EXACT LOCALIZED SOLUTIONS OF THE (2+1) DIMENSIONAL COMPLEX GINZBURG-LANDAU EQUATION

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ABSTRACT. The (2+1) dimensional (D) complex Ginzburg-Landau (CGL) equation is analytically studied using a method derived from the Painlevé test for integrability and Hirota's multilinear formalism. We show that 2D propagating pulse, 2D stationary pulse, 2D stationary dark and 2D propagating hole solutions are the structures that organize the dynamic of this extended systems. The degeneracies of this (2+1) dimensional CGL equation are examined as well as their solutions. This equation reveals, to exhibit rich repertoire of behaviors and, to be a generalization of an integrable system (nonlinear Schroedinger equation) and a variational equation (real Ginzburg-Landau equation).

KEYWORDS

(2+1) dimensional complex Ginzburg-Landau equation, Painlevé test for integrability, Hirota method, propagating pulses, stationary pulses, propagating hole solutions and stationary dark solutions.

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1. Introduction

Many spatially extended systems display the formation of patterns when driven sufficiently far from equilibrium. Since near threshold the form of the amplitude or envelope equation depends mainly on the symmetries and on the nature of the primary bifurcation, the amplitude description has become an important organizing principle of the theory of non-equilibrium pattern formation. This envelope equation plays a crucially important role in understanding of the power of nonlinearities in pattern selection and pattern dynamics. When subject to large external constraints, these systems may decay into spatio-temporal disorder, a state which is often reffered to weak turbulence. The best model which allows such phenomena is the complex Ginzburg-Landau (CGL) equation. Typical systems modelled by this equation include chemical reactions, liquid crystals, hydrodynamic convection, nonlinear optics, directional solidification or parametrically driven systems [1-6]. The (2+1) dimensional CGL equation model reads:

$$U_t + p\Delta U = aU - \delta|U|^2U, \tag{1}$$

which can be derived universally in the vicinity of a homogeneous Hopf bifurcation in extended systems [7-10] with the complex variable U(x,y,t) being the order parameter at the bifurcation. The parameters p, a, and δ are in general complex, that is of the form $\chi = \chi_r + i\chi_i$. This equation contains terms which account for most of the physical effects such as linear growth, nonlinear saturation, dispersion, diffusion, nonlinear renormalization of the frequency of temporal oscillations. By combining all these phenomena, CGL becomes a generalization of an integrable system (nonlinear Schroedinger equation) and of a variational equation (GL with real coefficients).

Since the study of solitons and the related issue of the construction of solution to a wide class of nonlinear equation has become one of the most exciting and extremely active areas of research investigation, the (1+1) dimensional CGL equation has been analytically solved. Using Hirota's method [11-12], Weiss-Tabor-Carnevale (WTC) [13] and Conte and Musette method [14], it appears that the (1+1) dimensional CGL equation possesses a rich variety of solutions. These solutions include coherent structures such as pulses, fronts, periodic unbounded waves, sinks and sources [11-19] and a transition to chaos [20-24]. The main interest of scientists were restricted to the (1+1) dimensional CGL equation case, because of the difficulty of finding exact analytically high dimensional solutions. Motivated by this, and since it is often put forward that a first step toward understanding complex dynamics in nonequilibrium systems is to find solution describing a basic pattern which play crucial role in long time behavior, in the present work, we investigate analytical solutions of the (2+1) dimensional CGL. Specifically, we focus, on Eq.(1) with $\vec{r} = (x, y)$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. We will use in our analysis, a method which is derived from the Painlevé test for integrability and Hirota's multilinear formalism. By the use of the method presented in [25-27] in which the process begins with Painlevé analysis developed by WTC [28] to obtain in natural way the ansatze which are then used in Hirota's method. The major contribution of our methodology lies in this use of one method to obtain the ansatze, which application has been intuitive. This gives an algorithmic way of obtaining the ansatze, thus making the use of Hirota's method considerably easier. The outline of this paper is as follows. In section 2, we present the formulation of the below mentioned problem. The strategy for obtaining special solutions is developed in section 3. Finally, the section 4 is devoted to the degeneracies of the (2+1)dimensional CGL equation.

2. Painlevé Analysis of the (2+1)Dimensional CGL Equation

Let us rewrite Eq.(1) as the following system of equations:

$$U_t + p(U_{xx} + U_{yy}) = aU - \delta U^2 V, \tag{2.a}$$

$$V_t + p^*(V_{xx} + V_{yy}) = a^* V - \delta^* V^2 U, \tag{2.b}$$

where a^*, p^*, δ^* and V are complex conjugates of a, p, δ and U respectively. In this analysis, U and V are assumed to be independent of each other.

The singular manifold expansion is introduced by assuming that U and V can be expanded as Laurent series:

$$U(x,y,t) = \phi^{\mu}(x,y,t) \sum_{j=0}^{\infty} U_j(x,y,t)\phi^j(x,y,t), \tag{3.a}$$

$$V(x,y,t) = \phi^{\eta}(x,y,t) \sum_{j=0}^{\infty} V_j(x,y,t)\phi^j(x,y,t) \tag{3.b}$$

where ϕ is arbitrary and analytic function which determines a singular manifold. In order to understand the singular structure of Eqs. (2.a) and (2.b), we carry out a leading order analysis by representing U and V as follows:

$$U(x,y,t) \sim U_0(x,y,t)\phi^{\mu}(x,y,t), V(x,y,t) \sim V_0(x,y,t)\phi^{\eta}(x,y,t), \tag{4}$$

where μ and η are the leading order exponents. The leading terms are:

