Working with Weighted Complete Intersections.

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1 Introduction.

This article contains the following:

I A presentation of the basic definitions, theorems and techniques of weighted complete intersections, along with many examples. This information was collected from a variety of sources (mainly [WPS]) but also includes some original results.

II Lists of various types of weighted complete intersections of dimensions 1, 2 and 3, i.e. with cyclic quotient canonical isolated singularities.

Weighted complete intersections occur naturally in many disguises. Enriques' famous example of a surface of general type such that ϕ_{4K_S} is not birational can be expressed as the weighted complete intersection S_{10} in $\mathbf{P}(1,1,2,5)$.

For certain classes of variety V of general type (e.g. minimal surfaces of general type) the canonical maps $\phi_{nK_V} : V \to \tilde{V}$, for large enough n, are birational onto the canonical model $\tilde{V}$. Define the canonical ring R_V by

$$R_V = \bigoplus_{n\geq 0} \mathrm{H}^0(V, nK_V).$$

The ring R_V is known to be finitely generated in these cases, although not necessarily in degree 1. So $\tilde{V} \cong \mathrm{Spec}\ R_V$ is a subvariety of some weighted projective space.

These weighted complete intersections are similar to the complete intersections of normal projective space $\mathbf{P}^n$ but are usually singular and hence have some pathologies.

However these weighted complete intersections are still very easy to visualise and to work with; their basic invariants are calculated using combinatorics. So they form a large quagmire of *good* examples. This article sets out to familarise the reader with weighted complete intersections and to give certain combinatoric conditions for their important properties. Some of these are already known (see [Da], [Di], [Du], [WPS], etc.) but some are new. This constitutes Chapter I.

In Chapter II we present various lists of weighted complete intersections of dimension 1, 2 and 3; all with at worst cyclic quotient isolated canonical singularities. The canonical 3-fold weighted complete intersections are interesting since they are all canonical models (see [R1], [R2], [R4, section 2.5]) and hence are of interest for classification purposes as well as in their own right. These were all calculated using a set of combinatoric conditions and a computer. We also give a complete list of the 95 families of weighted hypersurface K3 surfaces (see [R1, section 4.5]) found by Reid in 1979 after a long hand calculation. We also calculate the corresponding singularities.

Another method originally used by Reid to produce examples of K3 surfaces is to be found in section II.8. It is used to produce canonically and anti-canonically embedded canonical 3-folds.

From the Poincaré series of the graded ring corresponding to a weighted complete intersection, the degrees of the generators and the relations can be determined. This technique uses repeated differencing to evaluate the power series. Using the Riemann-Roch formula for canonical 3-folds (see section II.7) a Poincaré series can be produced from a list (or *record*) of invariants, which we hope will correspond to either a canonically or an anti-canonically embedded canonical 3-fold. Clearly there will be a large number of rejected records and hence this is very hit and miss. However in practice it works very well.

This article started life as the third chapter of my Ph.D. thesis [F2] and grew.

2 Acknowledgements.

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3 Notation.

All varieties will be assumed to be quasi-projective over an algebraically closed field $\mathbf{k}$ of characteristic zero. Let V be such a variety, of dimension m.

$\mathbf{k}^*$ is the multiplicative group of nonzero elements of $\mathbf{k}$.

$\mathbf{Z}$, $\mathbf{Q}$ are the rings of integers and rational numbers respectively.

$\mathbf{Z}_r$ is the Abelian group $\{0, 1, ..., r-1\}$ under addition modulo r.

$\mathbf{Z}_r^*$ is the group of units of $\mathbf{Z}_r$ under multiplication modulo r.

$\{a, ..., \hat{b}, ..., c\}$ is a list with the element b omitted.

$\mathbf{A}^m$ is affine m-space.

$\mathbf{P}^m$ is projective m-space.

$\mathbf{P}(a_0, ..., a_m)$ is used to denote the weighted projective space with weighting $a_0, ..., a_m$. When there is no ambiguity this will be denoted simply by $\mathbf{P}$.

V^0 is the nonsingular locus of V.

$\mathcal{O}_V$ is the sheaf of regular functions on V.

$\Omega_V^1 = \Omega_{V/\mathbf{k}}^1$ is the sheaf of regular 1-forms on V^0.

$\Omega_V^n = \Lambda^n \Omega_{V/\mathbf{k}}^1$ is the sheaf of regular n-forms on V^0.

$\omega_V = \Omega_V^m$ is the sheaf of regular canonical differentials on V^0.

K_V is the canonical divisor corresponding to $\omega_V = \mathcal{O}_V(K_V)$.

Let $\mathcal{L}$ be a coherent sheaf on V. Then

$$\mathrm{h}^i(\mathcal{L}) = \mathrm{h}^i(V, \mathcal{L}) = \dim \mathrm{H}^i(V, \mathcal{L}),$$
$$\chi(\mathcal{L}) = \textstyle\sum_i (-1)^i \mathrm{h}^i(\mathcal{L})$$

and $\phi_{\mathcal{L}}$ is the rational map corresponding to the sheaf $\mathcal{L}$.

Let D be a Cartier divisor on V. Then

$$\mathrm{h}^i(D) = \mathrm{h}^i(\mathcal{O}_V(D)),$$
$$\chi(D) = \textstyle\sum_i (-1)^i \mathrm{h}^i(\mathcal{O}_V(D)).$$

and ϕ_D is the rational map corresponding to the sheaf $\mathcal{O}_V(D)$.
In particular ϕ_{nK_V} is called the n^{th} canonical map.
$p_g(V) = \mathrm{h}^0(\omega_V)$ is the geometric genus of V.
$P_n(V) = \mathrm{h}^0(\omega_V^{\otimes n})$ is the n^{th} plurigenus of V. For negative n these are referred to as the anti-plurigenera.
The words smooth and non-singular will be used interchangeably.

I

Weighted complete intersections.

I.1 Preamble.

In this chapter we give a brief summary of the facts about weighted complete intersections along with many examples. We also prove necessary and sufficient conditions for a weighted hypersurface X_d in $\mathbf{P}(a_0, ..., a_n)$ to be quasismooth and well-formed.

Sections I.2 and I.3 recap the main definitions and theorems about weighted projective spaces and weighted complete intersections. Section I.4 sets out various facts about the cohomology of weighted complete intersections. Section I.5 contains necessary and sufficient conditions for quasismoothness in the hypersurface and codimension 2 cases. Information about cyclic quotient canonical singularities in dimensions 1, 2 and 3 is to be found in section I.6, along with two technical lemmas used to count points of intersection along singular strata of **P**. Examples of how to calculate the singularities of various weighted complete intersections are included in section I.7.

I.2 Definitions and theorems on weighted projective spaces.

We start by reviewing some definitions and notation concerned with weighted complete intersections.

I.2.1 Definition. Let a_0, ..., a_n be positive integers and define $S = S(a_0, ..., a_n)$ to be the graded polynomial ring $\mathbf{k}[x_0, ..., x_n]$, graded by $\deg x_i = a_i$. The *weighted projective space* $\mathbf{P}(a_0, ..., a_n)$ is defined by

$$\mathbf{P}(a_0, ..., a_n) = \text{Proj } S$$

I.2.2 Note. Let x_0, ..., x_n be affine coordinates on $\mathbf{A}^{n+1}$ and let the group $\mathbf{k}^*$ act via:

$$\lambda(x_0, ..., x_n) = (\lambda^{a_0} x_0, ..., \lambda^{a_n} x_n).$$

Then $\mathbf{P}(a_0, ..., a_n)$ is the quotient $(\mathbf{A}^{n+1} - \underline{0}) / \mathbf{k}^*$. Under this group action x_0, ..., x_n are the homogeneous coordinates on $\mathbf{P}(a_0, ..., a_n)$. Clearly $\mathbf{P}(a_0, ..., a_n)$ is a rational n-dimensional projective variety.

I.2.3 Affine coordinate pieces.

Let $\{x_0, ..., x_n\}$ be the homogeneous coordinates on $\mathbf{P}(a_0, ..., a_n)$. The affine piece $x_i \neq 0$ is isomorphic to $\mathbf{A}^n / \mathbf{Z}_{a_i}$. Let ϵ be a primitive $a_i{}^{\text{th}}$ root of unity. The group acts via:

$$z_j \mapsto \epsilon^{a_j} z_j$$

for all $j \neq i$, on the coordinates $\{z_0, ..., \hat{z}_i, ..., z_n\}$ of $\mathbf{A}^n$; here z_j is thought of as $x_j / \sqrt[a_i]{x_i}$. Compare this with the case of $\mathbf{P}^n$ where the affine coordinates on $x_i \neq 0$ are $z_j = x_j / x_i$.

I.2.4 Examples.

(i) $\mathbf{P}^n = \mathbf{P}(1, ..., 1)$.

(ii) Consider $\mathbf{P}(1,1,2)$ with homogeneous coordinates u, v and w. The affine piece $w = 1$ is $\mathbf{A}^2/\mathbf{Z}_2$ with group action

$$u \mapsto -u$$
$$v \mapsto -v$$

The coordinate ring R is given by:

$$\begin{aligned} R &= \mathbf{k}[u,v]^{\mathbf{Z}_2} \\ &= \mathbf{k}[u^2, v^2, uv] \\ &= \mathbf{k}[x,y,z]/(xy - z^2). \end{aligned}$$

So $\mathbf{P}(1,1,2)$ is the projective completion of the ordinary quadratic cone $xy = z^2$ in $\mathbf{A}^3$.

I.2.5 Lemma. *For all positive integers q we have*

$$\text{Proj } S(a_0, ..., a_n) \cong \text{Proj } S(qa_0, ..., qa_n).$$

Proof. This follows from the fact that the 2 graded rings are isomorphic.

□

From [EGA, Proposition 2.4.7] (also see [Hart, Exercise II.5.13]) we have:

I.2.6 Lemma. *Let S be a graded ring and define the truncation $S^{(q)} = \bigoplus_{m \geq 0} S_{qm}$ to be the graded subring with m^{th} graded part S_{qm}. Then there exists a canonical isomorphism* Proj $S^{(q)} \cong$ Proj S.

This is called the q-tuple Veronese embedding, and is used in the proof of the following:

I.2.7 Lemma. *Let a_0, ..., a_n be positive integers with no common factor. If $q = \text{hcf}(a_1, ..., a_n)$ then*

$$\text{Proj } S(a_0, ..., a_n) \cong \text{Proj } S(a_0, a_1/q, ..., a_n/q)$$

Proof. Define $S' = \bigoplus_{m \geq 0} S_{qm}$ with the same grading as S. So $S' \cong S^{(q)}$. By the previous lemma we have Proj $S' \cong$ Proj S.

Suppose $x_0^{p_0} ... x_n^{p_n}$ is a monomial of degree mq for any m. Hence $p_0 a_0 + ... + p_n a_n = qm$, and so $q \mid p_0 a_0$. As the $\{a_i\}$ have no common factor, $q \mid p_0$. Hence x_0 only appears in S' as x_0^q. Thus $S' = \mathbf{k}[x_0^q, x_1, ..., x_n]$, which is isomorphic to $S(qa_0, a_1, ..., a_n)$. Therefore

$$\text{Proj } S(a_0, ..., a_n) \cong \text{Proj } S' \cong \text{Proj } S(a_0, a_1/q, ..., a_n/q)$$

□

I.2.8 Quasi-reflections. Let G be a finite group acting on a variety X. A *quasi-reflection* is any element of G whose fixed locus is a hyperplane. No singularities are produced by the action of any group generated by quasi-reflections.

The cancelling which occurs in Lemma I.2.7 is nothing more than the elimination of quasi-reflections from the actions of each $\mathbf{Z}_{a_i}$ on the corresponding affine coordinate piece.

This lemma leads to the following corollary from [WPS, 1.3.1] (see also [De, Proposition 1.3]):

I.2.9 Corollary. $\mathbf{P}(a_0, ..., a_n) \cong \mathbf{P}(b_0, ..., b_n)$ *for some* $\{b_i\}$ *such that for each* i

$$\text{hcf}(b_0, ..., \hat{b_i}, ..., b_n) = 1.$$

Proof. By Lemma I.2.5 we can cancel any common factor of the $\{a_i\}$. By renumbering as necessary and by repeated applications of Lemma I.2.7 we can reduce $\mathbf{P}(a_0, ..., a_n)$ to the case $\mathbf{P}(b_0, ..., b_n)$. A maximum of $n+1$ applications of Lemma I.2.7 are required.

□

I.2.10 Examples.

(i) $\mathbf{P}(a, b) \cong \mathbf{P}^1$ for all a and b.

(ii) $\mathbf{P}(2,3,3) \cong \mathbf{P}(2,1,1)$.

(iii) Let $f = x^5 + y^3 + z^2 \in \mathbf{k}[x, y, z]$ with weights 6, 10 and 15 respectively. Define $X : (f = 0) \subset \mathbf{P} = \mathbf{P}(6, 10, 15)$. By the previous lemma $\mathbf{P} \cong \mathbf{P}^2$.

$$\mathbf{P}(6,10,15) \cong \mathbf{P}(6,2,3) \cong \mathbf{P}(3,1,3) \cong \mathbf{P}(1,1,1)$$

The monomials transform as:

$$(x^5, y^3, z^2) \mapsto (x, y^3, z^2) \mapsto (x, y^3, z) \mapsto (x, y, z)$$

Thus $X \subset \mathbf{P} \cong (x + y + z = 0) \subset \mathbf{P}^2 = \mathbf{P}^1 \subset \mathbf{P}^2$. Of course the coordinate rings of the affine cones (see III.2.14) over $X \subset \mathbf{P}$ and $\mathbf{P}^1 \subset \mathbf{P}^2$ are not isomorphic.

In view of Corollary I.2.9 we make the following:

I.2.11 Definition. The expression $\mathbf{P}(a_0, ..., a_n)$ is *well-formed* if for each i

$$\text{hcf}(a_0, ..., \hat{a_i}, ..., a_n) = 1.$$

I.2.12 The quotient map.

Let $T = \mathbf{k}[y_0, ..., y_n]$, where the $\{y_i\}$ all have weight 1, and so $\mathbf{P}^n \cong \text{Proj}\ T$. Consider the inclusion map $S \hookrightarrow T$ given by:

$$x_i \mapsto y_i^{a_i}$$

for all i. This induces a quotient map $\sigma : \mathbf{P}^n \to \mathbf{P}$. In terms of the coordinates $\{Y_i\}$ on $\mathbf{P}^n$

$$[Y_0, ..., Y_n] \mapsto [Y_0^{a_0}, ..., Y_n^{a_n}]$$

The map $\mathbf{P}^n \to \mathbf{P}$ is a ramified Galois covering with Galois group $\bigoplus_i \mathbf{Z}_{a_i}$.

I.2.13 Definition. Let $r > 0$, a_1, ..., a_n be integers and let x_1, ..., x_n be coordinates on $\mathbf{A}^n$. Suppose that $\mathbf{Z}_r$ acts on $\mathbf{A}^n$ via:

$$x_i \mapsto \epsilon^{a_i} x_i$$

for all i, where ϵ a primitive r^{th} root of unity. A singularity $Q \in X$ is of *type* $\frac{1}{r}(a_1, ..., a_n)$ if (X, Q) is isomorphic to an analytic neighbourhood of $(\mathbf{A}^n, 0)/\mathbf{Z}_r$.

I.2.14 Notation. Write $P_i \in \mathbf{P}$ for the point $[0, ..., 0, 1, 0, ..., 0]$, where the 1 is in the i^{th} position. We will call P_i a vertex, the 1-dimensional toric stratum P_iP_j an edge, etc.. The fundamental simplex (i.e. the union of all the coordinate hyperplanes $P_0...\hat{P}_i...P_n$) will be denoted by Δ.

I.2.15 The singular locus $\mathbf{P}_{sing}$ of P.

Define $h_{i,j,...} = \text{hcf}(a_i, a_j, ...)$. The vertex P_i is a singularity of type $\frac{1}{a_i}(a_0, ..., \hat{a}_i, ..., a_n)$. This singularity is not necessarily isolated. Each generic point P of the edge P_iP_j has an analytic neighbourhood $P \in U$ which is analytically isomorphic to $(0, Q) \in \mathbf{A}^1 \times Y$, where $Q \in Y$ is a singularity of type $\frac{1}{h_{i,j}}(a_0, ..., \hat{a}_i, ..., \hat{a}_j, ..., a_n)$. Similar results hold for higher dimensional toric strata. The singularities only occur on the fundamental simplex Δ.

Notice that $\text{codim}_{\mathbf{P}}(\mathbf{P}_{sing}) \geq 2$.

I.3 Definitions and theorems on weighted complete intersections.

The first few definitions come from [WPS].

I.3.1 Definition. Let X be a closed subvariety of a weighted projective space $\mathbf{P}$, and let $p : \mathbf{A}^{n+1} - \underline{0} \to \mathbf{P}$ be the canonical projection. The *punctured affine cone* C^*_X over X is given by $C^*_X = p^{-1}(X)$, and the *affine cone* C_X over X is the completion of C^*_X in $\mathbf{A}^{n+1}$.

Notice that $\mathbf{k}^*$ acts on C^*_X to give $X = C^*_X/\mathbf{k}^*$.

I.3.2 Lemma. *C^*_X has no isolated singularities.*

Proof. If $P \in C^*_X$ is singular then every point on the same fibre of the $\mathbf{k}^*$-action will be singular. □

I.3.3 Definition. X in $\mathbf{P}(a_0, ..., a_n)$ is *quasismooth* of dimension m if its affine cone C_X is smooth of dimension $m+1$ outside its vertex $\underline{0}$.

When $X \subset \mathbf{P}$ is quasismooth the singularities of X are due to the $\mathbf{k}^*$-action and hence are cyclic quotient singularities. Notice that this definition is not equivalent to the smoothness of the inverse image $\sigma^{-1}(X)$ under the quotient map of section I.2.12 (e.g. X_8 in $\mathbf{P}(2, 3, 5)$).

Another important fact ([WPS, Theorem 3.1.6]) is that a quasi-smooth subvariety X of $\mathbf{P}$ is a V-variety (i.e. a complex space which is locally isomorphic to the quotient of a complex manifold by a finite group of holomorphic automorphisms). This is used later to define the canonical sheaf of X, which is usually singular.

I.3.4 Definition. Let I be a homogeneous ideal of the graded ring S and define X_I to be:

$$X_I = \text{Proj } S/I \subset \mathbf{P}$$

Suppose furthermore that I is generated by a regular sequence $\{f_i\}$ of homogeneous elements of S. $X_I \subset \mathbf{P}$ is called a *weighted complete intersection* of multidegree $\{d_i = \deg f_i\}$. In this case, we denote by $X_{d_1,...,d_c}$ in $\mathbf{P} = \mathbf{P}(a_0, ..., a_n)$ a sufficiently general element of the family of all weighted complete intersections of multidegree $\{d_i\}$.

$X_{d_1,...,d_c}$ in $\mathbf{P}(a_0, ..., a_n)$ is of dimension $n - c$. In general we will write $C_{d_1,...,d_c}$ in $\mathbf{P}(a_0, ..., a_{c+1})$ for a dimension 1 complete intersection and $S_{d_1,...,d_c}$ in $\mathbf{P}(a_0, ..., a_{c+2})$ for a surface.

I.3.5 Definition. X_d in $\mathbf{P}(a_0, ..., a_n)$ will be said to be a *linear cone* if $d = a_i$ for some i (i.e. the defining equation f can be writen as $f = x_i + g$).

Clearly X_d in $\mathbf{P}(a_0, ..., a_n)$ in this case is isomorphic to $\mathbf{P}(a_0, ..., \hat{a}_i, ..., a_n)$.

I.3.6 Examples.

(i) X_{46} in $\mathbf{P}(4, 5, 6, 7, 23)$ is a general element in the family of all degree 46 hypersurfaces in $\mathbf{P}(4, 5, 6, 7, 23)$.

(ii) X_8 in $\mathbf{P}(1, 1, 1, 1, 4)$ is a double cover of $\mathbf{P}^3$ branched along a smooth octic surface.

I.3.7 The coefficient convention.

When a general polynomial of a given weighted homogeneous degree occurs in a calculation then it will usually be written without the non-zero coefficients. For example the defining polynomial for X_2 in $\mathbf{P}(1, 1, 1)$ is:

$$f = c_0x^2 + c_1xy + c_2xz + c_3y^2 + c_4yz + c_5z^2$$

and will be simply written as:

$$f = x^2 + xy + xz + y^2 + yz + z^2.$$

I.3.8 The canonical sheaf ω_X.

All weighted complete intersections (and weighed projective spaces) are V-manifolds (i.e. locally are quotients of $\mathbf{A}^n$ by a finite group action) and so the dualizing sheaf ω_X is given by:

$$\omega_X \cong i_*\omega_{X^0}$$

where $i : X^0 \hookrightarrow X$ is the inclusion of the smooth part X^0 into X. This sheaf is a divisorial sheaf (see [R1, appendix to section 1, Theorem 7]) and can be written as:

$$\omega_X \cong \mathcal{O}_X(K_X)$$

where K_X is a $\mathbf{Q}$-Cartier divisor (i.e. rK_X is a Cartier divisor for some nonzero integer r). In fact $K_X|_{X^0}$ is Cartier.

For the general definition of the canonical sheaf for varieties with at worst canonical singularities see [R4, section 1.4].

We now introduce an important concept which was not mentioned (and possibly missed) by Dolgachev in [WPS].

I.3.9 Definition. A subvariety $X \subset \mathbf{P}$ of codimension c is *well-formed* if the expression for $\mathbf{P}$ is well-formed (see Definition I.2.11) and X contains no codimension $c + 1$ singular stratum of $\mathbf{P}$.

This means that any codimension 1 stratum of X is either non-singular on $\mathbf{P}$, or an intersection $X \cap S$, where S is a codimension 1 stratum of $\mathbf{P}$, i.e. $\text{codim}_X(X \cap \mathbf{P}_{sing}) \geq 2$.

I.3.10 Well-formedness for hypersurfaces.

The hypersurface X_d in $\mathbf{P}(a_0, ..., a_n)$ is well-formed if and only if

(1) $\text{hcf}(a_0, ..., \hat{a}_i, ..., \hat{a}_j, ..., a_n) \mid d$

(2) $\text{hcf}(a_0, ..., \hat{a}_i, ..., a_n) = 1$

for all distinct i, j.

I.3.11 Well-formedness in codimension 2.

The codimension 2 weighted complete intersection X_{d_1,d_2} in $\mathbf{P}(a_0, ..., a_n)$ is well-formed if and only if

(1) for all distinct i, j and k, with $h = \text{hcf}(a_0, ..., \hat{a}_i, ..., \hat{a}_j, ..., \hat{a}_k, ..., a_n)$, either $h \mid d_1$ or $h \mid d_2$,

(2) for all distinct i and j, with $h = \text{hcf}(a_0, ..., \hat{a}_i, ..., \hat{a}_j, ..., a_n)$, then $h \mid d_1$ and $h \mid d_2$,

(3) for all i $\text{hcf}(a_0, ..., \hat{a}_i, ..., a_n) = 1$.

I.3.12 Well-formedness in higher codimensions.

The above conditions can be generalised to higher codimensions. $X_{d_1,...,d_c}$ in $\mathbf{P}(a_0, ..., a_n)$ is well-formed if and only if

(1) $\mathbf{P}(a_0, ..., a_n)$ is well-formed

(2) for all $\mu = 1, ..., c$ the highest common factor of any $(n - 1 - c + \mu)$ of the $\{a_i\}$ must divide at least μ of the $\{d_j\}$.

I.3.13 Note. Dimca also defines well-formedness (see [Di]) under a different name. He gives the following equivalent set of arithmetic conditions in the quasismooth case. Define:

$$\begin{aligned} m(h) &= |\{i : h \mid a_i\}| \\ k(h) &= |\{i : h \mid d_i\}| \\ q(h) &= \dim X + 1 - m(h) + k(h) \end{aligned}$$

for all $h \in \mathbf{Z}$. Then the quasismooth weighted complete intersection $X_{d_1,...,d_c}$ in $\mathbf{P}(a_0, ..., a_n)$ is well-formed if and only if $q(p) \geq 2$ for all primes p. This follows from a theorem essentially due to Hamm (see [Di, Proposition 2]).

In fact a weighted complete intersection (not necessarily quasismooth) is well-formed if and only if $q(h) \geq 2$ for all integers $h \geq 2$. This is easy to show from the conditions in section I.3.12.

I.3.14 The adjunction formula.

If $X_{d_1,...,d_c}$ in $\mathbf{P}(a_0, ..., a_n)$ is well-formed and quasismooth then $\omega_X \cong \mathcal{O}_X(\sum d_i - \sum a_i)$ (see [WPS, Theorem 3.3.4]). We define the *amplitude* to be this difference of sums, and will usually be denoted by α.

I.3.15 Note. The adjunction formula does not hold if the weighted complete intersection is not well-formed. We give two examples in dimensions 1 and 2 respectively.

(i) Consider the curve C_7 in $\mathbf{P}(1, 2, 3)$. Let $D \subset \mathbf{P}^2$ be the curve $\sigma^{-1}(C)$ where $\sigma : \mathbf{P}^2 \to \mathbf{P}$ is the quotient map (see section I.2.12). Then the curve D is non-singular of degree 7 and so is of genus 15. By Hurwitz Theorem (see [Hart, Corollary IV.2.4]) we calculate that $g(C) = 1$ and so $\omega_C \cong \mathcal{O}_C$. This contradicts the adjunction formula since the amplitude is 1.

(ii) An example in dimension 2 is the surface S_9 in $\mathbf{P}(1,2,2,3)$. A quick calculation shows that this surface is both quasismooth and non-singular. If it is well-formed then the amplitude $\alpha = 1$ and so $K_S^2 = \frac{3}{4}$. This contradicts the fact that $K_S^2 \in \mathbf{Z}$ whenever S is non-singular. In fact S_9 in $\mathbf{P}$ is a smooth K3 surface.

I.3.16 Well-formedness in dimensions greater than 2.

However we find that well-formedness only needs to be checked in dimensions 1 and 2. We have the following generalisation of a proposition due to Dimca (see [Di, Proposition 6]).

I.3.17 Theorem. *Let $X = X_{d_1,\dots,d_c}$ in $\mathbf{P}(a_0,\dots,a_n)$ be a quasismooth weighted complete intersection of dimension greater than 2. Then*

either (i) X is well-formed

or (ii) X is the intersection of a linear cone with other hypersurfaces (i.e. $a_i = d_\lambda$ for some i and λ).

I.3.18 Note.

(1) In case (ii) the weighted complete intersection is isomorphic to an intersection of lower codimension, i.e. $X_{d_1,\dots,\hat{d}_\lambda,\dots,d_c}$ in $\mathbf{P}(a_0,\dots,\hat{a}_i,\dots,a_n)$ or possibly a weighted projective space.

(2) Cases (i) and (ii) are not mutally exclusive. Consider the hypersurface X_2 in $\mathbf{P}(1,1,1,1,2)$ given by

$$f = z + \sum_{i,j} x_i x_j.$$

This is both a linear cone and well-formed, and is, of course, isomorphic to $\mathbf{P}^3$.

We need a preliminary result.

I.3.19 Lemma. *Let Z be the affine variety of all points P which satisfy the determinantal condition:*

$$\text{rank} \begin{pmatrix} g_1^1(P) & \dots & g_1^m(P) \\ \vdots & & \vdots \\ g_c^1(P) & \cdots & g_c^m(P) \end{pmatrix} \leq k$$

where $\{g_i^j\}$ are general weighted homogeneous non-zero polynomials. If Z is non-empty then $\operatorname{codim} Z \leq (m-k)(c-k)$.

