ON THE DIMENSION OF ANISOTROPIC FORMS IN I^n

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1.

Let k be a field of characteristic not 2, $W(k)$ be *Witt ring* of quadratic forms, and $I \subset W(k)$ be an ideal of even-dimensional forms. Ideal I generates the filtration $W(k) \supset I \supset I^2 \supset \dots \supset I^n \supset \dots$, significance of which is illustrated, for example, by the fact that the associated graded ring $gr^*_{I^\cdot}(W(k))$ is naturally isomorphic to the Milnor's ring $\mathrm{K}^M_*(k)/2$ and to the Galois cohomology $\mathrm{H}^*_{et}(k, \mathbb{Z}/2)$of the field k (by the result of V.Voevodsky). An important problem is to describe the possible dimensions of anisotropic forms from I^n.

Basic here is the following:

Hauptsatz (Arason-Pfister).
If $q \in I^n$ is anisotropic quadratic form, then $\dim(q)$ is either 0, or $\geqslant 2^n$.

The natural question arises: do we still have some restrictions on $\dim(q)$, if $\dim(q) \geqslant 2^n$.

The following Conjecture is well-known (was formulated, for example, by B.Kahn - see [6], Conjecture 9)

Conjecture 1.1 .
Let $q \in I^n(k)$ be anisotropic quadratic form. Then $\dim(q)$ is either 0, or 2^n, or $\geqslant 2^n + 2^{n-1}$.

This conjecture is settled for $n = 3$ by a theorem of A.Pfister - see [12], Satz 14 and Zusatz, and for $n = 4$ by a result of D.Hoffmann - see [4], Main Theorem.

In the present paper we will prove this Conjecture for all n for fields of characteristic 0 - see the Main Theorem .

The main tool here is the following lemma describing the possible dimension of "binary" direct summands in the motives of quadrics.

Lemma 1.2 .
Let $char(k) = 0$. Let Q be anisotropic quadric. Suppose L is a direct summand in $M(Q)$, and $L|_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(i)[2i]$ for some i (in other words, L is "binary"). Then $i = 2^r - 1$ for some r.

The proof of Lemma 1.2 will be given in the next section. This proof is based on the technique of V.Voevodsky.

Conjecture 1.1 implies some other conjectures. For example, we get the description of the so-called *twisted Pfister forms* - see [3], Conjecture 3.9, Proposition 3.11(i).

The same methods permit us to find all possible *splitting patterns* of forms of *height* 2 - see Theorem 3.1 , and to prove some interesting result about the degree of the maximal pure divisor of two symbols in Milnor's K-theory (mod 2) - see Corollary 4.8 . Finally we formulate the analogue of Conjecture 1.1 , describing all possible dimensions of anisotropic forms from I^n - see Conjecture 4.11 .

Some notations and conventions. Except for the proof of Lemma 1.2 , we will be working in the category of *effective Chow motives* $Chow^{eff.}(k)$ (see the Appendix). For smooth projective variety R we will denote it's motive as $M(R)$. We will denote Tate-motives as $\mathbb{Z}(i)[2i]$, in particular, $M(\mathrm{Spec}(k))$ will be denoted simply as $\mathbb{Z}$. By this reason, we use notation $\mathbf{Z}$ for groups and rings $\mathbb{Z}$.

In Section 2 we will be working in the *triangulated category of mixed motives* $DM^{-}_{eff.}(k)$. We refer the reader to [18] and [19] for basic facts and notations.

As $J^n \subset W(k)$ we will denote the ideal of forms of *degree* $\geqslant n$ (see [9]). From Voevodsky's methods it follows that $I^n = J^n$, but we will not use this fact here.

Everywhere in the text we will assume that fields have characteristic not 2.

Acknoledgements. I would want to thank Oleg Izhboldin for the very fruitfull discussions. Also, I want to thank D.Hoffmann and B.Kahn for suggesting me the name for this article. This work was done during my stay at Max-Planck-Institut für Mathematik, and I would like to express my gratitude to this institution for the support, excellent working conditions and the very stimulating atmosphere.

The Main result.

Main Theorem .

Let k be a field, s.t. $char(k) = 0$. Let $q \in I^n(k)$ be anisotropic quadratic form. Then $\dim(q)$ is either 0, or 2^n, or $\geqslant 2^n + 2^{n-1}$.

Now, we can show, how to deduce Main Theorem from Lemma 1.2 .

Lemma 1.3 .

Let k be any field of characteristic not 2.

Suppose q be some quadratic form, s.t. $M(Q)$ contains $\mathbb{Z}(l)[2l]$ as a direct summand. Suppose $m = \min(l, \dim(Q) - l)$. Then q is $m+1$-times isotropic.

Proof

If $M(Q)$ contains $\mathbb{Z}(l)[2l]$ as a direct summand, then it also contains $\mathbb{Z}(\dim(Q)-l)[2\dim(Q)-2l]$ (if $pr \in \mathrm{CH}^{\dim(Q)}(Q\times Q)$ is the corresponding projector, then just consider the dual one: $pr^{\vee}$, obtained by switching of factors in $Q \times Q$). So, we can assume that $m = l \leqslant \dim(Q)/2$.

Let $M(Q)$ contains $\mathbb{Z}(l)[2l]$ as a direct summand.

So, we have maps: $pr : M(Q) \to \mathbb{Z}(l)[2l]$ and $s : \mathbb{Z}(l)[2l] \to M(Q)$, s.t. $pr \circ s = id_{\mathbb{Z}(l)[2l]}$. Via identification: $\mathrm{Hom}(M(Q), \mathbb{Z}(l)[2l]) = \mathrm{CH}^l(Q)$ and $\mathrm{Hom}(\mathbb{Z}(l)[2l], M(Q)) = \mathrm{CH}_l(Q)$, our maps pr and s correspond to cycles $A \in \mathrm{CH}^l(Q)$, and $B \in \mathrm{CH}_l(Q)$. Then the composition $pr \circ s \in \mathrm{Hom}(\mathbb{Z}(l)[2l], \mathbb{Z}(l)[2l]) = \mathrm{CH}_0(\mathrm{Spec}(k)) = \mathbf{Z}$ is given by the degree of intersection $A \cap B \in \mathrm{CH}_0(Q)$. So, we have that $degree(A \cap B) = 1$. This implies: if $l < \dim(Q)/2$, then $degree(B)$ is odd, and if $l = \dim(Q)/2$ then at least one of $degree(A), degree(B)$ is odd. Now, everything follows from:

Sublemma 1.3.1 .

Let $0 \leqslant l \leqslant \dim(Q)/2$, and Q has an l-dimensional cycle of odd degree. Then q is $l+1$-times isotropic.

Proof

If $l = 0$ then, by Springer's Theorem (see [10], VII, Theorem 2.3), we get a rational point on Q. So, q is isotropic.

Suppose we proved the statement for any quadratic form p, and for any $0 \leqslant a < l$.

By taking intersection of A with plane section of codimension l, we get a zero-cycle of odd degree on Q. So, q is isotropic: $q = \mathbb{H} \perp q'$, for some form q'.

Let x be any rational point on q, then Q' can be identified with projective quadric of lines on Q, passing through x. And the union of all lines on Q passing through x is the cone over Q' with vertex x, and it is just the intersection $R := Q \cap T_x$, where T_x is a tangent space to Q at x. We have natural projection $\varphi : R\backslash x \to Q'$.

We can always take x outside A (since the set of rational points on isotropic quadric is dense). Then $\varphi_*(A \cap T_x)$ will be an $l-1$-cycle of odd degree on Q'. By induction, q' is l-times isotropic. So, q is $l+1$-times isotropic.

$\square$

$\square$

Lemma 1.4 .

Let k be a field of characteristic 0.

Let $q, p \in J^n(k)$ be anisotropic quadrics, $height(q) = height(p) = 2$, and $\dim(q) = \dim(p) < 2^{n+1}$. Suppose $q|_{k(P)}$ and $p|_{k(Q)}$ are isotropic. Then for arbitrary field extension K/k, $q|_K$ is hyperbolic if and only if $p|_K$ is.

Proof By Theorem A.16 , $M(Q)$, $M(P)$ contain isomorphic indecomposable direct summands U and V, s.t. $\mathrm{CH}_0(U|_{\overline{k}}) \neq 0$.

By Theorem A.17 , $M(Q)$ also contains direct summands $U(f)[2f]$ for any $0 \leqslant f < i_1(q)$. Since $U, \ldots, U(i_1(q)-1)[2i_1(q)-2]$ are not isomorphic, by Theorem A.12 , $\oplus_{0 \leqslant f < i_1(q)} U(f)[2f]$ is isomorphic to a direct summand of $M(Q)$. Since $i_1(q) < \dim(Q)/2$ (notice, that $height(q) = 2 > 1$), we have: $\mathrm{CH}_f(U|_{\overline{k}}) = 0$, for any $0 < f < i_1(q)$ (otherwise, $rank(\mathrm{CH}_f(Q|_{\overline{k}}))$ would be $\geqslant 2$). Since $U(i_1(q)-1)[2i_1(q)-2]$ is a direct summand in $M(Q)$, we have: $\mathrm{CH}_l(U) = 0$, for any $\dim(Q) - i_1(q) + 1 < l \leqslant \dim(Q)$.

If there is no $i_1(q) \leqslant l \leqslant \dim(Q) - i_1(q)$, s.t. $\mathrm{CH}_l(U|_{\overline{k}}) \neq 0$, then $\mathrm{CH}_s(U|_{\overline{k}})$ can be nonzero only for $s = 0$ and $\dim(Q) - i_1(q) + 1$, and $rank(\mathrm{CH}(U|_{\overline{k}})) \leqslant 2$.

By Lemma 1.2 , we have: $\dim(Q) - i_1(q) + 1 = 2^r - 1$ for some r. This is impossible, since if $\dim(q) = 2^n + 2i$, where $0 < i < 2^{n-1}$, then $i_1(q) = i$ (we remind that q has $height = 2$, and $q \in J^n(k)$), so, $\dim(Q) - i_1(q) + 1 = \dim(q) - i_1(q) - 1 = 2^n + i - 1 \neq 2^r - 1$ for any r.

This shows, that there exists $i_1(q) \leqslant l \leqslant \dim(Q) - i_1(q)$, s.t. $\mathrm{CH}_l(U|_{\overline{k}}) \neq 0$. This means, that for any field extension K/k the following conditions are equivalent:

1) $U|_K$ is a direct sum of Tate-motives.

2) $q|_K$ is hyperbolic.

Really, it is evident that 2) implies 1) (since the motive of a hyperbolic quadric is a sum of Tate-motives). In the other direction: if $U|_K$ is a direct sum of Tate-motives, then in $M(Q|_K)$ we have a direct summand, isomorphic to $\mathbb{Z}(l)[2l]$, where $i_1(q) \leqslant l \leqslant \dim(Q) - i_1(q)$. Let $m = \min(l, \dim(Q) - l)$. Since in $M(Q|_K)$ we have a direct summand, isomorphic to $\mathbb{Z}(l)[2l]$, by Lemma 1.3 , we get: $q|_K$ is $m+1$-times isotropic. Since $m + 1 > i_1(q)$, and $height(q) = 2$, we get: $q|_K$ is hyperbolic.

Applying similar considerations to p, we get that for any field extension K/k the following conditions are equivalent:

1) $V|_K$ is a direct sum of Tate-motives.

2) $p|_K$ is hyperbolic.

Since U is isomorphic to V we get that $q|_K$ is hyperbolic if and only if $p|_K$ is.

□

Theorem 1.5 .

Let k be a field of characteristic 0.

Suppose $q \in J^n(k)$ be of $\dim(q) < 2^n + 2^{n-1}$. *Then* $\dim(q_{an}) \leqslant 2^n$.

Proof

Suppose, it is not the case. Let us choose counterexample of the smallest possible dimension (among all forms over all fields of characteristic 0). Abusing notations, we will denote the corresponding field as k, and corresponding form as q.

Since q is counterexample of the smallest possible dimension, we have: $height(q) = 2$.

Let $s \subset q$ be any subform of dimension 2^{n-1}. By [2], Main Lemma, there exists field extension F/k, and some n-fold Pfister form π, that $s|_F$ is a subform of π, and for any anisotropic form φ over k, the restriction $\varphi|_F$ is anisotropic. In particular, if $p := (q|_F \perp -\pi)_{an}$, then $\dim(p) \leqslant \dim(q)$, and $q|_F$ is anisotropic.

Notice, that $p \in J^n(F)$, and $\dim(p) \leqslant d$.

Let E/F be any field extension. Then if $\dim((p|_E)_{an}) < \dim(q) = d$, then by definition of d, $\dim((p|_E)_{an}) \leqslant 2^n$, i.e.: $(p|_E)_{an}$ is proportional to an n-fold Pfister form over E. Then $(q|_E)_{an}$ is proportional to a *generalized Albert form*, and by the result of R.Elman and T.Y.Lam (see [1], Theorem 4.5), $\dim((q|_E)_{an})$ is either $\geqslant 2^n + 2^{n-1} > d$, or $\leqslant 2^n$. That means, that if $\dim((p|_E)_{an}) < d$, then $q|_E$ is isotropic.

In particular, $\dim(p) = d$, $height(p) = 2$, and $q|_{F(p)}$ is isotropic.

Changing q by p in the considerations above, we get that if E/F be any field extension, and $q|_E$ is isotropic, then $p|_K$ is isotropic. In particular, $p|_{F(q)}$ is isotropic.

By Lemma 1.4 , for any field extension E/F, $q|_E$ is hyperbolic if and only if $p|_E$ is.

Over the field $E = F(q)$, the forms $(q|_E)_{an}$ and $(p|_E)_{an}$ are proportional to some n-fold Pfister forms β and γ. We have: $c \cdot \gamma = b \cdot \beta - \pi|_E$ in $W(E)$.

Since $\dim(q) > 2^n = \dim(\pi)$, we have: $\pi|_E$ is anisotropic.

Take $K = E(\beta)$. We have: $q|_K$ is hyperbolic, but $p|_K = c\gamma|_K$, and this form is nonhyperbolic, since $\gamma \neq \beta$.

We get a contradiction. Theorem 1.5 is proven.

□

Proof of the Main Theorem

Theorem 1.5 , evidently, implies the Main Theorem in view of Hauptsatz , and the fact that $I^n \subset J^n$ (see [9]).

□

2. Proof of Lemma 1.2

Lemma 1.2 was, actually, proven (but not formulated) in [15] - see the proof of Statement 6.1 there. We will reproduce it's proof here for the reader's convenience.

Lemma 2.1 .

Let k be arbitrary field of characteristic not 2, and q be quadratic form over k.

Suppose L be a direct summand in $M(Q)$, s.t. $L|_{\overline{k}} \simeq \mathbb{Z}$. Then $L \simeq \mathbb{Z}$.

Proof

If Q is anisotropic, then by Sublemma A.17.8 , $\mathrm{CH}_{\dim(Q)-i_1(q)+1}(L|_{\overline{k}}) \neq 0$. Since $\mathrm{CH}_r(\mathbb{Z}) = 0$, for $r > 0$, we get: Q is isotropic, i.e.: $q = \mathbb{H} \perp q'$. Then, by [13], Proposition 1 (Theorem A.0), $M(Q) = \mathbb{Z} \oplus M(Q')(1)[2] \oplus \mathbb{Z}(\dim(Q))[2\dim(Q)]$, and $L = L_0 \oplus L' \oplus L_{\dim(Q)}$, where L_i is a direct summand of $\mathbb{Z}(i)[2i]$, and L' is a direct summand of $M(Q')(1)[2]$. Clearly, $L'|_{\overline{k}} = L_{\dim(Q)}|_{\overline{k}} = 0$, and $L_0|_{\overline{k}} = \mathbb{Z}$. Hence, $L_0 = \mathbb{Z}$, on the other hand, by Rost Nilpotence Theorem ([14], Corollary 10)(see Theorem A.1), $L' = L_{\dim(Q)} = 0$. So, $L = \mathbb{Z}$.

Lemma 2.1 is proven.

□

Proof of Lemma 1.2

The proof is based on the construction very close to that used by V.Voevodsky in [19].

Suppose, we have some direct summand N in $M(Q)$, s.t. $N|_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(n)[2n]$.

By Sublemma A.17.8 , $n = \dim(Q) - i_1(q) + 1$.

Consider plane section $P \subset Q$ of codimension $i_1(q) - 1$. Since from the existense of $i_1(q) - 1$-dimensional projective subspace on $Q|_E$ follows the existense of rational point on $P|_E$, from the later follows the existence of rational point on $Q|_E$, and first and third are equivalent (since $i_1(q) - 1 < i_1(q)$), we have that $q|_{k(P)}$ and $p|_{k(Q)}$ are isotropic. Then by Theorem A.16 , $M(P)$ contains a direct summand isomorphic to N. But now $n = \dim(P)$.

$Chow^{eff.}(k)$ is a full additive subcategory (closed under taking of direct summands) in the triangulated category $DM^{-}_{eff.}(k)$ - see [17]. The category $DM^{-}_{eff.}(k)$ contains the "motives" of all smooth simplicial schemes over k. If P is smooth projective variety over k, we denote as $\mathcal{X}^{\cdot}_P$ the *standard simplicial scheme*, corresponding to the pair $P \to \mathrm{Spec}(k)$ - see [18], or [15], Definition 2.3.1. $\mathcal{X}^m_P := P \times \cdots \times P$ ($m+1$ times), with faces and degenerations maps given by partial projection and diagonals.

From the natural projection: $\mathcal{X}^{\cdot}_P \to \operatorname{Spec}(k)$, we get a map: $M(\mathcal{X}^{\cdot}_P) \stackrel{pr}{\to} \mathbb{Z}$. One of the main properties of $M(\mathcal{X}^{\cdot}_P)$ is: pr is an isomorphism if and only if P has a 0-cycle of degree 1 - see [15], Theorem 2.3.4. In particular, $M(\mathcal{X}^{\cdot}_P|_{\overline{k}}) = \mathbb{Z}$. From this point, we will denote $M(\mathcal{X}^{\cdot}_P)$ simply as $\mathcal{X}_P$ (since we will not use simplicial schemes themselves any more).

Repeating the proof of [19], Theorem 4.4, we get that in $DM^{-}_{eff.}(k)$, $N = \operatorname{Cone}[-1](\mathcal{X}_P \to \mathcal{X}_P(n)[2n+1])$ (actually, this statement is the very particular case of [15], Lemma 3.23).

$N := Cone[-1](\mathcal{X}_P \xrightarrow{\mu'} \mathcal{X}_P(n)[2n+1])$, where μ' is some (actually, the only) nontrivial (since, otherwise, $\mathcal{X}_P$ would be a direct summand in $M(P)$, and by Lemma 2.1 , and Lemma 1.3 , p would be isotropic) element from $\operatorname{Hom}(\mathcal{X}_P, \mathcal{X}_P(n)[2n+1])$.

Let $pr : \mathcal{X}_P \to \mathbb{Z}$ be natural projection. Then, by [19], Lemma 4.7, $pr_* : \operatorname{Hom}(\mathcal{X}_P, \mathcal{X}_P(a)[b]) \to \operatorname{Hom}(\mathcal{X}_P, \mathbb{Z}(a)[b])$ is an isomorphism for any a, b.

Denote: $\mu := pr_*(\mu') \in \operatorname{Hom}(\mathcal{X}_P, \mathbb{Z}(n)[2n+1])$.

Sublemma 2.2 .

The map: $(\mu')^* : \operatorname{Hom}(\mathcal{X}_P, \mathbb{Z}(c)[d]) \to \operatorname{Hom}(\mathcal{X}_P, \mathbb{Z}(c+n)[d+2n+1])$ *coinsides with multiplication by* $\mu \in \operatorname{Hom}(\mathcal{X}_P, \mathbb{Z}(n)[2n+1])$.