$$p(U_{xx} + U_{yy}) \sim -\delta U^2 V, \tag{5.a}$$

$$p^*(V_{xx} + V_{yy}) \sim -\delta^* V^2 U, \tag{5.b}$$

Inserting Eq.(4) into Eqs.(5.a) and (5.b) and equating the exponents of the dominant terms, we find

$$\mu + \eta = -2. \tag{6}$$

To solve μ and η, we must equate the coefficients of the terms from Eqs.(5.a) and (5.b), we obtain

$$U_0 V_0 = -\frac{\mu(\mu-1)p}{\delta}(\phi_x^2 + \phi_y^2), \tag{7.a}$$

$$U_0 V_0 = -\frac{\eta(\eta-1)p^*}{\delta^*}(\phi_x^2 + \phi_y^2). \tag{7.b}$$

Eqs.(7.a) and (7.b) are combined, with help of Eq.(6), the leading order are easily seen to be

$$\mu = -1 - i\alpha, \eta = -1 + i\alpha, \tag{8}$$

where U_0, V_0 and α are solutions of the equations:

$$\alpha = \frac{3}{2}\frac{R}{I} \pm \left[2 + \frac{9}{4}\frac{R^2}{I^2}\right]^{\frac{1}{2}}, \alpha \neq 0, \tag{9.a}$$

$$U_0 V_0 = \frac{[3\alpha I - (2-\alpha^2)R](\phi_x^2 + \phi_y^2)}{|\delta|^2}, \tag{9.b}$$

and the intermediate variables R and I are defined as

$$R = R_e(p\delta^*) = p_r\delta_r + p_i\delta_i, \tag{10.a}$$

$$I = I_m(p\delta^*) = p_i\delta_r - p_r\delta_i. \tag{10.b}$$

We consider here the non-degenerate case $I \neq 0$. The case I = 0 will be treated at the last section.

3. Special Solutions of the (2+1) Dimensional CGL Equation

The leading-order exponent $1 + i\alpha$ is not an integer: the non integrable (2+1) dimensional CGL equation has already failed the Painlevé test. As expected, its general solutions can not be expressed in terms of elementary functions. However, partial integrability remains possible. An usual assumption to find these solutions, is to require the Laurent series to truncate after the power $\phi^{0\pm i\alpha}$ and to find a relation between (U_1, V_1) and (U,V). This is known as "WTC truncation procedure". Thus truncated expansions for Eq.(4) are:

$$U = U_0\phi^{-1-i\alpha} + U_1\phi^{-i\alpha}, V = V_0\phi^{-1+i\alpha} + V_1\phi^{i\alpha}. \tag{11}$$

We can progress by observing that (U_1, V_1) satisfies the same relation as (U,V) through the equations:

$$\phi^{-i\alpha} : U_{1t} + p(U_{1xx} + U_{1yy}) = aU_1 - \delta U_1^2 V_1, \tag{12.a}$$

$$\phi^{i\alpha} : V_{1t} + p^*(V_{1xx} + V_{1yy}) = a^* V_1 - \delta^* V_1^2 U_1. \tag{12.b}$$

Equations (12.a) and (12.b) justify the fact that the expansion can be truncated at the level $\phi^{\mp i\alpha}$. So that by choosing $U_1 = V_1 = 0$, the resulting zeroth order of equation is easier to solve.

Thus the truncated expansions take the form:

$$U = \frac{U_0}{\phi^{1+i\alpha}}, V = \frac{V_0}{\phi^{1-i\alpha}}. \tag{13}$$

By observing that the plane waves

$$U(x,y,t) = R_K e^{i(K_1 x + K_2 y - \omega t)}, \tag{14}$$

with

$$|R_K|^2 = \frac{p_r(K_1^2 + K_2^2) + a_r}{\delta_r}, \quad (\mathbf{15.a})$$

and the dispersion relation

$$\omega(K_1, K_2) = -p_i(K_1^2 + K_2^2) - a_i + \delta_i |R_K|^2, \quad (\mathbf{15.b})$$

where K_1 and K_2 are free parameters. The general solutions of the (2+1) dimensional CGL equation can be built by the help of the trivial solutions (14).

So the expressions (13) may be expressed as:

$$U = \frac{G}{F^{1+i\alpha}} e^{i(K_1 x + K_2 y - \Omega t)}, V = \frac{H}{F^{1-i\alpha}} e^{-i(K_1 x + K_2 y - \Omega t)}. \quad (\mathbf{16})$$

If we substitute relation (16) into Eqs.(2.a) and (2.b), then we convert our full set of trilinear equations:

$F\{[-i\Omega - a - p(K_1^2 + K_2^2) + \mathcal{D}_{\alpha,t} + 2ip(K_1 \mathcal{D}_{\alpha,x} + K_2 \mathcal{D}_{\alpha,y}) + p(\mathcal{D}_{\alpha,x}^2 + \mathcal{D}_{\alpha,y}^2)]\}(GF) - \frac{1}{2}(1 + i\alpha)(2 + i\alpha)pG[\mathcal{D}_{0,x}^2 + \mathcal{D}_{0,y}^2](F.F) = -\delta HG^2$, $(\mathbf{17.a})$

$F\{[i\Omega - a^* - p^*(K_1^2 + K_2^2) + \mathcal{D}_{\alpha,t}^* - 2ip^*(K_1 \mathcal{D}_{\alpha,x}^* + K_2 \mathcal{D}_{\alpha,y}^*) + p^*(\mathcal{D}_{\alpha,x}^{*2} + \mathcal{D}_{\alpha,y}^{*2})]\}(HF) - \frac{1}{2}(1 - i\alpha)(2 - i\alpha)p^* H[\mathcal{D}_{0,x}^2 + \mathcal{D}_{0,y}^2](F.F) = -\delta^* GH^2$, $(\mathbf{17.b})$

where $\mathcal{D}_{\alpha,z}$ is Hirota's modified bilinear operator defined by

$$\mathcal{D}_{\alpha,z}(GF) = \left\{ \left[\frac{\partial}{\partial z} - (1 + i\alpha)\frac{\partial}{\partial z'} \right] G(z)F(z') \right\}_{z=z'}. \quad (\mathbf{18})$$

Two families of solutions are found depending of the forms of expansions chosen for F, G and H.