This is an elementary fact (see [ACGH, P. 83]).

Proof of Theroem I.3.17. Let $X = (f_1,\dots,f_c) \subset \mathbf{P} = \mathbf{P}(a_0,\dots,a_n)$. Suppose that $\mathbf{P}$ is well-formed and assume that X is quasismooth with dim $X \geq 3$ but not well-formed. So there is a singular stratum $\tilde{\Pi}$ of $\mathbf{P}$ such that $\operatorname{codim}_X(\tilde{\Pi} \cap X) \leq 1$.

If $\operatorname{codim}_X(\tilde{\Pi} \cap X) = 0$ then $X \subset \tilde{\Pi}$ and so X is contained in some coordinate hyperplane. Thus some of the defining polynomials are of the form $f_\lambda = x_i$ for some λ and i. So X is the intersection of at least one linear cone with other hypersurfaces.

So assume that $\operatorname{codim}_X(\tilde{\Pi} \cap X) = 1$. By reordering we can assume that

$$\tilde{\Pi} = (x_k = \dots = x_n = 0) \subset \mathbf{P}$$

for some k. Let $\Pi = p^{-1}\tilde{\Pi} \subset \mathbf{A}^{n+1} - \{0\}$, where $p : \mathbf{A}^{n+1} - \{0\} \to \mathbf{P}$ is the natural projection. Since $\operatorname{codim}_X \tilde{\Pi} = 1$ then $k = \dim \Pi = n - c$. As Π is a fixed component of C_X then we can write the $\{f_\lambda\}$ as:

$$f_\lambda = \sum_{i=k}^{n} x_i g_\lambda^i(x_0, \ldots, x_{k-1}) + \left\{ \begin{matrix} \text{higher order terms} \\ \text{in } x_k, \ldots, x_n \end{matrix} \right\}$$

for all $\lambda = 1, \ldots, c$.

Define M_P to be the matrix

$$M_P = \begin{pmatrix} \partial f_1/\partial x_0(P) & \ldots & \partial f_1/\partial x_n(P) \\ \vdots & & \vdots \\ \partial f_c/\partial x_0(P) & \cdots & \partial f_c/\partial x_n(P) \end{pmatrix}.$$

Singular points on C_X occur whenever rank $M_P < c$. Consider this matrix restricted to Π:

$$M_{P\in\Pi} = \begin{pmatrix} 0, \ldots, 0 & g_1^k(P) & \ldots & g_c^k(P) \\ \vdots & \vdots & & \vdots \\ 0, \ldots, 0 & g_1^n(P) & \cdots & g_c^n(P) \end{pmatrix}.$$

So $P \in \Pi \cap C_X$ is singular whenever rank $(g_i^j) \leq c - 1$. Let Z be just this set.

If Z is empty then, in particular, $\underline{0} \notin Z$. As the entries of M_P are all weighted homogeneous polynomials, they must all be of degree 0. Thus, using the coefficient convention I.3.7,

$$f_\lambda = \sum x_i + \left\{ \begin{matrix} \text{higher order terms} \\ \text{in } x_k, \ldots, x_n \end{matrix} \right\}$$

for all $\lambda = 1, \ldots, c$. So X is the intersection of a linear cone with other hypersurfaces.

So assume that Z is non-empty. By the previous lemma, codim$Z \leq n - k - c + 2$. Remembering that $k = n - c$ we have

$$\dim Z \geq k - (n - k - c + 2) = n - c - 2 = \dim X - 2 \geq 1.$$

So $Z - \{0\}$ is non-empty and thus C_X is not smooth away from the origin, a contradiction. □

I.4 Cohomology of weighted complete intersections.

From [WPS, section 3.4.3] we have:

I.4.1 Lemma. *Let $X = (f_1, \ldots, f_c) \subset \mathbf{P}(a_0, \ldots, a_n)$ be a well-formed quasismooth weighted projective complete intersection. Let A be the graded ring $S(a_0, \ldots, a_n)/(f_1, \ldots, f_c)$ and A_n be the n^{th} graded part of A. Then*

$$\mathrm{H}^i(X, \mathcal{O}_X(n)) \cong \begin{cases} A_n & \text{if } i = 0 \\ 0 & \text{if } i = 1, \ldots, \dim X - 1 \\ A_{-n-\alpha} & \text{if } i = \dim X \end{cases}$$

for all $n \in \mathbf{Z}$.

In particular if S is a well-formed quasismooth weighted projective complete intersection of dimension 2 then the following are equivalent:

(i) S is a K3 surface.

(ii) $\omega_S \cong \mathcal{O}_S$.

(iii) the amplitude $\alpha = \sum_\lambda d_\lambda - \sum_i a_i = 0$.

For hypersurfaces we have the following result due to Steenbrink [S]:

I.4.2 Theorem. *Let* X *be the weighted hypersurface* X_d *in* $\mathbf{P}(a_0, ..., a_n)$ *with defining equation* f *and* $\alpha = d - \sum a_i$. *Then the Hodge structure is given by:*

$$h^{i,j}(X) = \begin{cases} 0 & \text{if } i+j \neq n-1 \text{ and } i \neq j \\ 1 & \text{if } i+j \neq n-1 \text{ and } i = j \\ \dim_{\mathbf{k}} \left(\frac{S(a_0,...,a_n)}{\theta_f} \right)_{jd+\alpha} & \text{if } i+j = n-1 \text{ and } i \neq j \\ \dim_{\mathbf{k}} \left(\frac{S(a_0,...,a_n)}{\theta_f} \right)_{jd+\alpha} + 1 & \text{if } i+j = n-1 \text{ and } i = j \end{cases}$$

where $\theta_f = (\partial f / \partial x_i)_{i=0,...,n}$ *is the Jacobian ideal of* f.

Proof. This follows from [WPS, section 4] and duality.

□

I.4.3 Note. The above formula satisfies the duality relations $h^{i,j} = h^{j,i} = h^{n-1-i,n-1-j}$ for all i and j because

$$\dim_{\mathbf{k}} \left(\frac{S(a_0, ..., a_n)}{\theta_f} \right)_{jd+\alpha} = \dim_{\mathbf{k}} \left(\frac{S(a_0, ..., a_n)}{\theta_f} \right)_{(n-1-j)d+\alpha}.$$

I.4.4 The Euler number.

The Euler number $e(V)$ of a variety V is defined by

$$e(V) = \sum_{i,j} (-1)^{i+j} h^{i,j}(V).$$

For a smooth curve C we have $e(C) = -\deg K_C = 2 - 2g$. For a surface S, with at worst Du Val singularities of types $\{Q_{n_i}\}_i$ where $Q = A, B$ or E, we have Noether's formula:

$$12\chi(\mathcal{O}_S) = K_S^2 + e(S) + \sum_i n_i.$$

In particular the case of a K3 surface S with Du Val singualrities of types $\{Q_{n_i}\}_i$ gives that $h^{1,1}(S) = 20 - \sum_i n_i$ and so $e(S) = 24 - \sum_i n_i$.

When X is a well-formed quasismooth weighted hypersurface of dimension 3 most of the Hodge numbers cancel or are zero and so

$$e(X) = 2(1 - h^{1,2}(X)).$$

I.4.5 Examples.

(i) The hypersurface S_3 in $\mathbf{P}(1,1,1,2)$ has Euler number 5. There are two ways to check this.

(a) It is easy to see that this surface has exactly one singularity, which is of type $\frac{1}{2}(1,1)$ (i.e. of Du Val type A_1). Also the amplitude is -2 and $K_S^2 = (-2)^2 \cdot \frac{3}{2} = 6$. By Noether's formula we have $e(S_3) = 5$.

(b) Alternatively, the Hodge numbers are simple to calculate. Let w, x, y and z be generators of weights 1, 1, 1 and 2 respectively in $S(1,1,1,2)$. Then

$$h^{1,1} = \dim\left(\frac{\mathbf{k}[w,x,y,z]}{(w^2,x^2,y^2,w+x+y)}\right)_1 = 2.$$

Thus the Hodge structure is:

$h^{i,j}$	$i=0$	$i=1$	$i=2$
$j=0$	1	0	0
$j=1$	0	3	0
$j=2$	0	0	1

Thus $e(S_3) = 1+3+1 = 5$.

(ii) The hypersurface X_{10} in $\mathbf{P}(1,1,1,2,5)$ has the following Hodge structure.

$h^{i,j}$	$i=0$	$i=1$	$i=2$	$i=3$
$j=0$	1	0	0	1
$j=1$	0	1	145	0
$j=2$	0	145	1	0
$j=3$	1	0	0	1

Let v, w, x, y and z be generators of weights 1, 1, 1, 2 and 5 respectively in $S(1,1,1,2,5)$. The only hard Hodge number is $h^{1,2}(X) = \dim_{\mathbf{k}}\left(\frac{\mathbf{k}[v,w,x,y,z]}{(v^9,w^9,x^9,y^4,z)}\right)_{20} = 145$. This gives an Euler number of -288.

I.5 Quasismoothness.

In this section we prove conditions for quasismoothness for hypersurfaces and codimension 2 weighted complete intersections.

First we consider the problem of a hypersurface.

I.5.1 Theorem. *The general hypersurface X_d in $\mathbf{P} = \mathbf{P}(a_0,...,a_n)$ of degree d, where $n \geq 1$ is quasismooth if and only if*

either (1) there exists a variable x_i for some i of weight d (i.e. X is a linear cone)

or (2) for every non-empty subset $I = \{i_0,...,i_{k-1}\}$ of $\{0,...,n\}$

either (a) there exists a monomial $x_I^M = x_{i_0}^{m_0}...x_{i_k-1}^{m_k-1}$ of degree d,

or (b) for $\mu = 1,...,k$, there exist monomials $x_I^{M_\mu}x_{e_\mu} = x_{i_0}^{m_{0,\mu}}...x_{i_k-1}^{m_{k-1,\mu}}x_{e_\mu}$ of degree d, where $\{e_\mu\}$ are k distinct elements.

I.5.2 Note. If X_d is a linear cone then f can be written as $f = x_i + g$ for some x_i and X_d is clearly quasismooth. So we need only consider the case where f is not linear in any of the variables (i.e. $\deg x_i = a_i \neq d$ for all i).

Proof. Assume that X_d in $\mathbf{P}$ is not a linear cone. Let F be the linear system of all homogeneous polynomials of degree d with respect to the weights a_i. Let $f \in F$ be a sufficiently general polynomial. Define $X_d : (f = 0) \subset \mathbf{P}$.

$$\begin{array}{ccc} C_X^* & \xrightarrow{i} & \mathbf{A}^{n+1} - \underline{0} \\ \downarrow & & \downarrow \\ X_d & \xrightarrow{i} & \mathbf{P} \end{array}$$

Note that the point $\underline{0}$ is a base point and is usually singular; as this point does not lie in C_X^* this does not affect quasismoothness. By Bertini's Theorem (see [Hart, Remark III.10.9.2]) the only singularities of the general C_X^* lie on the base locus of the linear system F. Any component of the base locus is just a coordinate k-plane for some $k = 0, ..., n$. So the general hypersurface X_d is quasismooth if and only if the general hypersurface C_X^* is non-singular at each point of its intersection with every coordinate k-plane contained in the base locus.

Let Π be a coordinate k-plane for some $k = 1, ..., n$. By renumbering, assume that Π is given by $x_k = ... = x_n = 0$, corresponding to the subset $I = \{0, ..., k-1\}$. Let $\Pi^0 \subset \Pi$ be the open toric stratum where x_0, ..., x_{k-1} are non-zero. Expand f in terms of the coordinates x_k, ..., x_n:

$$f = h(x_0, ..., x_{k-1}) + \sum_{i=k}^{n} x_i g_i(x_0, ..., x_{k-1}) + \left\{ \begin{array}{c} \text{higher order terms} \\ \text{in } x_k, ..., x_n \end{array} \right\}.$$

Assume that one of conditions (a) and (b) hold for I. If (a) holds (i.e. h is non-zero) then Π is not part of the base locus, and so by Bertini's Theorem Π^0 contains no singular points. Geometrically this means that C_X^* intersects Π^0 transversally and so Π^0 is normal to the hypersurface at the points of intersection.

Assume that only (b) holds. So $h \equiv 0$ and $\Pi \subset C_X^*$. By (b) there are at least k of the g_i which are non-zero. Singular points occur exactly on the locus $Z = \bigcap_i (g_i = 0) \subset \Pi^0$, which is an intersection of at least k free linear systems on Π^0. Thus $\dim Z \leq 0$. As Z is a quasicone, it is at worst the origin (compare Lemma I.3.2). Therefore C_X^* is non-singular along Π^0.

As one of these two conditions holds for every non-empty subset I, C_X^* is non-singular.

Conversely assume that conditions (a) and (b) do not hold for all I. Let I be a subset for which these two conditions fail. Without loss of generality assume that $I = \{0, ..., k-1\}$. Let Π be the corresponding coordinate k-plane $x_k = ... = x_n = 0$. As (a) and (b) do not hold

$$f = \sum_{i=k}^{n} x_i g_i(x_0, ..., x_{k-1}) + \left\{ \begin{array}{c} \text{higher order terms} \\ \text{in } x_k, ..., x_n \end{array} \right\}$$

and at most $k-1$ of the g_i are non-zero.

As above, singular points occur exactly on the intersection $Z = \bigcap_{i \geq k} (g_i = 0) \cap \Pi$. Since there are at most $k-1$ of the g_i which are non-zero, $\dim Z \geq k - (k-1) = 1$. Thus Z is non-empty and so C_X^* is singular on Π.

Therefore conditions (a) and (b) are both sufficient and necessary for quasismoothness when X_d in not a linear cone.

□

I.5.3 Note.

(i) The only quasismooth cones are the linear cones. Suppose a variable x_i does not occur in the defining equation f. So $C_X \cong C_{X'} \times \mathbf{A}^1$ where $X' : (f = 0) \subset \mathbf{P}(a_0, ..., \hat{a}_i, ..., a_n)$. Suppose that $C_{X'}$ has a singularity at the origin. Thus $C_{X'} \times \mathbf{A}^1$ has a line of singularities along $\underline{0} \times \mathbf{A}^1$; a contradiction. So $C_{X'}$ is non-singular at the origin and so f must be linear in a variable; this is the linear cone case.

(ii) Without loss of generality we can assume in (b) that $e_\mu \in \{0, ..., n\} - I$, since otherwise this is condition (a).

(iii) For $2|I| \geq n+1$ condition (b) implies condition (a), since there are simply not enough variables x_i.

(iv) Condition (b), with $|I| = 1$, of the theorem gives that for all $i = 0, ..., n$ there must exist a monomial $x_i^n x_{e_i}$, for some e_i, of degree d. This is equivalent to requiring that C_X^* is smooth along the coordinate axes (i.e. X_d is quasismooth at the vertices) and is in practice the most substantial case. Weighted hyperspaces (and polynomials) which satisfy this condition will be said to be *semi-quasismooth.*

(v) C_X contains no coordinate stratum of dimension $\geq (n+1)/2$ except possibly in the linear cone case.

So we have the following corollaries for curves, surfaces and 3-folds.

I.5.4 Corollary. *The curve C_d in $\mathbf{P}(a_0, a_1, a_2)$, where $d > a_i$, is quasismooth if and only if the following hold for all i:*

(1) there exists a monomial $x_i^n x_{e_i}$, for some e_i, of degree d.

(2) there exists a monomial of degree d which does not involve x_i.

Proof. Since $d > a_i$ for all i, X_d is not a linear cone. Conditions (1) and (2) come from considering the conditions of the above theorem for $|I| = 1$ and $|I| = 2$ respectively.

□

The proofs of the following corollaries are similar to the above.

I.5.5 Corollary. *The surface S_d in $\mathbf{P}(a_0, ..., a_3)$, where $d > a_i$, is quasismooth if and only if the following hold:*

(1) for all i there exists a monomial $x_i^n x_{e_i}$ for some e_i of degree d.

(2) for all distinct i, j

either there exists a monomial $x_i^m x_j^n$ of degree d,

or there exist monomials $x_i^{n_1} x_j^{m_1} x_{e_1}$ and $x_i^{n_2} x_j^{m_2} x_{e_2}$ of degree d such that e_1 and e_2 are distinct.

(3) there exists a monomial of degree d which does not involve x_i.

I.5.6 Corollary. *The 3-fold X_d in $\mathbf{P}(a_0, ..., a_4)$, where $d > a_i$, is quasismooth if and only if the following hold:*

(1) for all i there exists a monomial $x_i^n x_{e_i}$ of degree d.

(2) for all distinct i, j

either there exists a monomial $x_i^m x_j^n$ of degree d, *or there exist monomials $x_i^{n_1} x_j^{m_1} x_{e_1}$ and $x_i^{n_2} x_j^{m_2} x_{e_2}$ of degree d such that e_1 and e_2 are distinct.*

(3) *there exists a monomial of degree d which does not involve either x_i or x_j.*

In the codimension 2 case we have:

I.5.7 Theorem. *Suppose the general codimension 2 weighted complete intersection X_{d_1,d_2} in $\mathbf{P} = \mathbf{P}(a_0, ..., a_n)$, where $n \geq 2$, of multidegree $\{d_1, d_2\}$ is not the intersection of a linear cone with another hypersurface. X_{d_1,d_2} in $\mathbf{P}$ is quasismooth if and only if for each nonempty subset $I = \{i_0, ..., i_k - 1\}$ of $\{0, ..., n\}$ one of the following holds:*

(a) *there exists a monomial $x_I^{M_1}$ of degree d_1 and there exists a monomial $x_I^{M_2}$ of degree d_2*

(b) *there exists a monomial x_I^M of degree d_1, and for $\mu = 1, ..., k-1$ there exist monomials $x_I^{M_{mu}} x_{e_{mu}}$ of degree d_2, where $\{e_\mu\}$ are $k-1$ distinct elements.*

(c) *there exists a monomial x_I^M of degree d_2, and for $\mu = 1, ..., k-1$ there exist monomials $x_I^{M_{mu}} x_{e_{mu}}$ of degree d_1, where $\{e_\mu\}$ are $k-1$ distinct elements.*

(d) *for $\mu = 1, ..., k$, there exist monomials $x_I^{M_\mu^1} x_{e_\mu^1}$ of degree d_1, and $x_I^{M_\mu^2} x_{e_\mu^2}$ of degree d_1, such that $\{e_\mu^1\}$ are k distinct elements, $\{e_\mu^2\}$ are k distinct elements and $\{e_\mu^1, e_\mu^2\}$ contains at least $k+1$ distinct elements.*

Proof. Let F_1 and F_2 be linear systems of all homogeneous polynomials of degrees d_1 and d_2 respectively with respect to the weights a_0, ..., a_n. Let $f_1 \in F_1$ and $f_2 \in F_2$ be sufficiently general polynomials. Define

$$X = X_{d_1,d_2} : (f_1 = f_2 = 0) \subset \mathbf{P}.$$

We have the following commutative diagram:

$$\begin{array}{ccc} C_X^* & \xrightarrow{i} & \mathbf{A}^{n+1} - \underline{0} \\ \downarrow & & \downarrow \\ X & \xrightarrow{i} & \mathbf{P} \end{array}$$

The only singularities that can occur in the general member of the family occur on the coordinate strata. So as in the proof of quasismoothness for hypersurfaces, X is quasismooth if and only if C_X^* is smooth along all the coordinate strata.

Assume that one of conditions (a), (b), (c) or (d) holds for each nonempty subset I. Let Π be a coordinate k-plane for some k. By renumbering, we can assume that Π is given by $x_k = ... = x_n = 0$, corresponding to the subset $I = \{0, ..., k-1\}$. As before let Π^0 be the open toric strata where x_0, ..., x_{k-1} are all nonzero. Expand both f_1 and f_2 in terms of the coordinates x_k, ..., x_n:

$$f_\lambda = h_\lambda(x_0, ..., x_{k-1}) + \sum_{i=k}^{n} x_i g_\lambda^i(x_0, ..., x_{k-1}) + \left\{ \begin{array}{c} \text{higher order terms} \\ \text{in } x_k, ..., x_n \end{array} \right\}$$

for $\lambda = 1, 2$.

Suppose (a) holds. So h_1 and h_2 are non-zero on Π^0. If either h_1 or h_2 involves only one monomial then $\Pi^0 \cap C_X^*$ is empty. This includes the case when $k = 1$. So without loss of generality assume that h_1 and h_2 each involve at least 2 monomials and hence $k \geq 2$. Π^0 is not part of the base locus of F_1 or F_2. By Bertini's Theorem $(f_1 = 0)$ and $(f_2 = 0)$ are non-singular on Π^0. Since $(h_1 = 0)$ and $(h_2 = 0)$ are free linear systems on Π^0, $(h_1 = 0)$ and $(h_2 = 0)$ intersect transversally. Thus, at each point of $(h_1 = h_2 = 0) \cap \Pi^0$, there exist two distinct normals. Therefore C_X^* is non-singular along Π^0.

Suppose (b) holds. So h_1 is non-zero and there are at least $k-1$ of the $\{g_1^i\}$ which are non-zero. So Π^0 is not part of the base locus for F_1, and so by Bertini's Theorem we have that $(f_1 = 0)$ is non-singular on Π^0. Singular points occur exactly on the locus

$$Z = (h_1 = 0) \bigcap_i (g_2^i = 0) \subset \Pi^0,$$

which is an intersection of at least $k-1$ free linear systems on $(h_1 = 0) \cap \Pi^0$. Thus $\dim Z \leq 0$ and hence is at worst the origin. Therefore C_X^* is non-singular along Π^0.

The case where condition (c) holds is similar to the case for condition (b).

Suppose that only condition (d) holds. We have

$$f_\lambda = \sum_{i=k}^{n} x_i g_\lambda^i(x_0, \ldots, x_{k-1}) + \left\{ \begin{array}{c} \text{higher order terms} \\ \text{in } x_k, \ldots, x_n \end{array} \right\}$$

for $\lambda = 1, 2$. The normal directions, perpendicular to the plane Π, to the hypersurfaces are $(g_1^k, \ldots, g_1^n)$ and $(g_2^k, \ldots, g_2^n)$. Define the matrix M_P by

$$M_P = \begin{pmatrix} g_1^k(P) & \ldots & g_1^n(P) \\ g_2^k(P) & \ldots & g_2^n(P) \end{pmatrix}.$$

Singular points occur exactly on the locus $Z = \{P : \text{rank } M_P \leq 1\}$. As there are at least k monomials of the form $x_I^M x_e$ of degree d_λ, at least k of the $\{g_\lambda^i\}$ are non-zero. As these are free on Π^0, each row of the matrix M_P is non-zero for each $P \in \Pi^0$. Furthermore this matrix for any $P \in Z$ has at least $k+1$ non-zero columns, since there are at least $k+1$ distinct elements in $\{e_\mu^1, e_\mu^2\}$. By renumbering we can assume that the first $k+1$ columns of M^P are not identically zero on Π^0.

Fix $P \in \Pi^0$. Without loss of generality we can assume that $g_1^k(P) \neq 0$. If $g_2^k(P) = 0$ then $g_2^i(P) \neq 0$ for some $i > k$, and so M^P has rank 2. In this case $P \in C_X^*$ is non-singular. Suppose that $g_2^k(P) \neq 0$. Define $a = g_1^k(P)$, $b = g_2^k(P)$ and

$$Z_P = \bigcap_{i>k} (a g_2^i(Q) - b g_1^i(Q) = 0) \subset \Pi^0.$$

Notice that $P \in Z_P$ if and only if rank $M_P \leq 1$, which is equivalent to $P \in C_X^*$ being singular. Since Z_P is the intersection of k free linear systems on Π^0, $\dim Z_P \leq 0$ and so Z_P is at worst the origin. In particular $P \notin Z_P$ and hence $P \in C_X^*$ is non-singular. Therefore C_X^* is non-singular along Π^0.

As one of these four conditions holds for every non-empty subset I, C_X^* is non-singular.

Conversely assume that none of the conditions (a), (b), (c) or (d) hold for some non-empty subset I. Without loss of generality we can assume that $I = \{0, ..., k-1\}$. Let Π be the corresponding coordinate plane $x_k = ... = x_n = 0$. There are three cases:

(i) $\Pi \not\subset C_{X_{d_1}}$ So h_1 is non-zero and there are at most $k-2$ of the $\{g_2^i\}$ which are non-zero. The singular points are exactly the locus $Z = (h_1 = 0)\bigcap_i(g_2^i = 0)$. However

$$\dim Z \geq k - (k-2) - 1 = 1$$

and so Z contains more than the origin. Thus C_X^* is singular along Π.

(ii) $\Pi \not\subset C_{X_{d_2}}$ Similarly in this case C_X^* is singular along Π.

(iii) $\Pi \subset C_{X_{d_1}} \cap C_{X_{d_2}}$ In this case both h_1 and h_2 are identically zero. So

$$f_\lambda = \sum_{i=k}^{n} x_i g_\lambda^i(x_0, ..., x_{k-1}) + \left\{ \begin{matrix} \text{higher order terms} \\ \text{in } x_k, ..., x_n \end{matrix} \right\}$$

for $\lambda = 1, 2$. As condition (d) does not hold, one of two cases occurs:

either (1) for some λ there are at most $k-1$ of the $\{g_\lambda^i\}$ which are non-zero. Thus the intersection $Z_\lambda = \bigcap_i(g_\lambda^i = 0)$ has dimension at least 1 and so these $\{g_\lambda^i\}$ have a common solution. Therefore the matrix

$$M_P = \begin{pmatrix} g_1^k(P) & ... & g_1^n(P) \\ g_2^k(P) & ... & g_2^n(P) \end{pmatrix}$$

has rank less than 2 for some $P \in Z_\lambda$ and hence C_X^* is singular along Π.

or (2) there are at most k distinct elements in $\{e_\mu^1, e_\mu^2\}$. Thus there are at most k non-zero columns in the matrix M_P. Let $Z = \{P : \text{rank } M_P \leq 1\}$. Therefore

$$\dim Z \geq k - (k-1) = 1$$

and so contains more than just the origin. Therefore C_X^* is singular along Π.

So if one of these four conditions are not satisfied for every subset I then C_X^* is singular.

□

I.5.8 Corollary. *Suppose X_{d_1,d_2} in* **P** *is quasismooth and is not the intersection of a linear cone with another hypersurface. We have the following:*

(i) Every variable x_i occurs in at least one of the defining equations.

(ii) All but at most one variable are in both equations.

(iii) If x_i does not appear in one defining equation then there exists a monomial x_i^m occurring in the other equation.

Proof.

(i) This follows from the previous theorem with $|I| = 1$.

(ii) Suppose, after renumbering, that x_0 and x_1 are not involved in f_1. Then none of the conditions can hold for $I = \{0, 1\}$, a contradiction.

(iii) Suppose that x_i does not appear in f_1. Conditions (a), (b) and (d) cannot hold and so there must be a monomial x_i^m of degree d_2. Geometrically if one of the hypersurfaces is singular along a coordinate axis, because the equation f_i does not involve that variable, then the other hypersurface cannot pass through that axis.

□

I.6 Cyclic singularities and counting points.

In this section we give combinatorial conditions for cyclic quotient singularities to be isolated and canonical (see [R4, Definition 1.1] for the definitions of canonical and terminal singularities). The last two lemmas of this section are used to count the number of intersections along 1 and 2 dimensional strata. We also give an alternative proof of the first of these lemmas in terms of the Minkowski mixed volume of integral polyhedra.

I.6.1 Lemma. *A canonical curve point is smooth.*

This is clear since canonical singularities are normal. For dimension 2 we have:

I.6.2 Lemma. *The following are equivalent:*

(1) Q in S is a cyclic quotient canonical surface singularity.
(2) Q is of type $\frac{1}{r}(a,-a)$ for some index r and a coprime to r.
(3) Q is of type $\frac{1}{r}(1,-1)$ for some index r.