Proof It follows from the fact that $\Delta_{\mathcal{X}_P} : \mathcal{X}_P \to \mathcal{X}_P \otimes \mathcal{X}_P$ and $\pi_1 : \mathcal{X}_P \otimes \mathcal{X}_P \to \mathcal{X}_P$ are mutually inverse isomorphisms, $\mu \cdot u = \Delta_{\mathcal{X}_P}(\mu \otimes u)$,

$$\text{and} \quad \begin{array}{ccc} \mathcal{X}_P & \xrightarrow{\mu} & \mathbb{Z}(n)[2n+1] \\ \otimes & & \otimes \\ \mathcal{X}_P & \xrightarrow{u} & \mathbb{Z}(c)[d] \end{array} \quad \text{coincides with the composition:}$$

$$\begin{array}{ccccc} \mathcal{X}_P & \xrightarrow{\mu'} & \mathcal{X}_P(n)[2n+1] & \xrightarrow{pr(n)[2n+1]} & \mathbb{Z}(n)[2n+1] \\ \otimes & & \otimes & & \otimes \\ \mathcal{X}_P & \xrightarrow{id} & \mathcal{X}_P & \xrightarrow{u} & \mathbb{Z}(c)[d] \end{array}$$

which can be identified with the composition: $\mathcal{X}_P \stackrel{\mu'}{\to} \mathcal{X}_P(n)[2n+1] \stackrel{u}{\to} \mathbb{Z}(n+c)[2n+1+d]$, which is equal to $(\mu')^*(u)$.

$\square$

Sublemma 2.3 .

Multiplication by μ *performs an isomorphism:* $\operatorname{Hom}(\mathcal{X}_P, \mathbb{Z}(c)[d]) \to \operatorname{Hom}(\mathcal{X}_P, \mathbb{Z}(c+n)[d+2n+1])$ *for any* $d - c > 0$, *and is surjective for* $d = c$. *The same is true about cohomology with* $\mathbb{Z}/2$*-coefficients.*

Proof Since N is a direct summand in $M(P)$, and $\operatorname{Hom}(P, \mathbb{Z}(i)[j]) = 0$ for $j - i > n = \dim(P)$, we have: $\operatorname{Hom}(N, \mathbb{Z}(a)[b]) = 0$, for $b > 2a$, or $b-a>n$. Then the same is true about cohomology with $\mathbb{Z}/2$-coefficients.

Consider Hom's from the exact triangle $N \to \mathcal{X}_P \xrightarrow{\mu} \mathcal{X}_P(n)[2n+1] \to N[1]$ to $\mathbb{Z}(n+c)[2n+d+1]$. We have: $\mathrm{Hom}(N, \mathbb{Z}(n+c)[2n+d+1]) = 0$, if $d-c \geqslant 0$, and $\mathrm{Hom}(N, \mathbb{Z}(n+c)[2n+d]) = 0$, if $d-c>0$. This, combined with Sublemma 2.2 , implies the statement for $\mathbb{Z}$-coefficients. Considering Hom's to $\mathbb{Z}/2(n+c)[2n+d+1]$, we get the $\mathbb{Z}/2$-case.

$\square$

We can also consider $\tilde{\mathcal{X}}_P := \mathrm{Cone}[-1](\mathcal{X}_P \xrightarrow{pr} \mathbb{Z})$.

Sublemma 2.4 .
$\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(a)[b])$ is a 2-torsion group for $b>a$;
$\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(a)[b])$ embeds into $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(a)[b])$ for $b>a$.

Also, the natural map: $\tilde{\mathcal{X}}_P \xrightarrow{\delta} \mathcal{X}_P$ induces an isomorphism $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(a)[b]) \xrightarrow{=} \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(a)[b])$, for all $b>a$.

Proof For finite field extension E/k we have action of *transfers* on motivic cohomology. Transfer is a map

$$\mathrm{Tr}: \mathrm{Hom}(X|_E, \mathbb{Z}(a)[b]) = \mathrm{Hom}(X, M(E)(a)[b]) \to \mathrm{Hom}(X, \mathbb{Z}(a)[b]),$$

where the equality is duality, and the second map is induced by the projection $\pi: M(E) \to \mathbb{Z}$ (E/k-finite field extension); the main property of transfer is: $\mathrm{Tr} \circ j = \cdot[E:k]$, where

$$j: \mathrm{Hom}(X, \mathbb{Z}(a)[b]) \to \mathrm{Hom}(X, M(E)(a)[b]) = \mathrm{Hom}(X|_E, \mathbb{Z}(a)[b])$$

is induced by the map $j: \mathbb{Z} \to M(E)$ dual to π via duality.

Quadric P has a point E of degree 2, and over E, $\mathcal{X}_P$ becomes $\mathbb{Z}$ (see [19], Lemma 3.8, or [15], Theorem 2.3.4), so we have that $\mathrm{Hom}(\mathcal{X}_P|_E, \mathbb{Z}(a)[b]) = 0$ for $b-a>0$.

Now, considering the composition $\mathrm{Tr} \circ j = \cdot[E:k]$ we get that $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(a)[b])$ is a 2-torsion group for $b>a$.

In particular, this implies that the natural map $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(a)[b]) \to \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(a)[b])$ is an embedding, for $b>a$.

Since $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2(a)[b]) = 0$ for any $b>a$ (see [19], Corollary 2.3(1)), we also have that for $b>a$, $\delta^*: \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(a)[b]) \to \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(a)[b])$ is an isomorphism.

$\square$

We have the action of motivic cohomological operations Q_i on $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(*)[*'])$ and $\mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}(*)[*'])$ - see [19], Theorems 3.16 and 3.17. The differential Q_i acts without cohomology on $\mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(*)[*'])$ for any $i \leqslant [\log_2(n+1)]$ - see [19], Theorem 3.25+Lemma 4.11 (notice, that in the proof of the later no specific of Pfister case is used).

Denote $\eta := \mu(mod 2)$, i.e. the image of μ in cohomology with $\mathbb{Z}/2$ coefficients. From Sublemma 2.4 it follows that $\eta \neq 0$.

Denote $r = [\log_2(n)]$.

Sublemma 2.5 .

$Q_i(\eta) = 0$, for all $i \leqslant r$.

Proof Really, $Q_i(\eta) \in \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(n+2^i-1)[2n+2^{i+1}])$, and the later group is an extension of 2-cotorsion in $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(n+2^i-1)[2n+2^{i+1}])$, and 2-torsion in $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(n+2^i-1)[2n+2^{i+1}+1])$.

But, by Sublemma 2.3 , μ performs a surjections $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(2^i-1)[2^{i+1}-1]) \to \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(n+2^i-1)[2n+2^{i+1}])$, and $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(2^i-1)[2^{i+1}]) \to \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(n+2^i-1)[2n+2^{i+1}+1])$.

And the groups: $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(2^i-1)[2^{i+1}-1])$, $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(2^i-1)[2^{i+1}])$ are zero.

Really, from the exact triangle $N \to \mathcal{X}_P \to \mathcal{X}_P(n)[2n+1] \to N[1]$, we have exact sequence: $\mathrm{Hom}(N, \mathbb{Z}(2^i-1)[2^{i+1}-1]) \leftarrow \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(2^i-1)[2^{i+1}-1]) \leftarrow \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(2^i-1-n)[2^{i+1}-2n])$ The first group is zero since N is a *pure* motive (i.e.: a direct summand in the motive of some smooth projective variety), and (consequently) $\mathrm{Hom}(N, \mathbb{Z}(a)[b]) = 0$ for $b > 2a$ (see [19], Corollary 2.3(2)). The third group is zero, since $n > 2^i - 1$ (see [19], Corollary 2.2(1)). So, the second is zero as well. Analogously, in the case of $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(2^i-1)[2^{i+1}])$.

So, $Q_i(\eta) = 0$.

$\square$

Sublemma 2.6 .

Let $0 \leqslant j \leqslant r$.

Then Q_j is injective on $\mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(c)[d])$, if $d - c = n + 1 + 2^j$.

Proof

Let $\tilde{v} \in \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(c)[d])$, where $d - c = n + 1 + 2^j$. If $Q_j(\tilde{v}) = 0$, then $\tilde{v} = Q_j(\tilde{w})$, for some $\tilde{w} \in \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(c-2^j+1)[d-2^{j+1}+1])$ (since Q_j acts without cohomology on $\mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(*)[*']))$. Since $(d-2^{j+1}+1)-(c-2^j+1) = n+1 > 0$, we have that $\delta^* : \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(c-2^j+1)[d-2^{j+1}+1]) \to \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(c-2^j+1)[d-2^{j+1}+1])$ is an isomorphism, and there exists $w \in \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(c-2^j+1)[d-2^{j+1}+1])$, s.t. $\tilde{w} = \delta^*(w)$.

By Sublemma 2.3 , $w = \eta \cdot u$, for some $u \in \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(c-2^j+1-n)[d-2^{j+1}-2n])$. By [19], Theorem 3.17(2), $Q_j(\eta \cdot u) = Q_j(\eta) \cdot u + \eta \cdot Q_j(u) + \sum\{-1\}^{x_i}\varphi_i(\eta) \cdot \psi_i(u)$, where $x_i > 0$, and ϕ_i, ψ_i are cohomological operations of some bidegree $(*)[*']$, where $*' > 2* \geqslant 0$.

Notice, that $c-2^j+1-n = d-2^{j+1}-2n =: s$. But by [19], Proposition 2.7+Corollary 2.13(2)+Theorem 4.1, we have that $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2(a)[b]) \xrightarrow{pr^*}$

$\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(a)[b])$ is an isomorphism for $a \leqslant b$. So, $u = pr^*(u_0)$, where $u_0 \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2(s)[s]) = \mathrm{K}_s^M(k)/2$. We have: $Q_j(u_0) = 0$ and $\psi_i(u_0) = 0$ (since $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2(a)[b]) = 0$, for $b > a$).

But pr^* commutes with Q_j and ψ_i. So, $Q_j(u) = 0$ and $\psi_i(u) = 0$. That means: $Q_j(w) = Q_j(\eta \cdot u) = Q_j(\eta) \cdot u = 0$, by Sublemma 2.5 .

We get: $\tilde{v} = Q_j(\tilde{w}) = Q_j \circ \delta^*(w) = \delta^* \circ Q_j(w) = 0$. I.e., Q_j is injective on $\mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(c)[d])$.

$\square$

Denote: $\tilde{\eta} := \delta^*(\eta) \in \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(n)[2n+1])$. Since $\eta \neq 0$, we have: $\tilde{\eta} \neq 0$.

Sublemma 2.7 .

Let $0 \leqslant m < r$, *and* $\tilde{\eta} = Q_m \circ \cdots \circ Q_1 \circ Q_0(\tilde{\eta}_m)$, *for some* $\tilde{\eta}_m \in \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(n-2^{m+1}+m+2)[2n-2^{m+2}+m+4])$, *Then there exists* $\tilde{\eta}_{m+1}$, *s.t.* $\tilde{\eta}_m = Q_{m+1}(\tilde{\eta}_{m+1})$.

Proof Since Q_{m+1} acts without cohomology on $\mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(*)[*'])$, it is enough to show that $Q_{m+1}(\tilde{\eta}_m) = 0$.

Denote: $\tilde{v} := Q_{m+1}(\tilde{\eta}_m)$. We have: $\tilde{v} \in \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(n+m+1)[2n+m+3])$. Since Q_i commutes with Q_j (see [19], Theorem 3.17(1)), we have: $Q_m \circ Q_{m-1} \circ \cdots \circ Q_0(\tilde{v}) = Q_m \circ Q_{m-1} \circ \cdots \circ Q_0 \circ Q_{m+1}(\tilde{\eta}_m) = Q_{m+1} \circ Q_m \circ Q_{m-1} \circ \cdots \circ Q_0(\tilde{\eta}_m) = Q_{m+1}(\tilde{\eta}) = 0$, by Sublemma 2.5 .

But, for any $0 \leqslant t \leqslant m$, $Q_{t-1} \circ \cdots \circ Q_0(\tilde{v}) \in \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}(c)[d])$, where $d - c = n + 1 + 2^t$, and Q_t is injective on $\mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}(c)[d])$, by Sublemma 2.6 . So, from the equality: $Q_m \circ Q_{m-1} \circ \cdots \circ Q_0(\tilde{v}) = 0$, we get: $\tilde{v} = 0$.

$\square$

From Sublemma 2.7 it follows that $\tilde{\eta} = Q_r \circ \ldots Q_1 \circ Q_0(\tilde{\eta}_r)$. Denote: $\tilde{\gamma} := \tilde{\eta}_r$. We have: $\tilde{\gamma} \in \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(n-2^{r+1}+2+r)[2n-2^{r+2}+4+r])$.

But, $(2n - 2^{r+2} + 4 + r) - (n - 2^{r+1} + 2 + r) = n - 2^{r+1} + 2$, and $r = [\log_2(n)]$, so: $2^r \leqslant n < 2^{r+1}$. Since we know, that $\mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(a)[b]) = 0$, for $a \geqslant b$ ([19], Proposition 2.7+Corollary 2.13(2)+Theorem 4.1), and $\tilde{\eta} \neq 0$, the only possible choice for n is $n = 2^{r+1} - 1$.

Lemma 1.2 is proven.

$\square$

From the proof of Lemma 1.2 one can deduce more. Namely, since the group $\mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(n-2-2^{r+1}+2+r)[2n-2^{r+2}+4+r]) = \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(r+1)[r+2]) = \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(r+1)[r+2])$ can be identified with $\mathrm{Ker}(\mathrm{K}_{r+2}^M(k)/2 \to \mathrm{K}_{r+2}^M(k(P))/2)$ (by [18], Lemma 6.4 + [19], Theorem 4.1; or by [7], Theorem A.1), the nontrivial element $\tilde{\gamma} \in \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(r+1)[r+2])$ gives us nontrivial element $\alpha \in \mathrm{Ker}(\mathrm{K}_{r+2}^M(k)/2 \to \mathrm{K}_{r+2}^M(k(P))/2)$ (compare with [5], Theorem

3.1). It follows from the proof above, that this element α has the property that for arbitrary field extension E/k, $\alpha|_E$ is zero if and only if $\tilde{\gamma}|_E$ is zero, if and only if $\mu|_E$ is zero, if and only if P is isotropic. So, P is a *norm-variety* for $\alpha \in \mathrm{K}^M_{r+2}(k)/2$.

If α is a *pure symbol* $\{a_1, \ldots, a_{r+2}\}$, then it is easy to see that p must be proportional to a *neighbor* of the Pfister form $\langle\langle a_1, \ldots, a_{r+2}\rangle\rangle$ (simply because $\langle\langle a_1, \ldots, a_{r+2}\rangle\rangle|_{k(P)}$ is isotropic). Then our motive N must be isomorphic to a *Rost motive* (by [13], Proposition 4, and our Theorem A.9 (or [15], Lemma 3.21)).

Since it is quite difficult to imagine, that quadric can be a *norm-variety* for the element from $\mathrm{K}^M_{r+2}(k)/2$, which is not a pure symbol, the following conjecture is natural.

Conjecture 2.8 (compare with [5], Conjecture 3.2).

Let Q be anisotropic quadric, and N be a direct summand in $M(Q)$, s.t. $N|_{\overline{k}} = \mathbb{Z}(a)[2a] \oplus \mathbb{Z}(b)[2b]$. Then N is isomorphic to a (shifted) Rost motive $M_\alpha(c)[2c]$ for some pure symbol $\alpha \in \mathrm{K}^M_t(k)/2$, and $c = \min(a, b)$.

Moreover, if a or b is $< i_1(q)$ or $> \dim(Q) - i_1(q)$, then q is a neighbor of a Pfister form.

3. Splitting patterns of forms of height 2

Lemma 1.2 also permits to describe all possible dimensions of forms of *height* 2.

Theorem 3.1 .

Suppose k is a field of characteristic 0. Let q be anisotropic quadric over k, s.t. $height(q) = 2$.

Then the splitting pattern of q is:

1) for $q \in I^n \backslash I^{n+1}(k)$ for $n \geqslant 1$: either $(2^m - 2^n, 2^{n-1})$, for $m > n \geqslant 1$, or $(2^{n-1}, 2^{n-1})$, or $(2^{n-2}, 2^{n-1})$ (for $n \geqslant 2$),

2) for $q \notin I(k)$: $(2^m - 2^n + 1, 2^{n-1} - 1)$ for some $m \geqslant n > 1$.

Proof

Lemma 3.2 (compare with [15], Statement 6.2).

Let k be a field of characteristic not 2. Let q be anisotropic form over k, s.t. $height(q) = 2$ and $i_1(q)$ does not divide $i_2(q)$ (for example, $i_1(q) > i_2(q)$), then there exists an indecomposable direct summand N in $M(Q)$, s.t. $N|_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(\dim(Q) - i_1(q) + 1)[2\dim(Q) - 2i_1(q) + 2]$.

Proof

Let N be indecomposable direct summand in $M(Q)$, s.t. $a(N) = 0$ (see the definition of $a(N)$ before Theorem A.17). By Theorem A.17 , in $M(Q)$ we will have also direct summands $N(i)[2i]$ for any $0 \leqslant i <$

$i_1(q)$. Since $N(i)[2i]$ is not isomorphic to $N(j)[2j]$, if $i \neq j$, by Theorem A.12 , we get a direct summand isomorphic to $\oplus_{0 \leqslant i < i_1(q)} N(i)[2i]$.

If L is a complementary direct summand to $\oplus_{0 \leqslant i < i_1(q)} N(i)[2i]$ in $M(Q)$, then since $i_2(q)$ is not divisible by $i_1(q)$, we have that $i_1(q) \leqslant a(L) \leqslant \dim(Q)/2$. By Theorem A.17 , for any $i_1(q) \leqslant j \leqslant \dim(Q)/2$, $L(j - a(L))[2j - 2a(L)]$ will be a direct summand in $M(Q)$. Since $a(N(i)[2i]) = i \neq j = a(L(j - a(L))[2j - 2a(L)])$, for any $0 < i_1(q) \leqslant j \leqslant \dim(Q)/2$, we have: $L(j - a(L))[2j - 2a(L)]$ and $N(i)[2i]$ are not isomorphic. By Theorem A.12 , $(\oplus_{i_1(q) \leqslant j \leqslant \dim(Q)/2} L(j - a(L))[2j - 2a(L)]) \oplus (\oplus_{0 \leqslant i < i_1(q)} N(i)[2i])$ is isomorphic to a direct summand of $M(Q)$. This implies (using Corollary A.17.12): $\mathrm{CH}_r(N|_{\overline{k}}) = 0$, for any $r \neq 0$, or $\dim(Q) - i_1(q) + 1$, i.e.: $N|_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(\dim(Q) - i_1(q) + 1)[2\dim(Q) - 2i_1(q) + 2]$.

□

From Lemma 3.2 and Lemma 1.2 it follows, that if q is of *height* $= 2$, then either $\dim(Q) - i_1(q) + 1 = 2^r - 1$, or $i_1(q)$ divides $i_2(q)$.

Let us consider 2 cases: 1) $degree(q) = n$ for $n \geqslant 1$; 2) $q \notin I(k)$.

1) If $degree(q) = n \geqslant 1$, then $i_2(q) = 2^{n-1}$. So, either $i_1(q) = 2^l$ for $0 \leqslant l \leqslant n - 1$, or $\dim(q) - i_1(q) - 1 = 2^r - 1$.

In the former case, by Theorem 1.5 , we have: $l = n - 1$ or $n - 2$ (if $n \geqslant 2$), and we get splitting patterns $(2^{n-1}, 2^{n-1})$ and $(2^{n-2}, 2^{n-1})$.

In the later case, $\dim(q) - i_1(q) = 2^r$, and since $\dim(q) = 2(i_1(q) + i_2(q)) = 2i_1(q) + 2^n$, we have: $i_1(q) = 2^r - 2^n$ for some $r > n$. We get the splitting pattern $(2^r - 2^n, 2^{n-1})$.

2) In this case, $i_2(q) = 2^{n-1} - 1$ for some $n \geqslant 1$ (since if odd-dimensional quadratic form is "hyperbolic" over generic point of the corresponding quadric, then it is proportional to a hyperplane section in some Pfister form).

Let $p := q \perp -\det(q)$ (here under $\det(\langle a_1, \ldots, a_s \rangle)$ I mean $(-1)^{[\dim(q)/2]} \cdot \prod_{i=1}^{s} a_i$). Then $\det(p) = 1$, so: $p \in I^2(k)$.