3.1. **2D Pulse solutions.** Pulse solutions are recovered through the following assumption:

$$G = \gamma\epsilon g_1, H = \lambda\epsilon h_1, F = 1 + \epsilon^2 f_2, \quad (\mathbf{19})$$

where γ and λ are complex constant and h_1, g_1 and f_2 are real functions. Substituting Eq.(19) into Eq.(17.a) and collecting the coefficients of different powers of ϵ we obtain a set of three equations. Solving this set of equations, we find

$$g_1 = h_1 = exp\left\{kx + qy + \omega t + \xi^{(0)}\right\}, \quad (\mathbf{20})$$

$$f_2 = g_1^2 = h_1^2 = exp\left\{2[kx + qy + \omega t + \xi^{(0)}]\right\}, \quad (\mathbf{21})$$

where unknowns in these expressions are defined depending of the value of the coefficient p_r of the CGL equation as follows:

Case 1: when $p_r \neq 0$, we found that:

$$K_1 k + K_2 q = \alpha(k^2 + q^2), \quad (\mathbf{22})$$

this leads to two subcases:

Subcase 1: $K_1 = \alpha k, K_2 = \alpha q$.$(\mathbf{23})$

Then we obtain the following results:

$$\lambda\gamma = 4\frac{[(2-\alpha^2)R - 3\alpha I]}{|\delta|^2}(k^2+q^2) > 0 \qquad \textbf{(24.a)}$$

$$\omega = 0, \qquad \textbf{(24.b)}$$

$$\Omega = -a_i + [(1-\alpha^2)p_i + 2\alpha p_r](k^2+q^2) \qquad \textbf{(24.c)}$$

and

$$k^2 + q^2 = \mathcal{R}^2 \qquad \textbf{(24.d)}$$

with

$$\mathcal{R}^2 = \frac{a_r}{(1-\alpha^2)p_r - 2\alpha p_i} > 0. \qquad \textbf{(24.e)}$$

All parameters of the solutions are defined for this family of 2D stationary pulse solutions.

Subcase 2:

$$K_1 \neq \alpha k, K_2 \neq \alpha q, \qquad \textbf{(25)}$$

we find a two-parameter family of solutions. The unknowns $K_1, K_2, \gamma, \lambda, \omega, \Omega$ are parametrized by k, q:

$$\Omega = -a_i - p_i(K_1^2 + K_2^2) + 2p_r(K_1 k + K_2 q) + p_i(k^2+q^2), \qquad \textbf{(26.a)}$$

$$K_2 = \frac{\alpha}{q}(k^2+q^2) - \frac{k}{q}K_1, \qquad \textbf{(26.b)}$$

$\gamma\lambda$ and ω are given by Eq.(24.a) and Eq.(24.b) respectively. K_1 is determined by a quadratic equation

$[p_r(k^2+q^2)]K_1^2 - 2[\alpha p_r(k^2+q^2)k]K_1 + \left\{\alpha^2 p_r(k^2+q^2)^2 - q^2[(p_r - 2\alpha p_i)(k^2+q^2) - a_r]\right\} = 0.$(27)

Using Eqs.(16),(19), (20), (21) and (24.b), we obtain the 2D stationary pulse solutions as follows:

$$U(x,y,t) = \frac{\gamma e^{(kx+qy+\xi^{(0)})}}{[1+e^{2(kx+qy+\xi^{(0)})}]^{1+i\alpha}} e^{i(K_1 x + K_2 y - \Omega t)}, \qquad \textbf{(28.a)}$$

$$V(x,y,t) = \frac{\lambda e^{(kx+qy+\xi^{(0)})}}{[1+e^{2(kx+qy+\xi^{(0)})}]^{1-i\alpha}} e^{-i(K_1 x + K_2 y - \Omega t)}. \qquad \textbf{(28.b)}$$

Case 2: $p_r = 0$

In the non-degenerate case treated here ($I = p_i\delta_r - p_r\delta_i \neq 0$), this condition implies that $p_i\delta_r \neq 0$. We find a two-parameter family of solutions. The unknowns $K_1, K_2, \gamma, \lambda, \omega, \Omega$ are given by the following relations:

$$\lambda\gamma = \frac{4p_i}{|\delta|^2}[(2-\alpha^2)\delta_i - 3\alpha\delta_r](k^2+q^2), \qquad \textbf{(29.a)}$$

$$\omega = 2p_i(kK_1 + qK_2) - 2\alpha p_i(k^2+q^2), \qquad \textbf{(29.b)}$$

$$\Omega = -a_i - p_i(K_1^2 + K_2^2) + p_i(k^2+q^2), \qquad \textbf{(29.c)}$$

$$k^2 + q^2 = \mathcal{R}'^2, \qquad \textbf{(29.d)}$$

with

$$\mathcal{R}'^2 = \frac{-a_r}{2\alpha p_i} > 0. \tag{29.e}$$

K_1 and K_2 are free parameters. By using Eqs.(16), (19), (20) and (21), the propagating pulse solutions obtained in this case are recovered as:

$$U(x,y,t) = \frac{\gamma e^{(kx+qy+\omega t+\xi^{(0)})}}{[1+e^{2(kx+qy+\omega t+\xi^{(0)})}]^{1+i\alpha}} e^{i(K_1x+K_2y-\Omega t)}, \tag{30.a}$$

$$V(x,y,t) = \frac{\lambda e^{(kx+qy+\omega t+\xi^{(0)})}}{[1+e^{2(kx+qy+\omega t+\xi^{(0)})}]^{1-i\alpha}} e^{-i(K_1x+K_2y-\Omega t)}. \tag{30.b}$$