The above singularities are Du Val singularities of type A_{r-1}.

For 3-folds we have the following due to White, Morrison, Stevens, Danilov and Frumkin:

I.6.3 Lemma. *The following are equivalent:*

(1) S is an isolated cyclic quotient terminal 3-fold singularity.
(2) S is of type $\frac{1}{r}(b_0,b_1,b_2)$, for some positive integers r, b_0, b_1, b_2, with $r \geq 2$, r and b_i coprime and $r \mid b_i + b_j$ for a pair of distinct i, j.
(3) S is of the form $\frac{1}{r}(1,-1,b)$ for some $r \geq 2$ and b coprime to r.

The following two lemmas are very useful for calculating the number and arrangement of singularities on a complete intersection.

I.6.4 Lemma. *Let x and y be of weight a_0 and a_1 respectively, where* $\mathrm{hcf}(a_0,a_1)=1$. *Suppose $f(x,y)$ is a homogeneous polynomial of degree d, semi-quasismooth (see Note I.5.3(iv)) and sufficiently general. Let $P_0=[1,0]$ and $P_1=[0,1]$. Then $X_d:(f=0)$ in $\mathbf{P}(a_0,a_1)$ is a finite set and:*

(i) P_i is in X_d if and only if $a_i \nmid d$ for $i=0,1$,
(ii) there are exactly $\lfloor\frac{d}{a_0a_1}\rfloor$ other points in X_d.

Proof. Notice that x^{a_1}/y^{a_0} is an invariant of the group action of $\mathbf{k}^*$ on $\mathbf{A}^2-\underline{0}$ which defines $\mathbf{P}(a_0,a_1)$. There are four cases:

(i) $a_0 \mid d$ and $a_1 \mid d$. Then f is of the form

$$f=x^{d/a_0}+\ldots+y^{d/a_1},$$

written using the coefficient convention (see section I.3.7). So

$$\frac{f}{y^{d/a_1}}=\left(\frac{x_1^a}{y_0^a}\right)^{d/a_0a_1}+\ldots+1,$$

which has exactly $\frac{d}{a_0a_1}$ roots.

(ii) $a_0 \not| d$ and $a_1 \mid d$. Since X_d is semi-quasismooth, f is of the form

$$f = y(x^{(d-a_1)/a_0} + \dots + y^{(d-a_1)/a_1}).$$

The solution $y = 0$ gives the point P_0.

$$\frac{f}{y^{d/a_1}} = \left(\frac{x_1^a}{y_0^a}\right)^{(d-a_1)/a_0 a_1} + \dots + 1.$$

This has exactly $n = \frac{d-a_1}{a_0a_1}$ roots. So $d = na_0a_1 + a_1$. As $a_0 \not| d$ then $a_0 > 1$, and so $a_1 < a_0a_1$. Thus $n = \lfloor\frac{d}{a_0a_1}\rfloor$.

(iii) $a_0 \mid d$ and $a_1 \not| d$. Similar to (ii).

(iv) $a_0 \not| d$ and $a_1 \not| d$.

$$f = xy(x^{(d-a_0-a_1)/a_0} + \dots + y^{(d-a_0-a_1)/a_1})$$

So the two vertices P_0 and P_1 are solutions. Also

$$\frac{f}{xy^{d/a_1}} = \left(\frac{x_1^a}{y_0^a}\right)^{(d-a_0-a_1)/a_0 a_1} + \dots + 1,$$

which has exactly $n = \frac{d-a_0-a_1}{a_0a_1}$ roots on $\mathbf{P} - \{P_0, P_1\}$. So $d = na_0a_1 + (a_0 + a_1)$. As $a_0 \not| d$ and $a_1 \not| d$ then $a_0, a_1 \geq 2$ and not both equal to 2. Thus

$$a_0\, a_1 = (a_0 - 1)(a_1 - 1) - 1 + a_0 + a_1a_0 + a_1.$$

Therefore $n = \lfloor\frac{d}{a_0a_1}\rfloor$.

□

I.6.5 Lemma. *Let x_0, x_1 and x_2 have weights a_0, a_1 and a_2, where $hcf(a_0, a_1, a_2) = 1$. Let f and g be sufficiently general semi-quasismooth homogeneous polynomials in $\mathbf{k}[x_0, x_1, x_2]$ of degrees d and e respectively. Suppose that $X_{d,e} : (f = 0, g = 0)$ in $\mathbf{P}(a_0, a_1, a_2)$ is a finite set. Let*

$n_{i,j}$ be the number of points of $X_{d,e}$ along the edge P_iP_j,

$h_{i,j} = \mathrm{hcf}(a_i, a_j)$,

n_i be the number of points at the vertex P_i (i.e. $n_i = 0, 1$),

N be the number of points in $\mathbf{P} - \Delta$.

Then:

$$\frac{de}{a_0\, a_1\, a_2} = \sum_i \frac{n_i}{a_i} + \sum_{i>j} \frac{n_{i,j}}{h_{i,j}} + N$$

I.6.6 Note.

(1) $X_{d,e}$ in $\mathbf{P}$ is not automatically finite (consider $X_{5,9}$ in $\mathbf{P}(1, 2, 4)$).

(2) Similar results hold for higher codimensions and involve induction on the dimension.

(3) Notice that Lemma I.6.4 can be deduced from the above (consider $X_{d,1}$ in $\mathbf{P}(a_0, a_1, 1)$).
(4) This also has connections with the Minkowski mixed volumes of Newton polyhedra (see after proof).

Proof. Let $\sigma : \mathbf{P}^2 \to \mathbf{P}$ be the quotient map defined in section I.2.12. Let $F = \sigma^* f$ and $G = \sigma^* g$. Since $X_{d,e}$ is finite, $V(F)$ and $V(G)$ have no common components. By Bézout's theorem $Y = V(F,G)$ in $\mathbf{P}^2$ consists of exactly de points counted with multiplicity.

The restriction of σ to $\mathbf{P}^2 - \Delta$ is $a_0a_1a_2$-to-1, onto $\mathbf{P} - \Delta$. As there are N points on $\mathbf{P} - \Delta$ this accounts for $a_0a_1a_2N$ points on $\mathbf{P}^2 - \Delta$.

The restriction of σ to the line Q_iQ_j is $a_ia_j/h_{i,j}$-to-1, onto P_iP_j. Without loss of generality assume that $h_{i,j} \mid d$ but that $h_{i,j} \not| e$. Let k be such that $\{i,j,k\} = \{0,1,2\}$. Notice that $x_k \mid g$, or else there would exist a monomial $x_i^a x_j^b$ of degree e, contradicting $h_{i,j} \not| e$. Then f and g are of the form:

$$f = x_i^m x_j + x_j^m x_i + \dots.$$
$$g = x_k(x_i^{n'} + x_j^{m'} + \dots).$$

Thus F and G are of the form:

$$F = X_i^{ma_i} X_j^{a_j} + X_j^{na_j} X_i^{a_i} + \dots$$
$$G = X_k^{a_k}(X_i^{n'a_i} + X_j^{m'a_j} + \dots).$$

We localise F and G by setting $X_i = 1$, to give the corresponding affine equations $\overline{F}$ and $\overline{G}$. Let $[X_i, X_j, X_k] = [1, \xi, 0]$ be a point of intersection along the line Q_iQ_j. The multiplicity μ of the intersection is given by:

$$\begin{aligned}\mu &= \text{mult}(F, G, [1,\xi,0]) \\ &= \text{mult}(\overline{F}, \overline{G}, (\xi, 0)) \\ &= \text{mult}(X_i^{a_i} + X_i^{ma_i} + \dots, X_k^{a_k}, (\xi, 0)) \\ &= \text{mult}(X_i{'} + \dots, X_k^{a_k}, (0,0)) \\ &= a_k\end{aligned}$$

where $X_i' = X_i - \xi$. So this line contributes $(n_{i,j}a_k)a_ia_j/h_{i,j}$ points (counted with multiplicity) to Bézout's theorem.

Consider the vertex Q_i. If P_i is contained in X then $a_i \not| d$ and $a_i \not| e$. As X is semi-quasismooth, $a_i \mid d - a_j$ and $a_i \mid e - a_k$ for distinct i, j, and k. So f and g are of the form:

$$f = x_i^n x_j + \dots$$
$$g = x_i^m x_k + \dots$$

Thus:

$$F = X_i^{na_i} X_j + \dots$$
$$G = X_i^{ma_i} X_k + \dots$$

The intersection multiplicity μ at Q_i is:

$$\mu = \text{mult}(F, G, Q_i).$$

Localising at $X_i = 1$ gives:

$$\begin{aligned}\mu &= \mathrm{mult}(\overline{F}, \overline{G}, (0,0)) \\ &= \mathrm{mult}(X_j^{a_j} + ..., X_k^{a_k} + ..., (0,0)) \\ &= a_j a_k.\end{aligned}$$

Clearly $X_j^{a_j}$ and $X_k^{a_k}$ are the smallest degree monomials in $\overline{F}$ and $\overline{G}$. So this gives a contribution of $a_j a_k n_i$.

Combining the above gives:

$$de = \sum_{\text{distinct } i,j,k} n_i a_j a_k + \sum_{i>j,\ k\neq i,j} \frac{n_{i,j} a_i a_j a_k}{h_{i,j}} + N a_0 a_1 a_2,$$

which rearranges to give the formula in the lemma.

□

An alternative proof of the above two lemmas is via Newton polyhedra and the Minkowski mixed volume (see both [Be] and [Ku]).

I.6.7 Definition. An *integral polyhedron* S is a polyhedron in $\mathbf{R}^n$ with vertices in $\mathbf{Z}^n$. The n-dimensional volume of S will be denoted by $V_n(S)$, where the volume of the unit parallelepiped is 1.

I.6.8 Definition. For each $m = (m_1, ..., m_n) \in \mathbf{Z}^n$ define

$$x^m = x_1^{m_1} ... x_n^{m_n}.$$

Let $f \in \mathrm{k}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$ be a Laurent polynomial. Then

$$f = \sum_{m \in \mathbf{Z}^n} c_m x^m,$$

where all but a finite number of the $\{c_m\}$ are zero. The *Newton polyhedron* Newton(f) of f is the convex hull of $\{m \in \mathbf{Z}^n : c_m \neq 0\}$, and is an integral polyhedron.

I.6.9 Definition. Let $\mathcal{S} = \{S_i : i = 1, ..., n\}$ be a set of integral polyhedra. The *Minkowski mixed volume* $V(\mathcal{S})$ of $\mathcal{S}$ is given by:

$$V(\mathcal{S}) = (-1)^{n-1} \sum V_n(S_i) + (-1)^{n-2} \sum_{i>j} V_n(S_i + S_j) + ... + V_n(S_1 + ... + S_n)$$

where $S_i + S_j = \{s_i + s_j : s_i \in S_i, s_j \in S_j\}$.

This is the classical formula up to a multiple of $n!$

Let T^n be the n-dimensional torus $(\mathrm{k}^*)^n$. This corresponds to the open toric stratum in $\mathbf{P}$. Let $\mathcal{F}$ be a system of n sufficiently general Laurent polynomials $\{f_i : T^n \to \mathrm{k}\}$ with corresponding Newton polyhedra $\mathcal{S} = \{S_i\}$. The roots of these n polynomials in T^n are isolated. Let $L(\mathcal{F})$ be the number of such roots, counted with multiplicity. Then [Be, Theorem A] gives:

$$L(\mathcal{F}) = V(\mathcal{S}).$$

I.6.10 Alternative proof of Lemma I.6.4. Let T^1 be the torus $x_0 x_1 \neq 0$ in $\mathbf{P} = \mathbf{P}(a_0, a_1)$. Suppose that $a_0, a_1 \mid d$. Then $f = x_0^{d/a_0} + \ldots + x_1^{d/a_1}$. So

$$N_f = \text{Newton}(f) = [(d/a_0, 0), (0, d/a_1)],$$

where $[P, Q]$ denotes the line segment in $\mathbf{Z}^2$ from P to Q. So $V_1(N_f) + 1$ is the number of integral points on N_f, i.e. the number of solutions to

$$\{(\alpha, \beta) \in \mathbf{Z}^2 \; : \; \alpha \geq 0, \beta \geq 0, \alpha a_0 + \beta a_1 = d\}.$$

For a solution (α, β) we have $\alpha = (d - \beta a_1)/a_0 \in \mathbf{Z}$, i.e. $d \equiv \beta a_1 \bmod a_1$. As a_0 and a_1 are coprime, then a_1 is invertible modulo a_0, with inverse s. So $\beta \equiv ds \bmod a_0$, i.e. $\beta = ds + n a_0$ for some n. Also $0 \leq \beta \leq d/a_1$. So

$$-\frac{ds}{a_0} \leq n \leq \frac{d}{a_0\, a_1} - \frac{ds}{a_0}.$$

There are $\frac{d}{a_0 a_1} + 1$ such solutions. Thus f has $\frac{d}{a_0 a_1}$ roots on the torus T^1 in $\mathbf{P}$.

Similarly when $a_0 \not| \, d$, etc.. □

Lemma I.6.5 can be proved using analogous methods.

I.7 Determination of singularities on weighted complete intersections.

In this section we shall determine the singularities of three weighted complete intersections, presenting the calculations in detail. These examples are a good introduction to the theorems giving arithmetic conditions for weighted complete intersections to have at worst isolated canonical singularities.

I.7.1 The surface $S = S_{36}$ in $\mathbf{P}(7, 8, 9, 12)$.

We shall see that this surface has four singularities, one each of type A_2, A_3, A_6 and A_7. The Euler number of such a K3 surface is 6, which is the lowest Euler number found in any of the lists of weighted complete intersection K3 surfaces.

Let w, x, y and z be the homogeneous coordinates on $\mathbf{P} = \mathbf{P}(7, 8, 9, 12)$ of weights 7, 8, 9 and 12 respectively. Let f be a general polynomial of homogeneous degree 36. Using the coefficient convention (see section I.3.7) we have:

$$f = w^4 x + x^3 z + y^4 + z^3 + \text{others}.$$

So S is well-formed and, by Theorem I.5.1, is quasismooth. So the singularities of S arise only due to the singularities of $\mathbf{P}$ and occur only on the edges and vertices of $\mathbf{P}$. Consider the vertices.

P_0: f contains no monomial of the form w^n for any n and so $P_0 \in S$. Consider the affine piece $(w = 1)$. The point $P_0 \in S$ looks like:

$$(\tilde{f} = f(1, x, y, z) = x + \ldots = 0) \subset \mathbf{A}^3/\epsilon$$

where ϵ is a primitive 7^{th} root of unity and acts on the coordinates of $\mathbf{A}^3$ via:

$$x \mapsto \epsilon^8 x = \epsilon x$$
$$y \mapsto \epsilon^9 y = \epsilon^2 y$$
$$z \mapsto \epsilon^{12} z = \epsilon^5 z.$$

Notice that $\partial f/\partial x = w^4 + \dots$ is non-zero at P_0. By the Inverse Function Theorem y and z are local coordinates around $P_0 \in S$. This gives a singularity of type $\frac{1}{7}(2,5)$, which is Du Val of type A_6.

P_1: Again f contains no monomial of type x^n and so $P_1 \in S$. As above, this gives a Du Val singularity of type A_7.

P_2, P_3: Since f contains the monomials y^4 and z^3 then $P_2, P_3 \notin S$.

There are only two singular edges in $\mathbf{P}$, P_1P_3 which is analytically isomorphic to $\mathbf{k}^* \times \frac{1}{4}(3,1)$ and P_2P_3 which is $\mathbf{k}^* \times \frac{1}{3}(2,1)$.

P_1P_3: Since $f_{|P_1P_2} = x^3z + z^3 = z(x^3 + z^2)$ then S does not contain the edge P_1P_3. As $x \neq 0$ and $z \neq 0$ on the edge P_1P_3 then the affine piece $(z = 1)$ contains all of the intersection points. Since $(\partial f/\partial x)_{|z=1} = x^2 + \dots$ is non-zero then w and y are local coordinates on S at each of the points of $S \cap P_1P_3$. This is clear geometrically since S is a general element of all degree 36 hypersurfaces and so it must cross this line transversally. Thus each point is a singularity, which is analytically locally isomorphic to $\mathbf{A}^2/\epsilon$ where the coordinates of $\mathbf{A}^2$ are w and y and ϵ is a 4^{th} root acting via:

$$w \mapsto \epsilon^7 w = \epsilon^3 w$$
$$y \mapsto \epsilon^9 y = \epsilon y.$$

This gives a Du Val singularity of type A_3.

We must now count the number of intersection points on this edge. Each point of the intersection is given by the equation $x^3 + z^2 = 0$ in $\mathbf{P}(8,12)$. This is just X_{24} in $\mathbf{P}(8,12)$, i.e. X_6 in $\mathbf{P}(2,3)$. Either from first principles or from Lemma I.6.4 we can see that this is exactly one point.

P_2P_3: As above, there is exacltly one Du Val singularity, which is of type A_2, along this edge.

I.7.2 The 3-fold $X = X_{46}$ in $\mathbf{P}(4,5,6,7,23)$.

The hypersurface X_{46} in $\mathbf{P}(4,5,6,7,23)$ has the following singularities:

3 of type $\frac{1}{2}(1,1,1)$,
1 of type $\frac{1}{4}(3,1,1)$,
1 of type $\frac{1}{5}(4,1,2)$,
1 of type $\frac{1}{6}(5,1,1)$,
1 of type $\frac{1}{7}(6,1,3)$.

The singularities are checked as follows. Let v, w, x, y and z be the homogeneous coordinates of $\mathbf{P} = \mathbf{P}(4,5,6,7,23)$ of weights 4, 5, 6, 7 and 23 respectively. Let f be a general polynomial of homogeneous degree 46. Then f (using the coefficient convention) is of the form:

$$f = v^{10}x + w^8x + x^7v + y^6v + z^2 + \text{others}.$$

This is well-formed and quasismooth (see Theorem I.5.1). So the singularities of the hypersurface occur only on the edges and at the vertices of $\mathbf{P}$. Consider the vertices in reverse order:

P_4: Since f contains the monomial z^2 with a non-zero coefficient, $f(P_4) \neq 0$ and so $P_4 \notin X_{46}$.

P_3: There is no monomial of the form y^n for any n in f, and so $P_3 \in X_{46}$. Consider the affine piece $(y = 1)$. $P_3 \in X_{46}$ looks like:

$$(\tilde{f} = f(v, w, x, 1, z) = v + ... = 0) \subset \mathbf{A}^4/\epsilon$$

where ϵ is a primitive 7^{th} root of unity and acts as:

$$\begin{aligned} v &\mapsto \epsilon^4 v, \\ w &\mapsto \epsilon^5 w, \\ x &\mapsto \epsilon^6 x, \\ z &\mapsto \epsilon^{23} z. \end{aligned}$$

Notice that $\partial f/\partial v = y^6 + ...$ is non-zero at P_3. By the Inverse Function Theorem w, x and z are local coordinates on X_{46} around $P_3 \in X_{46}$. Thus the singularity here is of type $\frac{1}{7}(5, 6, 23)$. This is equivalent to $\frac{1}{7}(6, 1, 3)$, which is terminal.

P_2: Again there is no monomial of the form x^n for any n in f, and so $P_2 \in X_{46}$. Consider the affine piece $(x = 1)$. $P_2 \in X_{46}$ looks like:

$$(\tilde{f} = f(v, w, 1, y, z) = v + ... = 0) \subset \mathbf{A}^4/\epsilon$$

where ϵ is a primitive 6^{th} root of unity and acts as:

$$\begin{aligned} v &\mapsto \epsilon^4 v, \\ w &\mapsto \epsilon^5 w, \\ y &\mapsto \epsilon^7 y, \\ z &\mapsto \epsilon^{23} z. \end{aligned}$$

Notice that $\partial f/\partial v = x^7 + ...$ is non-zero at P_3. By the Inverse Function Theorem, w, y and z are local coordinates on X_{46} around $P_2 \in X_{46}$. Thus the singularity here is of type $\frac{1}{6}(5, 7, 23)$. This is equivalent to $\frac{1}{6}(5, 1, 1)$, which is terminal.

P_1: $P_1 \in X_{46}$ is locally $f = x + ... = 0$ and gives a terminal singularity of type $\frac{1}{5}(4, 1, 2)$.

P_0: $P_0 \in X_{46}$ is locally $f = x + ... = 0$ and gives a terminal singularity of type $\frac{1}{4}(3, 1, 1)$.

Consider the edges of $\mathbf{P}$. An edge P_iP_j is singular if and only if $h = \text{hcf}(a_i, a_j) \neq 1$. In which case it is analytically equivalent to $\mathbf{k}^* \times \frac{1}{h}(a_0, ..., \hat{a}_i, ..., \hat{a}_j, ..., a_4)$. So only the edge P_0P_2 is singular and looks like $\mathbf{k}^* \times \frac{1}{2}(1, 1, 1)$. Since $2 = \text{hcf}(4, 6) \mid 46$, the hypersurface does not contain this line. Lemma I.6.4 is used on X_{46} in $\mathbf{P}(4, 6)$, after cancelling the common factor, to give three points of intersection. Alternatively,

$$f_{|P_0P_2} = uxg_{36}(u, x) = uxg_3(u^3, x^2),$$

where g_{36} and g_3 are polynomials of degree 36 and 3 respectively. There are exactly three solutions to $g_3 = 0$, and so there are three points of intersection. So X_{46} crosses P_0P_2 transversally and hence there are three singularities, each of type $\frac{1}{2}(1,1,1)$, along P_0P_2.

I.7.3 The 3-fold $X_{12,14}$ in $\mathbf{P}(2,3,4,5,6,7)$.

The family $X_{12,14}$ in $\mathbf{P}(2,3,4,5,6,7)$ is an anticanonically embedded Fano 3-fold with only the following isolated terminal singularities: 1 of type $\frac{1}{5}(4,1,2)$, 2 of type $\frac{1}{3}(2,1,1)$ and 7 of type $\frac{1}{2}(1,1,1)$.

The singularities are checked as follows. Let u, v, w, x, y and z be the homogeneous coordinates of weights 2, 3, 4, 5, 6 and 7 respectively. Let f, g be homogeneous polynomials of degrees 12 and 14 respectively. Then $X = (f = g = 0) \subset \mathbf{P} = \mathbf{P}(2,3,4,5,6,7)$.

Consider the vertices of the weighted projective space $\mathbf{P}$. Since $5 \not| 12$ and $5 \not| 14$, $P_3 \in X$. So

$$f = x^2u + ...$$
$$g = x^2w + ...$$

Thus $\{v, y, z\}$ are local coordinates around P_3, which is therefore a singularity of type $\frac{1}{5}(3,6,7)$, i.e. $\frac{1}{5}(4,1,2)$. There are no other vertices contained in X.

Consider the 1-dimensional loci of $\mathbf{P}$.

P_0P_2: $h = \text{hcf}(2,4) = 2$ and

$$f = u^6 + w^3 + ...$$
$$g = u^7 + w^2y + ...$$

So the local coordinates are $\{v, x, z\}$ and the singularities are of type $\frac{1}{2}(1,1,1)$. There are three such intersection points (by Lemma I.6.4 applied to X_6 in $\mathbf{P}(1,2)$).

P_0P_4: Likewise $h = \text{hcf}(2,6) = 2$ and

$$f = u^6 + y^2 + ...$$
$$g = u^7 + u^5w + y^2u + ...$$

$(f = 0)$ in $\mathbf{P}(1,3)$ is two points by Lemma I.6.4. So there are two singularities, each of type $\frac{1}{2}(1,1,1)$, along P_0P_4.

P_2P_4: There is exactly one singularity, which is of type $\frac{1}{2}(1,1,1)$, on this line.

P_1P_4: This time $h = \text{hcf}(3,6) = 3$ and

$$f = v^4 + y^2 + ...$$
$$g = v^4u + y^2u + ...$$

So there are two of type $\frac{1}{3}(1,-1,1)$ on P_1P_4.

Consider the only singular 2-dimensional locus, $P_0P_2P_4$, of $\mathbf{P}$ where $h = \text{hcf}(2,4,6) = 2$. By Lemma I.6.5, there are seven intersection points (some of which have already been counted), all of type $\frac{1}{2}(1,1,1)$.

II

Lists of various weighted complete intersections.

II.1 Preamble.

The aim of this chapter is to produce lists of hypersurface and codimension 2 weighted complete intersections of dimension at most 3 with at worst isolated canonical singularities. We present various theorems giving conbinatoric conditions on the weights and degrees of such intersections. From these conditions we can produce lists of intersections (along with their corresponding singularities). In most cases a computer was used for its speed and inability to become bored.

Sections II.2 and II.3 treat the cases of dimension 1 and 2 respectively; and give corresponding lists. Section II.4 deals with the 3-fold case (both hypersurfaces and codimension 2) and sections II.5 and II.6 deal with the particular cases of canonical 3-folds and $\mathbf{Q}$-Fano 3-folds respectively. Section II.7 gives an alternative method for producing canonically and anticanonically embedded 3-fold complete intersections using the Poincaré series of a ring.

II.2 Weighted curve hypersurfaces.

II.2.1 Theorem. *A weighted curve complete intersection is smooth if and only if it is quasismooth.*

Proof. Any 1-dimensional cyclic quotient singularity is of type $\frac{1}{r}(a)$ for some coprime r and a. Let x be the coordinate on $\mathbf{A}^1$. The group $\mathbf{Z}_r$ acts via:

$$x \mapsto \epsilon^a x$$

where ϵ is a primitive r^{th} root of unity. So

$$\mathbf{A}^1/\mathbf{Z}_r \cong \text{Spec } \mathbf{k}[x]^{\mathbf{Z}_r} \cong \text{Spec } \mathbf{k}[x^r] \cong \text{Spec } \mathbf{k}[x] \cong \mathbf{A}^1$$

So this is non-singular. Notice that this group action is just a quasi-reflection (see section I.2.8).

□

From [O&W, Corollary 3.5] we have a formula for the genus of dimension 1 hypersurfaces.

II.2.2 Theorem. *Let C_d in $\mathbf{P}(a_0, a_1, a_2)$ be a non-singular curve. Then the genus g is given by:*

$$g = \frac{1}{2}\left(\frac{d^2}{a_0a_1a_2} - d\sum_{i>j}\frac{\mathrm{hcf}(a_i,a_j)}{a_ia_j} + \sum_{i=0}^{2}\frac{\mathrm{hcf}(d,a_i)}{a_i} - 1\right).$$

II.2.3 Theorem. *A weighted curve C_d in $\mathbf{P}(a_0, a_1, a_2)$ is well-formed, not a linear cone and quasismooth if and only if for each i the following three conditions hold:*

(1) $a_i < d$,

(2) $a_i \mid d$,

and (3) $\mathrm{hcf}(a_i, a_j) = 1$ for all distinct i, j.

Proof. C is well-formed if and only if $a_i \mid d$ for all i and $\mathrm{hcf}(a_i, a_j) = 1$ for all distinct i, j (see section I.3.10). These are conditions (2) and (3).

Suppose C is not a linear cone and quasismooth. Then conditions (1) holds. Also $a_i \mid d - a_e$ for some e. But this is already satisfied by condition (2).

The converse follows immediately from conditions (1), (2) and (3).

□

II.2.4 Smooth weighted curve hypersurfaces with amplitude $\alpha = d - \sum a_i = 0$.

We list the only smooth weighted curves of codimension 1 with $\alpha = 0$ satisfying the above conditions.