Let E/k be some field extension. Consider the following conditions: a) $\dim((p|_E)_{an}) < 2^n - 2$; b) $\dim((q|_E)_{an}) < 2^n - 1$; c) $\dim((q|_E)_{an}) = 1$; d) $\dim((p|_E)_{an}) = 0$.

Evidently, *a*) $\Rightarrow$ *b*), and *d*) $\Rightarrow$ *a*). Since $i_2(q) = 2^{n-1} - 1$, we get: *b*) $\Rightarrow$ *c*). Finally, since $p \in I^2(k)$ (and, consequently, the last *higher Witt index* of p is at least 2), we have: *c*) $\Rightarrow$ *d*). So, all the above conditions are equivalent. That means: the last *higher Witt index* of p is at least $2^{n-1} - 1$. Since such index is always a power of 2, we have $p \in J^n(k)$ (in the case $n = 2$ use the fact that $p \in I^2(q)$).

On the other hand, $p \notin J^{n+1}(k)$, since then over any field extension E/k we would have: $\dim((q|_E)_{an})$ is either $\geqslant 2^{n+1} - 1$, or $= 1$, which is not the case.

Then, by Theorem 1.5 , $\dim(p_{an})$ is either A) $\geqslant 2^{n+1}$, or B) $2^{n+1} > \dim(p_{an}) \geqslant 2^n + 2^{n-1}$, or C) $= 2^n$.

In the case A): $\dim(q) \geqslant 2^{n+1} - 1$, and $i_1(q) \geqslant 2^{n-1} > 2^{n-1} - 1 = i_2(q)$. So, by Lemma 3.2 and Lemma 1.2 , $\dim(q) - i_1(q) - 1 = 2^m - 1$, for some $m > n$. Since $\dim(q) = 2(i_1(q) + i_2(q)) + 1 = 2i_1(q) + 2^n - 1$, we get: $i_1(q) = 2^m - 2^n + 1$, and the splitting pattern of q is $(2^m - 2^n + 1, 2^{n-1} - 1)$.

In the case B): we have $n \geqslant 2$, and, over some field extension E/k, $height(p|_E) = 2$, then, by 1), $\dim((p|_E)_{an}) = 2^n + 2^{n-1}$. Since $\dim((q|_E)_{an}) > 2^n + 2^{n-1} - 1 > 2^n - 1$, we have that $height(q|_E) = 2$, i.e.: $q|_E$ is anisotropic. So, $\dim(q)$ is either $2^n + 2^{n-1} - 1$, or $2^n + 2^{n-1} + 1$, and $i_1(q)$ is either 2^{n-2}, or $2^{n-2} + 1$. We have two possibilities: I) $n > 2$; II) $n = 2$. I) In this case, $i_1(q)$ does not divide $i_2(q) = 2^{n-1} - 1$ So, by Lemma 3.2 and Lemma 1.2 , $\dim(q) - i_1(q) - 1 = 2^m - 1$ for some m, which leads to a contradiction. II) In this case, $\dim(p_{an}) = 6$, $i_2(q) = 1$. So, clearly, we must have: $i_1(q) = 1$ and $\dim(q) = 5$, since there are no $7-$*dimensional* forms with $i_1 = 2$ (one can use Lemma 3.2 and Lemma 1.2 here). We get the splitting pattern $(1, 1) = (2^2 - 2^2 + 1, 2^1 - 1)$.

In the case C): since $i_2(q) = 2^{n-1} - 1$ and $height(q) = 2$, we have $\dim(q) \geqslant 2^{n-1} + 1$. On the other hand, since $\dim(p_{an}) = 2^n$, we have: $\dim(q) \leqslant 2^{n-1} + 1$. So, q has splitting pattern: $(1, 2^{n-1} - 1) = (2^n - 2^n + 1, 2^{n-1} - 1)$.

Theorem 3.1 is proven.

□

4. Some other results

In this section we will give some other applications of Lemma 1.2 .

Theorem 4.1 .

Let k be a field of $char(k) = 0$. Let $q \in J^n(k)$ be anisotropic, and $\dim(q) < 2^{n+1}$.

Let N be indecomposable direct summand in $M(Q)$, s.t. $a(N) = 0$. Then $M(Q) = \oplus_{0 \leqslant i < i_1(q)} N(i)[2i]$.

Proof

We use induction on the $height(q)$. If $height(q) = 1$, then q is proportional to a Pfister form $\langle\langle \alpha \rangle\rangle$, and, by [13], Proposition 4, $M(Q) = \oplus_{0 \leqslant i < i_1(q)} M_\alpha(i)[2i]$, where M_α is a *Rost motive* (which is indecomposable)(instead, one can use Theorem A.17 , Theorem A.12 , and Sublemma A.17.8). Suppose now, $r \geqslant 2$, and the statement is true for all forms of $height \leqslant r - 1$ (over all fields). Let $q \in J^n(k)$ be anisotropic form of $height(q) = r$, and $\dim(q) < 2^{n+1}$. Then $p := (q|_{k(Q)})_{an}$ is of $height = r - 1$, so, $M(P) = \oplus_{0 \leqslant j < i_1(p)} L(j)[2j]$, where L is indecomoposable direct summand in $M(P)$, s.t. $a(L) = 0$.

Let N be indecomposable direct summand in $M(Q)$, s.t. $a(N) = 0$ (such summand is defined up to isomorphism, by Theorem A.9). Then, by Theorem A.17 , $N(i)[2i]$ will be a direct summand in $M(Q)$ for all $0 \leqslant i < i_1(q)$. By Theorem A.12 , $\oplus_{0\leqslant i<i_1(q)}N(i)[2i]$ will be a direct summand in $M(Q)$. Let M be a complimentary summand (i.e.: $M(Q) = M \oplus (\oplus_{0\leqslant i<i_1(q)}N(i)[2i])$). Then $a(M) \geqslant i_1(q)$. Then, by Corollary A.17.12 , $b(M) \leqslant \dim(Q) - i_1(q)$.

Suppose, $M \neq 0$.

By [13], Proposition 1 (Theorem A.0), $M(Q|_{k(Q)}) = \oplus_{0\leqslant i<i_1(q)}(\mathbb{Z}(i)[2i]\oplus \mathbb{Z}(\dim(Q)-i)[2\dim(Q)-2i])\oplus M(P)(i_1(q))[2i_1(q)]$. Since $a(L) \geqslant i_1(q)$, and $b(L) \leqslant \dim(Q) - i_1(q)$, we have that $M|_{k(Q)}$ is isomorphic a direct summand in $M(P)(i_1(q))[2i_1(q)]$. So, $M|_{k(Q)} = \oplus_{j\in\Lambda}L(i_1(q) + j)[2i_1(q) + 2j]$, where $\Lambda \subset [0, \dots, i_1(p) - 1]$ is some subset.

Let U be indecomposable direct summand in M, s.t. $a(U) = a(M)$. Clearly, $i_1(q) \leqslant a(U) < i_1(q) + i_1(p)$ (since $a(U) = i_1(q) + \min(j \in \Lambda)$). By Theorem A.17 , $U(i_1(q) + j - a(U))[2i_1(q) + 2j - 2a(U)]$ is a direct summand of $M(Q)$, for any $0 \leqslant j < i_1(p)$. Since $L(i_1(q))[2i_1(q)]$ is indecomposable, and $a(L(i_1(q))[2i_1(q)]) = i_1(q) = a(U(i_1(q)-a(L))[2i_1(q)-2a(L)]|_{k(Q)})$, we have: $L(i_1(q))[2i_1(q)]$ is a direct summand in $U(i_1(q) - a(L))[2i_1(q)-2a(L)]|_{k(Q)}$ (by Theorem A.9). Then $M(P)(i_1(q))[2i_1(q)] = \oplus_{0\leqslant j<i_1(p)}L(i_1(q) + j)[2i_1(q) + 2j]$ is isomorphic to a direct summand of $\oplus_{0\leqslant j<i_1(p)}U(i_1(q) + j - a(U))[2i_1(q) + 2j - 2a(U)]|_{k(Q)}$.

Since $a(N(i)[2i]) = i \neq l = a(U(l - a(U))[l - 2a(U)])$, for any $0 \leqslant i < i_1(q) \leqslant l < i_1(q) + i_1(p)$, by Theorem A.12 and above computations, $(\oplus_{0\leqslant i<i_1(q)}N(i)[2i])\oplus M(P)(i_1(q))[2i_1(q)]$ is isomorphic to a direct summand of $M(Q)$ (actually, to $M(Q)$ itself). Then $\mathrm{CH}_r(N|_{\overline{k}}) \neq 0$ only for $r = 0$ and $r = \dim(Q)-i_1(q)+1$. Hence, $N|_{\overline{k}} = \mathbb{Z}\oplus\mathbb{Z}(\dim(Q)-i_1(q) + 1)[2\dim(Q) - 2i_1(q) + 2]$.

By Lemma 1.2 , $\dim(Q) - i_1(q) + 1 = 2^t - 1$, for some t. This is impossible, since $q \in J^n$, and $\dim(q) < 2^{n+1}$. Contradiction. So, $M = 0$, and Theorem 4.1 is proven.

□

Definition 4.2 .

Let N be direct summand of $M(Q)$. We define Witt index $i_W(N)$ of N as the number of Tate motives in the decomposition of N into indecomposables, divided by 2.

Let $j_0 < j_1 < \cdots < j_{h(N)}$ be all possible values of $i_W(N|_F)$, for all field extensions F/k. We define the height of N as $h(N)$, and the splitting pattern of N as the sequence $(j_0, j_1 - j_0, \dots, j_{h(N)} - j_{h(N)-1})$.

Theorem 4.3 .

Let Q be anisotropic quadric, and N a direct summand of $M(Q)$. Then the splitting pattern of N consists of integers.

Proof

Let F be some field, and $\Lambda(F,q) := \{r|\,\text{s.t. either}\, 0 \leqslant r < i_W(q|_F),$ or $\dim(Q) - i_W(q|_F) < r \leqslant \dim(Q)\}$. Lemma 1.3 and [13], Proposition 1 (Theorem A.0), implies that $i_W(N|_F) = rank(\oplus_{r\in\Lambda(F,q)}\mathrm{CH}_r(N|_{\overline{k}}))$. This shows that to define *splitting pattern* of N, it is enough to consider F from the generalized splitting tower of M.Knebusch for q.

By induction, it is enough to show that $i_W(N|_{k(Q)})$ is integer. We can assume that N is indecomposable. In such a case, by Sublemma A.17.9 and Sublemma A.17.7 , either $i_W(N|_{k(Q)}) = 0$, or there exists subquadric $P \subset Q$ and direct summand L of $M(P)$, s.t. 1) $i_1(p) = 1$; 2) $p|_{k(Q)}$ and $q|_{k(P)}$ are isotropic; 3) $a(L) = 0$; 4) $N \simeq L(a(N))[2a(N)]$.

Clearly, $i_W(N|_{k(Q)}) = i_W(L|_{k(Q)}) = i_W(L|_{k(P)})$, and the later is equal to 1, by Sublemma A.17.5 (one should treat the evident case $\dim(P) = 0$ separately).

□

Lemma 1.3 and [13], Proposition 1 (Theorem A.0) imply the following:

Observation 4.4 .

If $N = M(Q)$, then $i_W(M(Q)) = i_W(q)$, $height(M(Q)) = height(q)$, and splitting pattern of $M(Q)$ coinsides with that for q.

□

Theorem 4.5 .

Let $char(k) = 0$, and anisotropic $q \in J^n(k)$ is of dimension $< 2^{n+1}$. Then splitting pattern of q is $(0, 2^{s_1}, 2^{s_2}, \ldots, 2^{s_h})$, where $0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_h = n-1$.

Proof

Sublemma 4.5.1 .

Suppose $char(k) = 0$, and anisotropic $q \in J^n(k)$ is of dimension $< 2^{n+1}$. Then $i_1(q) = 2^{s_1}$ divides $i_r(q)$ for $1 \leqslant r \leqslant h$.

Proof

By Theorem 4.1 , $M(Q) = \oplus_{0\leqslant i<i_1(q)} N(i)[2i]$, where N is indecomposable direct summand of $M(Q)$, s.t. $a(N) = 0$.

Clearly, if $(0, i_1(N), i_2(N), \ldots, i_h(N))$ is the *splitting pattern* of N, then the splitting pattern of $M(Q)$ will be $(0, i_1(q)\cdot i_1(N), i_1(q)\cdot i_2(N), \ldots, i_1(q)\cdot i_h(N))$. Since $i_h(q) = 2^{n-1}$, we have: $i_1(q) = 2^{s_1}$, and $i_r(q)$ is divisible by $i_1(q)$, for $1 \leqslant r \leqslant h$ (we use Theorem 4.3 here).

□

Since $i_{r+t}(q) = i_{1+t}((q|_{F_{r-1}})_{an})$, where F_{r-1} is $(r-1)$-st field in the *generalized splitting tower* of M.Knebusch for q, and $(q|_{F_{r-1}})_{an}$ satisfy the conditions of our theorem, we have by Sublemma 4.5.1 : $i_r(q) = 2^{s_r}$, and $i_r(q)$ divides $i_{r'}(q)$, for $r' \geqslant r$.

□

Main Theorem shows that in the sequence $0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_{h-1} \leqslant s_h = 2^{n-1}$ we nesessarily have: $s_{h-1} = s_h - 1$. This leads to the following natural:

Conjecture 4.6 .

Let $q \in J^n$ be anisotropic form of dimension $< 2^{n+1}$. Then the splitting pattern of q is $2^{n-height(q)} \cdot (0, 1, 2, 4, \ldots, 2^{height(q)-1})$

Theorem 4.5 permits to describe *splitting patterns* for (generalized) *Albert forms*:

Theorem 4.7 .

Let $char(k) = 0$. Let $\alpha, \beta \in \mathrm{K}_n^M(k)/2$ be pure symbols, and $A_{\alpha,\beta} := (\langle\langle\alpha\rangle\rangle - \langle\langle\beta\rangle\rangle)_{an}$ be generalized Albert form, corresponding to $\alpha + \beta$. Then the splitting pattern of $A_{\alpha,\beta}$ is $2^r(0, 1, 2, 4, \ldots, 2^{n-r-1})$, where r is the degree of common maximal pure divisor of α and β.

Proof

By the result of R.Elman and T.Y.Lam (see [1], Theorem 4.5), we have: $\dim((A_{\alpha,\beta}|_F|_{an}) = 2^{n+1} - 2^{r(F)+1}$, where $r(F)$ is the degree of common maximal pure divisor of $\alpha|_F$ and $\beta|_F$. By Theorem 4.5 , for all $1 \leqslant l \leqslant h$, there exists field F_l/k, s.t. $\dim((A_{\alpha,\beta}|_{F_l})_{an}) = 2 \cdot (2^{s_l} + \cdots + 2^{s_h})$. So, $2 \cdot (2^{s_l} + \cdots + 2^{s_h}) = 2^{n+1} - 2^{r(l)+1}$, for all $1 \leqslant l \leqslant h$, and some $r(l)$. It is not difficult to see that this is possible only if $(2^{s_1}, \ldots, 2^{s_h}) = 2^r(1, 2, 4, \ldots, 2^{n-r-1})$.

□

Corollary 4.8 .

Let $char(k) = 0$. Let $\alpha, \beta \in \mathrm{K}_n^M(k)/2$ be pure symbols, and r is the degree of common maximal pure divisor of α and β. Then for any $r \leqslant t \leqslant n$, there exists field extension E_t/k, s.t. the degree of common maximal pure divisor of $\alpha|_{E_t}$ and $\beta|_{E_t}$ is t.

Proof

Let $k = F_0 \subset F_1 \subset \cdots \subset F_h$ be *generalized splitting tower* of M.Knebusch for $A_{\alpha,\beta}$. Put $E_t := F_{t-r}$. Then, by Theorem 4.7 , $\dim((A_{\alpha,\beta}|_{E_t})_{an}) = 2^{n+1} - 2^{t+1}$. By [1], Theorem 4.5, the degree of common maximal pure divisor of $\alpha|_{E_t}$ and $\beta|_{E_t}$ is t.

□

Definition 4.9 .

Let $h \in \mathrm{K}_n^M(k)/2$. We define (algebraic) lenght of h as $length(h) := \min(s|\, h = \alpha_1 + \cdots + \alpha_s,\ \alpha_i - pure\ symbol)$.

Corollary 4.8 can be considered as a "derivative" of the following:

Question 4.10 .

Let $h \in \mathrm{K}_n^M(k)/2$, and $length(h) \geqslant s$. Is it true that there exists such field extension E/k, s.t. $length(h|_E) = s$?

By the result of V.Voevodsky, Question 4.10 has positive answer for $s = 1$, which implies Milnor's Conjecture on quadratic forms.

For the theory of quadratic forms it would be very useful to have positive answer for $s = 2$.

Remark

Corollary 4.8 can be considered as a "derivative" of Question 4.10 in the following sense. *length* gives increasing filtration on $\mathrm{K}_n^M(k)/2$. It is natural to call $h \in \mathrm{K}_n^M(k)/2$ infinitesimally close to 0, if $length(h) \leqslant 2$. In such neighborhood, we have more subtle filtration - by the *degree* of maximal pure divisor of $h = \alpha + \beta$ (which coinsides with the degree of maximal common pure divisor for α and β).

Finally, I will state the corrected version of Conjecture 1.1 .

Conjecture 4.11 .

Let $q \in I^n(k)$ be anisotropic. Then $\dim(q)$ is either $2^{n+1} - 2^{i+1}$, for some $0 \leqslant i \leqslant n$, or is even $\geqslant 2^{n+1}$.

Clearly, Conjecture 4.11 follows from Conjecture 4.6 . Also, it is not difficult to show:

Theorem 4.12 .

In the situation of Conjecture 4.11 , all the prescribed values of $\dim(q)$ are possible for suitable k and q.

Appendix

In this Appendix we will list some results on the motives of quadrics, we use in the main part of the paper. All results are from (or are slight modifications of results from) [15] and [16]. We reproduce proofs here for the reader's convenience.

All results in this section are valid under assumption: $char(k) \neq 2$, which we will assume everywhere.

We start by reminding some facts about $Chow^{eff.}(k)$ (see also [11]), and the structure of the motive of hyperbolic quadric.

An object of $Chow^{eff.}(k)$ is a pair (X, p_X), where $X = \sqcup_i X_i$, X_i is smooth connected projective variety over k, and $p_X \in \times_{i,j} \mathrm{CH}^{\dim(X_j)}(X_i \times X_j)$ is a proejctor: $p_X \circ p_X = p_X$, where the composition $\circ : \mathrm{CH}^{\dim(Z)}(Y \times$

$Z)\otimes\mathrm{CH}^{\dim(Y)}(X\times Y)\to\mathrm{CH}^{\dim(Z)}(X\times Z)$ is given by: $\pi_{X,Z_*}(\pi^*_{X,Y}(-)\cap\pi^*_{Y,Z}(-))$. One defines $\mathrm{Hom}_{Chow^{eff.}(k)}((X,p_X),(Y,P_Y))$ as $p_Y\circ\mathrm{CH}^{\dim(Y)}(X\times Y)\circ p_X\subset\mathrm{CH}^{\dim(Y)}(X\times Y)$ with the composition given by $\circ$. $Chow^{eff.}(k)$ is tensor additive category, where $(X,p_X)\oplus(Y,p_Y):=(X\sqcup Y,(p_X,p_Y))$, and $(X,p_X)\otimes(Y,p_Y):=(X\times Y,p_X\times p_Y)$.

If X is smooth projective variety over k, then one defines the *motive* $M(X)\in Ob(Chow^{eff.}(k))$ as the pair (X,Δ_X). We denote $M(\mathrm{Spec}(k))$ as $\mathbb{Z}$.

One observes that $M(\mathbb{P}^1)=(\mathbb{P}^1,\mathbb{P}^1\times\mathrm{Spec}(k))\oplus(\mathbb{P}^1,\mathrm{Spec}(k)\times\mathbb{P}^1)$. It is easy to see, that $(\mathbb{P}^1,\mathbb{P}^1\times\mathrm{Spec}(k))\simeq\mathbb{Z}$. One defines $\mathbb{Z}(1)[2]$ as $(\mathbb{P}^1,\mathrm{Spec}(k)\times\mathbb{P}^1)$. And $\mathbb{Z}(l)[2l]:=\mathbb{Z}(1)[2]^{\otimes l}$. If $U\in Ob(Chow^{eff.}(k))$, we denote: $U(l)[2l]:=U\otimes\mathbb{Z}(l)[2l]$. Also, we define: $\mathrm{CH}_r(U):=\mathrm{Hom}_{Chow^{eff.}(k)}(\mathbb{Z}(r)[2r],U)$. It is not difficult to show that $\mathrm{CH}_r(M(X)(l)[2l])$ can be naturally identified with $\mathrm{CH}_{r-l}(X)$.