After some arrangements: the 2D stationary pulse solutions (28) can be written as:

$$U(x,y,t) = \frac{\gamma}{2} sech(kx+qy+\xi^{(0)}) e^{i\{[(K_1-\alpha k)x+(K_2-\alpha q)y-\Omega t-\alpha\xi^{(0)}]+\alpha \ln[\frac{1}{2} sech(kx+qy+\xi^{(0)})]\}}, \tag{31.a}$$

$$U(x,y,t) = \frac{\lambda}{2} sech(kx+qy+\xi^{(0)}) e^{-i\{[(K_1-\alpha k)x+(K_2-\alpha q)y-\Omega t-\alpha\xi^{(0)}]+\alpha \ln[\frac{1}{2} sech(kx+qy+\xi^{(0)})]\}}, \tag{31.b}$$

and the 2D propagating pulse solutions (30) can be written:

$$U(x,y,t) = \frac{\gamma}{2} sech(kx+qy+\omega t+\xi^{(0)}) e^{i\{[(K_1-\alpha k)x+(K_2-\alpha q)y-(\Omega+\alpha\omega)t-\alpha\xi^{(0)}]+\alpha \ln[\frac{1}{2} sech(kx+qy+\omega t+\xi^{(0)})]\}}, \tag{32.a}$$

$$V(x,y,t) = \frac{\lambda}{2} sech(kx+qy+\omega t+\xi^{(0)}) e^{-i\{[(K_1-\alpha k)x+(K_2-\alpha q)y-(\Omega+\alpha\omega)t-\alpha\xi^{(0)}]+\alpha \ln[\frac{1}{2} sech(kx+qy+\omega t+\xi^{(0)})]\}}. \tag{32.b}$$

3.2. **2D Propagating hole solutions.** Next, we move on to the construction of the propagating hole solutions of the (2+1) dimensional CGL equation. To obtain the propagating hole solutions, we assume the following forms for G, H and F:

$$G = b(1+\epsilon^2 g_2), H = b^*(1+\epsilon^2 h_2), F = 1+\epsilon^2 f_2 \tag{33}$$

only f_2 is real function, g_2 and h_2 are complexe functions and b is complexe constant. Substituting relation (33) into Eq.(17.a), we obtain four equations at order $\epsilon^0, \epsilon^2, \epsilon^4$ and ϵ^6. These equations are solved and we find:

$$g_2 = \kappa e^{2(kx+qy+\omega t+\xi^{(0)})}, h_2 = \kappa^* e^{2(kx+qy+\omega t+\xi^{(0)})}, f_2 = e^{2(kx+qy+\omega t+\xi^{(0)})}. \tag{34}$$

The unknowns $b, \kappa, k, q, \omega, \Omega, K_1$ and K_2 can be computed from

$$|b|^2 = -\frac{2e_1}{|1-\kappa|^2|\delta|^2}(k^2+q^2), \tag{35.a}$$

$$\omega = 2p_i(kK_1+qK_2) - 2\alpha p_i(k^2+q^2) + \frac{\delta_i}{2\alpha}(|\kappa|^2-1)|b|^2, \tag{35.b}$$

$$\Omega = -p_i(K_1^2+K_2^2) - a_i + \delta_i|b|^2, \tag{35.c}$$

$$K_2 = \frac{1}{p_r q}\left[\alpha p_r + \frac{e_1\delta_r}{2\alpha}\frac{|\kappa|^2-1}{|1-\kappa|^2|\delta|^2}\right](k^2+q^2) - \frac{k}{q}K_1, \quad (\mathbf{35.d})$$

$$e_1 = 2\{\delta_r[(2-\alpha^2)p_r - 3\alpha p_i] + \delta_i[(2-\alpha^2)p_i + 3\alpha p_r]\} < 0. \quad (\mathbf{35.e})$$

K_1 is determined by a quadratic equation

$$e_2(k,q)K_1^2 - 2e_3(k,q)K_1 + e_4(k,q) = 0, \quad (\mathbf{36})$$

where

$$e_2(k,q) = p_r^2|\delta|^2|1-\kappa|^2(k^2+q^2), \quad (\mathbf{37.a})$$

$$e_3(k,q) = p_r|1-\kappa|^2|\delta|^2\left[\alpha p_r + \frac{e_1\delta_r}{2\alpha}\frac{|\kappa|^2-1}{|1-\kappa|^2|\delta|^2}\right]k(k^2+q^2), \quad (\mathbf{37.b})$$

$$e_4(k,q) = |1-\kappa|^2|\delta|^2\left[\alpha p_r + \frac{e_1\delta_r}{2\alpha}\frac{|\kappa|^2-1}{|1-\kappa|^2|\delta|^2}\right]^2(k^2+q^2)^2 + q^2[2e_1\delta_r p_r(k^2+q^2) + p_r|1-\kappa|^2|\delta|^2 a_r]. \quad (\mathbf{37.c})$$

k and q are free parameters. The remaining unknowns κ_r and κ_i should verify the following equations:

$$d_{2r}\kappa_r^3 + d_{3r}\kappa_r^2 + d_{4r}\kappa_r - d_{2i}\kappa_i^3 + d_{5r}\kappa_i^2 + d_{6i}\kappa_i - d_{2i}\kappa_i\kappa_r^2 + d_{2r}\kappa_r\kappa_i^2 + d_{7i}\kappa_r\kappa_i + d_{8r} = 0 \quad (\mathbf{38.a})$$

$$d_{2i}\kappa_r^3 + d_{3i}\kappa_r^2 + d_{4i}\kappa_r + d_{2r}\kappa_i^3 + d_{5i}\kappa_i^2 + d_{6r}\kappa_i + d_{2r}\kappa_i\kappa_r^2 + d_{2i}\kappa_r\kappa_i^2 + d_{7r}\kappa_r\kappa_i + d_{8i} = 0, \quad (\mathbf{38.b})$$

where

$$d_{2r} = d_{1r}|\delta|^2 - \frac{e_1\delta_i}{2\alpha}, d_{2i} = d_{1i}|\delta|^2 + \frac{e_1\delta_r}{2\alpha}, \quad (\mathbf{39.a})$$

$$d_{3r} = (\alpha d_{1i} - 3d_{1r})|\delta|^2 + \frac{e_1}{2}(\delta_r + \frac{\delta_i}{\alpha}), d_{3i} = -(\alpha d_{1r} + 3d_{1i})|\delta|^2 + \frac{e_1}{2}(\delta_i - \frac{\delta_r}{\alpha}), \quad (\mathbf{39.b})$$

$$d_{4r} = (3d_{1r} - 2\alpha d_{1i})|\delta|^2 - e_1(\delta_r - \frac{\delta_i}{2\alpha}), d_{4i} = (3d_{1i} + 2\alpha d_{1r})|\delta|^2 - e_1(\delta_i + \frac{\delta_r}{2\alpha}), \quad (\mathbf{39.c})$$

$$d_{5r} = (\alpha d_{1i} - d_{1r})|\delta|^2 + \frac{e_1}{2}(\delta_r + \frac{\delta_i}{\alpha}), d_{5i} = -(\alpha d_{1r} + d_{1i})|\delta|^2 + \frac{e_1}{2}(\delta_i - \frac{\delta_r}{\alpha}), \quad (\mathbf{39.d})$$

$$d_{6r} = d_{1r}|\delta|^2 + \frac{e_1\delta_i}{2\alpha}, d_{6i} = -d_{1i}|\delta|^2 + \frac{e_1\delta_r}{2\alpha}, \quad (\mathbf{39.e})$$

$$d_{7r} = -2d_{1r}|\delta|^2, d_{7i} = 2d_{1i}|\delta|^2, \quad (\mathbf{39.f})$$

$$d_{8r} = (\alpha d_{1i} - d_{1r})|\delta|^2 + \frac{e_1}{2}(\delta_r - \frac{\delta_i}{\alpha}), d_{8i} = -(d_{1i} + \alpha d_{1r})|\delta|^2 + \frac{e_1}{2}(\delta_i + \frac{\delta_r}{\alpha}), \quad (\mathbf{39.g})$$

$$d_{1r} = p_r - \alpha p_i, d_{1i} = p_i + \alpha p_r. \quad (\mathbf{39.h})$$

Another family of 2D propagating hole solutions is obtained, which satisfies the following conditions:

$$\kappa_r = 1, \kappa_i = \alpha, \quad (\mathbf{40.a})$$

$$|b|^2 = -\frac{2e_1}{\alpha^2|\delta|^2}(k^2+q^2), \quad (\mathbf{40.b})$$

$$\omega = 2p_i(kK_1 + qK_2) - 2(\frac{2}{\alpha}p_i + 3p_r)(k^2+q^2), \quad (\mathbf{40.c})$$

$$\Omega = -a_i - p_i(K_1^2 + K_2^2) + \delta_i|b|^2, \quad (\mathbf{40.d})$$

$$K_2 = \frac{1}{qp_r}\left[\frac{2}{\alpha}p_r - 3p_i\right](k^2+q^2) - \frac{k}{q}K_1, \qquad (\mathbf{40}.\mathbf{e})$$

Here K_1 satisfies the following quadratic equation

$[\alpha^2 p_r^2(k^2+q^2)]K_1^2 - 2[\alpha^2 p_r(\frac{2}{\alpha}p_r - 3p_i)k(k^2+q^2)]K_1 + \{\alpha^2(\frac{2}{\alpha}p_r - 3p_i)^2(k^2+q^2)^2 + q^2[4p_r[(2-\alpha^2)p_r - 3\alpha p_i](k^2+q^2) + \alpha^2 p_r a_r]\} = 0.(\mathbf{41})$

k and q are free parameters. Using Eqs.(16), (33) and (34), the 2D propagating hole solutions are given as follows

$$U(x,y,t) = \frac{b[1+\kappa e^{2(kx+qy+\omega t+\xi^{(0)})}]}{[1+e^{2(kx+qy+\omega t+\xi^{(0)})}]^{1+i\alpha}} e^{i(K_1x+K_2y-\Omega t)} \qquad (\mathbf{42}.\mathbf{a})$$

$$V(x,y,t) = \frac{b^*[1+\kappa^* e^{2(kx+qy+\omega t+\xi^{(0)})}]}{[1+e^{2(kx+qy+\omega t+\xi^{(0)})}]^{1-i\alpha}} e^{-i(K_1x+K_2y-\Omega t)} \qquad (\mathbf{42}.\mathbf{b})$$