Curve	D
C_3 in $\mathbf{P}(1,1,1)$	$3P$
C_4 in $\mathbf{P}(1,1,2)$	$2P$
C_6 in $\mathbf{P}(1,2,3)$	P

All are elliptic curves (i.e. $g = 1$ and $\omega \cong \mathcal{O}_C$) and are given by $\mathrm{Proj} R_C$ where R_C is:

$$R_C = \bigoplus_{n\geq 0} \mathrm{H}^0(\mathcal{O}_C(nD)),$$

and D is given in the above table.

II.2.5 The calculation. The above curves are the only ones satisfying the conditions of Theorem II.2.3. This is demomstrated as follows.

Order the $\{a_i\}$ by $a_0 \leq a_1 \leq a_2$. conditions (2) and (3) of Theorem II.2.3 give $a_0a_1a_2 \mid d$. Let $d = \lambda a_2$. As $\alpha = 0$ then $3a_2 \geq a_0 + a_1 + a_2 = d = \lambda a_2$. So $\lambda \leq 3$ (i.e. $\lambda = 2, 3$).

(i) $\lambda = 2$. So $a_0a_1 \mid 2$. Either $(a_0, a_1) = (1, 1)$ (i.e. C_4 in $\mathbf{P}(1,1,2)$) or $(a_0, a_1) = (1, 2)$ (i.e. C_6 in $\mathbf{P}(1,2,3)$).

(ii) $\lambda = 3$. So $a_0a_1 \mid 3$. Either $(a_0, a_1) = (1, 1)$ (i.e. C_3 in $\mathbf{P}(1,1,1)$) or $(a_0, a_1) = (1, 3)$ in which case $a_2 = 2 < a_1$, a contradiction.

II.2.6 The ring R_C. Consider an elliptic curve C and the divisor $D = 2P$, where P is any point on C. By Riemann-Roch,

$$h^0(nD) - h^1(nD) = \deg(nD) + (1 - g).$$

As $D > K \equiv 0$, then $h^1(nD) = 0$ for all $n \geq 1$. Also $g = 1$ and so

$$h^0(nD) = \deg(nD) = 2n.$$

Thus $h^0(D) = 2$ and $h^0(2D) = 4$. Let x_0, x_1 be a basis for $\mathrm{H}^0(D)$. Then x_0^2, x_0x_1 and x_1^2 are linearly independent elements of $\mathrm{H}^0(2D)$. As $h^0(2D) = 4$ then there exists an extra element y of degree 4.

Consider the map:

$$\phi_n : \mathrm{H}^0(D) \otimes \mathrm{H}^0((n-1)D) \to \mathrm{H}^0(nD).$$

Notice that x_0 and x_1 have no common base points. By the base-point-free pencil trick (see [ACGH, p. 126]),

$$\mathrm{Ker}\phi_n \cong \mathrm{H}^0((n-1)D - D) = \mathrm{H}^0((n-2)D)$$

which has dimension $2(n-2)$. Also $\mathrm{H}^0(D) \otimes \mathrm{H}^0((n-1)D)$ has dimension $2.2(n-1)$. So $\dim \mathrm{Im}\, \phi_n = 2n$, and hence ϕ_n is onto for all $n \geq 2$. This means that $\mathrm{H}^0(nD)$ is generated by $\mathrm{H}^0(D)$ and $\mathrm{H}^0((n-1)D)$.

So we have the following table of bases for the $\mathrm{H}^0(nD)$.

n	$h^0(nD)$	monomials
1	2	x_0, x_1.
2	4	x_0^2, x_0x_1, x_1^2, y.
3	6	x_0^3, $x_0^2x_1$, $x_0x_1^2$, x_1^3, x_0y, x_1y.
4	8	x_0^4, $x_0^3x_1$, $x_0^2x_1^2$, $x_0x_1^3$, x_1^4, x_0^2y, x_0x_1y, x_1^2y, y^2.

Notice that $\mathrm{H}^0(4D)$ has dimension 8, but there are 9 monomials. Since ϕ_4 is onto then the first eight in the list are linear independent. So there must be a relation of the form:

$$f = y^2 + yh_2(x_0, x_1) - g_4(x_0, x_1),$$

where h_2 and g_4 are homogeneous polynomials of degrees 4 and 2 respectively.

The number N_n of monomials in $\mathrm{H}^0(nD)$ is given by:

$$N_n = 1 + n \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Suppose that f was the only relation, then the dimension of the module generated by the monomials of degree n is $N_n - 1.N_{n-4} = 2n$, which is the same as $h^0(nD)$.

So the ring R is $\mathbf{k}[x_0, x_1, y]/(f)$, where x_i has weight 1 and y has weight 2, i.e. the curve is C_4 in $\mathbf{P}(1,1,2)$. This technique should be compared to that in [M, Lecture 1, p. 17 - 21] and to Weierstrass normal form.

II.2.7 Smooth weighted curve hypersurfaces with amplitude $\alpha = d - \sum a_i = 1$.

There are only two such curves which satisfy the conditions of Theorem II.2.3:

curve	genus	ω_C
C_4 in $\mathbf{P}(1,1,1)$	3	$\mathcal{O}_C(1)$
C_6 in $\mathbf{P}(1,1,3)$	2	$\mathcal{O}_C(1)$

These were calculated in a similar way to those of section II.2.5 and the genera by the formula in Theorem II.2.2.

II.3 Weighted surface complete intersections.

In this section we give necessary and sufficient conditions for surface weighted complete intersections of codimension 1 and 2 to be quasismooth, well-formed and have at worst canonical singularities. We also include lists of such intersections.

II.3.1 Theorem. *Let S_d in $\mathbf{P} = \mathbf{P}(a_0, a_1, a_2, a_3)$ be a general hypersurface of degree d and let $\alpha = d - \sum a_i$. S_d is quasi-smooth, well-formed with at worst canonical quotient singularities and is not a linear cone if and only if all the following hold:*

(1) For all i,
- *(i) $d > a_i$.*
- *(ii) there exists e such that $a_i \mid d - a_e$ (i.e. there exists a monomial $x_i^n x_e$ of degree d).*
- *(iii) there exists a monomial of degree d which does not involve x_i.*
- *(iv) if $a_i \nmid d$, then $a_i \mid \alpha$.*

(2) For all distinct i, j, with $h = \mathrm{hcf}(a_i, a_j)$, then
- *(i) $h \mid d$.*
- *(ii) $h \mid \alpha$.*
- *(iii) one of the following holds:*
  either there exists a monomial $x_i^m x_j^n$ of degree d,
  or there exist monomials $x_i^{n_1} x_j^{m_1} x_{e_1}$ and $x_i^{n_2} x_j^{m_2} x_{e_2}$ of degree d such that e_1 and e_2 are distinct.

(3) For all distinct i, j, k, $\mathrm{hcf}(a_i, a_j, a_k) = 1$.

II.3.2 Note. Since the hypersurface is well-formed then $\omega_S = \mathcal{O}_S(\alpha)$.

Proof. Let f be a general homogeneous polynomial of degree d in variables x_0, ..., x_3; define $S_d : (f = 0) \subset \mathbf{P}$.

S_d is quasismooth and not a linear cone if and only if conditions $(1i)$, $(1ii)$, $(1iii)$ and $(2iii)$ hold (see Corollary I.5.5).

Suppose furthermore that conditions $(1iv)$, $(2i)$, $(2ii)$ and (3) hold. As S_d is quasismooth the only singularities are due to the $\mathbf{k}^*$-action and hence are cyclic quotient singularities on the fundamental simplex $\Delta \subset \mathbf{P}$. By condition (3) only vertices and edges need be checked.

Consider $P_i \in S_d$. By renumbering we can assume that $i = 0$. So $a_0 \nmid d$. Condition $(1ii)$ gives that there exists an $e \neq 0$ such that $a_0 \mid d - a_e$. Without loss of generality we can assume that $e = 1$. So f is of the form $f = x_0^n x_1 + \ldots$. Thus $\partial f / \partial x_1$ is non-zero at P_0. By the Inverse Function Theorem x_2 and x_3 are local coordinates. So $P_0 \in S_d$ is of type $\frac{1}{a_0}(a_2, a_3)$. However $d = a_0 + \ldots + a_3 + \alpha$ and so $a_0 \mid a_2 + a_3 + \alpha$. By condition $(1iv)$, $a_0 \mid a_2 + a_3$. Let $h = \mathrm{hcf}(a_0, a_2)$. So $h \mid a_3$ and hence, by condition (3), $h = 1$. Therefore $P_0 \in S_d$ is a canonical singularity.

Consider the edge $P_i P_j$. Again by renumbering assume that $i = 0$ and $j = 1$. f restricted to $P_0 P_1$ is:

$$f = \sum x_0^n x_1^m$$

where the sum is taken over the set $\{(n,m) : na_0 + ma_1 = d\}$. If $a_0 \not| \ d$ then $a_0 \mid d - a_e$ for some $e \neq 0$. If $e \neq 1$ then $h = \mathrm{hcf}(a_0, a_1) \mid a_e$ and by condition (4) $h = 1$. Then P_0P_1 is non-singular. So assume that either $a_0 \mid d$ or $a_0 \mid d - a_1$. Hence f is not identically zero on P_0P_1, and so $S_d \cap P_0P_1$ is finite. Each point in this intersection is of type $\frac{1}{h}(a_2, a_3)$. Since $d = a_0 + ... + a_3 + \alpha$ and $h \mid \alpha$ then $h \mid a_2 + a_3$. Also $\mathrm{hcf}(h, a_2) = 1$. Thus each point is canonical.

Therefore S_d in $\mathbf{P}$ has at worst canonical singularities.

Conversely assume that S_d is quasismooth, well-formed, not a linear cone and has at worst only canonical singularities. Suppose $a_i \not| \ d$. By renumbering we can assume that $i = 0$. So $P_0 \in S_d$ and $a_0 \mid d - a_e$ for some e. Without loss of generality assume that $e = 1$. As above the singularity at $P_0 \in S_d$ is of type $\frac{1}{a_0}(a_2, a_3)$. Since this is canonical we have $a_0 \mid a_2 + a_3$ and so $a_0 \mid \alpha$. This is condition $(1iv)$.

Suppose $h = \mathrm{hcf}(a_i, a_j)$. By renumbering assume that $i = 0$ and $j = 1$. As S_d is well-formed then $h \mid d$, which is condition $(2i)$. So $P_0P_1 \cap S_d$ is a finite intersection, where each point is of type $\frac{1}{h}(a_2, a_3)$. This is canonical and so $h \mid \alpha$. This is condition $(2ii)$.

Suppose $h = \mathrm{hcf}(a_i, a_j, a_k)$. Without loss of generality assume that $i = 0$, $j = 1$ and $k = 2$. Let $h' = \mathrm{hcf}(a_0, a_1)$. So $h' \mid d$. Hence the line P_0P_1 contains singularities of type $\frac{1}{h'}(a_2, a_3)$. As these are canonical $h = \mathrm{hcf}(h', a_2) = 1$. This is condition (3).

□

II.3.3 Reid's 95 codimension 1 K3 surfaces.

In 1979, Reid produced the list of all families of codimension 1 weighted K3 surfaces; 95 in all (see [R1, section 4.5]). The full list follows along with their respective singularities.

Weighted K3 surface	Singularities	Weighted K3 surface	Singularities
X_4 in $\mathbf{P}(1,1,1,1)$		X_5 in $\mathbf{P}(1,1,1,2)$	A_1
X_6 in $\mathbf{P}(1,1,1,3)$		X_6 in $\mathbf{P}(1,1,2,2)$	$3{\times}A_1$
X_7 in $\mathbf{P}(1,1,2,3)$	A_1, A_2	X_8 in $\mathbf{P}(1,1,2,4)$	$2{\times}A_1$
X_8 in $\mathbf{P}(1,2,2,3)$	$4{\times}A_1$, A_2	X_9 in $\mathbf{P}(1,1,3,4)$	A_3
X_9 in $\mathbf{P}(1,2,3,3)$	A_1, $3{\times}A_2$	X_{10} in $\mathbf{P}(1,1,3,5)$	A_2
X_{10} in $\mathbf{P}(1,2,2,5)$	$5{\times}A_1$	X_{10} in $\mathbf{P}(1,2,3,4)$	$2{\times}A_1$, A_2, A_3
X_{11} in $\mathbf{P}(1,2,3,5)$	A_1, A_2, A_4	X_{12} in $\mathbf{P}(1,1,4,6)$	A_1
X_{12} in $\mathbf{P}(1,2,3,6)$	$2{\times}A_1$, $2{\times}A_2$	X_{12} in $\mathbf{P}(1,2,4,5)$	$3{\times}A_1$, A_4
X_{12} in $\mathbf{P}(1,3,4,4)$	$3{\times}A_3$	X_{12} in $\mathbf{P}(2,2,3,5)$	$6{\times}A_1$, A_4
X_{12} in $\mathbf{P}(2,3,3,4)$	$3{\times}A_1$, $4{\times}A_2$	X_{13} in $\mathbf{P}(1,3,4,5)$	A_2, A_3, A_4
X_{14} in $\mathbf{P}(1,2,4,7)$	$3{\times}A_1$, A_3	X_{14} in $\mathbf{P}(2,2,3,7)$	$7{\times}A_1$, A_2
X_{14} in $\mathbf{P}(2,3,4,5)$	$3{\times}A_1$, A_2, A_3, A_4	X_{15} in $\mathbf{P}(1,2,5,7)$	A_1, A_6
X_{15} in $\mathbf{P}(1,3,4,7)$	A_3, A_6	X_{15} in $\mathbf{P}(1,3,5,6)$	$2{\times}A_2$, A_5
X_{15} in $\mathbf{P}(2,3,5,5)$	A_1, $3{\times}A_4$	X_{15} in $\mathbf{P}(3,3,4,5)$	$5{\times}A_2$, A_3
X_{16} in $\mathbf{P}(1,2,5,8)$	$2{\times}A_1$, A_4	X_{16} in $\mathbf{P}(1,3,4,8)$	A_2, $2{\times}A_3$
X_{16} in $\mathbf{P}(1,4,5,6)$	A_1, A_4, A_5	X_{16} in $\mathbf{P}(2,3,4,7)$	$4{\times}A_1$, A_2, A_6
X_{17} in $\mathbf{P}(2,3,5,7)$	A_1, A_2, A_4, A_6	X_{18} in $\mathbf{P}(1,2,6,9)$	$3{\times}A_1$, A_2
X_{18} in $\mathbf{P}(1,3,5,9)$	$2{\times}A_2$, A_4	X_{18} in $\mathbf{P}(1,4,6,7)$	A_3, A_1, A_6
X_{18} in $\mathbf{P}(2,3,4,9)$	$4{\times}A_1$, $2{\times}A_2$, A_3	X_{18} in $\mathbf{P}(2,3,5,8)$	$2{\times}A_1$, A_4, A_7
X_{18} in $\mathbf{P}(3,4,5,6)$	$3{\times}A_2$, A_3, A_1, A_4	X_{19} in $\mathbf{P}(3,4,5,7)$	A_2, A_3, A_4, A_6
X_{20} in $\mathbf{P}(1,4,5,10)$	A_1, $2{\times}A_4$	X_{20} in $\mathbf{P}(2,3,5,10)$	$2{\times}A_1$, A_2, $2{\times}A_4$

X_{20} in $\mathbf{P}(2,4,5,9)$	$5\times A_1$, A_8	X_{20} in $\mathbf{P}(2,5,6,7)$	$3\times A_1$, A_5, A_6
X_{20} in $\mathbf{P}(3,4,5,8)$	A_2, $2\times A_3$, A_7	X_{21} in $\mathbf{P}(1,3,7,10)$	A_9
X_{21} in $\mathbf{P}(1,5,7,8)$	A_4, A_7	X_{21} in $\mathbf{P}(2,3,7,9)$	A_1, $2\times A_2$, A_8
X_{21} in $\mathbf{P}(3,5,6,7)$	$3\times A_2$, A_4, A_5	X_{22} in $\mathbf{P}(1,3,7,11)$	A_2, A_6
X_{22} in $\mathbf{P}(1,4,6,11)$	A_3, A_1, A_5	X_{22} in $\mathbf{P}(2,4,5,11)$	$5\times A_1$, A_3, A_4
X_{24} in $\mathbf{P}(1,3,8,12)$	$2\times A_2$, A_3	X_{24} in $\mathbf{P}(1,6,8,9)$	A_1, A_2, A_8
X_{24} in $\mathbf{P}(2,3,7,12)$	$2\times A_1$, $2\times A_2$, A_6	X_{24} in $\mathbf{P}(2,3,8,11)$	$3\times A_1$, A_{10}
X_{24} in $\mathbf{P}(3,4,5,12)$	$2\times A_2$, $2\times A_3$, A_4	X_{24} in $\mathbf{P}(3,4,7,10)$	A_1, A_6, A_9
X_{24} in $\mathbf{P}(3,6,7,8)$	$4\times A_2$, A_1, A_6	X_{24} in $\mathbf{P}(4,5,6,9)$	$2\times A_1$, A_4, A_2, A_8
X_{25} in $\mathbf{P}(4,5,7,9)$	A_3, A_6, A_8	X_{26} in $\mathbf{P}(1,5,7,13)$	A_4, A_6
X_{26} in $\mathbf{P}(2,3,8,13)$	$3\times A_1$, A_2, A_7	X_{26} in $\mathbf{P}(2,5,6,13)$	$4\times A_1$, A_4, A_5
X_{27} in $\mathbf{P}(2,5,9,11)$	A_1, A_4, A_{10}	X_{27} in $\mathbf{P}(5,6,7,9)$	A_4, A_5, A_2, A_6
X_{28} in $\mathbf{P}(1,4,9,14)$	A_1, A_8	X_{28} in $\mathbf{P}(3,4,7,14)$	A_2, A_1, $2\times A_6$
X_{28} in $\mathbf{P}(4,6,7,11)$	$2\times A_1$, A_5, A_{10}	X_{30} in $\mathbf{P}(1,4,10,15)$	A_3, A_1, A_4
X_{30} in $\mathbf{P}(1,6,8,15)$	A_1, A_2, A_7	X_{30} in $\mathbf{P}(2,3,10,15)$	$3\times A_1$, $2\times A_2$, A_4
X_{30} in $\mathbf{P}(2,6,7,15)$	$5\times A_1$, A_2, A_6	X_{30} in $\mathbf{P}(3,4,10,13)$	A_3, A_1, A_{12}
X_{30} in $\mathbf{P}(4,5,6,15)$	A_3, $2\times A_1$, $2\times A_4$, A_2	X_{30} in $\mathbf{P}(5,6,8,11)$	A_1, A_7, A_{10}
X_{32} in $\mathbf{P}(2,5,9,16)$	$2\times A_1$, A_4, A_8	X_{32} in $\mathbf{P}(4,5,7,16)$	$2\times A_3$, A_4, A_6
X_{33} in $\mathbf{P}(3,5,11,14)$	A_4, A_{13}	X_{34} in $\mathbf{P}(3,4,10,17)$	A_2, A_3, A_1, A_9
X_{34} in $\mathbf{P}(4,6,7,17)$	A_3, $2\times A_1$, A_5, A_6	X_{36} in $\mathbf{P}(1,5,12,18)$	A_4, A_5
X_{36} in $\mathbf{P}(3,4,11,18)$	$2\times A_2$, A_1, A_{10}	X_{36} in $\mathbf{P}(7,8,9,12)$	A_6, A_7, A_3, A_2
X_{38} in $\mathbf{P}(3,5,11,19)$	A_2, A_4, A_{10}	X_{38} in $\mathbf{P}(5,6,8,19)$	A_4, A_5, A_1, A_7
X_{40} in $\mathbf{P}(5,7,8,20)$	$2\times A_4$, A_6, A_3	X_{42} in $\mathbf{P}(1,6,14,21)$	A_1, A_2, A_6
X_{42} in $\mathbf{P}(2,5,14,21)$	$3\times A_1$, A_4, A_6	X_{42} in $\mathbf{P}(3,4,14,21)$	$2\times A_2$, A_3, A_1, A_6
X_{44} in $\mathbf{P}(4,5,13,22)$	A_1, A_4, A_{12}	X_{48} in $\mathbf{P}(3,5,16,24)$	$2\times A_2$, A_4, A_7
X_{50} in $\mathbf{P}(7,8,10,25)$	A_6, A_7, A_1, A_4	X_{54} in $\mathbf{P}(4,5,18,27)$	A_3, A_1, A_4, A_8
X_{66} in $\mathbf{P}(5,6,22,33)$	A_4, A_1, A_2, A_{10}		

However there are not so many dimension 2 weighted hypersurfaces with $\omega_S \cong \mathcal{O}_S(\pm 1)$:

II.3.4 Theorem. *There are exactly three families of dimension 2 weighted hypersurfaces with at worst canonical singularities and $\omega_S \cong \mathcal{O}_S(1)$, and exactly three families with $\omega_S \cong \mathcal{O}_S(-1)$,*

$\alpha = 1$	$\alpha = -1$
S_5 *in* $\mathbf{P}(1,1,1,1)$	S_3 *in* $\mathbf{P}(1,1,1,1)$
S_6 *in* $\mathbf{P}(1,1,1,2)$	S_4 *in* $\mathbf{P}(1,1,1,2)$
S_8 *in* $\mathbf{P}(1,1,1,4)$	S_6 *in* $\mathbf{P}(1,1,2,3)$

II.3.5 Note. These families are all non-singular.

Proof. Condition $(2ii)$ of Theorem II.3.1 is very strong when $\alpha = \pm 1$ and forces the a_i to be pairwise coprime. Similarly condition $(1iv)$ forces $a_i \mid d$ for each i. So $a_0a_1a_2a_3 \mid d$ and $d = a_0 + \ldots + a_3 + \alpha$. Order $a_3 \geq a_2 \geq a_1 \geq a_0 \geq 1$ and let $d = \lambda a_3$. Thus $a_0a_1a_2 \mid \lambda$ and $(\lambda - 1)a_3 = a_0 + \ldots + a_2 + \alpha$.

Suppose $\alpha = 1$. Then $2a_3 \leq \lambda a_3 = a_0 + \ldots + a_3 + 1 \leq 5a_3$. So $2 \leq \lambda \leq 5$. Running through the possible values of λ:

(i) $\lambda = 5$. So $a_0a_1a_2 \mid 5$. If $a_2 = 1$ then $a_4 = 1$ (i.e. S_5 in $\mathbf{P}(1,1,1,1)$). If $a_2 = 5$ then $a_3 = 2$, a contradiction.

(ii) $\lambda = 4$. So $a_0a_1a_2 \mid 4$. If $a_2 = 1$ then $a_4 = \frac{4}{3}$, a contradiction. If $a_2 = 2$ then $a_4 = \frac{5}{3}$, a contradiction. If $a_2 = 4$ then $a_4 = \frac{7}{3}$, a contradiction.

(iii) $\lambda = 3$. So $a_0a_1a_2 \mid 3$. If $a_2 = 1$ then $a_4 = 2$ (i.e. S_6 in $\mathbf{P}(1,1,1,2)$). If $a_2 = 3$ then $a_4 = 3$, a contradiction.

(iv) $\lambda = 2$. So $a_0a_1a_2 \mid 2$. If $a_2 = 1$ then $a_4 = 4$ (i.e. S_8 in $\mathbf{P}(1,1,1,4)$). If $a_2 = 2$ then $a_4 = \frac{5}{2}$, a contradiction.

So there are exactly three families.

Suppose that $\alpha = -1$. Then $2a_3 \leq \lambda a_3 = a_0 + ... + a_3 - 1 \leq 6a_3$. Thus $2 \leq \lambda \leq 6$. As above this gives rise to the following families: S_3 in $\mathbf{P}(1,1,1,1)$ in the case $\lambda = 3$, S_4 in $\mathbf{P}(1,1,1,2)$ and S_6 in $\mathbf{P}(1,1,2,3)$ in the case $\lambda = 2$.

□

Consider the case of codimension 2 complete intersections.

II.3.6 Theorem. *Suppose $S = S_{d_1,d_2}$ in $\mathbf{P} = \mathbf{P}(a_0,...,a_4)$ is quasismooth and is not the intersection of a linear cone with another hypersurface. Let $\alpha = \sum d_\lambda - \sum a_i$. S is well-formed and has at worst canonical singularities if and only if the following hold:*

(1) for all i, if $a_i \nmid d_1$ and $a_i \nmid d_2$ then $a_i \mid \alpha$.

(2) for all distinct i and j, with $h = \operatorname{hcf}(a_i, a_j)$, one of the following occurs:

(a) $h \mid d_1$ and $h \mid d_2$,

(b) $h \mid d_1$, $h \nmid d_2$ and $h \mid \alpha$, or

(c) $h \nmid d_1$, $h \mid d_2$ and $h \mid \alpha$.

(3) for all distinct i, j and k, with $h = \operatorname{hcf}(a_i, a_j, a_k)$, $h \mid d_1$, $h \mid d_2$ and $h \mid \alpha$.

(4) for all distinct i, j, k and l, $h = \operatorname{hcf}(a_i, a_j, a_k, a_l) = 1$

II.3.7 Note. Since the hypersurface is well-formed we have that $\omega_S = \mathcal{O}_S(\alpha)$.

Proof. Let f_1 and f_2 be sufficiently general homogeneous polynomials of degrees d_1 and d_2 respectively, in the variables x_0, ..., x_4 with respect to the weights a_0, ..., a_4. Define $S : (f_1 = 0, f_2 = 0) \subset \mathbf{P}$.

Since S is quasismooth the only singularities are due to the $\mathbf{k}^*$-action and hence are all cyclic quotient singularities occurring on the fundamental simplex Δ.

Assume conditions (1), ..., (4) hold. By conditions (2), (3) and (4) S is well-formed. By condition (4) only the vertices, edges and faces of Δ need be considered.

Suppose $P_i \in S$. By renumbering we can assume that $i = 0$. So $a_0 \nmid d_1$ and $a_0 \nmid d_2$. As S is quasismooth (and using $I = \{0\}$ in Theorem I.5.7) there exist monomials $x_0^n x_{e_1}$ and $x_0^m x_{e_2}$ of degrees d_1 and d_2, where $e_1 \neq e_2$. By renumbering we can write $e_1 = 1$ and $e_2 = 2$. So f_1 and f_2 are of the form:

$$f_1 = x_0^n x_1 + ...$$
$$f_2 = x_0^m x_2 + ...$$

Thus $\partial f_1/\partial x_1$ and $\partial f_2/\partial x_2$ are non-zero at P_0. By the Inverse Function Theorem, x_3 and x_4 are local coordinates around P_0. Hence $P_0 \in S$ is of type $\frac{1}{a_0}(a_3, a_4)$. As $d_1 + d_2 = a_0 + ... + a_4 + \alpha$ and $a_0 \mid \alpha$ then $a_0 \mid a_3 + a_4$. Let $h = \operatorname{hcf}(a_0, a_3)$. So $h \mid a_4$ and, by condition (3), $h \mid d_1$. Since $deg(x_0^n x_1) = d_1$, $h \mid a_1$ and so, by condition (4), $h = 1$. Thus $P_0 \in S$ is canonical.

Consider the edge P_iP_j. By renumbering we can assume that $i = 0$ and $j = 1$. Let $h = \operatorname{hcf}(a_0, a_1)$. Notice that $P_0P_1 \subset X_{d_\lambda}$ if and only if $h \nmid d_\lambda$ for $\lambda = 0, 1$. By condition (2), $h \mid d_\lambda$ for some λ. Without loss of generality assume that $h \mid d_1$. There are 2 cases:

(a) $h \mid d_2$. $P_0P_1 \cap (f_\lambda = 0)$ is a finite set of points for $\lambda = 0, 1$. Thus $P_0P_1 \cap S = \emptyset$.