More generally, let P, Q, and R be some smooth projective varieties over some field k of dimension l, m, and n, respectively. Then $\mathrm{Hom}_{Chow^{eff.}(k)}(M(P)(a)[2a],M(Q)(b)[2b])$ can be identified with $\mathrm{CH}^{m+b-a}(P\times Q)$. Under this identification, the composition:
$\mathrm{Hom}(M(P)(a)[2a],M(Q)(b)[2b])\times\mathrm{Hom}(M(Q)(b)[2b],M(R)(c)[2c])\to\mathrm{Hom}(M(P)(a)[2a],M(R)(c)[2c])$ corresponds to the map:
$(-,-):\mathrm{CH}^{m+b-a}(P\times Q)\times\mathrm{CH}^{n+c-b}(Q\times R)\to\mathrm{CH}^{n+c-a}(P\times R)$, $(\Phi,\Psi):=\pi_{1,3_*}(\pi_{1,2}{}^*(\Phi)\cap\pi_{2,3}{}^*(\Psi))$, where $\pi_{1,2}:P\times Q\times R\to P\times Q$, $\pi_{2,3}:P\times Q\times R\to Q\times R$, and $\pi_{1,3}:P\times Q\times R\to P\times R$ are natural projection.

In particular, taking $P=\mathrm{Spec}(k)$, we get pairing: $(-,-):\mathrm{CH}_a(Q)\times\mathrm{CH}^{n+c}(Q\times R)\to\mathrm{CH}_{a-c}(R)$. In this case, we will denote (v,Φ) as $\Phi(v)$.

Let now $char(k)\neq 2$, and Q be some quadric over k.

Choose some algebraic closure $\overline{k}\supset k$. Let $h^i\subset Q|_{\overline{k}}$ and $l_i\subset Q|_{\overline{k}}$, for $0\leqslant i<m/2$ be cycles, given by plane section of codimension i, and projective subspace of dimension i on $Q|_{\overline{k}}$, respectively. Let $l^1_{m/2},l^2_{m/2}\subset Q|_{\overline{k}}$ be projective subspaces of middle dimension from two different families. For $i=1,2$, let us define: $\overline{i}:=i$ if $m=4t$, and $\overline{i}:=3-i$, if $m=4t+2$.

Then in $\mathrm{CH}^m(Q|_{\overline{k}}\times Q|_{\overline{k}})$ we have an identity:
$\Delta_{Q|_{\overline{k}}}=\sum_{0\leqslant i<m/2}(h^i\times l_i+l_i\times h^i)+(l^{\overline{1}}_{m/2}\times l^1_{m/2}+l^{\overline{2}}_{m/2}\times l^2_{m/2})$.

Moreover, $h^i\times l_i$, and $l_i\times h^i$, for $0\leqslant i<m/2$, as well as $l^{\overline{1}}_{m/2}\times l^1_{m/2}$ and $l^{\overline{2}}_{m/2}\times l^2_{m/2}$ are mutually orthogonal idempotents.

This defines a decomposition: $M(Q|_{\overline{k}})=(\oplus_{i=0,\dots,m}\mathbb{Z}(i)[2i])\oplus\mathbb{Z}(m/2)[m]$. In particular, for $i\neq m/2$, $\mathrm{CH}_i(Q|_{\overline{k}})=\mathbf{Z}$, with generator l_i, if $i<m/2$,

and h^{m-i}, if $i > m/2$; $\mathrm{CH}_{m/2}(Q|_{\overline{k}}) = \mathbf{Z} \oplus \mathbf{Z}$, with generators $l^1_{m/2}$ and $l^2_{m/2}$.

We have an isomorphism: $\mathrm{End}(M(Q|_{\overline{k}})) \simeq \times_r \mathrm{End}(\mathrm{CH}_r(Q|_{\overline{k}}))$. The above choice of generators for $\mathrm{CH}_r(Q|_{\overline{k}})$ gives isomorphism $\mathrm{End}(\mathrm{CH}_r(Q|_{\overline{k}})) \simeq \mathbf{Z}$, for $r \neq m/2$, and $\mathrm{End}(\mathrm{CH}_{m/2}(Q|_{\overline{k}})) \simeq \mathrm{Mat}_{2\times 2}(\mathbf{Z})$. For $\psi \in \mathrm{End}(M(Q))$, we denote as $\psi_{(r)}$ the coordinate, corresponding to the action of $\psi|_{\overline{k}}$ on $\mathrm{CH}_r(Q|_{\overline{k}})$.

Under the isomorphism above, the element $\gamma \in \mathrm{CH}^m(Q|_{\overline{k}} \times Q|_{\overline{k}})$, where $\gamma = \sum_{0 \leqslant i < m/2}(\gamma_i \cdot h^i \times l_i + \gamma_{m-i} \cdot l_i \times h^i) + (\sum_{i=1,2;\, j=1,2} \gamma^{i,j}_{m/2} \cdot l^i_{m/2} \times l^j_{m/2})$ has coordinates: $\gamma_{(r)} = \gamma_r$, for $r \neq m/2$, and $\gamma^{i,j}_{(m/2)} = \gamma^{\overline{i},j}_{m/2}$.

Also, γ_i, resp. γ_{m-i}, is the degree of $\gamma \cap l_i \times h^i \in \mathrm{CH}_0(Q|_{\overline{k}} \times Q|_{\overline{k}})$, resp. of $\gamma \cap h^i \times l_i \in \mathrm{CH}_0(Q|_{\overline{k}} \times Q|_{\overline{k}})$, and $\gamma^{i,j}_{(m/2)}$ is the degree of $\gamma \cap l^i_{m/2} \times l^{\overline{j}}_{m/2} \in \mathrm{CH}_0(Q|_{\overline{k}} \times Q|_{\overline{k}})$.

If $q = l \cdot \mathbb{H} \perp q'$, then the motive of $M(Q)$ can be decomposed in the following way:

Theorem A.0 (M.Rost, [13], Proposition 1).
$M(Q) \simeq \oplus_{0 \leqslant i < l}(\mathbb{Z}(i)[2i] \oplus \mathbb{Z}(\dim(Q)-i)[2\dim(Q)-2i]) \oplus M(Q')(l)[2l]$.
$\mathrm{End}(M(Q)) = \times_{0 \leqslant i < l}(\mathrm{End}(\mathbb{Z}(i)[2i]) \times \mathrm{End}(\mathbb{Z}(\dim(Q)-i)[2\dim(Q)-2i])) \times \mathrm{End}(M(Q')(l)[2l])$.

Remark Actually, [13], Proposition 1 does not contain the second statement, but it follows from the fact that there are no Hom-s between different direct summands above.

If N is a direct summand of $M(Q)$, we will denote as $p_N \in \mathrm{End}(M(Q))$ the corresponding projector, and as $j_N : N \to M(Q)$, $\varphi_N : M(Q) \to N$ the natural maps, s.t. $\varphi_N \circ j_N = id_N$, and $j_N \circ \varphi_N = p_N$. We define: $\mathrm{CH}_r(N) := \mathrm{Hom}(\mathbb{Z}(r)[2r], N)$. The composition with j_N identifies this group with $p_N \cdot \mathrm{CH}_r(Q) \subset \mathrm{CH}_r(Q) = \mathrm{Hom}(\mathbb{Z}(r)[2r], M(Q))$. We choose generators e_w of $\mathrm{CH}_r(N|_{\overline{k}})$ in the following way: if $rank(\mathrm{CH}_r(N|_{\overline{k}})) = 1$, then e_w is arbitrary generator of this group, and if $rank(\mathrm{CH}_r(N|_{\overline{k}})) = 2$ (i.e.: $r = m/2$, and $\mathrm{CH}_r(N|_{\overline{k}}) = \mathrm{CH}_r(Q|_{\overline{k}})$), then we take $\varphi_N(l^1_{m/2})$ and $\varphi_N(l^2_{m/2})$ as generators (usually, we will denote $\varphi_N(l_i)$ and $\varphi_N(h^i)$ simply as l_i and h^i). We denote: $\mathrm{CH}(N) := \oplus_r \mathrm{CH}_r(N)$. We have: $\mathrm{CH}(N|_{\overline{k}}) = \oplus_{w \in \Omega(N)} e_w \cdot \mathbf{Z}$.

We define map $\overline{deg} : \mathrm{CH}(N|_{\overline{k}}) \to \mathbf{Z}/2$ by the rule: $\overline{deg}(e_w) := 1$.

We have a natural isomorphism $\mathrm{End}(N|_{\overline{k}}) \simeq \times_r \mathrm{End}(\mathrm{CH}_r(N|_{\overline{k}}))$. The above choice of generators for $\mathrm{CH}(N)$ identifies $\mathrm{End}(N|_{\overline{k}})$ with the product of matrix rings. For $\psi \in \mathrm{End}(N)$, we will denote the corresponding coordinates as $\psi_{(r)}$.

All the results we prove in this Appendix are, in some sense, corollaries of the following result (Rost Nilpotence Theorem):

Theorem A.1 (M.Rost, [14], Corollary 10).

1) Let $\varphi \in \mathrm{Hom}(M(Q), M(Q))$ be such that $\varphi|_{\overline{k}} = 0$. Then $\varphi^n = 0$ for some n.

2) Let $\varphi \in \mathrm{Hom}(M(Q), M(Q))$ be such that $\varphi|_{\overline{k}}$ is an isomorphism. Then φ is an isomorphism.

Remark In the case $char(k) = 0$, the nilpotence exponent n can be bounded by $\dim(Q) + 1$ - see [15], Lemma 3.10.

Corollary A.2 ([15], Lemma 3.12).

Let $\xi \in \mathrm{Hom}(M(Q), M(Q))$ be some map, s.t. $\xi|_{\overline{k}}$ is a projector. Then, for some d, ξ^{2^d} is a projector.

Proof

Let $x := \xi^2 - \xi \in \mathrm{Hom}(M(Q), M(Q)) = \mathrm{CH}^m(Q \times Q)$. Since $\xi|_{\overline{k}}$ is idempotent, $x|_{\overline{k}} = 0$. In particular, $2^s \cdot x = 0$, for some s, since Q is hyperbolic over some galois extension F/k of degree 2^s, and $\sum_{\sigma\in Gal(F/k)}(\sigma(x|_{\overline{k}})) = [F : k] \cdot x$.

By Theorem A.1 , we have: $x^t = 0$ for some t.

That means, that for some large d: $\binom{2^d}{j} \cdot x^j = 0$ for any $j > 0$.

From the equality: $\xi^2 = (\xi + x)$ (and the fact that ξ and x commute), we get: $\xi^{2^{d+1}} = \sum_{0 \leqslant j \leqslant 2^d} \binom{2^d}{j} \xi^{2^d - j} \cdot x^j = \xi^{2^d}$.

So, ξ^{2^d} is a projector.

$\square$

Corollary A.3 .

Let N be a direct summand in $M(Q)$, and $\psi \in \mathrm{Hom}(N, N)$ be such that:

1) $\psi|_{\overline{k}} = 0$. Then $\psi^n = 0$ for some n.

2) $\psi|_{\overline{k}}$ is a projector. Then ψ^n is a projector for some n.

3) $\psi|_{\overline{k}}$ is an isomorphism. Then ψ is an isomorphism.

Proof

Let $M(Q) = N \oplus M$. It is enough to consider $\varphi = \begin{pmatrix} \psi & 0 \\ 0 & \rho \end{pmatrix}$, where $\rho = 0$ in cases 1) and 2), and $= id_M$ in case 3). In case 1) apply Theorem A.1 (1), in case 2) - Corollary A.2 , in case 3) - Theorem A.1 (2).

$\square$

Lemma A.4 .

Let L and N be direct summands in $M(Q)$, s.t. $p_L|_{\overline{k}} \circ p_N|_{\overline{k}} = p_N|_{\overline{k}} \circ p_L|_{\overline{k}} = p_L|_{\overline{k}}$.

Then there exists direct summand $\tilde{L}$ in N, s.t. L is isomorphic to $\tilde{L}$, and $p_L|_{\overline{k}} = p_{\tilde{L}}|_{\overline{k}}$.

Proof

Let $j_L : L \to M(Q)$, $j_N : N \to M(Q)$, $\varphi_L : M(Q) \to L$, $\varphi_N : M(Q) \to N$ be s.t.: $\varphi_L \circ j_L = id_L$, $\varphi_N \circ j_N = id_N$, and $j_L \circ \varphi_L = p_L$, $j_N \circ \varphi_N = p_N$.

Take: $\alpha := \varphi_L \circ j_N : N \to L$, and $\beta := \varphi_N \circ j_L : L \to N$. If $\gamma := \alpha \circ \beta : L \to L$, then $\gamma|_{\overline{k}} = id_L$. By Corollary A.3 (2) and (1), $\gamma^s = id_L$, for some s. Consider $\psi := \varphi_N \circ p_L \circ j_N : N \to N$. Then ψ^s is a projector, $\psi^s = \beta \circ \tilde{\alpha}$, where $\tilde{\alpha} = \alpha \circ (\beta \circ \alpha)^{s-1}$, and $\tilde{\alpha} \circ \beta = id_L$. Then ψ^s defines direct summand $\tilde{L}$ in N, and for the corresponding projector in $M(Q)$, $p_{\tilde{L}} := j_N \circ \psi^s \circ \varphi_N$, we have: $p_{\tilde{L}}|_{\overline{k}} = p_L|_{\overline{k}}$.

□

Lemma A.5 (cf. [15], Lemma 3.13).

Let N be a direct summand in $M(Q)$, $\dim(Q) = m$.

1) Then there exists: $\kappa_{r,N} \in \mathrm{End}(N)$, s.t. $(\kappa_{r,N})_{(s)} = 0$, for any $s \neq r$, and $(\kappa_{r,N})_{(r)} = 2 \cdot id_{\mathrm{CH}_r(N|_{\overline{k}})}$.

2) If $rank(\mathrm{CH}_{m/2}(N|_{\overline{k}})) = 2$, then there exists: $\theta_{m/2,N} \in \mathrm{End}(N)$, s.t. $(\theta_{m/2,N})_{(m/2)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; $(\theta_{m/2,N})_{(r)} = 0$, for any $r \neq m/2$.

Proof 1) Let $\dim(Q) = m$. If $r \neq m/2$, then we can take $\kappa_{r,N} := \varphi_N \circ (h^r \times h^{m-r}) \circ j_N$.

If $r = m/2$, we can take $\kappa_{r,N} := \varphi_N \circ (2 \cdot id_{M(Q)} - \sum_{0 \leqslant i < m/2} (h^i \times h^{m-i} + h^{m-i} \times h^i)) \circ j_N$.

2) Take $\theta_{m/2,N} := \varphi_N \circ (h^{m/2} \times h^{m/2}) \circ j_N$.

□

Lemma A.6 .

Let N_i is a direct summand of $M(Q_i)$. Let for some odd number η, and some $\psi \in \mathrm{Hom}(N_1|_{\overline{k}}, N_2|_{\overline{k}})$, we have: $\eta \cdot \psi \in image(\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(N_1|_{\overline{k}}, N_2|_{\overline{k}}))$. Then $\psi \in image(\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(N_1|_{\overline{k}}, N_2|_{\overline{k}}))$.

Proof

Let F/k be Galois extension of degree 2^n, s.t. $N_i|_F$ is a sum of Tate-motives. We have: $\mathrm{Hom}(N_1|_F, N_2|_F) \to \mathrm{Hom}(N_1|_{\overline{k}}, N_2|_{\overline{k}})$ is an isomorphism. Let ψ_F be corresponding element of $\mathrm{Hom}(N_1|_F, N_2|_F)$.

We have: $\sigma(\psi_F) = \psi_F$, for any $\sigma \in Gal(F/k)$. Really, if $(\sigma(\psi_F) - \psi_F) \neq 0$, then $\eta \cdot (\sigma(\psi_F) - \psi_F) \neq 0$ too (since $\mathrm{Hom}(N_1|_F, N_2|_F)$ has no torsion), which contradicts to the fact that $\eta \cdot \psi \in image(\mathrm{Hom}(N_1, N_2) \to$

$\mathrm{Hom}(N_1|_{\overline{k}}, N_2|_{\overline{k}}))$. Then

$$2^n \cdot \psi_F = \sum_{\sigma \in Gal(F/k)} \sigma(\psi_F) \in image(\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(N_1|_F, N_2|_F)).$$

Since $\eta \cdot \psi_F$, $2^n \cdot \psi_F \in image(\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(N_1|_F, N_2|_F))$, we have: $\psi_F \in image(\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(N_1|_F, N_2|_F))$, which implies: $\psi \in image(\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(N_1|_{\overline{k}}, N_2|_{\overline{k}}))$.

$\square$

Definition A.7 .

Let N and N' be indecomposable direct summands in $M(Q)$. We say that N' is a normal form of N, if N' is isomorphic to N, and either $m = \dim(Q)$ is odd, or m is even and $(p_{N'})_{(m/2)}$ is of one of the following types:

1) 0; *2)* $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$; *3)* $\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$; *4)* id.

Theorem A.8 (cf. [15], the proof of Lemma 3.21).

Each direct summand in $M(Q)$ has a normal form.

Proof

The case of odd-dimensional quadric is trivial. So, we can assume that $m := \dim(Q)$ is even.

$(p_N)_{(m/2)}$ is an idempotent. So, if $rank((p_N)_{(m/2)})$ is 0, or 2, we get cases 1) and 4), respectively.

So, we can assume that $(p_N)_{(m/2)}$ is a projector in $\mathrm{Mat}_{2\times 2}(\mathbf{Z})$ of $rank = 1$ (equivalently, $det((p_N)_{(m/2)}) = 0$ and $tr((p_N)_{(m/2)}) = 1$).

Sublemma A.8.1 .

Let N be indecomposable direct summand in $M(Q)$, s.t. $(p_N)_{(m/2)} \neq 0$. Let $\psi \in \mathrm{End}(M(Q))$ be such that $\psi|_{\overline{k}} \circ p_N|_{\overline{k}} = p_N|_{\overline{k}} \circ \psi|_{\overline{k}} = \psi|_{\overline{k}}$, and $tr(\psi_{(m/2)})$ is odd. Then $2 \cdot \mathrm{End}(M(Q|_{\overline{k}})) \subset image(\mathrm{End}(M(Q)) \to \mathrm{End}(M(Q|_{\overline{k}})))$.

Proof If q is hyperbolic, then the map $\mathrm{End}(M(Q)) \to \mathrm{End}(M(Q|_{\overline{k}})$ is an isomorphism. So, we can assume that q is not hyperbolic.

Let $\tau \in \mathrm{Hom}(M(Q), M(Q)) = \mathrm{CH}^m(Q \times Q)$ be the morphism given by the graph of "reflection" τ_x (with any (rational) center $x \in \mathbb{P}^{m+1} \supset Q$). Then $\tau^2 = id_{M(Q)}$. Also, $\tau_{(i)} = 1$, for any $i \neq m/2$, and $\tau_{(m/2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Let $\psi_{(m/2)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $degree(\psi(h^{m/2})) = a + b + c + d$. Since $h^{m/2}$ is defined over k, we have that if $a + b + c + d$ is odd, then on

Q there is $m/2$-dimensional cycle of odd degree, which, by Sublemma 1.3.1 , implies that q is hyperbolic.

So, we can assume that $a+b+c+d$ is even. Then in each pair (a,d), (b,c), one element is odd and another is even.

Changing ψ to $\psi - \sum_{i\neq m/2}[\psi_{(i)}/2]\cdot\kappa_{i,Q}$, we can assume that $\psi_{(i)}$ are either 0's or 1's, for all $i\neq m/2$. Notice, that this new ψ still satisfy the conditions of the sublemma.