After some arrangements, the expressions (42.a) and (42.b) can be written in this final form

$$U(x,y,t) = \frac{b\kappa e^{(kx+qy+\omega t+\xi^{(0)})} + be^{-(kx+qy+\omega t+\xi^{(0)})}}{[e^{(kx+qy+\omega t+\xi^{(0)})} + e^{-(kx+qy+\omega t+\xi^{(0)})}]^{1+i\alpha}} e^{i\{[K_1'x+K_2'y-\Omega' t-\alpha\xi^{(0)}]+\alpha\ln[\frac{1}{2}sech(kx+qy+\omega t+\xi^{(0)})]\}}, \qquad (\mathbf{43}.\mathbf{a})$$

$$V(x,y,t) = \frac{b^*\kappa^* e^{(kx+qy+\omega t+\xi^{(0)})} + b^* e^{-(kx+qy+\omega t+\xi^{(0)})}}{[e^{-i\{(kx+qy+\omega t+\xi^{(0)})} + e^{-(kx+qy+\omega t+\xi^{(0)})}]^{1-i\alpha}} e^{-i\{[K_1'x+K_2'y-\Omega' t-\alpha\xi^{(0)}]+\alpha\ln[\frac{1}{2}sech(kx+qy+\omega t+\xi^{(0)})]\}}, \qquad (\mathbf{43}.\mathbf{b})$$

with

$$K_1' = K_1 - \alpha k, K_2' = K_2 - \alpha q, \Omega' = \Omega + \alpha\omega. \qquad (\mathbf{44})$$

It can then be easily seen that, by setting, $\zeta = kx + qy + \omega t + \xi^{(0)}$ we have the following results:

$$\lim_{\zeta \longrightarrow +\infty} U(\zeta) = b(\kappa_r + i\kappa_i)e^{i[k_1x+q_1y-\Omega_1' t-2\alpha\xi^{(0)}]}, \qquad (\mathbf{45}.\mathbf{a})$$

$$\lim_{\zeta \longrightarrow +\infty} V(\zeta) = b^*(\kappa_r - i\kappa_i)e^{-i[k_1x+q_1y-\Omega_1' t-2\alpha\xi^{(0)}]}, \qquad (\mathbf{45}.\mathbf{b})$$

$$\lim_{\zeta \longrightarrow -\infty} U(\zeta) = be^{i[k_2x+q_2y-\Omega_2' t]}, \qquad (\mathbf{45}.\mathbf{c})$$

$$\lim_{\zeta \longrightarrow -\infty} V(\zeta) = b^* e^{-i[k_2x+q_2y-\Omega_2' t]}, \qquad (\mathbf{45}.\mathbf{d})$$

with

$$k_1 = K_1' - \alpha k, q_1 = K_2' - \alpha q, \Omega_1' = \Omega' + \alpha\omega, \qquad (\mathbf{46}.\mathbf{a})$$

$$k_2 = K_1' + \alpha k, q_2 = K_2' + \alpha q, \Omega_2' = \Omega' - \alpha\omega. \qquad (\mathbf{46}.\mathbf{b})$$

Each solution connect two waves with different wave vectors. The solution behaves like a homogeneous wave of amplitude $b(\kappa_r + i\kappa_i)$ and wave vector $\vec{k} = (k_1, q_1)$ as $\zeta \longrightarrow +\infty$; for $\zeta \longrightarrow -\infty$, it corresponds to a wave of amplitude b and wave vector $\vec{k}' = (k_2, q_2)$. If we assume that b is real, the parameter $b\kappa$ can then be written as $b\kappa = b|\kappa|exp(i\theta)$

where

$$\cos(\theta) = \frac{\kappa_r}{\sqrt{\kappa_r^2 + \kappa_i^2}}, \quad \textbf{(47.a)}$$

$$\sin(\theta) = \frac{\kappa_i}{\sqrt{\kappa_r^2 + \kappa_i^2}}. \quad \textbf{(47.b)}$$

Eqs.(47.a) and (47.b) represent the phase shift $\theta \equiv \arg(b\kappa, b)$ between the two asymptotic waves.

3.3. **2D Stationary dark solutions.** The 2D stationary dark solutions are recovered under these special conditions:

$$\omega = 0, \kappa_r = -1, \kappa_i = 0. \quad \textbf{(48)}$$

Then the general solutions which take into account Eqs.(16), (33) and (34) have the form:

$$U(x,y,t) = \frac{b[1 - e^{2(kx+qy+\xi^{(0)})}]}{[1 + e^{2(kx+qy+\xi^{(0)})}]^{1+i\alpha}} e^{i(K_1 x + K_2 y - \Omega t)} \quad \textbf{(49.a)}$$

$$V(x,y,t) = \frac{b^*[1 - e^{2(kx+qy+\xi^{(0)})}]}{[1 + e^{2(kx+qy+\xi^{(0)})}]^{1-i\alpha}} e^{-i(K_1 x + K_2 y - \Omega t)}. \quad \textbf{(49.b)}$$

The unknowns in these solutions are determined depending on two cases:

Case 1: $K_1 = \alpha k, K_2 = \alpha q.$**(50)**

A discrete set of solutions is found with:

$$|b|^2 = \frac{3\alpha I - (2 - \alpha^2)R}{|\delta|^2}(k^2 + q^2), \quad \textbf{(51.a)}$$

$$\Omega = -a_i - (2p_i + 3\alpha p_r)(k^2 + q^2), \quad \textbf{(51.b)}$$

$$k^2 + q^2 = \mathcal{R}''^2, \quad \textbf{(51.c)}$$

$$\mathcal{R}''^2 = -\frac{a_r}{2p_r - 3\alpha p_i} > 0. \quad \textbf{(51.d)}$$

Case 2:

$$K_1 \neq \alpha k, K_2 \neq \alpha q. \quad \textbf{(52)}$$

We find a two-parameter family of solution defined by

$$K_2 = \frac{\alpha}{q}(k^2 + q^2) - \frac{k}{q}K_1, \quad \textbf{(53.a)}$$

$$\Omega = -a_i - p_i(K_1^2 + K_2^2) - [(2 - \alpha^2)p_i + 3\alpha p_r](k^2 + q^2). \quad \textbf{(53.b)}$$