(b) $h \nmid d_2$. In this case no monomial of the form $x_0^n x_1^m$ of degree d_2 exists (or else $h \mid d_2$). From Theorem III.3.7 (with $I = \{0,1\}$) there exists a monomial $x_0^n x_1^m x_e$ of degree d_2, where $e \neq 0, 1$. By renumbering we can assume that $e = 2$. Thus f_2 is of the form:

$$f_2 = x_0^n x_1^m x_2 + \ldots$$

and $\partial f_2 / \partial x_2$ is non-zero on $P_0P_1 \cap S$. By the Inverse Function Theorem, x_3 and x_4 are local coordinates around each point of $P_0P_1 \cap S$ and so each is of type $\frac{1}{h}(a_3, a_4)$. Condition (2b) gives $h \mid \alpha$ and so $h \mid a_3 + a_4$. Let $h' = \mathrm{hcf}(h, a_3)$. So $h' \mid a_4$ and thus by condition (4) $h' = 1$. Thus these points are canonical.

Therefore S has at worst canonical points along P_0P_1.

Consider the face $P_iP_jP_k$. As before assume $i = 0$, $j = 1$ and $k = 2$. By condition (3) $h = \mathrm{hcf}(a_0, a_1, a_2) \mid d_1$ and $h \mid d_2$. So $P_0P_1P_2$ intersects S transversally. Each point in the intersection is of type $\frac{1}{h}(a_3, a_4)$. As $h \mid \alpha$, $h \mid a_3 + a_4$. By condition (4) $\mathrm{hcf}(h, a_3) = 1$. Thus these points are canonical.

Therefore conditions (1), ..., (4) are sufficient.

Conversely assume that S is well-formed and has at worst canonical singularities. Suppose $a_i \nmid d_1$ and $a_i \nmid d_2$. By renumbering assume $i = 0$. Thus $P_0 \in S$. Since S is quasismooth there exist 2 monomials $x_0^n x_{e_1}$ and $x_0^m x_{e_2}$ of degrees d_1 and d_2, where $e_1 \neq e_2$. Without loss of generality we can assume that $e_1 = 1$ and $e_2 = 2$. As before we find that $P_0 \in S$ is of type $\frac{1}{a_0}(a_3, a_4)$. As this is canonical $a_0 \mid a_3 + a_4$ and so $a_0 \mid \alpha$. This is condition (1).

Suppose $h = \mathrm{hcf}(a_i, a_j)$ for distinct i and j. As usual we can renumber such that $i = 0$ and $j = 1$. As S is well-formed then $h \mid d_\lambda$ for some λ. Suppose $h \mid d_1$. If $h \mid d_2$ then this is condition (2a). So assume that $h \nmid d_2$. As above each point of $P_0P_1 \cap S$ is isolated and of type $\frac{1}{h}(a_3, a_4)$. Thus $h \mid a_3 + a_4$ and so $h \mid \alpha$. This is condition (2b). Likewise for the case when $h \mid d_2$ but $h \nmid d_1$. This gives condition (2c).

Suppose $h = \mathrm{hcf}(a_i, a_j, a_k)$ for distinct i, j and k. Renumber such that $i = 0$, $j = 1$ and $k = 2$. As S is well-formed then $h \mid d_1$ and $h \mid d_2$. Thus $P_0P_1P_2 \cap S$ is a finite number of points, all of type $\frac{1}{h}(a_3, a_4)$. As these are canonical $h \mid a_3 + a_4$ and so $h \mid \alpha$. This is condition (3). Also $\mathrm{hcf}(h, a_3) = \mathrm{hcf}(h, a_4) = 1$, which is condition (4).

So these conditions are necessary.

□

II.3.8 Codimension 2 Weighted K3 Surfaces.

There are 84 families of codimension 2 quasismooth, well-formed K3 surfaces with only canonical singularities and $\sum a_i \leq 100$. These were found by means of a computer search program.

Weighted K3 surfaces	Singularities	Weighted K3 surfaces	Singularities
$X_{2,3}$ in $\mathbf{P}(1,1,1,1,1)$		$X_{3,3}$ in $\mathbf{P}(1,1,1,1,2)$	A_1
$X_{3,4}$ in $\mathbf{P}(1,1,1,2,2)$	$2 \times A_1$	$X_{4,4}$ in $\mathbf{P}(1,1,1,2,3)$	A_2
$X_{4,4}$ in $\mathbf{P}(1,1,2,2,2)$	$4 \times A_1$	$X_{4,5}$ in $\mathbf{P}(1,1,2,2,3)$	$2 \times A_1$, A_2
$X_{4,6}$ in $\mathbf{P}(1,1,2,3,3)$	$2 \times A_2$	$X_{4,6}$ in $\mathbf{P}(1,2,2,2,3)$	$6 \times A_1$
$X_{5,6}$ in $\mathbf{P}(1,1,2,3,4)$	A_1, A_3	$X_{5,6}$ in $\mathbf{P}(1,2,2,3,3)$	$3 \times A_1$, $2 \times A_2$
$X_{6,6}$ in $\mathbf{P}(1,1,2,3,5)$	A_4	$X_{6,6}$ in $\mathbf{P}(1,2,2,3,4)$	$4 \times A_1$, A_3
$X_{6,6}$ in $\mathbf{P}(1,2,3,3,3)$	$4 \times A_2$	$X_{6,6}$ in $\mathbf{P}(2,2,2,3,3)$	$9 \times A_1$

$X_{6,7}$ in $\mathbf{P}(1,2,2,3,5)$	$3\times A_1,\ A_4$	$X_{6,7}$ in $\mathbf{P}(1,2,3,3,4)$	$A_1,\ 2\times A_2,\ A_3$
$X_{6,8}$ in $\mathbf{P}(1,1,3,4,5)$	A_4	$X_{6,8}$ in $\mathbf{P}(1,2,3,3,5)$	$2\times A_2,\ A_4$
$X_{6,8}$ in $\mathbf{P}(1,2,3,4,4)$	$2\times A_1,\ 2\times A_3$	$X_{6,8}$ in $\mathbf{P}(2,2,3,3,4)$	$6\times A_1,\ 2\times A_2$
$X_{6,9}$ in $\mathbf{P}(1,2,3,4,5)$	$A_1,\ A_3,\ A_4$	$X_{7,8}$ in $\mathbf{P}(1,2,3,4,5)$	$2\times A_1,\ A_2,\ A_4$
$X_{6,10}$ in $\mathbf{P}(1,2,3,5,5)$	$2\times A_4$	$X_{6,10}$ in $\mathbf{P}(2,2,3,4,5)$	$7\times A_1,\ A_3$
$X_{8,9}$ in $\mathbf{P}(1,2,3,4,7)$	$2\times A_1,\ A_6$	$X_{8,9}$ in $\mathbf{P}(1,3,4,4,5)$	$2\times A_3,\ A_4$
$X_{8,9}$ in $\mathbf{P}(2,3,3,4,5)$	$2\times A_1,\ 3\times A_2,\ A_4$	$X_{8,10}$ in $\mathbf{P}(1,2,3,5,7)$	$A_2,\ A_6$
$X_{8,10}$ in $\mathbf{P}(1,2,4,5,6)$	$3\times A_1,\ A_5$	$X_{8,10}$ in $\mathbf{P}(1,3,4,5,5)$	$A_2,\ 2\times A_4$
$X_{8,10}$ in $\mathbf{P}(2,3,4,4,5)$	$4\times A_1,\ A_2,\ 2\times A_3$	$X_{9,10}$ in $\mathbf{P}(1,2,3,5,8)$	$A_1,\ A_7$
$X_{9,10}$ in $\mathbf{P}(1,3,4,5,6)$	$A_2,\ A_3,\ A_5$	$X_{9,10}$ in $\mathbf{P}(2,2,3,5,7)$	$5\times A_1,\ A_6$
$X_{9,10}$ in $\mathbf{P}(2,3,4,5,5)$	$2\times A_1,\ A_3,\ 2\times A_4$	$X_{8,12}$ in $\mathbf{P}(1,3,4,5,7)$	$A_4,\ A_6$
$X_{8,12}$ in $\mathbf{P}(2,3,4,5,6)$	$4\times A_1,\ 2\times A_2,\ A_4$	$X_{9,12}$ in $\mathbf{P}(2,3,4,5,7)$	$3\times A_1,\ A_4,\ A_6$
$X_{10,11}$ in $\mathbf{P}(2,3,4,5,7)$	$2\times A_1,\ A_2,\ A_3,\ A_6$	$X_{10,12}$ in $\mathbf{P}(1,3,4,5,9)$	$A_2,\ A_8$
$X_{10,12}$ in $\mathbf{P}(1,3,5,6,7)$	$2\times A_2,\ A_6$	$X_{10,12}$ in $\mathbf{P}(1,4,5,6,6)$	$A_1,\ 2\times A_5$
$X_{10,12}$ in $\mathbf{P}(2,3,4,5,8)$	$3\times A_1,\ A_3,\ A_7$	$X_{10,12}$ in $\mathbf{P}(2,3,5,5,7)$	$2\times A_4,\ A_6$
$X_{10,12}$ in $\mathbf{P}(2,4,5,5,6)$	$5\times A_1,\ 2\times A_4$	$X_{10,12}$ in $\mathbf{P}(3,3,4,5,7)$	$4\times A_2,\ A_6$
$X_{10,12}$ in $\mathbf{P}(3,4,4,5,6)$	$2\times A_2,\ 3\times A_3,\ A_1$	$X_{11,12}$ in $\mathbf{P}(1,4,5,6,7)$	$A_1,\ A_4,\ A_6$
$X_{10,14}$ in $\mathbf{P}(1,2,5,7,9)$	A_8	$X_{10,14}$ in $\mathbf{P}(2,3,5,7,7)$	$A_2,\ 2\times A_6$
$X_{10,14}$ in $\mathbf{P}(2,4,5,6,7)$	$5\times A_1,\ A_3,\ A_5$	$X_{10,15}$ in $\mathbf{P}(2,3,5,7,8)$	$A_1,\ A_6,\ A_7$
$X_{12,13}$ in $\mathbf{P}(3,4,5,6,7)$	$2\times A_2,\ A_1,\ A_4,\ A_6$	$X_{12,14}$ in $\mathbf{P}(1,3,4,7,11)$	A_{10}
$X_{12,14}$ in $\mathbf{P}(1,4,6,7,8)$	$A_1,\ A_3,\ A_7$	$X_{12,14}$ in $\mathbf{P}(2,3,4,7,10)$	$4\times A_1,\ A_9$
$X_{12,14}$ in $\mathbf{P}(2,3,5,7,9)$	$A_2,\ A_4,\ A_8$	$X_{12,14}$ in $\mathbf{P}(3,4,5,7,7)$	$A_4,\ 2\times A_6$
$X_{12,14}$ in $\mathbf{P}(4,4,5,6,7)$	$3\times A_3,\ 2\times A_1,\ A_4$	$X_{12,15}$ in $\mathbf{P}(1,4,5,6,11)$	$A_1,\ A_{10}$
$X_{12,15}$ in $\mathbf{P}(3,4,5,6,9)$	$3\times A_2,\ A_1,\ A_8$	$X_{12,15}$ in $\mathbf{P}(3,4,5,7,8)$	$A_3,\ A_6,\ A_7$
$X_{12,16}$ in $\mathbf{P}(2,5,6,7,8)$	$4\times A_1,\ A_4,\ A_6$	$X_{14,15}$ in $\mathbf{P}(2,3,5,7,12)$	$A_1,\ A_2,\ A_{11}$
$X_{14,15}$ in $\mathbf{P}(2,5,6,7,9)$	$2\times A_1,\ A_5,\ A_8$	$X_{14,15}$ in $\mathbf{P}(3,4,5,7,10)$	$A_3,\ A_4,\ A_9$
$X_{14,15}$ in $\mathbf{P}(3,5,6,7,8)$	$2\times A_2,\ A_5,\ A_7$	$X_{14,16}$ in $\mathbf{P}(1,5,7,8,9)$	$A_4,\ A_8$
$X_{14,16}$ in $\mathbf{P}(3,4,5,7,11)$	$A_2,\ A_4,\ A_{10}$	$X_{14,16}$ in $\mathbf{P}(4,5,6,7,8)$	$A_1,\ 2\times A_3,\ A_4,\ A_5$
$X_{15,16}$ in $\mathbf{P}(2,3,5,8,13)$	$2\times A_1,\ A_{12}$	$X_{15,16}$ in $\mathbf{P}(3,4,5,8,11)$	$2\times A_3,\ A_{10}$
$X_{14,18}$ in $\mathbf{P}(2,3,7,9,11)$	$2\times A_2,\ A_{10}$	$X_{14,18}$ in $\mathbf{P}(2,6,7,8,9)$	$5\times A_1,\ A_2,\ A_7$
$X_{12,20}$ in $\mathbf{P}(4,5,6,7,10)$	$2\times A_1,\ 2\times A_4,\ A_6$	$X_{16,18}$ in $\mathbf{P}(1,6,8,9,10)$	$A_1,\ A_2,\ A_9$
$X_{16,18}$ in $\mathbf{P}(4,6,7,8,9)$	$2\times A_1,\ 2\times A_3,\ A_2,\ A_6$	$X_{18,20}$ in $\mathbf{P}(4,5,6,9,14)$	$2\times A_1,\ A_2,\ A_{13}$
$X_{18,20}$ in $\mathbf{P}(4,5,7,9,13)$	$A_6,\ A_{12}$	$X_{18,20}$ in $\mathbf{P}(5,6,7,9,11)$	$A_2,\ A_6,\ A_{10}$
$X_{18,22}$ in $\mathbf{P}(2,5,9,11,13)$	$A_4,\ A_{12}$	$X_{20,21}$ in $\mathbf{P}(3,4,7,10,17)$	$A_1,\ A_{16}$
$X_{18,30}$ in $\mathbf{P}(6,8,9,10,15)$	$2\times A_1,\ 2\times A_2,\ A_7,\ A_4$	$X_{24,30}$ in $\mathbf{P}(8,9,10,12,15)$	$A_1,\ A_3,\ A_8,\ A_2,\ A_4$

II.4 Weighted 3-fold complete intersections.

This section gives the corresponding conditions and lists for 3-folds.

II.4.1 Theorem. *Let X_d be a general hypersurface in $\mathbf{P} = \mathbf{P}(a_0, \dots, a_4)$ and let $\alpha = d - \sum a_i$. Then X_d is quasismooth with only isolated terminal quotient singularities and is not a linear cone if and only if all the following hold:*

(1) For all i,

(i) $d > a_i$.
(ii) there exists a monomial $x_i^m x_e$ *of degree d (i.e. there exists e such that* $a_i \mid d - a_e$*).*
(iii) if $a_i \not| d$*, there exists an* $m \neq i, e$ *such that* $a_i \mid a_m + \alpha$.

(2) For all distinct i, j, *with* $h = \operatorname{hcf}(a_i, a_j)$, *then*
(i) $h \mid d$.
(ii) there exists an $m \neq i, j$ *such that* $h \mid a_m + \alpha$.
(iii) one of the following holds:
either there exists a monomial $x_i^m x_j^n$ *of degree d,*
or there exist monomials $x_i^{n_1} x_j^{m_1} x_{e_1}$ *and* $x_i^{n_2} x_j^{m_2} x_{e_2}$ *of degree d such that* e_1 *and* e_2 *are distinct.*
(iv) there exists a monomial of degree d which does not involve x_i *or* x_j.

(3) For all distinct i, j, k, $\operatorname{hcf}(a_i, a_j, a_k) = 1$.

II.4.2 Note. Since the hypersurface is quasismooth and of dimension 3 then it is well-formed, and so $\omega_X = \mathcal{O}_X(\alpha)$.

Proof. Let f be a general homogeneous polynomial of degree d in variables $x_0, ..., x_3$; define $X_d : (f = 0) \subset \mathbf{P}$.

X_d is quasismooth and not a linear cone (and therefore well-formed) if and only if conditions ($1i$), ($1ii$), ($2i$), ($2iii$), ($2iv$) and (3) hold (see Corollary I.5.6). By calculating the types of the singularities on X_d we can show that conditions ($1iii$), ($2i$), ($2ii$) and (3) are equivalent to these singularities being terminal; the combinatorial conditions for which are found in Lemma I.6.3.

Suppose furthermore that conditions ($1iii$), ($2i$), ($2ii$) and (3) hold. As X_d is quasismooth the only singularities are due to the $\mathbf{k}^*$-action and hence are cyclic quotient singularities on the fundamental simplex $\Delta \subset \mathbf{P}$. By condition (3) only vertices and edges need be checked.

Consider $P_i \in X_d$. By renumbering we can assume that $i = 0$. So $a_0 \not| d$. By condition ($1ii$) there exists an $e \neq 0$ such that $a_0 \mid d - a_e$. Without loss of generality we can assume that $e = 1$. So f is of the form $f = x_0^n x_1 + ...$ Thus $\partial f / \partial x_1$ is nonzero at P_0. By the Inverse Function Theorem x_2, x_3 and x_4 are local coordinates around P_0. So $P_0 \in X_d$ is of type $\frac{1}{a_0}(a_2, a_3, a_4)$. However $d = a_0 + ... + a_4 + \alpha$ and so $a_0 \mid a_2 + a_4 + \alpha$. By condition ($1iv$), $a_0 \mid \alpha + a_m$ for some $m = 2, 3, 4$. Without loss of generality assume $m = 2$. By condition ($1iv$), $a_0 \mid a_3 + a_4$. Let $h = \operatorname{hcf}(a_0, a_3)$. So $h \mid a_3$ and hence, by condition (3), $h = 1$. Therefore $P_0 \in X_d$ is a terminal singularity.

Consider the edge $P_i P_j$. Again by renumbering assume that $i = 0$ and $j = 1$. f restricted to $P_0 P_1$ is:

$$f = \sum x_0^n x_1^m$$

where the sum is taken over the set $\{(n, m) : na_0 + ma_1 = d\}$. If $a_0 \not| d$ then $a_0 \mid d - a_e$ for some $e \neq 0$. If $e \neq 1$ then $h = \operatorname{hcf}(a_0, a_1) \mid a_e$ and by condition (4) $h = 1$. Then $P_0 P_1$ is nonsingular. So assume that either $a_0 \mid d$ or $a_0 \mid d - a_1$. Hence f is not identically zero on $P_0 P_1$, and so $X_d \cap P_0 P_1$ is finite. Each point in this intersection is of type $\frac{1}{h}(a_2, a_3, a_4)$. By condition ($2ii$) $h \mid \alpha + a_m$ for some $m = 2, 3, 4$. By renumbering assume $m = 2$. Since $d = a_0 + ... + a_4 + \alpha$, then $h \mid a_3 + a_4$. Also $\operatorname{hcf}(h, a_3) = 1$. Thus each point is terminal.

Therefore X_d in $\mathbf{P}$ has at worst terminal singularities.

Conversely assume that X_d is quasismooth, not a linear cone and has at worst only terminal singularities. Suppose $a_i \not| d$. By renumbering we can assume that $i = 0$. So $P_0 \in X_d$ and $a_0 \mid d - a_e$ for some e. Without loss of generality assume that $e = 1$. As above the singularity at $P_0 \in X_d$ is of type $\frac{1}{a_0}(a_2, a_3, a_4)$. Since this is terminal we have, after renumbering, $a_0 \mid a_2 + a_3$ and so $a_0 \mid \alpha + a_m$ for some m. This is condition $(1iv)$.

Suppose $h = \mathrm{hcf}(a_i, a_j)$. By renumbering assume that $i = 0$ and $j = 1$. As X_d is well-formed then $h \mid d$, which is condition $(2i)$. So $P_0P_1 \cap X_d$ is a finite intersection, where each point is of type $\frac{1}{h}(a_2, a_3, a_4)$. This is terminal and so $h \mid \alpha + a_m$ for $m = 2, 3, 4$. This is condition $(2ii)$.

Suppose $h = \mathrm{hcf}(a_i, a_j, a_k)$. Without loss of generality assume that $i = 0$, $j = 1$ and $k = 2$. Let $h' = \mathrm{hcf}(a_0, a_1)$. So $h' \mid d$. Hence the line P_0P_1 contains singularities of type $\frac{1}{h'}(a_2, a_3, a_4)$. As these are terminal $h = \mathrm{hcf}(h', a_2) = 1$. This is condition (3).

□

II.4.3 Theorem. *There are exactly four families of quasismooth 3-fold weighted hypersurfaces with only terminal isolated quotient singularities and $\omega_X \cong \mathcal{O}_X$:*

$$X_5 \textit{ in } \mathbf{P}(1,1,1,1,1)$$
$$X_6 \textit{ in } \mathbf{P}(1,1,1,1,2)$$
$$X_8 \textit{ in } \mathbf{P}(1,1,1,1,4)$$
$$X_{10} \textit{ in } \mathbf{P}(1,1,1,2,5)$$

Notice that the above are all non-singular.

Proof. As $K_X \cong \mathcal{O}_X$ then $\alpha = 0$. Suppose $h = \mathrm{hcf}(a_i, a_j) \neq 1$ for distinct i, j. By Theorem II.4.1 $(2ii)$ there exists an $m \neq i, j$ such that $h \mid a_m + \alpha$. However $\alpha = 0$ and so $h \mid a_m$. By (3) $h = 1$, a contradiction. Hence a_i and a_j are coprime for distinct i, j.

Suppose that $a_i \not| d$. Then there exists an $m \neq i, e_i$ such that $a_i \mid a_m + \alpha$. Thus $a_i = \mathrm{hcf}(a_i, a_m) = 1$, contradicting $a_i \not| d$. Thus $a_i \mid d$ for all i.

Order the $\{a_i\}$ such that $a_4 \geq ... \geq a_0$. So $5a_4 \geq d \geq 2a_4$. Let $d = \lambda a_4$. Thus $\lambda = 2, 3, 4$ or 5. As the $\{a_i\}$ are pairwise coprime then $a_0a_1a_2a_3a_4 \mid d$ and so $a_0a_1a_2a_3 \mid \lambda$. Also $a_0 + ... + a_3 = (\lambda - 1)a_4$. There are four cases:

(i) $\lambda = 5$. Either $(a_0, a_1, a_2, a_3) = (1,1,1,1)$ giving $a_4 = 1$ (i.e. X_5 in $\mathbf{P}(1,1,1,1,1)$) or $(a_0, a_1, a_2, a_3) = (1,1,1,5)$ giving $a_4 = 2 < a_3$.

(ii) $\lambda = 4$. So $\lambda - 1 = 3$ and divides $a_0 + ... + a_3$. There are three possibilities:
 (a) $(a_0, a_1, a_2, a_3) = (1,1,1,1)$, giving $3 \mid 4$.
 (b) $(a_0, a_1, a_2, a_3) = (1,1,1,2)$, giving $3 \mid 5$,
 (c) $(a_0, a_1, a_2, a_3) = (1,1,1,4)$, giving $3 \mid 7$.
 All of these possibilities give contradictions.

(iii) $\lambda = 3$. Either $(a_0, a_1, a_2, a_3) = (1,1,1,1)$ giving $a_4 = 2$ (i.e. X_6 in $\mathbf{P}(1,1,1,1,2)$), or $(a_0, a_1, a_2, a_3) = (1,1,1,3)$ giving $a_4 = 3$, contradicting the coprime condition.

(iv) $\lambda = 2$. Either $(a_0, a_1, a_2, a_3) = (1,1,1,1)$ giving $a_4 = 4$ (i.e. X_8 in $\mathbf{P}(1,1,1,1,4)$), or $(a_0, a_1, a_2, a_3) = (1,1,1,2)$ giving $a_4 = 5$ (i.e. X_{10} in $\mathbf{P}(1,1,1,2,5)$).

□

Consider the case of codimension 2 complete intersections.

II.4.4 Theorem. *Suppose $X = X_{d_1,d_2}$ in $\mathbf{P} = \mathbf{P}(a_0, ..., a_5)$ is quasismooth and not the intersection of a linear cone with another hypersurface. Let $\alpha = \sum d_\lambda - \sum a_i$. X has at worst* *terminal singularities if and only if the following hold:*

(1) for all i, if $a_i \not| d_1$ and $a_i \not| d_2$ then there exists e_1, e_2 and m such that $a_i \mid d_1 - a_{e_1}$, $a_i \mid d_2 - a_{e_2}$ and $a_i \mid \alpha + a_m$, where $\{i, e_1, e_2, m\}$ are distinct.

(2) for all distinct i and j, with $h = \mathrm{hcf}(a_i, a_j)$, at least one of the following occurs:

(a) $h \mid d_1$ and $h \mid d_2$,

(b) $h \mid d_1$, $h \not| d_2$ and $h \mid \alpha + a_m$ for some $m \neq i, j$, or

(c) $h \not| d_1$, $h \mid d_2$ and $h \mid \alpha + a_m$ for some $m \neq i, j$.

(3) for all distinct i, j and k, with $h = \mathrm{hcf}(a_i, a_j, a_k)$, $h \mid d_1$, $h \mid d_2$ and $h \mid \alpha + a_m$ for some $m \neq i, j, k$.

(4) for all distinct i, j, k and l, $h = \mathrm{hcf}(a_i, a_j, a_k, a_l) = 1$.

II.4.5 Note. Since X is quasismooth, of dimension 3 and not the intersect of a linear cone with other hypersurfaces then X is well-formed. Thus $\omega_X = \mathcal{O}_X(\alpha)$.

Proof. Let f_1 and f_2 be sufficiently general homogeneous polynomials of degrees d_1 and d_2 respectively, in the variables x_0, ..., x_4 with respect to the weights a_0, ..., a_4. Define $X : (f_1 = 0, f_2 = 0) \subset \mathbf{P}$.

Since X is quasismooth the only singularities are due to the $\mathbf{k}^*$-action and hence are all cyclic quotient singularities occurring on the fundamental simplex Δ.

Assume conditions (1), ..., (4) hold. By condition (4) only the vertices, edges and faces of Δ need be considered.

Suppose $P_i \in X$. By renumbering we can assume that $i = 0$. So $a_0 \not| d_1$ and $a_0 \not| d_2$. By condition (1), there exist monomials $x_0^{n_1} x_{e_1}$ and $x_0^{n_2} x_{e_2}$ of degrees d_1 and d_2, where $e_1 \neq e_2$. Note that this is really quasismoothness. By renumbering we can write $e_1 = 1$ and $e_2 = 2$. So f_1 and f_2 are of the form:

$$f_1 = x_0^{n_1} x_1 + \dots$$

$$f_2 = x_0^{n_2} x_2 + \dots$$

Thus $\partial f_1 / \partial x_1$ and $\partial f_2 / \partial x_2$ are non-zero at P_0. By the Inverse Function Theorem, x_3, x_4 and x_5 are local coordinates. Hence $P_0 \in X$ is of type $\frac{1}{a_0}(a_3, a_4, a_5)$. By condition (1) $a_0 \mid \alpha + a_m$ for some $m \neq 0, 1, 2$. Without loss of generality assume $m = 3$. As $d_1 + d_2 = a_0 + \dots + a_5 + \alpha$ then $a_0 \mid a_4 + a_5$. Let $h = \mathrm{hcf}(a_0, a_4)$. So $h \mid a_5$ and, by condition (3), $h \mid d_1$. Since $\deg x_0^n x_1 = d_1$, $h \mid a_1$ and so, by condition (4), $h = 1$. Thus $P_0 \in X$ is terminal.

Consider the edge $P_i P_j$. By renumbering we can assume that $i = 0$ and $j = 1$. Let $h = \mathrm{hcf}(a_0, a_1)$. Notice that $P_0 P_1 \subset X_{d_\lambda}$ if and only if $h \not| d_\lambda$ for $\lambda = 0, 1$. By condition (2), $h \mid d_\lambda$ for some λ. Without loss of generality assume that $h \mid d_1$. There are two cases:

(a) $h \mid d_2$. $P_0 P_1 \cap (f_\lambda = 0)$ is a finite set of points for $\lambda = 0, 1$. Thus $P_0 P_1 \cap X = \emptyset$.