We have two cases: A) a and b, or c and d are odd; B) a and c, or b and d are odd.

Let $\psi^{\vee}$ be the dual morphism. Denote: $\tilde{\psi}:=\psi^{\vee}$, if $m=4t+2$, and $:=\tau\circ\psi^{\vee}\circ\tau$, if $m=4t$. Then $\tilde{\psi}_{(m/2)}=\begin{pmatrix} d & b\\ c & a\end{pmatrix}$

A) Put $\varepsilon:=\psi\circ\tilde{\psi}-\kappa_{m/2,Q}\circ(ad\cdot id+ab\cdot\tau))$.

B) Put $\varepsilon:=\tilde{\psi}\circ\psi-\kappa_{m/2,Q}\circ(ad\cdot id+ac\cdot\tau))$.

It is easy to see, that: $\varepsilon_{(m/2)}=\begin{pmatrix} 0 & 0\\ 2(cd-ab) & 0\end{pmatrix}$, in case A), and $=\begin{pmatrix} 0 & 2(bd-ac)\\ 0 & 0\end{pmatrix}$, in case B).

Clearly, $\varepsilon_{(i)}=\psi_{(i)}\cdot{\psi^{\vee}}_{(i)}$, for any $i\neq m/2$. At the same time, ${\varepsilon^2}_{(m/2)}=0$.

Since $\varepsilon_{(i)}\in\{0,1\}$, $\varepsilon^2|_{\overline{k}}$ is a projector. By Corollary A.3 (2), ε^{2r} is a projector. Since $\varepsilon^{2r}|_{\overline{k}}=\varepsilon^2|_{\overline{k}}$, and $\varepsilon^2|_{\overline{k}}\circ p_N|_{\overline{k}}=p_N|_{\overline{k}}\circ\varepsilon^2|_{\overline{k}}=\varepsilon^2|_{\overline{k}}$, by Lemma A.4 , we get a direct summand $\tilde{L}$ in N, s.t. $p_{\tilde{L}}|_{\overline{k}}=\varepsilon^2|_{\overline{k}}$. Since ${\varepsilon^2}_{(m/2)}=0$, and $(p_N)_{(m/2)}\neq 0$, we have: $N\neq\tilde{L}$. Since N is indecomposable, we have: $\tilde{L}=0$. In particular, $\varepsilon^2|_{\overline{k}}=p_{\tilde{L}}|_{\overline{k}}=0$. This implies: $\varepsilon_{(i)}=0$, for any $i\neq m/2$.

Since $(cd-ab)$ and $(bd-ac)$ are odd in respective cases, we have, using ε, $\tau\circ\varepsilon$, $\varepsilon\circ\tau$, $\tau\circ\varepsilon\circ\tau$, and Lemma A.6 , that for any $u\in 2\cdot\mathrm{Mat}_{2\times 2}(\mathbf{Z})$, there exists $\varphi\in\mathrm{Hom}(M(Q),M(Q))$, s.t. $\varphi_{(i)}=0$, for any $i\neq m/2$, and $\varphi_{(m/2)}=u$.

Since we also have $\kappa_{i,Q}:=h^i\times h^{m-i}$ for $i\neq m/2$, we get the statement.

$\square$

Changing p_N by $\tau\circ p_N\circ\tau$, if nesessary (which does not change the isomorphism class of N), we can assume that in case A): a and b are odd, and in case B): b and d are odd. Since $(p_N)_{(m/2)}$ is an idempotent of $rank=1$, we have: $(p_N)_{(m/2)}=\gamma\alpha\gamma^{-1}$, where $\alpha,\gamma\in\mathrm{Mat}_{2\times 2}(\mathbf{Z})$, and α is of type (2) in case (A), and of type (3) in case (B). Then it is easy to see that $u:=(\gamma-id)$ is in $2\cdot\mathrm{Mat}_{2\times 2}(\mathbf{Z})$.

That means that $p_N|_{\overline{k}} = \overline{\varphi}\circ\pi\circ\overline{\varphi}^{-1}$, where $\pi_{(m/2)} = \alpha$ is of type (2) in case (A), and of type (3) in case (B), and: $(\overline{\varphi} - id) \in 2\cdot \mathrm{End}(M(Q|_{\overline{k}}))$.

Take $\psi := p_N$, then $tr(\psi_{(m/2)}) = 1$, and we can apply Sublemma A.8.1 . From Sublemma A.8.1 it follows that $\overline{\varphi}$ is defined over k by some morphism φ. Then φ is an isomorphism, by Theorem A.1 (2), since it is over $\overline{k}$. $\rho := \varphi^{-1}\circ p_N\circ\varphi$ is a projector (since p_N is), so, $\rho = p_{N'}$ for some N'. Clearly, φ defines an isomorphism between N and N', and $(p_{N'})_{(m/2)} = \alpha$ is of type (2), or (3).

$\square$

Theorem A.9 (cf. [15], Lemma 3.21).

Let N_1 and N_2 be nonisomorphic indecomposable direct summands in $M(Q)$. Let N_1' and N_2' be corresponding normal forms.

Then: $p_{N_1'}|_{\overline{k}}\circ p_{N_2'}|_{\overline{k}} = p_{N_2'}|_{\overline{k}}\circ p_{N_1'}|_{\overline{k}} = 0$.

Proof

Let $\varphi := p_{N_1'}\circ p_{N_2'}$.

Since N_a' is a *normal form*, we have: that $\varphi_{(m/2)}$ is a projector in $\mathrm{Mat}_{2\times 2}(\mathbf{Z})$, and $\varphi_{(m/2)}\cdot(p_{N_a'})_{(m/2)} = (p_{N_a'})_{(m/2)}\cdot\varphi_{(m/2)} = \varphi_{(m/2)}$.

That means, $\varphi|_{\overline{k}}$ is a projector, and $\varphi|_{\overline{k}}\circ p_{N_a'}|_{\overline{k}} = p_{N_a'}|_{\overline{k}}\circ\varphi|_{\overline{k}} = \varphi|_{\overline{k}}$.

By Corollary A.3 , φ^s is a projector for some s, and if $\varphi^s = p_L$, for some L, then by Lemma A.4 , L is isomorphic to a direct summand in N_a'. Since N_a is indecomposable, we have that either L is isomorphic to N_a', or $L = 0$. Since N_1' is not isomorphic to N_2', we have: $L = 0$. This implies: $p_{N_1'}|_{\overline{k}}\circ p_{N_2'}|_{\overline{k}} = \varphi|_{\overline{k}} = \varphi^s|_{\overline{k}} = 0$

In the same way, considering $\psi := p_{N_2'}\circ p_{N_1'}$, we get: $p_{N_2'}|_{\overline{k}}\circ p_{N_1'}|_{\overline{k}} = 0$.

$\square$

Lemma A.10 .

Let L and N be direct summands in $M(Q)$, s.t. $p_L|_{\overline{k}} = p_N|_{\overline{k}}$. Then L is isomorphic to N.

Proof

Consider $\alpha := \varphi_N\circ j_L$, and $\beta := \varphi_L\circ j_N$. Then $(\beta\circ\alpha)|_{\overline{k}} = id_{N|_{\overline{k}}}$ and $(\alpha\circ\beta)|_{\overline{k}} = id_{L|_{\overline{k}}}$. By Corollary A.3 (3), L is isomorphic to N.

$\square$

Lemma A.11 .

Let L_1, L_2 be direct summands in $M(Q)$, s.t. $p_{L_1}|_{\overline{k}}\circ p_{L_2}|_{\overline{k}} = p_{L_2}|_{\overline{k}}\circ p_{L_1}|_{\overline{k}} = 0$. Then there exists direct summand M in $M(Q)$, s.t. M is isomorphic to $L_1\oplus L_2$, and $p_M|_{\overline{k}} = p_{L_1}|_{\overline{k}} + p_{L_2}|_{\overline{k}}$.

Proof

Consider $\pi := p_{L_1} + p_{L_2}$. Then $\pi|_{\overline{k}}$ is a projector, and by Corollary A.3 , π^r is a projector for some r, i.e. there exists direct summand M of $M(Q)$, s.t. $\pi^r = p_M$.

By Lemma A.4 , there exists direct summand $\tilde{L_1}$ in M, s.t. $\tilde{L_1}$ is isomorphic to L_1, and $p_{\tilde{L_1}}|_{\overline{k}} = p_{L_1}|_{\overline{k}}$. Then, for complimentary projector $p_{\tilde{L_2}} := p_M - p_{L_1}$, we have: $p_{\tilde{L_2}}|_{\overline{k}} = p_{L_2}|_{\overline{k}}$. By Lemma A.10 , $\tilde{L_2}$ is isomorphic to L_2, and so, $M \simeq L_1 \oplus L_2$.

□

Theorem A.12 .

Let $N_1, \ldots, N_s$ be pairwise nonisomorphic indecomposable direct summands in $M(Q)$. Then $\oplus_{i=1}^{s} N_i$ is isomorphic to a direct summand in $M(Q)$.

Proof

Let $N'_1, \ldots, N'_s$ be *normal forms* of $N_1, \ldots, N_s$. The statement follows from Theorem A.9 and the inductive application of Lemma A.11 .

□

We remind, that if N is a direct summand in $M(Q)$, then $N|_{\overline{k}} = \oplus_{w\in\Omega}\mathbb{Z}(w)[2w]$, and $\mathrm{CH}(N|_{\overline{k}}) := \oplus_r \mathrm{CH}_r(N|_{\overline{k}}) = \oplus_{w\in\Omega} e_w \cdot \mathbf{Z}$, where e_w is the generator of $\mathrm{CH}_r(N|_{\overline{k}})$, if $rank(\mathrm{CH}_r(N|_{\overline{k}})) = 1$, and $\varphi_N(l^1_{m/2}), \varphi_N(l^2_{m/2})$, if $rank(\mathrm{CH}_r(N|_{\overline{k}})) = 2$. We have the natural map $\overline{deg} : \mathrm{CH}(N|_{\overline{k}}) \to \mathbf{Z}/2$, defined by the rule: $\overline{deg}(e_w) = 1$.

Theorem A.13 .

Let N be an indecomposable direct summand in $M(Q)$, and $\psi \in \mathrm{End}(N)$ be some morphism. Then either $\overline{deg} \circ \psi = \overline{deg}$, or $\overline{deg} \circ \psi = 0$.

Proof

Sublemma A.13.1 .

Let N be a direct summand in $M(Q)$. Let $\psi \in \mathrm{End}(N)$. Then either there exists idempotent $\varepsilon \in \mathrm{End}(N)$, s.t. $(\varepsilon - \psi)|_{\overline{k}} \in \{2 \cdot \mathrm{End}(N|_{\overline{k}}) + \theta_{m/2,N} \cdot \mathbf{Z}\}$, or Q is (even-dimensional) hyperbolic.

Proof

Let $\psi_{(r,\mathbf{Z}/2)} \in \mathrm{End}(\mathrm{CH}_r(N|_{\overline{k}})/2)$ be the map induced by ψ.

If $rank(\mathrm{CH}_r(N|_{\overline{k}})) = 1$, then $\psi_{(r,\mathbf{Z}/2)}$ is always a projector.

If $rank(\mathrm{CH}_r(N|_{\overline{k}})) = 2$, then $r = m/2$, and either $\overline{deg}(\psi(h^{m/2})) = 1$, or one of: $\psi_{(m/2,\mathbf{Z}/2)}$, $(\psi + \theta_{m/2,N})_{(m/2,\mathbf{Z}/2)}$ is a projector. In the former case, Q is hyperbolic, by Sublemma 1.3.1 , since $\psi(h^{m/2}) \in \mathrm{CH}_{m/2}(Q)$ has odd degree.

Taking into account, that the map $idempotents(\mathrm{End}(N|_{\overline{k}})) \to idempotents(\mathrm{End}(N|_{\overline{k}}) \otimes \mathbf{Z}/2)$ is surjective, we have: if Q is not (even-dimensional) hyperbolic, then there exists idempotent $\overline{\varepsilon} \in \mathrm{End}(N|_{\overline{k}})$, s.t. $(\psi|_{\overline{k}} - \overline{\varepsilon}) \in \{2 \cdot \mathrm{End}(N|_{\overline{k}}) + \theta_{m/2,N} \cdot \mathbf{Z}\}$.

Let there exists r (equal to $m/2$, certainly), s.t. $rank(\mathrm{CH}_r(N|_{\overline{k}})) = 2$. Changing ψ by $\psi + \theta_{m/2,N}$, if nesessary, we can assume that $\psi|_{\overline{k}} \equiv \overline{\varepsilon} \pmod 2$. Then either 1) $tr(\psi_{(m/2)})$ is odd, or 2) $\psi_{(m/2,\mathbf{Z}/2)} = id$.

1) In this case, by Sublemma A.8.1 , there exists $\varepsilon \in \mathrm{End}(N)$, s.t. $\varepsilon|_{\overline{k}} = \overline{\varepsilon}$.

2) In this case, $tr(\psi_{(m/2)})$ is even, and $det(\psi_{(m/2)})$ is odd. Take: $\varepsilon' := -\psi \circ (\psi - tr(\psi_{(m/2)}) \cdot id_N) - (det(\psi_{(m/2)}) - 1) \cdot id_N \in \mathrm{End}(N)$, then $(\varepsilon')_{(m/2)} = id$, and $(\psi - \varepsilon')_{(r)} \in 2 \cdot \mathrm{End}(\mathrm{CH}_r(N|_{\overline{k}}))$, for $r \neq m/2$.

If $rank(\mathrm{CH}_r(N|_{\overline{k}}) \leqslant 1$, for all r, take: $\varepsilon' := \psi$.

Put $\varepsilon := \varepsilon' - \sum_{r \neq m/2}([\varepsilon'_{(r)}/2] \cdot \kappa_{r,N})$, then $\varepsilon|_{\overline{k}} = \overline{\varepsilon}$.

$\square$

Let $\varepsilon \in \mathrm{End}(N)$ be an idempotent from Sublemma A.13.1 .

Since N is indecomposable, ε is either 0, or id_N, which implies: that either $\overline{deg} \circ \varepsilon = \overline{deg}$, or $\overline{deg} \circ \varepsilon = 0$. By Sublemma A.13.1 , the same is true for ψ. The hyperbolic case is evident.

$\square$

Theorem A.14 (cf. [15], Lemma 3.25).

Let N_1 be indecomposable direct summand in $M(Q_1)$, and N_2 be indecomposable direct summand in $M(Q_2)$. Suppose $\alpha \in \mathrm{Hom}(N_1, N_2)$, and $\beta \in \mathrm{Hom}(N_2, N_1)$ be such maps, that the composition: $\overline{deg} \circ \beta \circ \alpha : \mathrm{CH}(N_1|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. Then $N_1 \backsimeq N_2$.

Proof

Sublemma A.14.1 .

In the situation of Theorem A.14 , for $\gamma := \beta \circ \alpha \in \mathrm{End}(N_1)$, we have: $\gamma|_{\overline{k}} \in \{id_{N_1|_{\overline{k}}} + 2 \cdot \mathrm{End}(N_1|_{\overline{k}}) + \theta_{m_1/2,N_1}|_{\overline{k}} \cdot \mathbf{Z}\}$.

Proof Let $\varepsilon \in \mathrm{End}(N_1)$ be projector from Sublemma A.13.1 , s.t. $(\varepsilon - \gamma)|_{\overline{k}} \in \{2 \cdot \mathrm{End}(N_1|_{\overline{k}}) + \theta_{m_1/2,N_1} \cdot \mathbf{Z}\}$.

Clearly, $\overline{deg} \circ \varepsilon = \overline{deg} \circ \gamma \neq 0$. Hence, $\varepsilon \neq 0$. Since N_1 is indecomposable, we get: $\varepsilon = id_{N_1}$.

$\square$

As an evident consequence of Sublemma A.14.1 , we get:

Sublemma A.14.2 .

In the situation of Theorem A.14 , $(\beta \circ \alpha)|_{\overline{k}} : \mathrm{CH}(N_1|_{\overline{k}})/2 \to \mathrm{CH}(N_1|_{\overline{k}})/2$ is an isomorphism.

Sublemma A.14.3 .

In the situation of Theorem A.14 , $\overline{deg}(\beta \circ \alpha(y)) = \overline{deg}(y)$, for any $y \in \mathrm{CH}(N_1|_{\overline{k}})$, and $\overline{deg}(\alpha \circ \beta(z)) = \overline{deg}(z)$, for any $z \in \mathrm{CH}(N_2|_{\overline{k}})$. In particular, the conditions of Theorem A.14 are symmetric with respect to N_1 and N_2.

Proof

From Sublemma A.14.1 it follows that $\overline{deg}(\beta\circ\alpha(y)) = \overline{deg}(y)$.

Since $(\beta\circ\alpha)|_{\overline{k}} : \mathrm{CH}(N_1|_{\overline{k}})/2 \to \mathrm{CH}(N_1|_{\overline{k}})/2$ is an isomorphism (by Sublemma A.14.2), we have: $\alpha\circ\beta(\mathrm{CH}(N_2|_{\overline{k}})/2) = (\alpha\circ\beta)^2(\mathrm{CH}(N_2|_{\overline{k}})/2)$ is isomorphic to $\mathrm{CH}(N_1|_{\overline{k}})/2$. Let $\varepsilon\in\mathrm{End}(N_2)$ be projector from Sublemma A.13.1 , s.t. $(\varepsilon-\alpha\circ\beta)|_{\overline{k}}\in\{2\cdot\mathrm{End}(N_2|_{\overline{k}})+\theta_{m_2/2,N_2}\cdot\mathbf{Z}\}$. Then, since N_2 is indecomposable, ε is either 0, or id_{N_2}.

If $\varepsilon=0$, then $(\alpha\circ\beta)^2|_{\overline{k}}\in 2\cdot\mathrm{End}(N_2|_{\overline{k}})$, and $\mathrm{CH}(N_1|_{\overline{k}})/2=0$, which clearly contradicts to the assumptions of Theorem A.14 . So, $\varepsilon=1$. Then $\overline{deg}(\alpha\circ\beta(z)) = \overline{deg}(z)$, for any $z\in\mathrm{CH}(N_2|_{\overline{k}})$.

□

Sublemma A.14.4 .

In the situation of Theorem A.14 , for any r,
$rank(\mathrm{CH}_r(N_1|_{\overline{k}})) = rank(\mathrm{CH}_r(N_2|_{\overline{k}}))$.

Proof

It follows from Sublemma A.14.2 , and the fact that the conditions of Theorem A.14 are symmetric with respect to N_1 and N_2 (Sublemma A.14.3).

□

Sublemma A.14.5 .

In the situation of Theorem A.14 , $\overline{deg}(\alpha(y)) = \overline{deg}(y)$, for any $y\in\mathrm{CH}(N_1|_{\overline{k}})$, and $\overline{deg}(\beta(z)) = \overline{deg}(z)$, for any $z\in\mathrm{CH}(N_2|_{\overline{k}})$.

Proof

By Sublemma A.14.3 , it is enough to show that if $\overline{deg}(y)=0$, then $\overline{deg}(\alpha(y))=0$, and if $\overline{deg}(z)=0$, then $\overline{deg}(\beta(z))=0$.

$\mathrm{Ker}(\overline{deg})\subset\mathrm{CH}(N_1|_{\overline{k}})$ is generated by: $\{2\cdot e_w\}_{w\in\Omega_1}$ and $l^1_{m_1/2}+l^2_{m_1/2}=h^{m_1/2}$. So, if for some $y\in\mathrm{CH}(N_1|_{\overline{k}})$, we have: $\overline{deg}(y)=0$, and $\overline{deg}(\alpha(y))=1$, then $rank(\mathrm{CH}_{m_1/2}(N_1|_{\overline{k}}))=2$ and $\overline{deg}(\alpha(h^{m_1/2}))=1$. By Sublemma A.14.4 , $rank(\mathrm{CH}_{m_1/2}(N_2|_{\overline{k}}))=2$, which implies: $m_1=m_2$, and $\overline{deg}:\mathrm{CH}_{m_2/2}(N_2|_{\overline{k}})\to\mathbf{Z}/2$ coinsides with the usual degree $(mod 2)$. Then on Q_2 there is $m_2/2$-dimensional cycle of odd degree. By Sublemma 1.3.1 , Q_2 is hyperbolic. Then any indecomposable direct summand in $M(Q_2)$ is a Tate-motive, which contradicts to the fact that $rank(\mathrm{CH}_{m_2/2}(N_2|_{\overline{k}}))=2$.