Here K_1 is determined by a quadratic equation

$$[p_r(k^2+q^2)]K_1^2 - 2[\alpha p_r k(k^2+q^2)]K_1 + \left\{\alpha^2 p_r(k^2 + q^2)^2 + q^2\left[[(2 - \alpha^2)p_r - 3\alpha p_i](k^2 + q^2) + a_r\right]\right\} = 0. \quad \textbf{(54)}$$

$|b|^2$ still satisfies Eq.(51.a). k and q are free parameters.

4. Degeneracies of (2+1) Dimensional CGL Equation

In the limiting case of $I_m(\frac{2p}{\delta}) = 0$, the 2D CGL equation degenerates. This puts the following constraints on the coefficients of the equation.

$$I = p_i\delta_r - p_r\delta_i = 0. \quad \textbf{(55)}$$

Leading-order analysis then leads to:

$$U_0V_0 = \frac{2R}{|\delta|^2}(\phi_x^2 + \phi_y^2). \quad \textbf{(56)}$$

As before, looking for special solutions of the (2+1) dimensional CGL equation with the restrictions I = 0, we obtain that its systematic resolution leads to 2D pulses, 2D propagating hole solutions and 2D stationary dark solutions. The method is identical to that presented in the previous section.

4.1. **2D Pulse solutions.** The pulse solutions read

$$U(x,y,t) = \frac{\gamma}{2}sech(kx + qy + \omega t + \xi^{(0)})e^{i[K_1x+K_2y-\Omega t]}, \quad \textbf{(57.a)}$$

$$V(x,y,t) = \frac{\lambda}{2}sech(kx + qy + \omega t + \xi^{(0)})e^{-i[(K_1x+(K_2y-\Omega t]}, \quad \textbf{(57.b)}$$

and necessarily respect $p_r(K_1k + K_2q) = 0$. We distinguish two cases:

Case 1: $P_r = 0$ The parameters are determined by

$$K_2 = -\frac{k}{q}K_1, \quad \textbf{(58.a)}$$

$$\lambda\gamma = \frac{8R}{|\delta|^2}(k^2 + q^2), R > 0, \quad \textbf{(58.b)}$$

$$\omega = 0, \quad \textbf{(58.c)}$$

$$\Omega = -a_i + \frac{p_i}{q^2}(k^2 + q^2)(q^2 - K_1^2), \quad \textbf{(58.d)}$$

$$K_1^2 = \frac{p_rq^2(k^2 + q^2) - a_rq^2}{p_r(k^2 + q^2)} > 0. \quad \textbf{(58.e)}$$

Here k and q are chosen arbitrarily.

Such restrictions (I=0) include the case of real Ginzburg-Landau equation (2D RGL:$p_i = \delta_i = a_i = 0$):

$$U_t + p_r(U_{xx} + U_{yy}) = a_rU - \delta_r|U|^2U. \quad \textbf{(59)}$$

All parameters are parametrized by k and q.

$$K_2 = -\frac{k}{q}K_1, \quad \textbf{(60.a)}$$

$$\lambda\gamma = \frac{8p_r}{\delta_r}(k^2 + q^2), p_r\delta_r > 0, \quad \textbf{(60.b)}$$

$$\omega = 0, \quad \textbf{(60.c)}$$

$$\Omega = 0, \quad \textbf{(60.d)}$$

$$K_1^2 = \frac{p_r q^2(k^2+q^2) - a_r q^2}{p_r(k^2+q^2)},. \qquad (\mathbf{60.e})$$

Case 2: $p_r = 0$.

A four-parameter family of pulses is found for the 2D Schroedinger equation ($p_r = \delta_r = a_r$):

$$-iU_t + p_i(U_{xx} + U_{yy}) = a_i U - \delta_i |U|^2 U. \qquad (\mathbf{61})$$

The free parameters are chosen to be k, q, K_1 and K_2:

$$\lambda\gamma = \frac{8p_i}{\delta_i}(k^2+q^2), p_i\delta_i > 0, \qquad (\mathbf{62.a})$$

$$\omega = 2p_i(kK_1 + qK_2), \qquad (\mathbf{62.b})$$

$$\Omega = -a_i + p_i[k^2 + q^2 - (K_1^2 + K_2^2)]. \qquad (\mathbf{62.c})$$

4.2. **2D Propagating hole solutions.** Propagating hole solutions are obtained:

$$U(x,y,t) = \frac{b\kappa e^{(kx+qy+\omega t+\xi^{(0)})} + be^{-(kx+qy+\omega t+\xi^{(0)})}}{[e^{(kx+qy+\omega t+\xi^{(0)})} + e^{-(kx+qy+\omega t+\xi^{(0)})}]} e^{i(K_1 x + K_2 y - \Omega t)}, \qquad (\mathbf{63.a})$$

$$V(x,y,t) = \frac{b^*\kappa^* e^{(kx+qy+\omega t+\xi^{(0)})} + b^* e^{-(kx+qy+\omega t+\xi^{(0)})}}{[e^{(kx+qy+\omega t+\xi^{(0)})} + e^{-(kx+qy+\omega t+\xi^{(0)})}]} e^{-i(K_1 x + K_2 y - \Omega t)}. \qquad (\mathbf{63.b})$$