(b) $h \not| d_2$. In this case no monomial of the form $x_0^{n_0} x_1^{n_1}$ of degree d_2 exists (or else $h \mid d_2$). From Theorem I.5.7 (with $I = \{0, 1\}$) there exists a monomial $x_0^{n_0} x_1^{n_1} x_e$ of degree d_2, where $e \neq 0, 1$. By renumbering we can assume that $e = 2$. Thus f_2 is of the form:

$$f_2 = x_0^{n_0} x_1^{n_1} x_2 + \dots$$

and $\partial f_2 / \partial x_2$ is non-zero on $P_0 P_1 \cap X$. By the Inverse Function Theorem, x_3, x_4 and x_5 are local coordinates and so each point of $P_0 P_1 \cap X$ is of type $\frac{1}{h}(a_3, a_4, a_5)$. Condition (2$b$) gives $h \mid \alpha + a_m$ for some $m \neq 0, 1, 2$. Assume that $m = 3$, and hense $h \mid a_4 + a_5$.

Let $h' = \text{hcf}(h, a_4)$. So $h \mid a_4$ and thus by condition (4) $h = 1$. Thus these points are terminal.

Therefore X has at worst terminal points along P_0P_1.

Consider the face $P_iP_jP_k$. As before assume $i = 0$, $j = 1$ and $k = 2$. By condition (3) $h = \text{hcf}(a_0, a_1, a_2) \mid d_1$ and $h \mid d_2$. So $P_0P_1P_2$ intersects X transversally. Each point in the intersection is of type $\frac{1}{h}(a_3, a_4, a_5)$. As $h \mid \alpha + a_m$ for some $m \neq 0, 1, 2$, after renumbering, $h \mid a_3 + a_4$. By condition (4) $\text{hcf}(h, a_3) = 1$. Thus these points are terminal.

Therefore condition (1), ..., (4) are sufficient.

Conversely assume that X has at worst terminal singularities. Suppose $a_i \not| d_1$ and $a_i \not| d_2$. By renumbering assume $i = 0$. Thus $P_0 \in X$. Since X is quasismooth there exist 2 monomials $x_0^n x_{e_1}$ and $x_0^m x_{e_2}$ of degrees d_1 and d_2, where $e_1 \neq e_2$. This gives the first part of condition (1). Without loss of generality we can assume that $e_1 = 1$ and $e_2 = 2$. As before we find that $P_0 \in X$ is of type $\frac{1}{a_0}(a_3, a_4, a_5)$. As this is terminal, after renumbering, $a_0 \mid a_3 + a_4$ and so $a_0 \mid \alpha + a_5$. This is condition (1).

Suppose $h = \text{hcf}(a_i, a_j)$ for distinct i and j. As usual we can renumber such that $i = 0$ and $j = 1$. As X is well-formed then $h \mid d_\lambda$ for some λ. Suppose $h \mid d_1$. If $h \mid d_2$ then this is condition ($2a$). So assume that $h \not| d_2$. As above each point of $P_0P_1 \cap X$ is isolated and of type $\frac{1}{h}(a_3, a_4, a_5)$. After renumbering, $h \mid a_3 + a_4$ and so $h \mid \alpha + a_5$. This is condition ($2b$). Likewise for the case when $h \mid d_2$ but $h \not| d_1$. This gives condition ($2c$).

Suppose $h = \text{hcf}(a_i, a_j, a_k)$ for distinct i, j and k. Renumber such that $i = 0$, $j = 1$ and $k = 2$. Since X is well-formed $h \mid d_1$ and $h \mid d_2$. $P_0P_1P_2 \cap X$ is a finite number of points, all of type $\frac{1}{h}(a_3, a_4, a_5)$. As these are terminal, after renumbering, $h \mid a_3 + a_4$ and so $h \mid \alpha + a_5$. This is condition (3). Condition (4) follows from the fact that $\text{hcf}(h, a_3) = \text{hcf}(h, a_4) = 1$.

So these conditions are necessary.

□

II.4.6 Codimension 2 weighted 3-fold complete intersection with trivial canonical bundle.

The four families of 3-fold codimension 2 quasismooth complete intersections with at worst terminal singularities, $\omega_X \cong \mathcal{O}_X$ and $\sum a_i < 100$ are:

$X_{2,4}$ in $\mathbf{P}(1,1,1,1,1,1)$
$X_{3,3}$ in $\mathbf{P}(1,1,1,1,1,1)$
$X_{3,4}$ in $\mathbf{P}(1,1,1,1,1,2)$
$X_{4,4}$ in $\mathbf{P}(1,1,1,1,2,2)$

Again the above are all non-singular and were found using a computer search based on the conditions of Theorem II.4.4.

II.5 Canonically embedded weighted 3-folds.

II.5.1 Canonically embedded 3-fold weighted hypersurfaces.

There are 23 families of 3-fold quasismooth weighted hypersurfaces with only terminal isolated quotient singularities with $\omega_X \cong \mathcal{O}_X(1)$ and $\sum a_i \leq 100$.

Hypersurface.	K_X^3	p_g	Singularities.
X_6 in $\mathbf{P}(1,1,1,1,1)$	6	5	
X_7 in $\mathbf{P}(1,1,1,1,2)$	7/2	4	$\frac{1}{2}(1,-1,1)$
X_8 in $\mathbf{P}(1,1,1,2,2)$	2	3	$4 \times \frac{1}{2}(1,-1,1)$

X_9 in $\mathbf{P}(1,1,1,2,3)$	3/2	3	$\frac{1}{2}(1,-1,1)$
X_{10} in $\mathbf{P}(1,1,1,1,5)$	2	4	
X_{10} in $\mathbf{P}(1,1,2,2,3)$	5/6	2	$5\times\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
X_{12} in $\mathbf{P}(1,1,1,2,6)$	1	3	$2\times\frac{1}{2}(1,-1,1)$
X_{12} in $\mathbf{P}(1,1,2,3,4)$	1/2	2	$3\times\frac{1}{2}(1,-1,1)$
X_{12} in $\mathbf{P}(1,2,2,3,3)$	1/3	1	$6\times\frac{1}{2}(1,-1,1)$, $4\times\frac{1}{3}(1,-1,1)$
X_{14} in $\mathbf{P}(1,1,2,2,7)$	1/2	2	$7\times\frac{1}{2}(1,-1,1)$
X_{15} in $\mathbf{P}(1,2,3,3,5)$	1/6	1	$\frac{1}{2}(1,-1,1)$, $5\times\frac{1}{3}(1,-1,1)$
X_{16} in $\mathbf{P}(1,1,2,3,8)$	1/3	2	$2\times\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
X_{16} in $\mathbf{P}(1,2,3,4,5)$	2/15	1	$4\times\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$
X_{18} in $\mathbf{P}(1,2,2,3,9)$	1/6	1	$9\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$
X_{18} in $\mathbf{P}(2,3,3,4,5)$	1/20	0	$4\times\frac{1}{2}(1,-1,1)$, $6\times\frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$
X_{20} in $\mathbf{P}(2,3,4,5,5)$	1/30	0	$5\times\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $4\times\frac{1}{5}(1,-1,2)$
X_{21} in $\mathbf{P}(1,3,4,5,7)$	1/20	1	$\frac{1}{4}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$
X_{22} in $\mathbf{P}(1,2,3,4,11)$	1/12	1	$5\times\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$
X_{28} in $\mathbf{P}(1,3,4,5,14)$	1/30	1	$\frac{1}{3}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$
X_{28} in $\mathbf{P}(3,4,5,7,8)$	1/120	0	$\frac{1}{3}(1,-1,1)$, $3\times\frac{1}{4}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$, $\frac{1}{8}(1,-1,3)$
X_{30} in $\mathbf{P}(2,3,4,5,15)$	1/60	0	$7\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$, $2\times\frac{1}{5}(1,-1,2)$
X_{40} in $\mathbf{P}(3,4,5,7,20)$	1/210	0	$\frac{1}{3}(1,-1,1)$, $2\times\frac{1}{4}(1,-1,1)$, $2\times\frac{1}{5}(1,-1,2)$, $\frac{1}{7}(1,-1,2)$
X_{46} in $\mathbf{P}(4,5,6,7,23)$	1/420	0	$\frac{1}{4}(1,-1,1)$, $3\times\frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$, $\frac{1}{6}(1,-1,1)$, $\frac{1}{7}(1,-1,3)$

II.5.2 Conjecture. This list was produced using a computer program. In fact the program was run much further but produced no more examples. I conjecture that the lists in this section and in sections II.5.3, II.6.5, and II.6.6 are complete lists, and not limited by $\sum a_i \leq 100$.

II.5.3 Interesting Example. The family X_{46} in $\mathbf{P}(4,5,6,7,23)$ has p_g, P_2 and P_3 all zero. It is interesting to find canonical 3-folds with as many of their first plurigenera equal to zero as possible (see also [F1, section 4.9]). This is the best such weighted complete intersections example found in these lists.

II.5.4 Canonically embedded codimension 2 weighted 3-folds.

There are 59 families of 3-fold codimension 2 weighted complete intersections satisfying the conditions of Theorem II.4.4 with $\omega_X \cong \mathcal{O}_X(1)$ and $\sum a_i \leq 100$.

Complete Intersection	K_X^3	p_g	Singularities.
$X_{2,5}$ in $\mathbf{P}(1,1,1,1,1,1)$	10	6	
$X_{3,4}$ in $\mathbf{P}(1,1,1,1,1,1)$	12	6	
$X_{3,5}$ in $\mathbf{P}(1,1,1,1,1,2)$	15/2	5	$\frac{1}{2}(1,-1,1)$
$X_{4,4}$ in $\mathbf{P}(1,1,1,1,1,2)$	8	5	
$X_{3,6}$ in $\mathbf{P}(1,1,1,1,2,2)$	9/2	4	$3\times\frac{1}{2}(1,-1,1)$
$X_{4,5}$ in $\mathbf{P}(1,1,1,1,2,2)$	5	4	$2\times\frac{1}{2}(1,-1,1)$
$X_{2,8}$ in $\mathbf{P}(1,1,1,1,1,4)$	4	5	
$X_{4,6}$ in $\mathbf{P}(1,1,1,1,2,3)$	4	4	
$X_{4,6}$ in $\mathbf{P}(1,1,1,2,2,2)$	3	3	$6\times\frac{1}{2}(1,-1,1)$
$X_{3,8}$ in $\mathbf{P}(1,1,1,1,2,4)$	3	4	$2\times\frac{1}{2}(1,-1,1)$
$X_{4,7}$ in $\mathbf{P}(1,1,1,2,2,3)$	7/3	3	$\frac{1}{3}(1,-1,1)$, $2\times\frac{1}{2}(1,-1,1)$
$X_{5,6}$ in $\mathbf{P}(1,1,1,2,2,3)$	5/2	3	$3\times\frac{1}{2}(1,-1,1)$
$X_{6,6}$ in $\mathbf{P}(1,1,1,2,3,3)$	2	3	

$X_{4,8}$ in $\mathbf{P}(1,1,2,2,2,3)$	4/3	2	$\frac{1}{3}(1,-1,1)$, $8\times\frac{1}{2}(1,-1,1)$
$X_{6,6}$ in $\mathbf{P}(1,1,2,2,2,3)$	3/2	2	$9\times\frac{1}{2}(1,-1,1)$
$X_{3,10}$ in $\mathbf{P}(1,1,1,2,2,5)$	3/2	3	$5\times\frac{1}{2}(1,-1,1)$
$X_{4,9}$ in $\mathbf{P}(1,1,2,2,3,3)$	1	2	$2\times\frac{1}{2}(1,-1,1)$, $3\times\frac{1}{3}(1,-1,1)$
$X_{6,7}$ in $\mathbf{P}(1,1,2,2,3,3)$	7/6	2	$3\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{4,10}$ in $\mathbf{P}(1,1,1,2,3,5)$	4/3	3	$\frac{1}{3}(1,-1,1)$
$X_{4,10}$ in $\mathbf{P}(1,1,2,2,2,5)$	1	2	$10\times\frac{1}{2}(1,-1,1)$
$X_{6,8}$ in $\mathbf{P}(1,1,2,2,3,4)$	1	2	$6\times\frac{1}{2}(1,-1,1)$
$X_{6,8}$ in $\mathbf{P}(1,2,2,2,3,3)$	2/3	1	$12\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{6,9}$ in $\mathbf{P}(1,1,2,3,3,4)$	3/4	2	$\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$
$X_{6,9}$ in $\mathbf{P}(1,2,2,3,3,3)$	1/2	1	$3\times\frac{1}{2}(1,-1,1)$, $6\times\frac{1}{3}(1,-1,1)$
$X_{4,12}$ in $\mathbf{P}(1,1,2,2,3,6)$	2/3	2	$4\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{6,10}$ in $\mathbf{P}(1,1,2,3,3,5)$	2/3	2	$2\times\frac{1}{3}(1,-1,1)$
$X_{6,10}$ in $\mathbf{P}(1,2,2,2,3,5)$	1/2	1	$15\times\frac{1}{2}(1,-1,1)$
$X_{6,10}$ in $\mathbf{P}(1,2,2,3,3,4)$	5/12	1	$\frac{1}{4}(1,-1,1)$, $7\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{4,14}$ in $\mathbf{P}(1,2,2,2,3,7)$	1/3	1	$\frac{1}{3}(1,-1,1)$, $14\times\frac{1}{2}(1,-1,1)$
$X_{6,12}$ in $\mathbf{P}(1,2,2,3,4,5)$	3/10	1	$\frac{1}{5}(1,-1,2)$, $9\times\frac{1}{2}(1,-1,1)$
$X_{8,10}$ in $\mathbf{P}(1,2,2,3,4,5)$	1/3	1	$\frac{1}{3}(1,-1,1)$, $10\times\frac{1}{2}(1,-1,1)$
$X_{6,12}$ in $\mathbf{P}(1,2,3,3,4,4)$	1/4	1	$3\times\frac{1}{2}(1,-1,1)$, $3\times\frac{1}{4}(1,-1,1)$
$X_{6,12}$ in $\mathbf{P}(2,2,3,3,3,4)$	1/6	0	$9\times\frac{1}{2}(1,-1,1)$, $8\times\frac{1}{3}(1,-1,1)$
$X_{6,13}$ in $\mathbf{P}(1,2,3,3,4,5)$	13/60	1	$\frac{1}{4}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$, $\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{9,10}$ in $\mathbf{P}(1,2,3,3,4,5)$	1/4	1	$\frac{1}{4}(1,-1,1)$, $2\times\frac{1}{2}(1,-1,1)$, $3\times\frac{1}{3}(1,-1,1)$
$X_{6,14}$ in $\mathbf{P}(1,2,2,3,4,7)$	1/4	1	$\frac{1}{4}(1,-1,1)$, $10\times\frac{1}{2}(1,-1,1)$
$X_{8,12}$ in $\mathbf{P}(1,2,3,4,4,5)$	1/5	1	$\frac{1}{5}(1,-1,1)$, $6\times\frac{1}{2}(1,-1,1)$
$X_{6,14}$ in $\mathbf{P}(2,2,2,3,3,7)$	1/6	0	$21\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{8,12}$ in $\mathbf{P}(2,2,3,3,4,5)$	2/15	0	$\frac{1}{5}(1,-1,2)$, $12\times\frac{1}{2}(1,-1,1)$, $4\times\frac{1}{3}(1,-1,1)$
$X_{6,15}$ in $\mathbf{P}(2,3,3,3,4,5)$	1/12	0	$\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $10\times\frac{1}{3}(1,-1,1)$
$X_{6,16}$ in $\mathbf{P}(1,2,3,3,4,8)$	1/6	1	$2\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$, $2\times\frac{1}{4}(1,-1,1)$
$X_{10,12}$ in $\mathbf{P}(1,2,3,4,5,6)$	1/6	1	$5\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{10,12}$ in $\mathbf{P}(2,2,3,4,5,5)$	1/10	0	$15\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{5}(1,-1,2)$
$X_{10,12}$ in $\mathbf{P}(2,3,3,4,4,5)$	1/12	0	$6\times\frac{1}{2}(1,-1,1)$, $4\times\frac{1}{3}(1,-1,1)$, $3\times\frac{1}{4}(1,-1,1)$
$X_{8,15}$ in $\mathbf{P}(2,3,3,4,5,5)$	1/15	0	$2\times\frac{1}{2}(1,-1,1)$, $5\times\frac{1}{3}(1,-1,1)$, $3\times\frac{1}{5}(1,-1,2)$
$X_{6,18}$ in $\mathbf{P}(1,2,3,3,5,9)$	2/15	1	$\frac{1}{5}(1,-1,2)$, $4\times\frac{1}{3}(1,-1,1)$
$X_{6,18}$ in $\mathbf{P}(2,2,3,3,4,9)$	1/12	0	$\frac{1}{4}(1,-1,1)$, $13\times\frac{1}{2}(1,-1,1)$, $4\times\frac{1}{3}(1,-1,1)$
$X_{10,14}$ in $\mathbf{P}(2,2,3,4,5,7)$	1/12	0	$\frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$, $17\times\frac{1}{2}(1,-1,1)$
$X_{6,20}$ in $\mathbf{P}(1,2,3,4,5,10)$	1/10	1	$3\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{5}(1,-1,2)$
$X_{12,14}$ in $\mathbf{P}(2,3,4,4,5,7)$	1/20	0	$\frac{1}{5}(1,-1,2)$, $9\times\frac{1}{2}(1,-1,1)$, $3\times\frac{1}{4}(1,-1,1)$
$X_{12,15}$ in $\mathbf{P}(1,3,4,5,6,7)$	1/14	1	$\frac{1}{7}(1,-1,2)$, $\frac{1}{2}(1,-1,1)$
$X_{10,18}$ in $\mathbf{P}(2,3,4,5,6,7)$	1/28	0	$\frac{1}{4}(1,-1,1)$, $\frac{1}{7}(1,-1,3)$, $7\times\frac{1}{2}(1,-1,1)$, $3\times\frac{1}{3}(1,-1,1)$
$X_{12,16}$ in $\mathbf{P}(2,3,4,5,6,7)$	4/105	0	$\frac{1}{5}(1,-1,2)$, $\frac{1}{7}(1,-1,2)$, $8\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{8,22}$ in $\mathbf{P}(2,3,4,4,5,11)$	1/30	0	$\frac{1}{3}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$, $10\times\frac{1}{2}(1,-1,1)$, $2\times\frac{1}{4}(1,-1,1)$
$X_{12,18}$ in $\mathbf{P}(2,3,4,5,6,9)$	1/30	0	$\frac{1}{5}(1,-1,1)$, $9\times\frac{1}{2}(1,-1,1)$, $4\times\frac{1}{3}(1,-1,1)$
$X_{12,18}$ in $\mathbf{P}(3,4,4,5,6,7)$	3/140	0	$\frac{1}{5}(1,-1,1)$, $\frac{1}{7}(1,-1,2)$, $3\times\frac{1}{4}(1,-1,1)$, $3\times\frac{1}{2}(1,-1,1)$
$X_{10,21}$ in $\mathbf{P}(3,4,5,5,6,7)$	1/60	0	$\frac{1}{4}(1,-1,1)$, $\frac{1}{6}(1,-1,1)$, $3\times\frac{1}{3}(1,-1,1)$, $2\times\frac{1}{5}(1,-1,2)$

$X_{12,21}$ in $\mathbf{P}(3,4,5,6,7,7)$	1/70	0	$\frac{1}{5}(1,-1,2)$, $\frac{1}{2}(1,-1,1)$, $3\times\frac{1}{7}(1,-1,2)$
$X_{12,28}$ in $\mathbf{P}(3,4,5,6,7,14)$	1/105	0	$\frac{1}{5}(1,-1,1)$,$2\times\frac{1}{3}(1,-1,1)$,$2\times\frac{1}{2}(1,-1,1)$,$2\times\frac{1}{7}(1,-1,2)$

II.6 Q-Fano 3-folds.

In [R4, section 4.3] Reid conjectures that if X is a $\mathbf{Q}$-Fano 3-fold then $\mathcal{O}_X(-K_X)$ has a global section. This is false as shown by the following example:

II.6.1 Example.

The family $X_{12,14}$ in $\mathbf{P}(2,3,4,5,6,7)$ is an anticanonically embedded Fano 3-fold with only the following isolated terminal singularities: 1 of type $\frac{1}{5}(4,1,2)$, 2 of type $\frac{1}{3}(2,1,1)$ and 7 of type $\frac{1}{2}(1,1,1)$. These singularities were determined eariler.

Since it is quasismooth and of dimension 3, $\omega_X \cong \mathcal{O}_X(-1)$ and $K_X^3 = -\frac{1}{30}$. By an unpublished result of Barlow (see [R4, Corollary 10.3]) we have

$$K_X.c_2(X) = \sum_{\text{singularities } Q} \frac{r_Q^2-1}{r_Q} - 24\chi(\mathcal{O}_X)$$

where r_Q is the index of the singularity Q of type $\frac{1}{r_Q}(1,-1,b_Q)$. So $K_X.c_2 = -\frac{101}{30} < 0$. However $\mathcal{O}_X(-K_X) \cong \mathcal{O}_X(1)$ has no global sections.

Experimentation leads to the following:

II.6.2 Conjecture. *Every weighted hypersurface $\mathbf{Q}$-Fano 3-fold X, with canonical singularities, has a global section of ω_X^{-1}.*

This is clear in one particular case.

II.6.3 Lemma. *Consider X_d in $\mathbf{P}(a_0,...,a_4)$ be a family of $\mathbf{Q}$-Fano 3-folds with only isolated terminal singularities. Suppose also that $a_0 \le ... \le a_4$ and $a_4 \nmid d$. Then ω_X^{-1} has a global section.*

Proof. As $a_4 \nmid d$, the vertex P_4 is contained in X. The condition for a terminal singularity at P_i gives that there exists an a_m such that $a_4 \mid a_m + \alpha$. So $a_m = \mu a_4 + (-\alpha)$ for some integer μ. Since $\alpha < 0$ and $a_4 \ge a_m$, then $\mu \le 0$. Thus $\deg(x_4^{(-\mu)}x_m) = -\alpha$ and so $\dim \mathrm{H}^0(\mathcal{O}_X(-\alpha)) \ge 1$. But $\mathrm{H}^0(\omega_X^{-1}) \cong \mathrm{H}^0(\mathcal{O}_X(-\alpha))$, and so ω_X^{-1} has a global section.

□

Notice that when $\alpha = -1$, there exists a generator x_i with deg $(x_i) = 1$, i.e. $a_0 = 1$.

II.6.4 Lemma. *There is a bijection between the following:*

(i) the set of families of quasismooth, well-formed weighted surface hypersurfaces S_d in $\mathbf{P}(a_1,...,a_4)$ with only canonical singularities and trivial canonical class.

(ii) the set of families of quasismooth weighted 3-folds hypersurfaces X_d in $\mathbf{P}(1,a_1,...,a_4)$ with only terminal singularities and $\omega_X \cong \mathcal{O}_X(-1)$.

Proof. Suppose that S_d in $\mathbf{P} = \mathbf{P}(a_1,...,a_4)$ is a K3 surface, with at worst canonical singularities. By comparing the conditions in Theorems II.3.1 and II.4.1 it is clear that the conditions of the latter are satisfied for $X = X_d$ in $\mathbf{P}(1,a_1,...,a_n)$. Thus X is quasismooth with at worst terminal singularities.

Conversely suppose X_d in $\mathbf{P}(1, a_1, ..., a_n)$ is quasismooth and has at worst terminal singularities. It can be seen from Theorems II.3.1 and II.4.1 that only condition $(1ii)$ of Theorem II.3.1 needs proof (the others being either trivially satisfied or equivalent in both the surface and the 3-fold case).

Set $a_0 = 1$ and consider a_i for $i \neq 0$. Suppose that condition $(1ii)$ does not hold. So $a_i \not| d - a_e$ for all $e = 1, ..., 4$. In particular $a_i \not| d$. Thus $a_i \mid d - a_0$, i.e. $a_i \mid d - 1$. Since $a_i \not| d$ then Theorem II.4.1 $(1iv)$ gives that there exists an $m \neq 0, i$ such that $a_i \mid a_m - 1$. Hence $a_i \mid (d-1) - (a_m - 1)$, i.e. $a_i \mid d - a_m$, a contradiction. So $a_i \mid d - a_e$ for some $e \neq 0, i$, which is condition $(1ii)$ of Theorem II.3.1.

□

II.6.5 Note. Each singularity on the K3 surface is of type $\frac{1}{r}(a, -a)$ for some r and a, with respect to some pair of the coordinates x_1, ..., x_4. Forming the corresponding **Q**-Fano 3-fold results in an extra local coordinate x_0 at each singularity, which is thus of type $\frac{1}{r}(a, -a, 1)$. A similar result holds for higher codimensions.

II.6.6 List of anti-canonically embedded (Q-Fano) weighted 3-folds.

The previous lemma gives a bijection between Reid's list of 95 families of weighted K3 surfaces (see section II.3.3 or [R4, section 4.5]) and the 95 families of quasismooth weighted hypersurface **Q**-Fano 3-folds, with $\alpha = -1$ and $\sum a_i \leq 100$. These were found by a computer search and are listed below.

