Y}(B_t) = Re([V, V], F(B)) + \|\overline{\partial}_B(V)\|^2.$$

2.16. REMARK. There is a natural identification $f : A^{0,1}(\mathfrak{g}_{\mathbf{C}}) \to A^1(\mathfrak{g})$ defined by $f(B) = B - B^\dagger$, where $A^1(\mathfrak{g})$ is the affine space of "unitary connections." Then the inner product on these spaces are related by $(f(B_1), f(B_2) = 2Re(B_1, B_2)$ for $B_i \in A^{0,1}(\mathfrak{g}_{\mathbf{C}})$.

3. Main theorems. A smooth vector bundle $E \to M$ is called (smoothly) *indecomposable* if it is not a direct sum of two proper subbundles. If E is indecomposable, then $\mathrm{rk}(E) \leq \dim(M)$. Recall that a unitary connection on E is said to be *irreducible* if the holonomy group acts irreducibly on each fiber.

3.1. LEMMA. *Let E be a smooth complex vector bundle over a connected manifold. Then the following conditions are equivalent.*

(1) *E is smoothly indecomposable.*

(2) *Every unitary connection on E is irreducible.*

(3) *For any unitary connection on E, every parallel endomorphism of E is a constant multiple of the identity endomorphism.*

3.2. VANISHING THEOREM. *Let E be a smooth indecomposable vector bundle over a connected complex manifold M and let D'' be a semi-connection on E. Then any endomorphism f of E such that $D''(f) := D'' \circ f - f \circ D'' = 0$ is a constant multiple of the identity endomorphism.*

PROOF: We fix any hermitian metric on E. Then there is a unique unitary connection D with D'' as its (0,1) part. Then for any $f \in A^0(\mathrm{End}\, E)$,

$$(D''f)^\dagger = D'(f^\dagger) = 0.$$

Thus if we put $f = f_1 + \sqrt{-1} f_2$, where f_i's are skew-hermitian endomorphism of (E, h), then f_i's are parallel and hence f is parallel. Thus by the lemma (3.1), f is constant. ∎

Now let $(\mathcal{E}, h)$ be a holomorphic hermitian vector bundle over a compact complex hermitian manifold M. We assume that the underlying smooth vector bundle $E = |\mathcal{E}|$ of $\mathcal{E}$ is smoothly indecomposable. Let $\mathfrak{g}_{\mathbf{C}}$ be the bundle of trace-free endomorphisms of $\mathcal{E}$. Note that the group $\mathrm{SL}(E)$ of smooth endomorphisms of E with determinant 1 acts on $A^{0,1}(\mathfrak{g}_{\mathbf{C}})$:

$$(g, B) \mapsto -\overline{\partial} g \cdot g^{-1} + g \circ B \circ g^{-1}.$$

We define the *Hermitian-Yang-Mills functional*

$$\mathcal{Y} : A^{0,1}(\mathfrak{g}_{\mathbf{C}}) \to \mathbb{R}$$

by $\mathcal{Y}(B) = \frac{1}{2} \| F(B) \|^2$ as in (2.11). Then $\mathcal{Y}$ is invariant under the subgroup $\mathrm{SU}(E)$ of $\mathrm{SL}(E)$. But the zero set of $\mathcal{Y}$ is invariant under the whole group $\mathrm{SL}(E)$.

3.3. THEOREM. *If $(\mathcal{E}, h)$ is a smoothly indecomposable holomorphic hermitian vector bundle over a hermitian complex surface M with the trivial canonical line bundle, then* *the critical points of the Hermitian-Yang-Mills functional $\mathcal{Y}$ are the Cauchy-Riemann operators.*

PROOF: It suffices to show that for any $B \in A^{0,1}(\mathfrak{g}_{\mathbf{C}})$,

$$(*) \qquad \overline{\partial}_B : A^{0,1}(\mathfrak{g}_{\mathbf{C}}) \to A^{0,2}(\mathfrak{g}_{\mathbf{C}})$$

is surjective (cf. (2.14)). But by the Serre duality (B.5), the cokernel of (*) is isomorphic to the kernel of

$$\overline{\partial}_B : A^{2,0}(\mathfrak{g}_{\mathbf{C}}) \to A^{2,1}(\mathfrak{g}_{\mathbf{C}}).$$

Since the canonical line bundle Ω^2 is trivial, we are done by the vanishing theorem (3.2). ∎