□

Sublemma A.14.6 .

In the situation of Theorem A.14 , for any $u\in\mathrm{Hom}(N_1,N_2)$, either $\overline{deg}\circ u=\overline{deg}$, or $\overline{deg}\circ u=0$. The same holds for any $v\in\mathrm{Hom}(N_2,N_1)$.

Proof

If $\overline{deg} \circ u \neq 0$, then the pair (u, β) satisfy the conditions of Theorem A.14 , by Sublemma A.14.5 . The statement follows now again from Sublemma A.14.5 .

$\square$

Sublemma A.14.7 .

Let $rank(\mathrm{CH}_r(N_1|_{\overline{k}})) = 2$. *Suppose* α, *and* β *are as in Theorem A.14 . Then there exists* α', β', *s.t.* ${\beta'}_{(r)} = \lambda \cdot \beta_{(r)}$, *where* λ *is odd,* $det({\alpha'}_{(r)}) = det({\beta'}_{(r)})$, *and* $(\beta' \circ \alpha')_{(r)} = l \cdot id_{\mathrm{CH}_r(N_1|_{\overline{k}})}$, *where* l *is odd.*

Proof

Let $rank(\mathrm{CH}_r(N_1|_{\overline{k}})) = 2$. Then, by Sublemma A.14.1 , $\gamma_{(r)} \in \{id + 2 \cdot \mathrm{Mat}_{2\times 2}(\mathbf{Z}) + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \mathbf{Z}\}$. This implies that $det(\gamma_{(r)}) = \mu$ is odd, and $tr(\gamma_{(r)})$ is even.

In $\mathrm{End}(\mathrm{CH}_r(N_1|_{\overline{k}}))$ we have an equality: $-\gamma_{(r)} \circ (\gamma_{(r)} - tr(\gamma_{(r)})) = \mu \cdot id_{\mathrm{CH}_r(N_1|_{\overline{k}})}$. Take $\alpha'' := -\alpha \circ (\beta \circ \alpha - tr(\gamma_{(r)}) \cdot id_{N_1})$. It is easy to see that $(\beta \circ \alpha'')_{(r)} = \mu \cdot id_{\mathrm{CH}_r(N_1|_{\overline{k}})}$.

Let $\alpha' := l \cdot \alpha''$, and $\beta' := det({\alpha''}_{(r)}) \cdot \beta$. Then $det({\alpha'}_{(r)}) = det({\beta'}_{(r)})$, and $(\beta' \circ \alpha')_{(r)} = (\mu^2 \cdot det({\alpha''}_{(r)})) \cdot id_{\mathrm{CH}_r(N_1|_{\overline{k}})}$.

$\square$

Sublemma A.14.8 .

In the situation of Theorem A.14 , for any r, *s.t.* $rank(\mathrm{CH}_r(N_1|_{\overline{k}})) = 1$, *there exists* $\kappa_{r,1\to 2} \in \mathrm{Hom}(N_1, N_2)$, *s.t.* $(\kappa_{r,1\to 2})_{(r)} = 2$, *and* $(\kappa_{r,1\to 2})_{(s)} = 0$, *for any* $s \neq r$.

If for some r, $rank(\mathrm{CH}_r(N_i|_{\overline{k}})) = 2$, *then there exist* $\theta_{r,1\to 2}$ *and* $\kappa_{r,1\to 2} \in \mathrm{Hom}(N_1, N_2)$, *s.t.* $(\theta_{r,1\to 2})_{(r)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $(\kappa_{r,1\to 2})_{(r)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, *and* $(\theta_{r,1\to 2})_{(s)} = 0 = (\kappa_{r,1\to 2})_{(s)}$, *for any* $s \neq r$.

Proof

Let e_i be the generator of $\mathrm{CH}_r(N_i|_{\overline{k}})$. We can assume that N_i is in the *normal form* (see Definition A.7).

Using $h^{m_i - r}$ (and Sublemma A.8.1 (with $\psi = p_{N_i}$), if $r = m_i/2$), we have: $2 \cdot \mathrm{CH}_r(M(Q_i)|_{\overline{k}}) \subset image(\mathrm{CH}_r(M(Q_i)) \to \mathrm{CH}_r(M(Q_i)|_{\overline{k}}))$. Since $\varphi_{N_i} \circ j_{N_i} = id_{N_i}$, we have: $2 \cdot \mathrm{CH}_r(N_i|_{\overline{k}}) \subset image(\mathrm{CH}_r(N_i) \to \mathrm{CH}_r(N_i|_{\overline{k}}))$. In particular, $2 \cdot e_i \in image(\mathrm{CH}_r(N_i) \to \mathrm{CH}_r(N_i|_{\overline{k}}))$. Let $2 \cdot e_i = \overline{g_i}$.

By Sublemma A.14.4 , $\overline{deg}(\alpha(e_1)) = \overline{deg}(e_1) = 1$, and $\overline{deg}(\beta(e_2)) = \overline{deg}(e_2) = 1$.

That means: $e_1 \in image(\mathrm{CH}_r(N_1) \to \mathrm{CH}_r(N_1|_{\overline{k}}))$ if and only if $e_2 \in image(\mathrm{CH}_r(N_2) \to \mathrm{CH}_r(N_2|_{\overline{k}}))$.

We have two cases: A) $e_i \in image(\mathrm{CH}_r(N_i) \to \mathrm{CH}_r(N_i|_{\overline{k}}))$, and B) $e_i \notin image(\mathrm{CH}_r(N_i) \to \mathrm{CH}_r(N_i|_{\overline{k}}))$.

Define $u \in \mathrm{Hom}(N_1, \mathbb{Z}(r)[2r])$ as a composition: $N_1 \stackrel{j_{N_1}}{\to} M(Q_1) \stackrel{h^{m_1-r}}{\to} \mathbb{Z}(r)[2r]$. Define $v \in \mathrm{Hom}(\mathbb{Z}(r)[2r], N_2)$ as: f_2, where $\overline{f_2} = e_2$, in case A), and as g_2 in case B).

In case A): either $r > m_i/2$, or $r = m_i/2$ and N_i is of type (3) (see Definition A.7). In any case, we have: $j_{N_1}(e_1) = \pm h^{m_1-r}$.

In case B): either $r < m_i/2$, or $r = m_i/2$ and N_i is of type (2) (see Definition A.7). In any case, we have: $j_{N_1}(e_1) = \pm l_r$.

Clearly, $u(e_1)$ (as a map from $\mathbb{Z}(r)[2r]$ to $\mathbb{Z}(r)[2r]$) is equal to id times the degree of intersection of $j_{N_1}(e_1)$ and h^{m_1-r}. So, it is ± 2 in case A), and ± 1 in case B).

Define $\kappa_{r,1\to 2} := \pm v \circ u$. Then $\kappa_{r,1\to 2}(e_1) = \pm u(e_1)\cdot \overline{v}$ and it is equal to $2\cdot \overline{f_2}$ in case A), and to $1\cdot \overline{g_2}$ in case B). So, in any case, $\kappa_{r,1\to 2}(e_1) = 2\cdot e_2$, and so, $(\kappa_{r,1\to 2})_{(r)} = 2$. Since $u(\mathrm{CH}_s(N_1|_{\overline{k}})) = 0$, for any $s \neq r$ (since $\mathrm{Hom}(\mathbb{Z}(s)[2s], \mathbb{Z}(r)[2r]) = 0$, for $s \neq r$), we have: $(\kappa_{r,1\to 2})_{(s)} = 0$, for any $s \neq r$.

Let now, for some r, $rank(\mathrm{CH}_r(N_i|_{\overline{k}})) = 2$. Then $m_1 = m_2$, and we can define: $\theta_{r,1\to 2} := \varphi_{N_2} \circ (h^{m_1/2} \times h^{m_1/2}) \circ j_{N_1}$. It is easy to see that $\theta_{r,1\to 2}$ has needed properties.

To define $\kappa_{r,1\to 2}$, observe that if α' and β' are as in Sublemma A.14.7 , then ${\beta'}_{(r)} = \lambda\cdot\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, and ${\alpha'}_{(r)} = \lambda\cdot\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$.

If the pair (α, β) satisfy the conditions of Theorem A.14 , then the pair $(\alpha, \beta + \theta_{r,2\to 1})$ satisfy the conditions of Theorem A.14 too (by Sublemma A.14.6).

Then there exists odd μ, and $\alpha'' \in \mathrm{Hom}(N_1, N_2)$, $\beta'' \in \mathrm{Hom}(N_2, N_1)$, s.t. ${\beta''}_{(r)} = \mu\cdot(\beta + \theta_{r,2\to 1})_{(r)}$, and $det({\alpha''}_{(r)}) = det({\beta''}_{(r)})$, and $(\beta'' \circ \alpha'')_{(r)} = l'' \cdot id_{\mathrm{CH}_r(N_1|_{\overline{k}})}$.

Take: $\varepsilon := \lambda\cdot\alpha'' - \mu\cdot\alpha' + \lambda\mu\cdot\theta_{r,1\to 2}$, then: $\varepsilon_{(r)} = \lambda\mu\cdot\begin{pmatrix} 2 & 0 \\ 0 & 2\end{pmatrix}$. The composition $\overline{deg}\circ\varepsilon : \mathrm{CH}_r(N_1|_{\overline{k}}) \to \mathbf{Z}/2$ is zero. By Sublemma A.14.6 , for any s, $\overline{deg}\circ\varepsilon : \mathrm{CH}_s(N_1|_{\overline{k}}) \to \mathbf{Z}/2$ is zero. I.e. $\varepsilon_{(s)} \in \mathbf{Z}$ is even. Define: $\varepsilon' := \varepsilon - \sum_{s\neq r}(\varepsilon_{(s)}/2)\cdot\kappa_{s,1\to 2}$. Then ${\varepsilon'}_{(r)} = \varepsilon_{(r)}$, and $\varepsilon_{(s)} = 0$, for $s \neq r$. By Lemma A.6 , there exists $\kappa_{r,1\to 2} \in \mathrm{Hom}(N_1, N_2)$, s.t. $\lambda\mu\cdot\kappa_{r,1\to 2}|_{\overline{k}} = \varepsilon'$. Clearly, $\kappa_{r,1\to 2}$ has desired properties.

$\square$

Sublemma A.14.9 .

Suppose $rank(\mathrm{CH}_r(N_i)) = 2$, and α, β satisfy the conditions of Theorem A.14 . Then there exists $\alpha_2 \in \mathrm{Hom}(N_1, N_2)$, s.t. $(\alpha - \alpha_2)_{(r)} \in$

$\{2\cdot \mathrm{Hom}(\mathrm{CH}_r(N_1|_{\overline{k}}), \mathrm{CH}_r(N_2|_{\overline{k}})) + \theta_{r,1\to 2}\cdot \mathbf{Z}\}$, *and* $(\alpha_2)_{(r)} = \eta\cdot A$, *where* η *is odd and* $A : \mathrm{CH}_r(N_1|_{\overline{k}}) \to \mathrm{CH}_r(N_2|_{\overline{k}})$ *is an isomorphism.*

Proof

Since $rank(\mathrm{CH}_r(N_1|_{\overline{k}})) = rank(\mathrm{CH}_r(N_2|_{\overline{k}})) = 2$, we clearly have: $r = m_1/2 = m_2/2$.

For any morphism $u \in \mathrm{Hom}(N_i, N_j)$, we denote as $u^{\vee}$ *the dual* morphism $\varphi_{N_i} \circ (j_{N_j} \circ u \circ \varphi_{N_i})^{\vee} \circ j_{N_j} \in \mathrm{Hom}(N_j, N_i)$.

Let us denote as $\tilde{u} \in \mathrm{Hom}(N_j, N_i)$ the following morphism: $\tilde{u} := u^{\vee}$, if m is not divisible by 4, and $:= \tau_i \circ u^{\vee} \circ \tau_j$, if m is divisible by 4.

If $u_{(r)} = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, then $\tilde{u}_{(r)} = \begin{pmatrix} t & y \\ z & x \end{pmatrix}$.

Suppose α' and β' be as in Sublemma A.14.7 . Let $f := g.c.d.(c-b, d-a)$, and $f = u\cdot(d-a) + v\cdot(c-b)$. Then for $\psi := (\nu(u - v\tau_2)) \circ \alpha' + (\nu(u+v\tau_2)) \circ \tilde{\beta}' - \lambda(ua+vb)\kappa_{r,1\to 2}$, we have: $\psi_{(r)} = \lambda\nu\cdot \begin{pmatrix} 2f & 0 \\ 0 & 0 \end{pmatrix}$.

Let $\alpha_1 := \lambda\nu\cdot\alpha - \psi\circ([(a-d)/2f] + [(b-c)/2f]\cdot\tau_1)$. Then $\alpha_1 = \lambda\nu\cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, where $(a_1 - d_1), (b_1 - c_1) \in \{0, f\}$.

I.e.: either 1) $a_1 = d_1$, or 2) $b_1 = c_1$, or 3) $a_1 - d_1 = b_1 - c_1$.

Considering $\alpha_1 \circ \tau_1$ (and $\alpha_2 \circ \tau_1$), we can reduce case 2) to case 1). So, it is enough to consider cases 1) and 3).

1) Take $\alpha_2 := \alpha_1 - \lambda\nu((a_1-1)\cdot\theta_{r,1\to 2} - (b_1-a_1+1)/2\cdot\kappa_{r,1\to 2}\circ\tau_1)$. Then $\alpha_{2(r)} = \lambda\nu\cdot \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ (notice that $(b_1-a_1+1) \equiv (b-a+1) \equiv 0\,(\,mod\, 2)$).

3) take $\alpha_2 := \alpha_1 - \lambda\nu(\kappa_{r,1\to 2}\circ([(a_1+d_1)/4] + [(b_1+c_1)/4]\cdot\tau_1) - [\{(a_1+d_1)/4\} + \{(b_1+c_1)/4\}]\cdot\theta_{r,1\to 2})$. Then $(\alpha_2)_{(r)} = \lambda\nu \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$, where $a_2 - d_2 = b_2 - c_2$; $(a_2+d_2), (b_2+c_2) \in \{0, 2\}$, and $a_2+b_2+c_2+d_2 = 2$.

Hence, in any case, $det((\alpha_2)_{(r)}/\lambda\nu) = \pm 1$.

$\square$

Sublemma A.14.10 .

In the situation of Theorem A.14 , there exist $\alpha_3 \in \mathrm{Hom}(N_1, N_2)$, *and some odd number* η, *that for all* s, $(\alpha_3)_{(s)} = \eta\cdot A_s$, *where* A_s *is invertible.*

Proof

If $rank(\mathrm{CH}_s(N_1|_{\overline{k}})) \leqslant 1$, for all s, take: $\alpha' := \alpha$, and $\eta := 1$. If $rank(\mathrm{CH}_r(N_1|_{\overline{k}})) = 2$, for some r, take: $\alpha' := \alpha_2$, and take η from the Sublemma A.14.9 (here α_2, A are also from Sublemma A.14.9).

By Sublemma A.14.9 , $\overline{deg} \circ \alpha' : \mathrm{CH}_r(N_1|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. By Sublemma A.14.6 , for all s, s.t. $rank(\mathrm{CH}_s(N_1|_{\overline{k}})) = 1$, $(\alpha')_{(s)} = \lambda_s$ is odd.

Let r be such that $rank(\mathrm{CH}_s(N_1|_{\overline{k}})) = 2$. Define $\alpha_3 := \alpha' - \sum_{s \neq r}(\lambda_s - \eta)/2 \cdot \kappa_{s,1\to 2}$, where $\kappa_{s,1\to 2}$ are elements from Sublemma A.14.8 . Then $(\alpha_3)_{(s)} = \eta$, for any $s \neq r$, and $(\alpha_3)_{(r)} = \eta \cdot A'$, where A' is invertible.

□

From Sublemma A.14.10 and Lemma A.6 it follows that, in the situation of Theorem A.14 , there exists $\alpha_4 \in \mathrm{Hom}(N_1, N_2)$, s.t. $\alpha_4|_{\overline{k}}$ is an isomorphism. Since conditions of Theorem A.14 are symmetric with respect to N_1 and N_2 (by Sublemma A.14.3), we also have some $\beta_4 \in \mathrm{Hom}(N_2, N_1)$, s.t. $\beta_4|_{\overline{k}}$ is an isomorphism. Then $(\beta_4 \circ \alpha_4)|_{\overline{k}}$ and $(\alpha_4 \circ \beta_4)|_{\overline{k}}$ are isomorphisms, and then $\beta_4 \circ \alpha_4$ and $\alpha_4 \circ \beta_4$ are, by Corollary A.3 (3).

Theorem A.14 is proven.

□

Theorem A.15 .

Let N_1 is indecomposable direct summand in $M(Q_1)(d_1)[2d_1]$, and N_2 is indecomposable direct summand in $M(Q_2)(d_2)[2d_2]$, for some d_1, d_2. Suppose $\alpha \in \mathrm{Hom}(N_1, N_2)$ and $\beta \in \mathrm{Hom}(N_2, N_1)$ are such that the map: $\overline{deg} \circ \beta \circ \alpha : \mathrm{CH}(N_1|_{\overline{k}}) \to \mathbb{Z}/2$ is nonzero. Then $N_1 \simeq N_2$.

Proof

Statement follows from Theorem A.14 , taking into account that $M(Q_i)(d_i)[2d_i]$ is a direct summand in $M(Q_i')$, where $q_i' := q_i \perp d_i \cdot \mathbb{H}$, by [13], Proposition 1 (Theorem A.0).

□

Theorem A.16 ([15], Lemma 3.26).

Suppose Q_1, Q_2 be some smooth projective quadrics over k, and

$$\alpha \in \mathrm{Hom}(M(Q_1)(d_1)[2d_1], M(Q_2)(d_2)[2d_2]),$$

$$\beta \in \mathrm{Hom}(M(Q_2)(d_2)[2d_2], M(Q_1)(d_1)[2d_1])$$

be such morphisms, that the composition

$$\overline{deg} \circ \beta \circ \alpha : \mathrm{CH}_r(M(Q_1)(d_1)[2d_1]|_{\overline{k}}) \to \mathbf{Z}/2$$

is nonzero for some r. Then there exist indecomposable direct summands N_1 of $M(Q_1)(d_1)[2d_1]$, and N_2 of $M(Q_2)(d_2)[2d_2]$, such that $N_1 \simeq N_2$, and $\mathbb{Z}(r)[2r]$ is a direct summand in $N_1|_{\overline{k}}$.

Proof

Lemma A.16.1 .

Suppose N and L be indecomposable direct summands of $M(Q)$ in normal form, and $\gamma \in \mathrm{Hom}(N, L)$ be such map, that the composition $\overline{deg} \circ j_L \circ \gamma : \mathrm{CH}(N|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. Then $N \simeq L$.

Proof

Let $\overline{deg} \circ j_L \circ \gamma : \mathrm{CH}_r(N|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero for some r. In particular, $\mathrm{CH}_r(N|_{\overline{k}}) \neq 0 \neq \mathrm{CH}_r(L|_{\overline{k}})$. By Theorem A.12 , either $N \simeq L$, or $r = \dim(Q)/2$ (since only for such r, $rank(\mathrm{CH}_r(M(Q)|_{\overline{k}})) = 2$), and $rank(\mathrm{CH}_r(N|_{\overline{k}})) = rank(\mathrm{CH}_r(L|_{\overline{k}})) = 1$.