Hypersurface.	K_X^3	Singularities.
X_4 in $\mathbf{P}(1,1,1,1,1)$	-4	
X_5 in $\mathbf{P}(1,1,1,1,2)$	-5/2	$\frac{1}{2}(1,-1,1)$
X_6 in $\mathbf{P}(1,1,1,1,3)$	-2	
X_6 in $\mathbf{P}(1,1,1,2,2)$	-3/2	$3 \times \frac{1}{2}(1,-1,1)$
X_7 in $\mathbf{P}(1,1,1,2,3)$	-7/6	$\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
X_8 in $\mathbf{P}(1,1,1,2,4)$	-1	$2 \times \frac{1}{2}(1,-1,1)$
X_8 in $\mathbf{P}(1,1,2,2,3)$	-2/3	$4 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
X_9 in $\mathbf{P}(1,1,1,3,4)$	-3/4	$\frac{1}{4}(1,-1,1)$
X_9 in $\mathbf{P}(1,1,2,3,3)$	-1/2	$\frac{1}{2}(1,-1,1)$, $3 \times \frac{1}{3}(1,-1,1)$
X_{10} in $\mathbf{P}(1,1,1,3,5)$	-2/3	$\frac{1}{3}(1,-1,1)$
X_{10} in $\mathbf{P}(1,1,2,2,5)$	-1/2	$5 \times \frac{1}{2}(1,-1,1)$
X_{10} in $\mathbf{P}(1,1,2,3,4)$	-5/12	$2 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$
X_{11} in $\mathbf{P}(1,1,2,3,5)$	-11/30	$\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$
X_{12} in $\mathbf{P}(1,1,1,4,6)$	-1/2	$\frac{1}{2}(1,-1,1)$
X_{12} in $\mathbf{P}(1,1,2,3,6)$	-1/3	$2 \times \frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{3}(1,-1,1)$
X_{12} in $\mathbf{P}(1,1,2,4,5)$	-3/10	$3 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$
X_{12} in $\mathbf{P}(1,1,3,4,4)$	-1/4	$3 \times \frac{1}{4}(1,-1,1)$
X_{12} in $\mathbf{P}(1,2,2,3,5)$	-1/5	$6 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$
X_{12} in $\mathbf{P}(1,2,3,3,4)$	-1/6	$3 \times \frac{1}{2}(1,-1,1)$, $4 \times \frac{1}{3}(1,-1,1)$
X_{13} in $\mathbf{P}(1,1,3,4,5)$	-13/60	$\frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$
X_{14} in $\mathbf{P}(1,1,2,4,7)$	-1/4	$3 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$
X_{14} in $\mathbf{P}(1,2,2,3,7)$	-1/6	$7 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
X_{14} in $\mathbf{P}(1,2,3,4,5)$	-7/60	$3 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$
X_{15} in $\mathbf{P}(1,1,2,5,7)$	-3/14	$\frac{1}{2}(1,-1,1)$, $\frac{1}{7}(1,-1,3)$

X_{15} in $\mathbf{P}(1,1,3,4,7)$	-5/28	$\frac{1}{4}(1,-1,1)$, $\frac{1}{7}(1,-1,2)$
X_{15} in $\mathbf{P}(1,1,3,5,6)$	-1/6	$2 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{6}(1,-1,1)$
X_{15} in $\mathbf{P}(1,2,3,5,5)$	-1/10	$\frac{1}{2}(1,-1,1)$, $3 \times \frac{1}{5}(1,-1,2)$
X_{15} in $\mathbf{P}(1,3,3,4,5)$	-1/12	$5 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$
X_{16} in $\mathbf{P}(1,1,2,5,8)$	-1/5	$2 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$
X_{16} in $\mathbf{P}(1,1,3,4,8)$	-1/6	$\frac{1}{3}(1,-1,1)$, $2 \times \frac{1}{4}(1,-1,1)$
X_{16} in $\mathbf{P}(1,1,4,5,6)$	-2/15	$\frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$, $\frac{1}{6}(1,-1,1)$
X_{16} in $\mathbf{P}(1,2,3,4,7)$	-2/21	$4 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{7}(1,-1,2)$
X_{17} in $\mathbf{P}(1,2,3,5,7)$	-17/210	$\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$, $\frac{1}{7}(1,-1,3)$
X_{18} in $\mathbf{P}(1,1,2,6,9)$	-1/6	$3 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
X_{18} in $\mathbf{P}(1,1,3,5,9)$	-2/15	$2 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$
X_{18} in $\mathbf{P}(1,1,4,6,7)$	-3/28	$\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{7}(1,-1,1)$
X_{18} in $\mathbf{P}(1,2,3,4,9)$	-1/12	$4 \times \frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$
X_{18} in $\mathbf{P}(1,2,3,5,8)$	-3/40	$2 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$, $\frac{1}{8}(1,-1,3)$
X_{18} in $\mathbf{P}(1,3,4,5,6)$	-1/20	$3 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$
X_{19} in $\mathbf{P}(1,3,4,5,7)$	-19/420	$\frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$, $\frac{1}{7}(1,-1,2)$
X_{20} in $\mathbf{P}(1,1,4,5,10)$	-1/10	$\frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{5}(1,-1,1)$
X_{20} in $\mathbf{P}(1,2,3,5,10)$	-1/15	$2 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $2 \times \frac{1}{5}(1,-1,2)$
X_{20} in $\mathbf{P}(1,2,4,5,9)$	-1/18	$5 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{9}(1,-1,2)$
X_{20} in $\mathbf{P}(1,2,5,6,7)$	-1/21	$3 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{6}(1,-1,1)$, $\frac{1}{7}(1,-1,3)$
X_{20} in $\mathbf{P}(1,3,4,5,8)$	-1/24	$\frac{1}{3}(1,-1,1)$, $2 \times \frac{1}{4}(1,-1,1)$, $\frac{1}{8}(1,-1,3)$
X_{21} in $\mathbf{P}(1,1,3,7,10)$	-1/10	$\frac{1}{10}(1,-1,3)$
X_{21} in $\mathbf{P}(1,1,5,7,8)$	-3/40	$\frac{1}{5}(1,-1,2)$, $\frac{1}{8}(1,-1,1)$
X_{21} in $\mathbf{P}(1,2,3,7,9)$	-1/18	$\frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{9}(1,-1,4)$
X_{21} in $\mathbf{P}(1,3,5,6,7)$	-1/30	$3 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$, $\frac{1}{6}(1,-1,1)$
X_{22} in $\mathbf{P}(1,1,3,7,11)$	-2/21	$\frac{1}{3}(1,-1,1)$, $\frac{1}{7}(1,-1,2)$
X_{22} in $\mathbf{P}(1,1,4,6,11)$	-1/12	$\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{6}(1,-1,1)$
X_{22} in $\mathbf{P}(1,2,4,5,11)$	-1/20	$5 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$
X_{24} in $\mathbf{P}(1,1,3,8,12)$	-1/12	$2 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$
X_{24} in $\mathbf{P}(1,1,6,8,9)$	-1/18	$\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{9}(1,-1,1)$
X_{24} in $\mathbf{P}(1,2,3,7,12)$	-1/21	$2 \times \frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{7}(1,-1,3)$
X_{24} in $\mathbf{P}(1,2,3,8,11)$	-1/22	$3 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{11}(1,-1,4)$
X_{24} in $\mathbf{P}(1,3,4,5,12)$	-1/30	$2 \times \frac{1}{3}(1,-1,1)$, $2 \times \frac{1}{4}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$
X_{24} in $\mathbf{P}(1,3,4,7,10)$	-1/35	$\frac{1}{2}(1,-1,1)$, $\frac{1}{7}(1,-1,2)$, $\frac{1}{10}(1,-1,3)$
X_{24} in $\mathbf{P}(1,3,6,7,8)$	-1/42	$4 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{7}(1,-1,1)$
X_{24} in $\mathbf{P}(1,4,5,6,9)$	-1/45	$2 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{9}(1,-1,2)$
X_{25} in $\mathbf{P}(1,4,5,7,9)$	-5/252	$\frac{1}{4}(1,-1,1)$, $\frac{1}{7}(1,-1,3)$, $\frac{1}{9}(1,-1,2)$
X_{26} in $\mathbf{P}(1,1,5,7,13)$	-2/35	$\frac{1}{5}(1,-1,2)$, $\frac{1}{7}(1,-1,1)$
X_{26} in $\mathbf{P}(1,2,3,8,13)$	-1/24	$3 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{8}(1,-1,3)$
X_{26} in $\mathbf{P}(1,2,5,6,13)$	-1/30	$4 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$, $\frac{1}{6}(1,-1,1)$
X_{27} in $\mathbf{P}(1,2,5,9,11)$	-3/110	$\frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$, $\frac{1}{11}(1,-1,5)$
X_{27} in $\mathbf{P}(1,5,6,7,9)$	-1/70	$\frac{1}{5}(1,-1,1)$, $\frac{1}{6}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{7}(1,-1,3)$
X_{28} in $\mathbf{P}(1,1,4,9,14)$	-1/18	$\frac{1}{2}(1,-1,1)$, $\frac{1}{9}(1,-1,2)$
X_{28} in $\mathbf{P}(1,3,4,7,14)$	-1/42	$\frac{1}{3}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{7}(1,-1,2)$

X_{28} in $\mathbf{P}(1,4,6,7,11)$	-1/66	$2 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{6}(1,-1,1)$, $\frac{1}{11}(1,-1,3)$
X_{30} in $\mathbf{P}(1,1,4,10,15)$	-1/20	$\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$
X_{30} in $\mathbf{P}(1,1,6,8,15)$	-1/24	$\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{8}(1,-1,1)$
X_{30} in $\mathbf{P}(1,2,3,10,15)$	-1/30	$3 \times \frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$
X_{30} in $\mathbf{P}(1,2,6,7,15)$	-1/42	$5 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{7}(1,-1,1)$
X_{30} in $\mathbf{P}(1,3,4,10,13)$	-1/52	$\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{13}(1,-1,4)$
X_{30} in $\mathbf{P}(1,4,5,6,15)$	-1/60	$\frac{1}{4}(1,-1,1)$, $2 \times \frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{5}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
X_{30} in $\mathbf{P}(1,5,6,8,11)$	-1/88	$\frac{1}{2}(1,-1,1)$, $\frac{1}{8}(1,-1,3)$, $\frac{1}{11}(1,-1,2)$
X_{32} in $\mathbf{P}(1,2,5,9,16)$	-1/45	$2 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$, $\frac{1}{9}(1,-1,4)$
X_{32} in $\mathbf{P}(1,4,5,7,16)$	-1/70	$2 \times \frac{1}{4}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$, $\frac{1}{7}(1,-1,3)$
X_{33} in $\mathbf{P}(1,3,5,11,14)$	-1/70	$\frac{1}{5}(1,-1,1)$, $\frac{1}{14}(1,-1,5)$
X_{34} in $\mathbf{P}(1,3,4,10,17)$	-1/60	$\frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{10}(1,-1,3)$
X_{34} in $\mathbf{P}(1,4,6,7,17)$	-1/84	$\frac{1}{4}(1,-1,1)$, $2 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{6}(1,-1,1)$, $\frac{1}{7}(1,-1,2)$
X_{36} in $\mathbf{P}(1,1,5,12,18)$	-1/30	$\frac{1}{5}(1,-1,2)$, $\frac{1}{6}(1,-1,1)$
X_{36} in $\mathbf{P}(1,3,4,11,18)$	-1/66	$2 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{11}(1,-1,3)$
X_{36} in $\mathbf{P}(1,7,8,9,12)$	-1/168	$\frac{1}{7}(1,-1,3)$, $\frac{1}{8}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
X_{38} in $\mathbf{P}(1,3,5,11,19)$	-2/165	$\frac{1}{3}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$, $\frac{1}{11}(1,-1,4)$
X_{38} in $\mathbf{P}(1,5,6,8,19)$	-1/120	$\frac{1}{5}(1,-1,1)$, $\frac{1}{6}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{8}(1,-1,3)$
X_{40} in $\mathbf{P}(1,5,7,8,20)$	-1/140	$2 \times \frac{1}{5}(1,-1,2)$, $\frac{1}{7}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$
X_{42} in $\mathbf{P}(1,1,6,14,21)$	-1/42	$\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{7}(1,-1,1)$
X_{42} in $\mathbf{P}(1,2,5,14,21)$	-1/70	$3 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$, $\frac{1}{7}(1,-1,3)$
X_{42} in $\mathbf{P}(1,3,4,14,21)$	-1/84	$2 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{7}(1,-1,2)$
X_{44} in $\mathbf{P}(1,4,5,13,22)$	-1/130	$\frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$, $\frac{1}{13}(1,-1,3)$
X_{48} in $\mathbf{P}(1,3,5,16,24)$	-1/120	$2 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$, $\frac{1}{8}(1,-1,3)$
X_{50} in $\mathbf{P}(1,7,8,10,25)$	-1/280	$\frac{1}{7}(1,-1,2)$, $\frac{1}{8}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$
X_{54} in $\mathbf{P}(1,4,5,18,27)$	-1/180	$\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$, $\frac{1}{9}(1,-1,2)$
X_{66} in $\mathbf{P}(1,5,6,22,33)$	-1/330	$\frac{1}{5}(1,-1,2)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{11}(1,-1,2)$

II.6.7 Codimension 2 Q-Fano weighted complete intersections.

There are 85 codimension 2 quasi-smooth **Q**-Fano weighted complete intersections which satisfy the conditions of Theorem II.4.4, $\alpha = -1$ and $\sum a_i \leq 100$.

Complete intersection	K_X^3	Singularities.
$X_{2,3}$ in $\mathbf{P}(1,1,1,1,1,1)$	-6	
$X_{3,3}$ in $\mathbf{P}(1,1,1,1,1,2)$	-9/2	$\frac{1}{2}(1,-1,1)$
$X_{3,4}$ in $\mathbf{P}(1,1,1,1,2,2)$	-3	$2 \times \frac{1}{2}(1,-1,1)$
$X_{4,4}$ in $\mathbf{P}(1,1,1,1,2,3)$	-8/3	$\frac{1}{3}(1,-1,1)$
$X_{4,4}$ in $\mathbf{P}(1,1,1,2,2,2)$	-2	$4 \times \frac{1}{2}(1,-1,1)$
$X_{4,5}$ in $\mathbf{P}(1,1,1,2,2,3)$	-5/3	$\frac{1}{3}(1,-1,1)$, $2 \times \frac{1}{2}(1,-1,1)$
$X_{4,6}$ in $\mathbf{P}(1,1,1,2,3,3)$	-4/3	$2 \times \frac{1}{3}(1,-1,1)$
$X_{4,6}$ in $\mathbf{P}(1,1,2,2,2,3)$	-1	$6 \times \frac{1}{2}(1,-1,1)$
$X_{5,6}$ in $\mathbf{P}(1,1,1,2,3,4)$	-5/4	$\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$
$X_{5,6}$ in $\mathbf{P}(1,1,2,2,3,3)$	-5/6	$3 \times \frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{3}(1,-1,1)$
$X_{6,6}$ in $\mathbf{P}(1,1,1,2,3,5)$	-6/5	$\frac{1}{5}(1,-1,2)$
$X_{6,6}$ in $\mathbf{P}(1,1,2,2,3,4)$	-3/4	$\frac{1}{4}(1,-1,1)$, $4 \times \frac{1}{2}(1,-1,1)$
$X_{6,6}$ in $\mathbf{P}(1,1,2,3,3,3)$	-2/3	$4 \times \frac{1}{3}(1,-1,1)$

$X_{6,6}$ in $\mathbf{P}(1,2,2,2,3,3)$	-1/2	$9 \times \frac{1}{2}(1,-1,1)$
$X_{6,7}$ in $\mathbf{P}(1,1,2,2,3,5)$	-7/10	$\frac{1}{5}(1,-1,2)$, $3 \times \frac{1}{2}(1,-1,1)$
$X_{6,7}$ in $\mathbf{P}(1,1,2,3,3,4)$	-7/12	$\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{3}(1,-1,1)$
$X_{6,8}$ in $\mathbf{P}(1,1,1,3,4,5)$	-4/5	$\frac{1}{5}(1,-1,1)$
$X_{6,8}$ in $\mathbf{P}(1,1,2,3,3,5)$	-8/15	$\frac{1}{5}(1,-1,2)$, $2 \times \frac{1}{3}(1,-1,1)$
$X_{6,8}$ in $\mathbf{P}(1,1,2,3,4,4)$	-1/2	$2 \times \frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{4}(1,-1,1)$
$X_{6,8}$ in $\mathbf{P}(1,2,2,3,3,4)$	-1/3	$6 \times \frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{3}(1,-1,1)$
$X_{6,9}$ in $\mathbf{P}(1,1,2,3,4,5)$	-9/20	$\frac{1}{4}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$, $\frac{1}{2}(1,-1,1)$
$X_{7,8}$ in $\mathbf{P}(1,1,2,3,4,5)$	-7/15	$\frac{1}{3}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$, $2 \times \frac{1}{2}(1,-1,1)$
$X_{6,10}$ in $\mathbf{P}(1,1,2,3,5,5)$	-2/5	$2 \times \frac{1}{5}(1,-1,2)$
$X_{6,10}$ in $\mathbf{P}(1,2,2,3,4,5)$	-1/4	$\frac{1}{4}(1,-1,1)$, $7 \times \frac{1}{2}(1,-1,1)$
$X_{8,9}$ in $\mathbf{P}(1,1,2,3,4,7)$	-3/7	$\frac{1}{7}(1,-1,2)$, $2 \times \frac{1}{2}(1,-1,1)$
$X_{8,9}$ in $\mathbf{P}(1,1,3,4,4,5)$	-3/10	$\frac{1}{5}(1,-1,1)$, $2 \times \frac{1}{4}(1,-1,1)$
$X_{8,9}$ in $\mathbf{P}(1,2,3,3,4,5)$	-1/5	$\frac{1}{5}(1,-1,2)$, $2 \times \frac{1}{2}(1,-1,1)$, $3 \times \frac{1}{3}(1,-1,1)$
$X_{8,10}$ in $\mathbf{P}(1,1,2,3,5,7)$	-8/21	$\frac{1}{3}(1,-1,1)$, $\frac{1}{7}(1,-1,3)$
$X_{8,10}$ in $\mathbf{P}(1,1,2,4,5,6)$	-1/3	$\frac{1}{6}(1,-1,1)$, $3 \times \frac{1}{2}(1,-1,1)$
$X_{8,10}$ in $\mathbf{P}(1,1,3,4,5,5)$	-4/15	$\frac{1}{3}(1,-1,1)$, $2 \times \frac{1}{5}(1,-1,1)$
$X_{8,10}$ in $\mathbf{P}(1,2,3,4,4,5)$	-1/6	$\frac{1}{3}(1,-1,1)$, $4 \times \frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{4}(1,-1,1)$
$X_{9,10}$ in $\mathbf{P}(1,1,2,3,5,8)$	-3/8	$\frac{1}{8}(1,-1,3)$, $\frac{1}{2}(1,-1,1)$
$X_{9,10}$ in $\mathbf{P}(1,1,3,4,5,6)$	-1/4	$\frac{1}{4}(1,-1,1)$, $\frac{1}{6}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
$X_{9,10}$ in $\mathbf{P}(1,2,2,3,5,7)$	-3/14	$\frac{1}{7}(1,-1,3)$, $5 \times \frac{1}{2}(1,-1,1)$
$X_{9,10}$ in $\mathbf{P}(1,2,3,4,5,5)$	-3/20	$\frac{1}{4}(1,-1,1)$, $2 \times \frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{5}(1,-1,2)$
$X_{8,12}$ in $\mathbf{P}(1,1,3,4,5,7)$	-8/35	$\frac{1}{5}(1,-1,1)$, $\frac{1}{7}(1,-1,2)$
$X_{8,12}$ in $\mathbf{P}(1,2,3,4,5,6)$	-2/15	$\frac{1}{5}(1,-1,1)$, $4 \times \frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{3}(1,-1,1)$
$X_{9,12}$ in $\mathbf{P}(1,2,3,4,5,7)$	-9/70	$\frac{1}{5}(1,-1,2)$, $\frac{1}{7}(1,-1,2)$, $3 \times \frac{1}{2}(1,-1,1)$
$X_{10,11}$ in $\mathbf{P}(1,2,3,4,5,7)$	-11/84	$\frac{1}{3}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$, $\frac{1}{7}(1,-1,3)$, $2 \times \frac{1}{2}(1,-1,1)$
$X_{10,12}$ in $\mathbf{P}(1,1,3,4,5,9)$	-2/9	$\frac{1}{9}(1,-1,2)$, $\frac{1}{3}(1,-1,1)$
$X_{10,12}$ in $\mathbf{P}(1,1,3,5,6,7)$	-4/21	$\frac{1}{7}(1,-1,1)$, $2 \times \frac{1}{3}(1,-1,1)$
$X_{10,12}$ in $\mathbf{P}(1,1,4,5,6,6)$	-1/6	$\frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{6}(1,-1,1)$
$X_{10,12}$ in $\mathbf{P}(1,2,3,4,5,8)$	-1/8	$\frac{1}{8}(1,-1,3)$, $3 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$
$X_{10,12}$ in $\mathbf{P}(1,2,3,5,5,7)$	-4/35	$\frac{1}{7}(1,-1,3)$, $2 \times \frac{1}{5}(1,-1,2)$
$X_{10,12}$ in $\mathbf{P}(1,2,4,5,5,6)$	-1/10	$5 \times \frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{5}(1,-1,1)$
$X_{10,12}$ in $\mathbf{P}(1,3,3,4,5,7)$	-2/21	$\frac{1}{7}(1,-1,2)$, $4 \times \frac{1}{3}(1,-1,1)$
$X_{10,12}$ in $\mathbf{P}(1,3,4,4,5,6)$	-1/12	$2 \times \frac{1}{3}(1,-1,1)$, $3 \times \frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$
$X_{11,12}$ in $\mathbf{P}(1,1,4,5,6,7)$	-11/70	$\frac{1}{5}(1,-1,1)$, $\frac{1}{7}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$
$X_{10,14}$ in $\mathbf{P}(1,1,2,5,7,9)$	-2/9	$\frac{1}{9}(1,-1,4)$
$X_{10,14}$ in $\mathbf{P}(1,2,3,5,7,7)$	-2/21	$\frac{1}{3}(1,-1,1)$, $2 \times \frac{1}{7}(1,-1,3)$
$X_{10,14}$ in $\mathbf{P}(1,2,4,5,6,7)$	-1/12	$\frac{1}{4}(1,-1,1)$, $\frac{1}{6}(1,-1,1)$, $5 \times \frac{1}{2}(1,-1,1)$
$X_{10,15}$ in $\mathbf{P}(1,2,3,5,7,8)$	-5/56	$\frac{1}{7}(1,-1,3)$, $\frac{1}{8}(1,-1,3)$, $\frac{1}{2}(1,-1,1)$
$X_{12,13}$ in $\mathbf{P}(1,3,4,5,6,7)$	-13/210	$\frac{1}{5}(1,-1,1)$, $\frac{1}{7}(1,-1,2)$, $2 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$
$X_{12,14}$ in $\mathbf{P}(1,1,3,4,7,11)$	-2/11	$\frac{1}{11}(1,-1,3)$
$X_{12,14}$ in $\mathbf{P}(1,1,4,6,7,8)$	-1/8	$\frac{1}{8}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{4}(1,-1,1)$
$X_{12,14}$ in $\mathbf{P}(1,2,3,4,7,10)$	-1/10	$\frac{1}{10}(1,-1,3)$, $4 \times \frac{1}{2}(1,-1,1)$
$X_{12,14}$ in $\mathbf{P}(1,2,3,5,7,9)$	-4/45	$\frac{1}{5}(1,-1,2)$, $\frac{1}{9}(1,-1,4)$, $\frac{1}{3}(1,-1,1)$

$X_{12,14}$ in $\mathbf{P}(1,3,4,5,7,7)$	-2/35	$\frac{1}{5}(1,-1,2)$, $2 \times \frac{1}{7}(1,-1,2)$
$X_{12,14}$ in $\mathbf{P}(1,4,4,5,6,7)$	-1/20	$\frac{1}{5}(1,-1,1)$, $3 \times \frac{1}{4}(1,-1,1)$, $2 \times \frac{1}{2}(1,-1,1)$
$X_{12,14}$ in $\mathbf{P}(2,3,4,5,6,7)$	-1/30	$\frac{1}{5}(1,-1,2)$, $7 \times \frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{3}(1,-1,1)$
$X_{12,15}$ in $\mathbf{P}(1,1,4,5,6,11)$	-3/22	$\frac{1}{11}(1,-1,2)$, $\frac{1}{2}(1,-1,1)$
$X_{12,15}$ in $\mathbf{P}(1,3,4,5,6,9)$	-1/18	$\frac{1}{9}(1,-1,2)$, $3 \times \frac{1}{3}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$
$X_{12,15}$ in $\mathbf{P}(1,3,4,5,7,8)$	-3/56	$\frac{1}{7}(1,-1,2)$, $\frac{1}{8}(1,-1,3)$, $\frac{1}{4}(1,-1,1)$
$X_{12,16}$ in $\mathbf{P}(1,2,5,6,7,8)$	-2/35	$\frac{1}{5}(1,-1,2)$, $\frac{1}{7}(1,-1,1)$, $4 \times \frac{1}{2}(1,-1,1)$
$X_{14,15}$ in $\mathbf{P}(1,2,3,5,7,12)$	-1/12	$\frac{1}{12}(1,-1,5)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
$X_{14,15}$ in $\mathbf{P}(1,2,5,6,7,9)$	-1/18	$\frac{1}{6}(1,-1,1)$, $\frac{1}{9}(1,-1,4)$, $2 \times \frac{1}{2}(1,-1,1)$
$X_{14,15}$ in $\mathbf{P}(1,3,4,5,7,10)$	-1/20	$\frac{1}{4}(1,-1,1)$, $\frac{1}{10}(1,-1,3)$, $\frac{1}{5}(1,-1,2)$
$X_{14,15}$ in $\mathbf{P}(1,3,5,6,7,8)$	-1/24	$\frac{1}{6}(1,-1,1)$, $\frac{1}{8}(1,-1,3)$, $2 \times \frac{1}{3}(1,-1,1)$
$X_{14,16}$ in $\mathbf{P}(1,1,5,7,8,9)$	-4/45	$\frac{1}{5}(1,-1,2)$, $\frac{1}{9}(1,-1,1)$
$X_{14,16}$ in $\mathbf{P}(1,3,4,5,7,11)$	-8/165	$\frac{1}{3}(1,-1,1)$, $\frac{1}{5}(1,-1,2)$, $\frac{1}{11}(1,-1,3)$
$X_{14,16}$ in $\mathbf{P}(1,4,5,6,7,8)$	-1/30	$\frac{1}{5}(1,-1,2)$, $\frac{1}{6}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{4}(1,-1,1)$
$X_{15,16}$ in $\mathbf{P}(1,2,3,5,8,13)$	-1/13	$\frac{1}{13}(1,-1,5)$, $2 \times \frac{1}{2}(1,-1,1)$
$X_{15,16}$ in $\mathbf{P}(1,3,4,5,8,11)$	-1/22	$\frac{1}{11}(1,-1,4)$, $2 \times \frac{1}{4}(1,-1,1)$
$X_{14,18}$ in $\mathbf{P}(1,2,3,7,9,11)$	-2/33	$\frac{1}{11}(1,-1,5)$, $2 \times \frac{1}{3}(1,-1,1)$
$X_{14,18}$ in $\mathbf{P}(1,2,6,7,8,9)$	-1/24	$\frac{1}{8}(1,-1,1)$, $5 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
$X_{12,20}$ in $\mathbf{P}(1,4,5,6,7,10)$	-1/35	$\frac{1}{7}(1,-1,2)$, $2 \times \frac{1}{2}(1,-1,1)$, $2 \times \frac{1}{5}(1,-1,1)$
$X_{16,18}$ in $\mathbf{P}(1,1,6,8,9,10)$	-1/15	$\frac{1}{10}(1,-1,1)$, $\frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
$X_{16,18}$ in $\mathbf{P}(1,4,6,7,8,9)$	-1/42	$\frac{1}{7}(1,-1,1)$,$2\times \frac{1}{2}(1,-1,1)$,$2\times\frac{1}{4}(1,-1,1)$,$\frac{1}{3}(1,-1,1)$
$X_{18,20}$ in $\mathbf{P}(1,4,5,6,9,14)$	-1/42	$\frac{1}{14}(1,-1,3)$, $2 \times \frac{1}{2}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$
$X_{18,20}$ in $\mathbf{P}(1,4,5,7,9,13)$	-2/91	$\frac{1}{7}(1,-1,3)$, $\frac{1}{13}(1,-1,3)$
$X_{18,20}$ in $\mathbf{P}(1,5,6,7,9,11)$	-4/231	$\frac{1}{7}(1,-1,3)$, $\frac{1}{11}(1,-1,2)$, $\frac{1}{3}(1,-1,1)$
$X_{18,22}$ in $\mathbf{P}(1,2,5,9,11,13)$	-2/65	$\frac{1}{5}(1,-1,1)$, $\frac{1}{13}(1,-1,6)$
$X_{20,21}$ in $\mathbf{P}(1,3,4,7,10,17)$	-1/34	$\frac{1}{17}(1,-1,5)$, $\frac{1}{2}(1,-1,1)$
$X_{18,30}$ in $\mathbf{P}(1,6,8,9,10,15)$	-1/120	$\frac{1}{8}(1,-1,1)$,$2\times\frac{1}{2}(1,-1,1)$,$2\times\frac{1}{3}(1,-1,1)$,$\frac{1}{5}(1,-1,1)$
$X_{24,30}$ in $\mathbf{P}(1,8,9,10,12,15)$	-1/180	$\frac{1}{9}(1,-1,1)$,$\frac{1}{2}(1,1,1)$,$\frac{1}{4}(1,-1,1)$,$\frac{1}{3}(1,-1,1)$,$\frac{1}{5}(1,-1,2)$

II.6.8 Note. $X_{12,14}$ in $\mathbf{P}(2,3,4,5,6,7)$ is the only element in the above list with $a_i \geq 2$ for all i (see Example II.6.1).