3.4. REMARK. The trace part of $\operatorname{End}\mathcal{E}$ is not important in the above theorem. Namely, if we consider $A^{0,1}(\operatorname{End}\mathcal{E})$ as a domain of $\mathcal{Y}$, then the same conclusion is true. If we identify $A^{0,1}(\operatorname{End}\mathcal{E}) \simeq A^1(u(E))$, where $u(E)$ is the real bundle of skew hermitian endomorphisms of $(\mathcal{E}, h)$, then

$$(3.5) \qquad \mathcal{Y}(A) = \frac{1}{4}\|R_A^+ + \frac{\sqrt{-1}}{2}\Phi K_A\|^2, \quad A \in A^1(u(E))$$

where $R_A = R + d_D(A) + \frac{1}{2}[A, A]$ is the curvature tensor of $D + A$ (D being the canonical connection on $(\mathcal{E}, h)$), R_A^+ is the *self-dual* part of R_A, $K_A = \sqrt{-1}\Lambda R_A$ is the *mean curvature tensor* [6], and Φ is the fundamental 2-form of M. Then the first variational formula becomes

$$d\mathcal{Y}(A)(v) = \frac{1}{4}(v, d_A^*(R_A + \sqrt{-1}\Phi K_A)), \quad v \in A^1(u(E))$$

where d_A^* is the adjoint of $d_A := d_D + A : A^1(u(E)) \to A^2(u(E))$.

APPENDIX : HODGE THEORY

A. Real case. Let (M, g) be a compact oriented Riemannian manifold of dimension m and let (E, h) be a smooth Riemannian vector bundle over M of rank r. Let D be a metric connection on E. The induced *exterior derivatives*

$$(A.1) \qquad d_D : A^p(E) \to A^{p+1}(E)$$

do not form a complex, unless E is flat, and the obstruction is the curvature R_D. The adjoint of the operator (A.1) is denoted by

$$\delta_D : A^{p+1} \to A^p(E).$$

Then

$$(A.2) \qquad \delta_D = -(-1)^{mp} \star d_D \star \qquad \text{on } A^{p+1}(E),$$

where $\star$ is the Hodge star. We put

$$\Delta_D = d_D \delta_D + \delta_D d_D,$$

which is a self-adjoint elliptic operator, and let

$$H^p_D(E) = Ker(\Delta_D | A^p(E)).$$

A.3. THEOREM (POINCARÉ DUALITY). $H^p_D(E) \simeq H^{m-p}_D(E)$.

PROOF: Immediate consequence of (A.2). ∎

A.4. THEOREM. *Let r be the rank of E and $e(M)$ be the Euler characteristic of M. Then*
(1) $h^p_D(E) := \dim H^p_D(E) < \infty$.
(2) $\sum_{p=0}^{m} (-1)^p h^p_D(E) = r \cdot e(M)$.

PROOF: (1) is standard. Note that the operator

$$d_D + \delta_D : A^\bullet(E) \to A^\bullet(E) \tag{A.5}$$

is a self-adjoint elliptic operator with the same kernel as Δ_D. Thus $\sum (-1)^p h^p$ is equal to the index of

$$d_D + \delta_D : A^{even}(E) \to A^{odd}(E),$$

and the theorem follows from the Atiyah-Singer index theorem [1]. ∎

The Riemannian structure h on E induces canonically an isomorphism

$$\flat : E \to E^\vee$$

onto the dual vector bundle $E^\vee$ of E, by *lowering indices*. This *musical isomorphism* induces an isomorphism

$$\flat : A^\bullet(E) \to A^\bullet(E^\vee)$$

of $A^\bullet$-modules. The Riemannian structure $h^\vee$ on $E^\vee$ is the one making $\flat$ an isometry.

The connection D on E induces a connection $D^\vee$ on $E^\vee$ and $D^\vee$ is also compatible with $h^\vee$. Thus we have

$$d_{D^\vee} : A^p(E^\vee) \to A^{p+1}(E^\vee)$$

and its adjoint

$$\delta_{D^\vee} : A^{p+1}(E^\vee) \to A^p(E^\vee).$$

A.6. THEOREM. *The musical isomorphism $\flat : A^\bullet(E) \to A^\bullet(E^\vee)$ commutes with $\star$, d and δ, i.e.,*
(1) $\flat \star (\xi) = \star \flat(\xi)$

(2) $\flat d_D(\xi) = d_{D^\vee}\flat(\xi)$

(3) $\flat\delta_D(\xi) = \delta_{D^\vee}\flat(\xi)$

for any $\xi \in A^p(E)$.

PROOF: (1) and (2) is easy. (3) follows from (1), (2) and (A.2). ∎

A.7. COROLLARY (POINCARÉ DUALITY). $H^p_D(E) \simeq H^p_{D^\vee}(E^\vee)$.