If N is not isomorphic to L, then, in the notations of Definition A.7 , N and L are of types (2), or (3). If L is of type (3), then $\overline{deg} \circ j_L : \mathrm{CH}_r(L|_{\overline{k}}) \to \mathbf{Z}/2$ is zero (since in such case $\mathrm{CH}_r(L|_{\overline{k}})$ is generated by the class of h^r). This is, clearly, not the case, so L is of type (2). Since L is not isomorphic to N, and N,L are indecomposable direct summands in *normal form*, we have by Theorem A.9 : N is of type (3). But then the generator h^r of $\mathrm{CH}_r(N|_{\overline{k}})$ is defined over k. That means: the map $\overline{deg} \circ j_L \circ \gamma : \mathrm{CH}_r(N) \to \mathbf{Z}/2$ is nonzero. This implies that on Q there exists r-dimensional cycle of odd degree (namely, $j_L \circ \gamma(h^r)$). By Sublemma 1.3.1 , Q is hyperbolic. Then N and L must be Tate-motives. Since $rank(\mathrm{CH}_r(N|_{\overline{k}})) = rank(\mathrm{CH}_r(L|_{\overline{k}})) = 1$, we have: $N \simeq \mathbb{Z}(r)[2r] \simeq L$. Contradiction. So, $N \simeq L$.

□

Lemma A.16.2 .

Let N is a direct summand of $M(Q_2)(d_2)[2d_2]$, and the maps: $\alpha \in \mathrm{Hom}(M(Q_1)(d_1)[2d_1], N)$, $\beta \in \mathrm{Hom}(N, M(Q_1)(d_1)[2d_1])$ are such that the composition $\overline{deg} \circ \beta \circ \alpha : \mathrm{CH}(M(Q_1)(d_1)[2d_1]|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. Then there exists direct summand L of $M(Q_1)(d_1)[2d_1]$ isomorphic to N.

Proof

Let $M(Q_1)(d_1)[2d_1] = \oplus_{a \in \Lambda_1} L^a$, where L^a are indecomposable direct summands in *normal form* (by Theorem A.12 , we can always find such decompositions). Let $\alpha_a := \alpha \circ j_{L^a} \in \mathrm{Hom}(L^a, N)$, and $\beta_c := \varphi_{L^c} \circ \beta$. We have: $\beta \circ \alpha = \sum_{a,c \in \Lambda_1} (p_{L^c} \circ \beta \circ \alpha \circ p_{L^a}) = \sum_{a,c \in \Lambda_1} (j_{L^c} \circ \beta_c \circ \alpha_a \circ \varphi_{L^a})$.

Since the composition $\overline{deg} \circ \beta \circ \alpha : \mathrm{CH}(M(Q_1)(d_1)[2d_1]|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero, we have: for some a, c, the composition $\overline{deg} \circ j_{L^c} \circ \beta_c \circ \alpha_a : \mathrm{CH}(L^a|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero.

By Lemma A.16.1 , there exists an isomorphism $\psi : L^c \to L^a$. Take: $u := \alpha_a \circ \psi \in \mathrm{Hom}(L^c, N)$, and $v := \beta_c \in \mathrm{Hom}(N, L^c)$. We have: $\overline{deg} \circ j_{L^c} \circ v \circ u : \mathrm{CH}_r(L^c|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. Since the composition $\mathrm{CH}_r(L^c|_{\overline{k}}) \stackrel{j_{L^c}}{\to} \mathrm{CH}_r(M(Q_1)(d_1)[2d_1]|_{\overline{k}}) \stackrel{\overline{deg}}{\to} \mathbf{Z}/2$ either coinsides with

$\overline{deg} : \mathrm{CH}_r(L^c|_{\overline{k}}) \to \mathbf{Z}/2$, or is zero, we get: $\overline{deg} \circ v \circ u : \mathrm{CH}_r(L^c|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. By Theorem A.15 , $L^c \simeq N$.

$\square$

Let $M(Q_2)(d_2)[2d_2] = \oplus_{b \in \Lambda_2} N_2^b$, where N^b are indecomposable direct summands. Let $\alpha^b := \varphi_{N^b} \circ \alpha$, and $\beta^b := \beta \circ j_{N^b}$. We have: $\beta \circ \alpha = \sum_{b \in \Lambda_2}(\beta \circ p_{N^b} \circ \alpha) = \sum_{b \in \Lambda_2}(\beta_b \circ \alpha_b)$.

So, if $\overline{deg} \circ \beta \circ \alpha : \mathrm{CH}_r(M(Q_1)(d_1)[2d_1]|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero, then for some $b \in \Lambda_2$, the map: $\overline{deg} \circ \beta_b \circ \alpha_b : \mathrm{CH}_r(M(Q_1)(d_1)[2d_1]|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. By Lemma A.16.2 , there exists some indecomposable direct summand L of $M(Q_1)(d_1)[2d_1]$ isomorphic to N^b. Clearly, $rank(\mathrm{CH}_r(N^b|_{\overline{k}})) \neq 0$, so, $\mathbb{Z}(r)[2r]$ is a direct summand in $N^b|_{\overline{k}}$.

$\square$

Let N be a direct summand in $M(Q)$. We define:
$a(N) := \min(s|\, \mathrm{CH}_s(N|_{\overline{k}}) \neq 0)$, $b(N) := \max(s|\, \mathrm{CH}_s(N|_{\overline{k}}) \neq 0)$, and $c(N) := \dim(Q) - b(N)$.

Let $k = F_0 \subset F_1 \subset \cdots \subset F_h$ be *generalized splitting tower* of M.Knebusch for Q.

We have the following application of Theorem A.16 :

Theorem A.17 ([16], Corollary 2).

Suppose Q is smooth projective quadric of dimension m, and $i_W(q|_{F_t}) \leqslant i, j < i_W(q|_{F_{t+1}})$. Let N be indecomposable direct summand of $M(Q)$, such that $a(N) = i$. Then in $M(Q)$ there exists direct summand isomorphic to $N(j-i)[2j-2i]$.

Proof

Sublemma A.17.1 .

Let N be a direct summand in $M(Q)$, and $\pi : M(Q) \to \mathbb{Z}$ be the map induced by the natural projection $Q \to \mathrm{Spec}(k)$. Then the map $N \otimes M(Q) \stackrel{id \otimes \pi}{\to} N$ has a section $\nu : N \to N \otimes M(Q)$.

Proof

It is easy to see that the composition: $N \stackrel{j_N}{\to} M(Q) \stackrel{\Delta}{\to} M(Q) \otimes M(Q) \stackrel{\varphi_N \otimes id}{\to} N \otimes M(Q)$ gives desired section.

$\square$

Sublemma A.17.2 .

Let $i < j \leqslant m/2$. Let N be a direct summand of $M(Q)$, s.t. $a(N) = i$. Suppose over $k(Q)$ there exists map $\tilde{\alpha} : N(j-i)[2j-2i]|_{k(Q)} \to M(Q)|_{k(Q)}$, s.t. the composition $\overline{deg} \circ \tilde{\alpha} : \mathrm{CH}_j(N(j-i)[2j-2i]|_{\overline{k(Q)}}) \to \mathbf{Z}/2$ is nonzero. Then there exists map $\alpha : N(j-i)[2j-2i] \to M(Q)$, s.t. the map $\overline{deg} \circ \alpha : \mathrm{CH}_j(N(j-i)[2j-2i]|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero.

Proof

We have natural group homomorphism $\theta : \mathrm{Hom}_{Chow(k)}(N(j-i)[2j-2i]\otimes M(Q), M(Q)) \to \mathrm{Hom}_{Chow(k(Q))}(N(j-i)[2j-2i]|_{k(Q)}, M(Q)|_{k(Q)})$, which corresponds to the natural map $\vartheta_2 : \mathrm{CH}^{m+i-j}(Q\times Q\times Q) \to \mathrm{CH}^{m+i-j}(Q\times Q|_{k(Q)})$ (notice, that $\mathrm{Hom}_{Chow(k)}(N(j-i)[2j-2i]\otimes M(Q), M(Q))$ is naturally a direct summand in $\mathrm{CH}^{m+i-j}(Q\times Q\times Q)$, and $\mathrm{Hom}_{Chow(k(Q))}(N(j-i)[2j-2i]|_{k(Q)}, M(Q)|_{k(Q)})$ is naturally a direct summand in $\mathrm{CH}^{m+i-j}(Q\times Q|_{k(Q)})$). θ is surjective, since ϑ_2 is. So, there exists $\gamma \in \mathrm{Hom}_{Chow(k)}(N(j-i)[2j-2i]\otimes M(Q), M(Q))$, s.t. $\theta(\gamma) = \tilde{\alpha}$. Define $\alpha := \gamma\circ\nu(j-i)[2j-2i]$.

Since $M(Q)|_{\overline{k}}$, and $N|_{\overline{k}}$ are direct sums of Tate-motives, we have: $\mathrm{CH}_i(N\otimes M(Q)|_{\overline{k}}) = \oplus_{s+t=i}\mathrm{CH}_s(N|_{\overline{k}})\otimes\mathrm{CH}_t(M(Q)|_{\overline{k}}) = \mathrm{CH}_i(N|_{\overline{k}})\otimes\mathrm{CH}_0(M(Q)|_{\overline{k}})$. In particular, $id\otimes\pi : \mathrm{CH}_i(N\otimes M(Q)|_{\overline{k}}) \to \mathrm{CH}_i(N|_{\overline{k}})$ is an isomorphism. Hence, $\nu(j-i)[2j-2i] : \mathrm{CH}_j(N(j-i)[2j-2i]|_{\overline{k}}) \to \mathrm{CH}_j(N(j-i)[2j-2i]\otimes M(Q)|_{\overline{k}})$ is an isomorphism too. So, taking l_i as a generator of $\mathrm{CH}_i(N|_{\overline{k}}) = \mathrm{CH}_j(N(j-i)[2j-2i]|_{\overline{k}})$ (notice, that $0\leqslant i<m/2$), we have: $\nu(j-i)[2j-2i](l_i) = l_i\otimes l_0$. By the definition of θ, $\overline{deg}(\gamma(l_i\otimes l_0)) = \overline{deg}(\theta(\gamma)(l_i)) = \overline{deg}(\tilde{\alpha}(l_i)) \neq 0$. So, the composition $\overline{deg}\circ\alpha : \mathrm{CH}_j(N(j-i)[2j-2i]|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero.

□

Sublemma A.17.3 .

Let $i_W(q|_{F_t}) \leqslant i < j < i_W(q|_{F_{t+1}})$. Let N be a direct summand of $M(Q)$, s.t. $a(N) = i$. Then there exists map $\alpha : N(j-i)[2j-2i] \to M(Q)$, s.t. the map $\overline{deg}\circ\alpha : \mathrm{CH}_j(N(j-i)[2j-2i]|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero.

Proof

We can use induction on $\dim(q)$. If $\dim(q)\leqslant 1$, then there is nothing to prove.

Suppose the statement is proven for all forms of dimension $<\dim(q)$. Changing k by $k(Q)$, and using Sublemma A.17.2 , we can assume that $i_W(q)>0$.

Then, by Theorem A.0 , $M(Q) = \oplus_{0\leqslant s<i_W(q)}(\mathbb{Z}(s)[2s]\oplus\mathbb{Z}(m-s)[2m-2s])\oplus M(Q_{an})(i_W(q))[2i_W(q)]$, and $\mathrm{End}(M(Q)) = \times_{0\leqslant s<i_W(q)}(\mathrm{End}(\mathbb{Z}(s)[2s])\times\mathrm{End}(\mathbb{Z}(m-s)[2m-2s]))\times\mathrm{End}(M(Q_{an})(i_W(q))[2i_W(q)])$. This implies, that if N is a direct summand of $M(Q)$, then $N = \oplus_{0\leqslant s<i_W(q)}(N_s\oplus N_{m-s})\oplus N_{an}$, where N_{an} is a direct summand of $M(Q_{an})(i_W(q))[2i_W(q)]$, and N_t is a direct summand of $\mathbb{Z}(t)[2t]$ (i.e. either $\mathbb{Z}(t)[2t]$, or 0).

We have two cases: 1) $0\leqslant i<i_W(q)$; 2) $i_W(q)\leqslant i<m/2$.

1) In this case, $N_i = \mathbb{Z}(i)[2i]$, since $\mathrm{CH}_i(N|_{\overline{k}})\neq 0$. Since $0\leqslant i<i_W(q)$, we have: $j<i_W(q)$, and cycle $h^i\times l_j\in\mathrm{CH}^{m+i-j}(Q\times Q|_{\overline{k}})$ is defined over k, so it gives us the map $\alpha\in\mathrm{Hom}(M(Q)(j-i)[2j-2i], M(Q)) = \mathrm{CH}^{m+i-j}(Q\times Q)$. We can take $l_i\in\mathrm{CH}_i(Q|_{\overline{k}}) =$

$\mathrm{CH}_j(M(Q)(j-i)[2j-2i]|_{\overline{k}})$ as a generator of $\mathrm{CH}_j(N(j-i)[2j-2i]|_{\overline{k}})$. Clearly, $\alpha(l_i) = l_j$. So, the composition $\overline{deg} \circ \alpha : \mathrm{CH}_j(N(j-i)[2j-2i]|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero.

2) Since $\mathrm{CH}_j(\mathbb{Z}(s)[2s]) = 0$, for $j \neq s$, we have: $\mathrm{CH}_j(N(j-i)[2j-2i]|_{\overline{k}}) = \mathrm{CH}_j(N_{an}(j-i)[2j-2i]|_{\overline{k}})$. Taking $i' := i - i_W(q)$, $j' := j - i_W(q)$, $q' = q_{an}$, and $N' := N_{an}(-i_W(q))[-2i_W(q)]$. Clearly, N', q', i, j satisfy the conditions of Theorem A.17 . Since $i_W(q) > 0$, we have: $\dim(q') < \dim(q)$. By inductive assumption, there exists $\alpha' \in \mathrm{Hom}(N'(j'-i')[2j'-2i'], M(Q'))$, s.t. the composition $\overline{deg} \circ \alpha' : \mathrm{CH}_{j'}(N'(j'-i')[2j'-2i']|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. Let $\psi : M(Q')(i_W(q))[2i_W(q)] \to M(Q)$, and $\rho : N \to N'(i_W(q))[2i_W(q)]$ be natural “embedding” and “projection”.

Take: $\alpha := \psi \circ \alpha'(i_W(q))[2i_W(q)] \circ \rho(j-i)[2j-2i] \in \mathrm{Hom}(N(j-i)[2j-2i], M(Q))$.

Since ψ and ρ act isomorphically on CH_j, and $\overline{deg} \circ \psi = \overline{deg} : \mathrm{CH}_j(M(Q')(i_W(q))[2i_W(q)]|_{\overline{k}}) \to \mathbf{Z}/2$, we have: the composition $\overline{deg} \circ \alpha : \mathrm{CH}_j(N(j-i)[2j-2i]) \to \mathbf{Z}/2$ is nonzero.

$\square$

Sublemma A.17.4 (cf. [15], Lemma 4.5).

Let $i_W(q|_{F_t}) \leqslant i < j < i_W(q|_{F_{t+1}})$. Let N be indecomposable direct summand of $M(Q)$, s.t. $a(N) = i$. Then there exists direct summand of $M(Q)$, isomorphic to $N(j-i)[2j-2i]$.

Proof

Let $\Delta_{h^{j-i}} \in \mathrm{CH}^{m+j-i}(Q \times Q)$ be the cycle given by the diagonal embedding of the $(j-i)$-codimensional plane section. Denote corresponding element of $\mathrm{Hom}(M(Q), M(Q)(j-i)[2j-2i]) = \mathrm{CH}^{m+j-i}(Q \times Q)$ as ε_{j-i}. Define: $\beta := \varphi_N(j-i)[2j-2i] \circ \varepsilon_{j-i} \in \mathrm{Hom}(M(Q), N(j-i)[2j-2i])$.

Clearly, $\varepsilon_{j-i}(l_j) = l_i \in \mathrm{CH}_i(M(Q)|_{\overline{k}}) = \mathrm{CH}_j(M(Q)(j-i)[2j-2i]|_{\overline{k}})$, and it is a generator of the later group.

Since $i < m/2$, and $\mathrm{CH}_i(N|_{\overline{k}}) = \mathbb{Z}$, the map $\varphi_N(j-i)[2j-2i] : \mathrm{CH}_j(M(Q)(j-i)[2j-2i]|_{\overline{k}}) \to \mathrm{CH}_j(N(j-i)[2j-2i]|_{\overline{k}})$ is an isomorphism. So, the map $\beta : \mathrm{CH}_j(M(Q)|_{\overline{k}}) \to \mathrm{CH}_j(N(j-i)[2j-2i]|_{\overline{k}})$ is surjective.

By Sublemma A.17.3 , we also have a map: $\alpha \in \mathrm{Hom}(N(j-i)[2j-2i], M(Q))$, s.t. the composition $\overline{deg} \circ \alpha : \mathrm{CH}_j(N(j-i)[2j-2i]|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. By above, the composition $\overline{deg} \circ \alpha \circ \beta : \mathrm{CH}_j(M(Q)|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. By Lemma A.16.2 , $M(Q)$ has a direct summand isomorphic to $N(j-i)[2j-2i]$.

$\square$

Sublemma A.17.5 (cf. [16], proof of Statement).

Let Q be anisotropic quadric of dimension m, s.t. $i_1(q) = 1$. Let N be direct summand of $M(Q)$, s.t. $\mathrm{CH}_m(N|_{\overline{k}}) \neq 0$. Then $\mathrm{CH}_0(N|_{\overline{k}}) \neq 0$.

Proof

Suppose it is not the case. So, $\mathrm{CH}_m(N|_{\overline{k}}) \neq 0$ and $\mathrm{CH}_0(N|_{\overline{k}}) = 0$.

By Sublemma A.17.1 , the natural projection $N \otimes M(Q) \to N$ has a section ν. So, $\pi \circ \nu = id_N$.

Let $\beta_0 \in \mathrm{Hom}(\mathbb{Z}(m)[2m], M(Q)) = \mathrm{CH}_m(Q)$ be the morphism corresponding to the "generic cycle" on Q. Let $u := (\varphi_N \otimes id) \circ (\beta_0 \otimes id) \in \mathrm{Hom}(M(Q)(m)[2m], N \otimes M(Q))$, $v := \Delta^\vee \circ (j_N \otimes id) \in \mathrm{Hom}(N \otimes M(Q), M(Q)(m)[2m])$, where $\Delta^\vee \in \mathrm{Hom}(M(Q \times Q), M(Q))$ is dual to the "diagonal embedding" Δ via duality: $\mathrm{CH}^*(A \times B) \simeq \mathrm{CH}^*(B \times A)$.

Since $j_N \circ \varphi_N \circ \beta_0 = \beta_0$ (since $\mathrm{CH}_m(N|_{\overline{k}}) = \mathbb{Z}$, and hence $j_N \circ \varphi_N : \mathrm{CH}_m(Q|_{\overline{k}}) \to \mathrm{CH}_m(Q|_{\overline{k}})$ is identity map), we have: $v \circ u = id \in \mathrm{End}(M(Q)(m)[2m])$. So, $N \otimes M(Q) = M(Q)(m)[2m] \oplus X$. Let $\varphi_X \in \mathrm{Hom}(N \otimes M(Q), X)$, and $j_X \in \mathrm{Hom}(X, N \otimes M(Q))$ be the corresponding projection and embedding. I.e.: $\varphi_X \circ j_X = id_X$, $j_X \circ \varphi_X + u \circ v = id_{N \otimes M(Q)}$, and $\varphi_X \circ u = 0$, $v \circ j_X = 0$.

In particular, $\pi \circ \nu = (\pi \circ j_X) \circ (\varphi_X \circ \nu) + (\pi \circ u) \circ (v \circ \nu)$. So, there exist maps $\alpha_1 : N \to M(Q)(m)[2m]$, $\beta_1 : M(Q)(m)[2m] \to N$, and $\alpha_2 : N \to X$, $\beta_2 : X \to N$, s.t. $\beta_1 \circ \alpha_1 + \beta_2 \circ \alpha_2 = id_N$.