II.7 The plurigenera formulas.

Before we describe the Ried's table method for producing examples of weighted complete intersection we must state the plurigenera formulas for canonical and **Q**-Fano 3-folds.

II.7.1 Definition. For a singularity Q of type $\frac{1}{r}(1,-1,b)$ define:

$$l(Q,n) = \begin{cases} 0 & \text{if } n = 0,1 \\ \sum_{k=1}^{n-1} \frac{\overline{bk}(r-\overline{bk})}{2r} & \text{if } n \geq 2 \end{cases}$$

where $\overline{x}$ denotes the smallest non-negative residue of x modulo r. This is extended to negative integers via:

$$l(-n) = -l(n+1)$$

for all $n \geq 0$. This is for consistancy with Serre duality. For a collection (or *basket*) $\mathcal{B}$ of singularities define:

$$l(n) = \sum_{Q \in \mathcal{B}} l(q, n)$$

for all $n \in \mathbf{Z}$.

From [F1, Theorem 2.5, equation (4)] (see also [R4, Chapter III]) we have the following:

II.7.2 Theorem. *For any projective 3-fold X, with at worst canonical singularities, there exists a basket $\mathcal{B}$ of singularities such that*

$$\chi(\mathcal{O}_X(nK_X)) = \frac{(2n-1)n(n-1)}{12r} K_X^3 - (2n-1)\chi(\mathcal{O}_X) + l(n)$$

for all $n \in \mathbf{Z}$.

II.7.3 Canonical 3-folds.

Let X be a canonical 3-fold. Then K_X is ample and we have:

$$P_n = \chi(\mathcal{O}_X(nK_X)) = \frac{(2n-1)n(n-1)}{12r} K_X^3 - (2n-1)\chi(\mathcal{O}_X) + l(n)$$

for all $n \geq 2$. This formula is Reid's exact plurigenera formula.

II.7.4 Q-Fano 3-fold complete intersections.

If X is a $\mathbf{Q}$-Fano 3-fold then $-K_X$ is ample. Moreover if X is also a complete intersection then $\chi(\mathcal{O}_X) = 1$. So:

$$P_{-n} = \chi(\mathcal{O}_X(-nK_X)) = \frac{(2n+1)n(n+1)}{12r}(-K_X)^3 + (2n+1) - l(n+1)$$

for all $n \geq 1$.

II.8 The Reid table method.

Consider a complete intersection $X_{d_1,\ldots,d_c}$ in $\mathbf{P}(a_0, \ldots, a_n)$. The Poincaré series (see [WPS, section 3.4] and compare [A&M, 11.1]) corresponding to the coordinate ring R of X is:

$$\mathcal{P}(t) = \sum_{n=0}^{\infty} \mathrm{h}^0(X, \mathcal{O}_X(n)) t^n$$

$$= \frac{\prod_{i=1}^{i=c} (1 - t^{d_i})}{\prod_{i=0}^{i=n} (1 - t^{a_i})}$$

Moreover if $\omega_X \cong \mathcal{O}_X(1)$ then $\mathcal{P}(t) = \sum_{n=0}^{\infty} P_n(X)t^n$, where $P_n(X)$ are the plurigenera of X. In the case of a **Q**-Fano 3-fold with $\omega_X \cong \mathcal{O}_X(-1)$ then $\mathcal{P}(t) = \sum_{n=0}^{\infty} P_{-n}(X)t^n$, where $P_{-n}(X)$ are the anti-plurigenera of X.

II.8.1 Example. X_6 in $\mathbf{P}^4$ has Poincaré series

$$\mathcal{P}(t) = \frac{(1-t^6)}{(1-t)^5} = 1 + t + 5t^2 + 15t^3 + ...$$

So $p_g = 1$, $P_2 = 5$, $P_3 = 15$, etc.

II.8.2 Question. Given a list of plurigenera (which could arise from a record of pluridata) does there exist a complete intersection with $\omega_X \cong \mathcal{O}_X(\pm 1)$?

The following lemma due to Reid helps answer the above.

II.8.3 Lemma. *Given a sequence $p_0 = 1, p_1, p_2, ...$ such that*

$$\sum_{i=0}^{\infty} p_i t^i = \frac{\prod_{i=1}^{i=c} (1-t^{d_i})}{\prod_{i=0}^{i=n} (1-t^{a_i})}$$

for some $\{d_i, a_i\}$. Then these $\{d_i, a_i\}$ are unique up to $a_i \neq d_j$ and are determinable.

Proof. The following is a constructive proof. Let $q_i^0 = p_i$. So

$$\sum_{i=0}^{\infty} q_i^0 t^i = \frac{\prod(1-t^{d_i})}{\prod(1-t^{a_i})}.$$

Without loss of generality assume that $d_c \geq ... \geq d_1$ and $a_n \geq ... \geq a_0$. Clearly we may assume $a_0 \neq d_1$ or else these two terms would cancel. There are two cases:

(i) $a_0 < d_1$. Let a_0 occur with multiplicity μ. Then $\mathcal{P}(t) = 1 + \mu t^{a_0}$ + higher order terms. So the first non-zero q_i^0 is $q_{a_0}^0 = \mu < 0$. Define $q_i^1 = q_i^0 - q_{i-a_0}^0$, where $q_i^0 = 0$ if $i < 0$. Then $q_{a_0}^1 = q_{a_0}^0 - 1$. Thus

$$\sum_{i=0}^{\infty} q_i^1 t^i = \sum_{i=0}^{\infty} (q_i^0 - q_{i-a_0}^0) t^i$$

$$= (1-t^{a_0}) \sum_{i=0}^{\infty} q_i^0 t^i$$

$$= \frac{\prod_{i=1}^{c} (1-t^{d_i})}{\prod_{i=1}^{n} (1-t^{a_i})}.$$

This involves one less a_i.

(ii) $d_1 < a_0$. Let d_1 occur with multiplicity μ. Then $\mathcal{P}(t) = 1 - \mu t^{d_1}$ + higher order terms. So the first non-zero q_i^0 is $q_{d_1}^0 = -\mu < 0$. Define $q_i^1 = q_i^0 + q_{i-d_1}^1$, for $i = 1, 2, \ldots$ where $q_i^1 = 0$ if i¡ 0. This corresponds to:

$$\sum_{i=0}^{\infty} q_i^1 t^i = \sum_{i=0}^{\infty} (q_i^0 + q_{i-d_1}^1) t^i$$

$$= \sum_{i=0}^{\infty} (q_i^0 + q_{i-d_1}^0 + q_{i-2d_1}^0 + \ldots) t^i$$

$$= \frac{\prod_{i=2}^{c} (1 - t^{d_i})}{\prod_{i=0}^{n} (1 - t^{a_i})}.$$

This involves one less d_i.

Repetition of the above steps clearly terminates when

$$\sum_{i=0}^{\infty} q_i^b t^i = 1$$

By induction on the number of a_i and d_j it is clear that the process uniquely determines the a_i and d_j.

□

II.8.4 The table method. So the proof of the above lemma allows us to construct a weighted complete intersection from a list of 'plurigenera'. This construction is easily set out in the form of a table. In the first column write down the integers $\{0, 1, 2, \ldots\}$ and in the second the list $\{1, P_1, P_2, \ldots\}$. Let the n^{th} column be denoted by q_i^n for $i = 0, 1, \ldots$. Each successive column is obtained as follows. Look down the list $\{q_i^n\}$ of the n^{th} column to find the postion of the first non-zero entry (disregard the initial 1 at the top of the column). Suppose this is in row r. There are 2 cases:

(i) this entry is postive. First enter (r) at the head of this column. This will keep a record of the degrees of the generators. The $(n+1)^{\text{th}}$ column is obtained by the rule:

$$q_i^{n+1} = q_i^n - q_{i-r}^n,$$

assuming that $q_i^n = 0$ for all $i < 0$.

(ii) this entry is negative. First enter $(-r)$ at the head of this column. This will keep a record of the degrees of the relations. The $(n+1)^{\text{th}}$ column is obtained by the rule:

$$q_i^{n+1} = q_i^n - q_{i-r}^{n+1},$$

assuming that $q_i^{n+1} = 0$ and for all $i < 0$.

The process is clearly defined and the integers at the head of each column keep track of the a_i and $-d_i$.

II.8.5 Example. Consider the record of pluridata $K^3 = \frac{1}{6}$, $\chi = 1$, $p_g = 0$, 9 singularities of type $\frac{1}{2}(1,1,1)$ and 8 singularities of type $\frac{1}{3}(2,1,1)$. Using Reid's plurigenera formula (see section II.7) the plurigenera P_n corresponding to this record was calculated and is given below. The table obtained is the following:

n	P_n	(2)	(2)	(3)	(3)	(3)	(4)	(-6)	(-12)
0	1	1	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0	0	0
2	2	1	0	0	0	0	0	0	0
3	3	3	3	2	1	0	0	0	0
4	4	2	1	1	1	1	0	0	0
5	6	3	0	0	0	0	0	0	0
6	11	7	5	2	0	-1	-1	0	0
7	12	6	3	2	1	0	0	0	0
8	19	8	1	1	1	1	0	0	0
9	25	13	7	2	0	0	0	0	0
10	32	13	5	2	0	-1	0	0	0
11	41	16	3	2	1	0	0	0	0
12	54	22	9	2	0	0	-1	-1	0
13	64	23	7	2	0	0	0	0	0
14	81	27	5	2	0	-1	0	0	0
15	98	34	11	2	0	0	0	0	0
16	117	36	9	2	0	0	0	0	0
17	139	41	7	2	0	0	0	0	0
18	166	49	13	2	0	0	1	0	0
19	191	52	11	2	0	0	0	0	0
20	224	58	9	2	0	0	0	0	0

This gives $X_{6,12}$ in $\mathbf{P}(2,2,3,3,3,4)$, which has the above record.

II.8.6 Note. Of course this method cannot tell the difference between X_6 in $\mathbf{P}(1,1,1,2)$ and the example of V. Iliev $X_{3,6}$ in $\mathbf{P}(1,1,1,2,3)$, in which the cubic relation does not involve the degree 3 generator.

However in this section we are only interested in the general member of a family of weighted complete intersections and so Iliev's example does not occur.

II.8.7 Warning. Although in general it is clear when this process stops, it is not clear when it is worth continuing with a particular list of integers.

II.8.8 The analysis.

This process is basically the same as that in section II.2.6 on the coordinate ring

$$R = \bigoplus_{m \geq 0} R_m.$$

Starting from the dimensions of each R_m the degrees of the generators and relations can be found. At each stage it is assumed that the monomials are linearly independent unless

(i) there already exist relations of a lower degree, or
(ii) a relation is forced by the dimension not being large enough.

For the above example we have the following analysis:

Degree	Dimension	Monomials	
0	1	1	
1	0	$\emptyset$	
2	2	x_0, x_1.	
3	3	y_0, y_1, y_2.	
4	4	x_0^2, x_0x_1, x_1^2, z.	
5	6	x_0y_0, x_0y_1, x_0y_2, x_1y_0, x_1y_1, x_1y_2.	
6	11	x_0^3, $x_0^2x_1$, $x_0x_1^2$, x_1^3, y_0^2, y_0y_1, y_0y_2, y_1^2, y_1y_2, y_2^2, x_0z, x_1z.	1 relation.

If this calculation is continued only one more relation is found, which is of degree 12

II.8.9 Canonical 3-fold complete intersections.

The formula:

$$P_2 = \frac{1}{2}K_X^3 - 3(1 - p_g) + l(2)$$

limits the value of p_g (since $K_X^3 > 0$) and defines K_X^3 in terms of a particular basket of singularities and P_2.

II.8.10 Q-Fano complete intersections.

The formula:

$$P_{-1} = -\frac{1}{2}K_X^3 + 3 - l(2)$$

defines K_X^3 in terms of a particular basket of singularities and P_{-1}.

II.8.11 The search. The search through all combinations of $P \geq 0$ ($P_2 = P$ for canonical 3-folds and $P_{-1} = P$ for the Fano case) and baskets will give every possible list of plurigenera (respectively anti-plurigenera). Hence a list of canonically (respectively anti-canonically) embedded complete intersections can be found. Of course this is not a finite search, and requires a computer to make any resonable progress.

The order of the search was as follows. Let Q_i for $i = 0, 1, \ldots$ be a list of the types of 3-fold cyclic quotient singularity $\frac{1}{r}(1, -1, a)$ in order of increasing index r and increasing a within each index. So $Q_0 = \frac{1}{2}(1, 1, 1)$, $Q_1 = \frac{1}{3}(1, -1, 1)$, etc.. The program took 2 integer arguments l and u, and searched through all baskets $\{n_i \times Q_i\}$ such that $l \leq \sum_{i=0}^{\infty} n_i(i + 2) < u$.

II.8.12 The raw list.

Here is the first part of the list produced by the search program (with arguments 0 8).

X_6 in $\mathbf{P}(1,1,1,1,3)$
X_{12} in $\mathbf{P}(1,1,1,4,6)$
X_4 in $\mathbf{P}(1,1,1,1,1)$
X_5 in $\mathbf{P}(1,1,1,1,2)$
X_8 in $\mathbf{P}(1,1,1,2,4)$
X_{10} in $\mathbf{P}(1,1,1,3,5)$
$X_{2,3}$ in $\mathbf{P}(1,1,1,1,1,1)$
$X_{3,3}$ in $\mathbf{P}(1,1,1,1,1,2)$
$X_{3,4}$ in $\mathbf{P}(1,1,1,1,2,2)$

X_6 in $\mathbf{P}(1,1,1,2,2)$
$X_{4,4}$ in $\mathbf{P}(1,1,1,1,2,3)$
X_7 in $\mathbf{P}(1,1,1,2,3)$
X_9 in $\mathbf{P}(1,1,1,3,4)$
$X_{2,2,2}$ in $\mathbf{P}(1,1,1,1,1,1,1)$
$X_{6,6}$ in $\mathbf{P}(1,1,1,2,3,3)$
X_{12} in $\mathbf{P}(1,1,2,3,4)$
$X_{4,4}$ in $\mathbf{P}(1,1,1,2,2,2)$
X_{10} in $\mathbf{P}(1,1,2,2,5)$
$X_{4,5}$ in $\mathbf{P}(1,1,1,2,2,3)$
X_{18} in $\mathbf{P}(1,1,2,6,9)$
$X_{4,6}$ in $\mathbf{P}(1,1,1,2,3,3)$
$X_{5,6}$ in $\mathbf{P}(1,1,1,2,3,4)$
$X_{6,8}$ in $\mathbf{P}(1,1,1,3,4,5)$

II.8.13 Refinement.

Of course this list contains complete intersections already obtained in other ways (see sections II.5 and II.6) and some intersections which do not meet the requirements; i.e.

(1) dimension 3,
(2) quasismooth but not the intersection of a linear cone with other hypersurfaces,
(3) canonically or anti-canonically embedded,
(4) and have at worst terminal singularities.

The example $X_{6,22}$ in $\mathbf{P}(2,2,3,4,5,11)$ from the raw list is not quasismooth, since the polynomial of degree 6 does not involve the generator of weight 5. We use the following lemma to cut out a large number of elements from the raw list produced by the search program.

II.8.14 Lemma. *Let $X_{d_1,\dots,d_c}$ in $\mathbf{P}(a_0,\dots,a_n)$ be quasismooth but not an intersection of a linear cone with other hypersurfaces. Suppose also that $d_1,\dots,d_c$ and $a_0,\dots,a_n$ are in increasing order. Then:*

(i) $d_c > a_n$, $d_{c-1} > a_{n-1}$, ..., $d_1 > a_{n-c+1}$.
(ii) if $d_{c-1} < a_n$ then $a_n \mid d_c$.

Proof. (i). Suppose $d_c > a_n$, ..., $d_{c-k+1} > a_{n-k+1}$ and $d_{c-k} < a_{n-k}$ for some $k = 0,\dots,c-1$. So $d_i < a_{n-k}$ for all $i \le c-k$. Therefore the polynomials f_1, ..., f_{n-k} do not involve the variables x_{n-k}, ..., x_n.

Let Π be the coordinate $(k+1)$-plane in $\mathbf{A}^n + 1$ given by $x_0 = \dots = x_{n-k-1} = 0$. So f_1, ..., f_{n-k} are identically zero on Π. Define $Z = (f_{c-k+1} = \dots = f_c = 0) \cap \Pi$. Thus $\dim Z \ge 1$ and so $Z - \underline{0}$ is non-empty. Let $Q \in Z - \underline{0}$. Then $\partial f_i/\partial x_j$ are zero at Q for all $i \le c-k$ and for all j. Therefore

$$\operatorname{rank}\begin{pmatrix} \partial f_1/\partial x_0(Q) & \dots & \partial f_1/\partial x_n(Q) \\ \vdots & & \vdots \\ \partial f_c/\partial x_0(Q) & \cdots & \partial f_c/\partial x_n(Q) \end{pmatrix} \le k - c.$$

Thus $Q \in C_X^*$ is singular and so X is not quasismooth.

(ii) is treated likewise.

□

II.8.15 Example. So a codimension 2 complete intersection X_{d_1,d_2} in $\mathbf{P}(a_1,...,a_n)$, which is quasismooth and not the intersection of a linear cone with another hypersurface, satisfies:

(i) $d_2 > a_n$ and $d_1 > a_{n-1}$.

(ii) if $d_1 < a_n$ then $a_n \mid d_2$.

So this lemma gives extra combinatoric conditions to help remove *nasty* complete intersections.

II.8.16 The final list.

The program was run between the limits 0 and 32 and gave the following list (after cutting out repetitions and nasty complete intersections):

Complete Intersection	K_X^3	p_g	Singularities.
$X_{2,2,2}$ in $\mathbf{P}(1,1,1,1,1,1,1)$	-8	0	
$X_{2,2,4}$ in $\mathbf{P}(1,1,1,1,1,1,1)$	16	7	
$X_{2,2,6}$ in $\mathbf{P}(1,1,1,1,1,1,3)$	8	6	
$X_{2,3,3}$ in $\mathbf{P}(1,1,1,1,1,1,1)$	18	7	
$X_{3,3,3}$ in $\mathbf{P}(1,1,1,1,1,1,2)$	27/2	6	$\frac{1}{2}(1,-1,1)$
$X_{3,3,4}$ in $\mathbf{P}(1,1,1,1,1,2,2)$	9	5	$2\times\frac{1}{2}(1,-1,1)$
$X_{3,4,4}$ in $\mathbf{P}(1,1,1,1,2,2,2)$	6	4	$4\times\frac{1}{2}(1,1,1)$
$X_{4,4,4}$ in $\mathbf{P}(1,1,1,1,2,2,3)$	16/3	4	$\frac{1}{3}(1,-1,1)$
$X_{4,4,4}$ in $\mathbf{P}(1,1,1,2,2,2,2)$	4	3	$8\times\frac{1}{2}(1,-1,1)$
$X_{4,4,5}$ in $\mathbf{P}(1,1,1,2,2,2,3)$	10/3	3	$\frac{1}{3}(1,-1,1)$, $4\times\frac{1}{2}(1,1,1)$
$X_{4,4,6}$ in $\mathbf{P}(1,1,1,2,2,3,3)$	8/3	3	$2\times\frac{1}{3}(1,1-,1)$
$X_{4,4,6}$ in $\mathbf{P}(1,1,2,2,2,2,3)$	2	2	$12\times\frac{1}{2}(1,1,1)$
$X_{4,5,6}$ in $\mathbf{P}(1,1,2,2,2,3,3)$	5/3	2	$2\times\frac{1}{3}(1,-1,1)$, $6\times\frac{1}{2}(1,1,1)$
$X_{4,6,6}$ in $\mathbf{P}(1,1,2,2,3,3,3)$	4/3	2	$4\times\frac{1}{3}(1,-1,1)$
$X_{4,6,6}$ in $\mathbf{P}(1,2,2,2,2,3,3)$	1	1	$18\times\frac{1}{2}(1,1,1)$
$X_{5,6,6}$ in $\mathbf{P}(1,1,2,2,3,3,4)$	5/4	2	$\frac{1}{4}(1,-1,1)$, $4\times\frac{1}{2}(1,1,1)$
$X_{5,6,6}$ in $\mathbf{P}(1,2,2,2,3,3,3)$	5/6	1	$4\times\frac{1}{3}(1,-1,1)$, $9\times\frac{1}{2}(1,1,1)$
$X_{6,6,10}$ in $\mathbf{P}(2,2,2,3,3,4,5)$	1/4	0	$\frac{1}{4}(1,-1,1)$, $22\times\frac{1}{2}(1,1,1)$
$X_{6,6,6}$ in $\mathbf{P}(1,2,2,2,3,3,4)$	3/4	1	$\frac{1}{4}(1,-1,1)$, $13\times\frac{1}{2}(1,1,1)$
$X_{6,6,6}$ in $\mathbf{P}(1,2,2,3,3,3,3)$	2/3	1	$8\times\frac{1}{3}(1,-1,1)$
$X_{6,6,6}$ in $\mathbf{P}(2,2,2,2,3,3,3)$	1/2	0	$27\times\frac{1}{2}(1,1,1)$
$X_{6,6,7}$ in $\mathbf{P}(1,2,2,3,3,3,4)$	7/12	1	$\frac{1}{4}(1,-1,1)$, $4\times\frac{1}{3}(1,-1,1)$, $4\times\frac{1}{2}(1,1,1)$
$X_{6,6,8}$ in $\mathbf{P}(1,1,2,3,3,4,5)$	4/5	2	$\frac{1}{5}(1,-1,2)$
$X_{6,6,8}$ in $\mathbf{P}(1,2,2,3,3,4,4)$	1/2	1	$\frac{1}{4}(1,-1,1)$, $8\times\frac{1}{2}(1,1,1)$
$X_{6,6,8}$ in $\mathbf{P}(2,2,2,3,3,3,4)$	1/3	0	$18\times\frac{1}{2}(1,1,1)$, $4\times\frac{1}{3}(1,-1,1)$
$X_{6,7,8}$ in $\mathbf{P}(1,2,2,3,3,4,5)$	7/15	1	$\frac{1}{5}(1,-1,2)$, $2\times\frac{1}{3}(1,-1,1)$, $6\times\frac{1}{2}(1,1,1)$
$X_{6,8,10}$ in $\mathbf{P}(1,2,3,3,4,5,5)$	4/15	1	$2\times\frac{1}{5}(1,-1,2)$, $2\times\frac{1}{3}(1,-1,1)$
$X_{6,8,10}$ in $\mathbf{P}(2,2,3,3,4,4,5)$	1/6	0	$2\times\frac{1}{4}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$, $14\times\frac{1}{2}(1,1,1)$
$X_{6,8,9}$ in $\mathbf{P}(1,2,3,3,4,4,5)$	3/10	1	$\frac{1}{5}(1,-1,2)$, $2\times\frac{1}{4}(1,-1,1)$, $2\times\frac{1}{2}(1,1,1)$
$X_{8,10,12}$ in $\mathbf{P}(2,3,4,4,5,5,6)$	1/15	0	$2\times\frac{1}{5}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$, $10\times\frac{1}{2}(1,1,1)$
$X_{8,9,10}$ in $\mathbf{P}(2,3,3,4,4,5,5)$	1/10	0	$2\times\frac{1}{5}(1,-1,2)$, $2\times\frac{1}{4}(1,-1,1)$, $3\times\frac{1}{3}(1,-1,1)$, $4\times\frac{1}{2}(1,1,1)$
$X_{9,10,12}$ in $\mathbf{P}(2,3,3,4,5,6,7)$	1/14	0	$\frac{1}{7}(1,-1,2)$, $6\times\frac{1}{3}(1,-1,1)$, $5\times\frac{1}{2}(1,1,1)$
$X_{10,11,12}$ in $\mathbf{P}(2,3,4,5,5,6,7)$	11/210	0	$5\times\frac{1}{2}(1,1,1)$, $2\times\frac{1}{3}(1,-1,1)$, $2\times\frac{1}{5}(1,-1,2)$, $\frac{1}{7}(1,-1,3)$

$X_{10,12,14}$ in $\mathbf{P}(2,3,4,5,6,7,8)$	1/24	0	$\frac{1}{8}(1,-1,3)$, $\frac{1}{4}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$, $8\times\frac{1}{2}(1,1,1)$
$X_{10,12,18}$ in $\mathbf{P}(3,4,5,5,6,7,9)$	2/105	0	$\frac{1}{7}(1,-1,1)$, $2\times\frac{1}{5}(1,-1,1)$, $4\times\frac{1}{3}(1,-1,1)$
$X_{12,14,15}$ in $\mathbf{P}(3,4,5,6,7,7,8)$	1/56	0	$\frac{1}{2}(1,1,1)$, $\frac{1}{4}(1,-1,1)$, $2\times\frac{1}{7}(1,-1,2)$, $\frac{1}{8}(1,-1,3)$
$X_{12,15,16}$ in $\mathbf{P}(3,4,5,6,7,8,9)$	1/63	0	$2\times\frac{1}{2}(1,1,1)$, $3\times\frac{1}{3}(1,-1,1)$, $\frac{1}{7}(1,-1,2)$, $\frac{1}{9}(1,-1,2)$
$X_{12,16,18}$ in $\mathbf{P}(4,5,6,6,7,8,9)$	1/105	0	$\frac{1}{7}(1,-1,1)$, $\frac{1}{5}(1,-1,1)$, $2\times\frac{1}{3}(1,-1,1)$, $6\times\frac{1}{2}(1,1,1)$

II.8.17 Note. After refinement there are no codimension 2 or 1 complete intersections left in the list.

II.8.18 Extra example. The family of intersections $X_{2,2,2,2,2}$ in $\mathbf{P}^8$ is smooth, $K_X^3 = 16$, $p_g = 9$ and $\chi(\mathcal{O}_X) = -8$.

If the search were continued this would eventually appear; however the program becomes painfully slow.

II.8.19 Conjecture.

(1) There are no canonical complete intersections with codimension greater than 5.

(2) There are no $\mathbf{Q}$-Fano complete intersections with codimension greater than 3.

II.8.20 K3 surfaces. Reid has done a similar search to produce lists of K3 surface weighted complete intersections; using Riemann-Roch for $\mathcal{O}_S(1)$ (see [R4, Theorem 9.1]). This time the search is finite due to the following theorem pointed out by Reid:

II.8.21 Theorem. *Let S be a K3 surface with canonical (Du Val) singularities of types A_{n_i}, D_{n_i} or E_{n_i} for $i = 1, ..., n$. So $\sum n_i \leq 19$. This limits the singularities present on the K3 surface to a finite list.*

Proof. Let $f : T \to S$ be a minimal resolution. T is still a K3 surface. By [BP&V, Proposition VIII.3.3] $\mathrm{h}^{1,1} = \mathrm{h}^1(\Omega^1_T) = 20$. By the Signature Theorem [BP&V, Theorem IV.2.13] we have that the cup product restricted to $\mathrm{H}^2(T, \mathbf{R})$ is non-degenerate of type $(1, \mathrm{h}^{1,1} - 1) = (1, 19)$. Via the Néron-Severi group, the exceptional (-2)-curves of the resolution f are linearly independent in $\mathrm{H}^{1,1}$, each with negative self-intersection.

It is well known that a Du Val singularity of type A_n, D_n or E_n contributes exactly n (-2)-curves to T. Thus $\sum n_i \leq 19$.

□

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