B. Complex case. Now let (M, g) be a compact complex hermitian manifold of dimension n. Thus $m = 2n$ is the real dimension of M. Let (E, h) be a smooth hermitian vector bundle over M and let $D = D' + D''$ be a unitary connection on E. Then

$$d_D = d'_D + d''_D$$

and

$$\delta_D = \delta'_D + \delta''_D.$$

Now from (A.2), we have

$$\delta'_D = - \star d''_D \star, \qquad \delta''_D = - \star d'_D \star . \tag{B.1}$$

We put

$$\Delta'_D = d'_D\delta'_D + \delta'_D d'_D, \qquad \Delta''_D = d''_D\delta''_D + \delta''_D d''_D.$$

They are self-adjoint elliptic operators. We put

$$H^{p,q}_{D'}(E) = Ker(\Delta'_D | A^{p,q}(E))$$

and

$$H^{p,q}_{D''}(E) = Ker(\Delta''_D | A^{p,q}(E)).$$

B.2. THEOREM (POINCARÉ DUALITY).

$$H^{p,q}_{D'}(E) \simeq H^{n-q,n-p}_{D''}(E)$$

PROOF: Obvious from (B.1). ∎

B.3. THEOREM. (1) $h^{p,q}_{D''}(E) := \dim_{\mathbf{C}} H^{p,q}_{D''}(E) < \infty$

(2) *For each* p, $\sum_{q=0}^{n}(-1)^q h^{p,q}_{D''}(E) = \int_M ch(\Omega^p \otimes E) \cdot todd(M)$.

PROOF: Similar to the proof of A.4. ∎

The hermitian structure h on E induces canonically a *conjugate* linear isomorphism

$$\flat : E \to E^\vee$$

onto the dual vector bundle $E^\vee$ of E and this induces a conjugate linear isomorphism

$$\flat : A^{p,q}(E) \to A^{q,p}(E^\vee).$$

B.4. THEOREM. *For $\xi \in A^{p,q}(E)$,*

(1) $\flat \star (\xi) = \star \flat(\xi)$

(2) $\flat d'_D(\xi) = d''_{D^\vee}\flat(\xi), \qquad \flat d''_D(\xi) = d'_{D^\vee}\flat(\xi)$

(3) $\flat \delta'_D(\xi) = \delta''_{D^\vee}\flat(\xi), \qquad \flat \delta''_D(\xi) = \delta'_{D^\vee}\flat(\xi)$

PROOF: This follows from theorem (A.6). ▮

B.5. COROLLARY (SERRE DUALITY). (1) *Let D'' be a semi-connection on E. Then for any hermitian structure on E, $H^{p,q}_{D''}(E)^\vee \simeq H^{n-p,n-q}_{D^\vee{}''}(E^\vee)$.*

(2) *Let Ω^p be the p-th exterior power of the holomorphic cotangent bundle of (M, g), equipped with the canonical connection compatible with the hermitian structure and the holomorphic structure, and let ∇ be the induced connection on $\Omega^p \otimes E$ from the one on Ω^p and D on E. Then $H^{p,q}_{D''}(E) \simeq H^{0,q}_{\nabla''}(\Omega^p \otimes E)$.*

PROOF: From theorem (B.4), we have a conjugate linear isomorphism

$$H^{p,q}_{D''}(E) \simeq H^{q,p}_{D^\vee{}'}(E^\vee).$$

By applying the Hodge star or the Poincaré duality (B.2), we get (1). (2) is more or less trivial. ▮

If we assume that (M, g) is Kähler, then

$$\sqrt{-1}[\Lambda, d''_D] = \delta'_D, \quad -\sqrt{-1}[\Lambda, d'_D] = \delta''_D \tag{B.6}$$

and hence

$$\Delta_D = \Delta'_D + \Delta''_D. \tag{B.7}$$

In particular, Δ_D preserves the bi-grade. The Laplacians Δ'_D and Δ''_D are not in general equal and their difference is an algebraic operator

$$\sqrt{-1}[\Lambda, R_D] = \Delta'_D - \Delta''_D, \tag{B.8}$$

where $R_D = d_D \circ d_D : A^{p,q}(E) \to A^{p+1,q+1}(E)$ is the curvature operator of D.

B.9. THEOREM (HODGE DECOMPOSITION). *Suppose $[\Lambda, R_D] = 0$ on $A^k(E)$. Then*

$$H^k_D(E) = \sum_{p+q=k} H^{p,q}_{D''}(E).$$

The proof is obvious and we also have Lefschetz decomposition as in the ordinary case. (cf. [9]).

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