If $\overline{deg} \circ \beta_1 \circ \alpha_1 : \mathrm{CH}_m(N|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero, then, by Theorem A.16 , we have a direct summand N' of N, which is isomorphic to a direct summand of $M(Q)(m)[2m]$, and $\mathrm{CH}_m(N') \neq 0$. Since $\mathrm{CH}_m(Q) \to \mathrm{CH}_m(Q|_{\overline{k}})$ is an isomorphism, and N' is a direct summand of $M(Q)$, we have: $\mathrm{CH}_m(N') \to \mathrm{CH}_m(N'|_{\overline{k}})$ is an isomorphism. Since N' is also a direct summand in $M(Q)(m)[2m]$, and $\mathrm{CH}_m(N') = \mathbf{Z}$, we have: $\mathrm{CH}_m(N') = \mathrm{CH}_m(M(Q)(m)[2m])$, and $\mathrm{CH}_m(M(Q)(m)[2m]) \to \mathrm{CH}_m(M(Q)(m)[2m]|_{\overline{k}})$ is an isomorphism. This implies that on Q there is a 0-cycle of degree 1. By Springer's Theorem, Q is isotropic - a contradiction. So, $\overline{deg} \circ \beta_1 \circ \alpha_1 : \mathrm{CH}_m(N|_{\overline{k}}) \to \mathbf{Z}/2$ is zero.

Consider $E = k(Q)$. We have: $q|_E = \mathbb{H} \oplus q'$, where q' is anisotropic (since $i_1(q) = 1$). By [13], Proposition 1 (Theorem A.0), $M(Q|_E) = \mathbb{Z} \oplus M(Q')(1)[2] \oplus \mathbb{Z}(m)[2m]$. And then $N|_E = N_{an} \oplus \mathbb{Z}(m)[2m]$, where N_{an} is a direct summand of $M(Q')(1)[2]$ (since $\mathrm{CH}_m(N|_{\overline{k}}) = \mathbf{Z}$, and $\mathrm{CH}_0(N|_{\overline{k}}) = 0$). Moreover, the corresponding map $j_{\mathbb{Z}(m)[2m] \to N|_E} : \mathbb{Z}(m)[2m] \to N|_E$ coinsides with $(\varphi_N \circ \beta_0)|_E$. Then the map $j_{\mathbb{Z}(m)[2m] \to N|_E} \otimes id_{M(Q)} : M(Q)(m)[2m]|_E \to N \otimes M(Q)|_E$ coinsides with $u|_E$. This implies that the complimentary direct summand $N_{an} \otimes M(Q)$ (in $(N \otimes M(Q))|_E$) is isomorphic to $X|_E$. Notice that $N_{an} \otimes M(Q)$ is a direct summand in $M(Q')(1)[2] \otimes M(Q)$. So, $\alpha_2|_E$ and $\beta_2|_E$ give us maps $\alpha_2' : N|_E \to M(Q' \times Q)(1)[2]$, and $\beta_2' : M(Q' \times Q)(1)[2] \to N|_E$, s.t. $\beta_2' \circ \alpha_2' = (\beta_2 \circ \alpha_2)|_E$.

If $\alpha'_2 \circ j_{\mathbb{Z}(m)[2m]\to N|_E} \in \mathrm{Hom}(\mathbb{Z}(m)[2m], M(Q'\times Q)(1)[2]) = \mathrm{CH}_{m-1}(Q'\times Q)$ is represented by the cycle A, and $\varphi_{N|_E\to\mathbb{Z}(m)[2m]} \circ \beta'_2 \in \mathrm{Hom}(M(Q'\times Q)(1)[2], \mathbb{Z}(m)[2m]) = \mathrm{CH}^{m-1}(Q'\times Q)$ is represented by the cycle B, then the composition $(\varphi_{N|_E\to\mathbb{Z}(m)[2m]} \circ \beta'_2) \circ (\alpha'_2 \circ j_{\mathbb{Z}(m)[2m]\to N|_E}) \in \mathrm{End}(\mathbb{Z}(m)[2m]) = \mathbf{Z}$ is given by the degree of the 0-cycle $A \cap B \in \mathrm{CH}_0(Q'\times Q)$. Since Q' is anisotropic, this number is even, by Springer's Theorem. Since $j_{\mathbb{Z}(m)[2m]\to N|_E}$ and $\varphi_{N|_E\to\mathbb{Z}(m)[2m]}$ are isomorphisms on CH_m, we have: the composition $\overline{deg}\circ\beta'_2\circ\alpha'_2 = \overline{deg}\circ\beta_2\circ\alpha_2 : \mathrm{CH}_m(N|_{\overline{k}}) \to \mathbf{Z}/2$ is zero. Since also $\overline{deg}\circ\beta_1\circ\alpha_1 : \mathrm{CH}_m(N|_{\overline{k}}) \to \mathbf{Z}/2$ is zero, we get a contradiction with: $\beta_1\circ\alpha_1 + \beta_2\circ\alpha_2 = id_N$.

□

Sublemma A.17.6 .

Let Q be anisotropic quadric, and L is indecomposable direct summand in $M(Q)$, s.t. $a(L) = 0$. Then for any subquadric $P \subset Q$, s.t $\dim(P) = \dim(Q) - i_1(q) + 1$, we have:

1) $M(P)$ contains a direct summand isomorphic to L;

2) $p|_{k(Q)}$ and $q|_{k(P)}$ are isotropic.

Proof

For any field extension E/k, we have: $p|_E$ is isotropic if and only if $q|_E$ is. In particular we get 2). So, we have rational (algebro-geometric) maps $f : Q \to P$, and $g : P \to Q$. Let $\alpha \in \mathrm{CH}^{\dim(P)}(Q\times P) = \mathrm{Hom}(M(Q), M(P))$, and $\beta \in \mathrm{CH}^{\dim(Q)}(P\times Q) = \mathrm{Hom}(M(P), M(Q))$ be closures of the graphs of f and g, respectively. Clearly, $\alpha(l_0) = l_0$, and $\beta(l_0) = l_0$. So, the composition $\overline{deg}\circ\beta\circ\alpha : \mathrm{CH}_0(Q|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. By Theorem A.16 , $M(P)$ contains a direct summand isomorphic to L (notice that if M is indecomposable direct summand of $M(Q)$, s.t. $\mathrm{CH}_0(M) \neq 0$, then $M \simeq L$, by Theorem A.9).

□

Sublemma A.17.7 .

Let Q be anisotropic quadric, and L is indecomposable direct summand in $M(Q)$, s.t. $a(L) = 0$. Then there exists subquadric $P \subset Q$, s.t.:

1) $i_1(p) = 1$;

2) $M(P)$ contains a direct summand isomorphic to L;

3) $p|_{k(Q)}$ and $q|_{k(P)}$ are isotropic.

Proof

Use induction on the dimension of Q. The case of $\dim(Q) = 0$ is trivial. Suppose the statement is true for all quadrics of dimension $< \dim(Q)$. Consider P from Sublemma A.17.6 . Either $i_1(q) = 1$, in which case statement is trivial, or $\dim(P) < \dim(Q)$. Then there exists

P', s.t. P' satisfy 1) and 2), and $p'|_{k(P)}$, $p|_{k(P')}$ are isotropic. Since we also have: $p|_{k(Q)}$, $q|_{k(P)}$ are isotropic, we get: P' satisfy 3) (since if $q_2|_{k(q_1)}$, $q_3|_{k(q_2)}$ are isotropic, then $q_3|_{k(q_1)}$ is).

$\square$

Sublemma A.17.8 .

Let Q be anisotropic quadric of dimension m, and L is indecomposable direct summand in $M(Q)$, s.t. $a(L) = 0$. Then $c(L) = i_1(q) - 1$.

Proof

Let P be quadric from Sublemma A.17.7 . Then L is also a direct summand in $M(P)$. By Sublemma A.17.5 , $b(L) = \dim(P)$. In particular, by [13], Proposition 1 (Theorem A.0), $L|_{k(P)}$ contains $\mathbb{Z}$ and $\mathbb{Z}(\dim(P))[2\dim(P)]$ as direct summands.

Since Q is anisotropic, P is also anisotropic. If $\dim(P) = 0$, then $rank(\mathrm{CH}_0(L|_{\overline{k}})) = 2$, and hence, $m = 0$ (since L is a direct summand in $M(Q)$). In this case everything is evident. If $b(L) = \dim(P) > 0$, then the map $\mathrm{CH}_{b(L)}(P) \to \mathrm{CH}_{b(L)}(P|_{\overline{k}})$ is surjective, and $\mathrm{CH}_{b(L)}(L|_{\overline{k}}) = \mathrm{CH}_{b(L)}(P|_{\overline{k}})$. So, the map $\mathrm{CH}_{b(L)}(L) \to \mathrm{CH}_{b(L)}(L|_{\overline{k}})$ is surjective. And the map $\overline{deg} : \mathrm{CH}_{b(L)}(L) \to \mathbf{Z}/2$ is nonzero. If $b(L) < m/2$, then $\overline{deg} = \overline{deg} \circ j_L : \mathrm{CH}_{b(L)}(L) \to \mathbf{Z}/2$, and we get $b(L)$-dimensional cycle of odd degree on Q. By Sublemma 1.3.1 , Q is isotropic - contradiction. So, $b(L) \geqslant m/2$. Since $L|_{k(P)}$ contains $\mathbb{Z}(b(L))[2b(L)]$ as a direct summand, $L|_{k(Q)}$ also contains $\mathbb{Z}(b(L))[2b(L)]$ as a direct summand, by Sublemma A.17.7 (3). Then $b(L) > m - i_1(q)$, by Lemma 1.3 (since $b(L) \geqslant m/2$).

By Sublemma A.17.4 , $M(Q)$ contains direct summand isomorphic to $L(i_1(q) - 1)[2i_1(q) - 2]$. This implies: $b(L) \leqslant \dim(Q) - i_1(q) + 1$. So, $b(L) = \dim(Q) - i_1(q) + 1$, and $c(L) = i_1(q) - 1$.

$\square$

Sublemma A.17.9 .

Let Q be anisotropic quadric of dimension m, and N is indecomposable direct summand in $M(Q)$, s.t. $a(N) < i_1(q)$. Let L be indecomposable direct summand in $M(Q)$, s.t $a(L) = 0$. Then $N \simeq L(a(N))[2a(N)]$.

Proof

We can assume that $a(N) > 0$. By Sublemma A.17.4 , $L(a(N))[2a(N)]$ is isomorphic to a direct summand of $M(Q)$ (since $a(N) < i_1(q)$). We have: $\mathrm{CH}_{a(N)}(N|_{\overline{k}}) \neq 0 \neq \mathrm{CH}_{a(N)}(L|_{\overline{k}})$. Consider L, $L(a(N))[2a(N)]$, and N. Since $a(N) > 0$, we have: L is not isomorphic to $L(a(N))[2a(N)]$. By the same reason, L is not isomorphic to N (since $a(L) \neq a(N)$). If $L(a(N))[2a(N)]$ is not isomorphic to N, then $N \oplus L \oplus L(a(N))[2a(N)]$ is a direct summand of $M(Q)$, by Theorem A.12 . Since $rank(\mathrm{CH}_{a(N)}(N \oplus$

$L(a(N))[2a(N)]|_{\overline{k}})) \geqslant 2$, we get: $a(N) = m/2$, which implies: $i_1(q) = m/2+1$, and $b(L) = c(L) = m/2 = a(N)$. In this case, $\mathrm{CH}_{a(N)}(L|_{\overline{k}}) \neq 0$, and $rank(\mathrm{CH}_{a(N)}(N \oplus L \oplus L(a(N))[2a(N)]|_{\overline{k}})) \geqslant 3$ - contradiction.

$\square$

Sublemma A.17.10 .

Let Q be anisotropic quadric of dimension m, and N is indecomposable direct summand in $M(Q)$, s.t. either $a(N)$, or $c(N)$ is $< i_1(q)$. Then $a(N) + c(N) = i_1(q) - 1$.

Proof

Changing N by $N^{\vee}$, if nesessary, we can assume that $a(N) \leqslant c(N)$, in particular, $0 \leqslant a(N) < i_1(q)$. By Sublemma A.17.9 , $N \simeq L(a(N))[2a(N)]$. In this case, $a(N) + c(N) = a(L) + c(L) = c(L) = i_1(q) - 1$.

$\square$

Sublemma A.17.11 .

Let Q be anisotropic quadric of dimension m. Let N be indecomposable direct summand in $M(Q)$. Then there exists $0 \leqslant t < h = height(q)$, s.t. for fields $F_t \subset F_{t+1}$ from the generalized splitting tower of M.Knebusch for Q, we have: $i_W(q|_{F_{t+1}}) > a(N), c(N) \geqslant i_W(q|_{F_t})$, and $a(N) + c(N) \leqslant i_W(q|_{F_t}) + i_W(q|_{F_{t+1}}) - 1$.

Proof

Let $k = F_0 \subset F_1 \subset \cdots \subset F_h$ be the *generalized splitting tower* for Q. Let $1 \leqslant t < h$ be the minimal number such that $N|_{F_{t+1}}$ contains either $\mathbb{Z}(a(N))[2a(N)]$, or $\mathbb{Z}(b(N))[2b(N)]$ as a direct summand (since $q|_{F_h}$ is hyperbolic, and $\mathrm{CH}_{a(N)}(N|_{\overline{k}}) \neq 0$, such t exists). Changing N by $N^{\vee}$, if nesessary, we can assume that $N|_{F_{t+1}}$ contains $\mathbb{Z}(a(N))[2a(N)]$ as a direct summand (notice, that $a(N) + c(N)$ will not change under this procedure). Since $N|_{F_t}$ does not contain Tate-motives, we have: $a(N), c(N) \geqslant i_W(q|_{F_t})$. This shows that $N|_{F_t}$ is a direct summand in $M((Q|_{F_t})_{an})(i_W(q|_{F_t}))[2i_W(q|_{F_t})]$. Let N' be indecomposable direct summand of $N|_{F_t}$, s.t. $\mathrm{CH}_{a(N)}(N'|_{\overline{F_t}}) \neq 0$. Then $a(N') + c(N') = 2i_W(q|_{F_t}) + i_1((q|_{F_t})_{an}) - 1$, by Sublemma A.17.10 . Since $a(N') = a(N)$, and $c(N') \leqslant c(N)$, we get: $a(N) + c(N) \leqslant i_W(q|_{F_t}) + i_W(q|_{F_{t+1}}) - 1$.

$\square$

We have the following easy corollary of Sublemma A.17.11 .

Corollary A.17.12 ([16], Statement).

Let Q be anisotropic quadric of dimension m. Let N be indecomposable direct summand in $M(Q)$.

Then there exists $0 \leqslant t < h$, s.t. $i_W(q|_{F_{t+1}}) > a(N), c(N) \geqslant i_W(q|_{F_t})$, and $a(N) + c(N) = i_W(q|_{F_t}) + i_W(q|_{F_{t+1}}) - 1$.

Proof

Let $l := i_W(q|_{F_{t+1}}) - 1 - a(N)$. By Sublemma A.17.4 , $N(l)[2l]$ is isomorphic to a direct summand M of $M(Q)$. Clearly, $a(M) = a(N) + l = i_W(q|_{F_{t+1}}) - 1$, and $c(M) = c(N) - l$. Since $a(M) \geqslant i_W(q|_{F_t})$, we have, by Sublemma A.17.11 , $c(M) \geqslant i_W(q|_{F_t})$. This implies: $a(N) + c(N) \geqslant i_W(q|_{F_t}) + i_W(q|_{F_{t+1}}) - 1$. Combined with Sublemma A.17.11 , this gives required result.

$\square$

Now we can prove Theorem A.17 . Let Q, i, j, N be as in Theorem A.17 . Clearly, $i = a(N)$. If $j > i$, then everything is contained in Sublemma A.17.4 .

Let $(i-j) > 0$. Let $F = F_r$ be a field in the *generalized splitting tower* of M.Knebusch, given by Sublemma A.17.11 . Then $i_W(q|_{F_{r+1}}) > a(N) \geqslant i_W(q|_{F_r})$. By conditions of Theorem A.17 , $i_W(q|_{F_{r+1}}) > a(N) - (i-j) \geqslant i_W(q|_{F_r})$. Also, $i_W(q|_{F_{r+1}}) > c(N) \geqslant i_W(q|_{F_r})$. On the other hand, from Sublemma A.17.11 it follows that $i_W(q|_{F_{r+1}}) > c(N) + (i-j) \geqslant i_W(q|_{F_r})$. That means that the pair $(i', j') := (c(N), c(N) + (i-j))$ satisfy the conditions of Sublemma A.17.4 . But $c(N) = a(N^{\vee})$. By Sublemma A.17.4 , in $M(Q)$ there exists a direct summand L, isomorphic to $N^{\vee}(i-j)[2i-2j]$. Then $L^{\vee}$ will be isomorphic to $N(j-i)[2j-2i]$.

Theorem A.17 is proven.

$\square$

We proved all the results we use in the main part of the paper. Let us just show simple consequence of Sublemma A.17.10 .

Corollary A.18 ([16], Corollary 3).
1) If $q_1|_{k(Q_2)}$ and $q_2|_{k(Q_1)}$ are isotropic, then $\dim(q_1) - i_1(q_1) = \dim(q_2) - i_1(q_2)$.
2) If $P \subset Q$ is a subquadric, s.t. $p|_{k(Q)}$ is isotopic, then codim$(P \subset Q) < i_1(q)$.
3) In the situation of 2), $i_1(p) = i_1(q) -$ codim$(P \subset Q)$.

Proof
1) Since $q_1|_{k(Q_2)}$ and $q_2|_{k(Q_1)}$ are isotropic, by Theorem A.14 , there exist isomorphic direct summands N_1 of $M(Q_1)$, and N_2 of $M(Q_2)$, s.t. $a(N_i) = 0$. By Sublemma A.17.10 , $\dim(N_i) := b(N_i) - a(N_i) = \dim(Q_i) - a(N_i) - c(N_i) = \dim(Q_i) - i_1(q_i) + 1$. Since $N_1 \simeq N_2$, we get the equality.
2) and 3) Follow from 1), taking into account that $i_1(p) \geqslant 1$.

$\square$

References

[1] R.Elman, T.Y.Lam, *Pfister forms and K-theory of fields.*, J.Algebra **23** (1972), 181-213.

[2] D.W.Hoffmann, *Isotropy of quadratic forms over the function field of a quadric.*, Math. Zeit., 220 (1995), 461-476.

[3] D.W.Hoffmann, *Twisted Pfister Forms.*, Documenta Math. **1**, (1996) 67-102.

[4] D.W.Hoffmann, *On the dimensions of anisotropic quadratic forms in I^4.*, Invent. Math. **131** (1998), 185-198.

[5] O.Izhboldin, *Quadratic Forms with Maximal Splitting, II.*, Preprint., February 1999.

[6] B.Kahn, *Formes quadratiques de hauteur et de degré* 2., Indag. Math. **7** (1996), 47-66.

[7] B.Kahn, *Motivic cohomology of smooth geometrically cellular varieties.*, Preprint. (http://www.math.uiuc.edu/K-theory/0218)

[8] M.Knebusch, *Generic splitting of quadratic forms, I.*, Proc. London Math. Soc. **33** (1976), 65-93.

[9] M.Knebusch, *Generic splitting of quadratic forms, II.*, Proc. London Math. Soc. **34** (1977), 1-31.

[10] T.Y.Lam, *Algebraic theory of quadratic forms.*, Massachusets, Benjamin 1973.

[11] Yu.I.Manin, *Correspondences, motifs and monoidal transformations.*, Mat. Sb. **77** (1968), no. 4, 475-507; English transl. in Math. USSR-Sb. **6** (1968), no. 4, 439-470.

[12] A.Pfister, *Quadratische Formen in biliebigen Körpern.*, Invent. Math. **1** (1966), 116-132.

[13] M.Rost, *Some new results on the Chow-groups of quadrics.*, Preprint, Regensburg, 1990.

[14] M.Rost, *The motive of a Pfister form.*, Preprint, 1998 (see www.physik.uniregensburg.de/ rom03516/papers.html).

[15] A.Vishik, *Integral Motives of Quadrics*, MPIM-preprint, 1998 (13).

[16] A.Vishik, *Direct summands in the motives of quadrics.*, Preprint, 1999.

[17] V.Voevodsky, *Triangulated category of motives over a field*, K-Theory Preprint Archives, Preprint 74, 1995 (see www.math.uiuc.edu/K-theory/0074/).

[18] V.Voevodsky, *Bloch-Kato conjecture for $\mathbb{Z}/2$-coefficients and algebraic Morava K-theories.*, Preprint, June 1995. (www.math.uiuc.edu/K-theory/0076/)

[19] V.Voevodsky, *Milnor conjecture.*, MPIM-preprint, 1997 (8).