The unknowns are determined as:

$$|b|^2 = -\frac{8R}{|\delta|^2|1-\kappa|^2}(k^2+q^2), \kappa \neq 1, R < 0 \qquad (\mathbf{64.a})$$

$\omega = \frac{8R(\kappa_r-1)\{\delta_r[(\kappa_r-1)p_r-\kappa_i p_i]+\delta_i[(\kappa_r-1)p_i+\kappa_i p_r]\}-2|\delta|^2|1-\kappa|^2\{[(\kappa_r-1)p_r-\kappa_i p_i]^2+[(\kappa_r-1)p_i+\kappa_i p_r]^2\}}{|\delta|^2|1-\kappa|^2 p_r[(\kappa_r-1)^2+\kappa_i^2]}(k^2 + q^2), (64.b)$

$$K_2 = \frac{e}{q}(k^2+q^2) - \frac{k}{q}K_1, \qquad (\mathbf{64.c})$$

$$\Omega = -a_i - p_i(K_1^2 + K_2^2) - \frac{8p_i}{|1-\kappa|^2}(k^2+q^2), \qquad (\mathbf{64.d})$$

$$e = \frac{4(\kappa_r - 1)R[\delta_r(\kappa_r-1) - \kappa_i\delta_i] - |\delta|^2|1-\kappa|^2 p_i[(\kappa_r-1)^2+\kappa_i^2]}{|\delta|^2|1-\kappa|^2 p_r[(\kappa_r-1)^2+\kappa_i^2]}. \qquad (\mathbf{64.e})$$

Here K_1 satisfies the following quadratic equation:

$[|1-\kappa|^2 p_r(k^2+q^2)]K_1^2 - 2[|1-\kappa|^2 e p_r k(k^2+q^2)]K_1 + \{e^2(k^2+q^2)^2|1-\kappa|^2 p_r + q^2[8p_r(k^2 + q^2) + |1-\kappa|^2 a_r]\} = 0.(65)$

κ_r and κ_i should lie on the following circle:

$$\kappa_r^2 + \kappa_i^2 = 1, \qquad (\mathbf{66})$$

and k and q are free parameters.

The case includes the 2D nonlinear Schroedinger equation (Eq.(61)) when $p_r = \delta_r = a_r = 0$. All coefficients can be parametrized by k, q, K_1 and K_2.

$$|b|^2 = -\frac{8p_i}{\delta_i|1-\kappa|^2}(k^2+q^2), p_i\delta_i < 0, \kappa \neq 1, \qquad \textbf{(67.a)}$$

$$\omega = 2p_i\left[kK_1 + qK_2 + \frac{\kappa_i}{\kappa_r - 1}(k^2+q^2)\right], \qquad \textbf{(67.b)}$$

$$\Omega = -a_i - p_i(K_1^2 + K_2^2) - \frac{8p_i}{|1-\kappa|^2}(k^2+q^2), \qquad \textbf{(67.c)}$$

$$\kappa_r^2 + \kappa_i^2 = 1. \qquad \textbf{(67.d)}$$

This case also includes the 2D RGL equation (Eq.(59)) when $p_i = \delta_i = a_i = 0$. A two-parameter family of 2D propagating hole solutions is found. We use k and q as free parameters:

$$|b|^2 = -\frac{8p_r}{\delta_r|1-\kappa|^2}(k^2+q^2), p_r\delta_r < 0, \qquad \textbf{(68.a)}$$

$$\omega = -2p_r\left[k^2 + q^2 - \frac{\kappa_r - 1}{\kappa_i}(kK_1 + qK_2)\right], \qquad \textbf{(68.b)}$$

$$\Omega = 0, \qquad \textbf{(68.c)}$$

$$K_2 = \frac{e'}{q}(k^2+q^2) - \frac{k}{q}K_1, \qquad \textbf{(68.d)}$$

$$e' = \frac{4(\kappa_r - 1)}{[(\kappa_r-1)^2 + \kappa_i^2]|1-\kappa|^2}, \qquad \textbf{(68.e)}$$

$$\kappa_r^2 + \kappa_i^2 = 1. \qquad \textbf{(68.f)}$$

Where K_1 satisfies the following equation

$[|1-\kappa|^2 p_r(k^2+q^2)]K_1^2 - 2[|1-\kappa|^2 e' p_r k(k^2+q^2)]K_1 + \{e'^2(k^2+q^2)^2|1-\kappa|^2 p_r + q^2[8p_r(k^2+q^2) + |1-\kappa|^2 a_r]\} = 0.(69)$

4.3. **2D stationary dark solutions.** These solutions read

$$U(x,y,t) = \frac{b[1 - e^{2(kx+qy+\xi^{(0)})}]}{[1 + e^{2(kx+qy+\xi^{(0)})}]}e^{i(K_1x + K_2y - \Omega t)}, \qquad \textbf{(70.a)}$$

$$V(x,y,t) = \frac{b^*[1 - e^{2(kx+qy+\xi^{(0)})}]}{[1 + e^{2(kx+qy+\xi^{(0)})}]}e^{-i(K_1x + K_2y - \Omega t)}. \qquad \textbf{(70.b)}$$

A two-parameter of such family of solutions is found

$$|b|^2 = -\frac{2R}{|\delta|^2}(k^2+q^2), R < 0, \qquad \textbf{(71.a)}$$

$$\Omega = -a_i - p_i[K_1^2 + K_2^2 + 2(k^2+q^2)], \qquad \textbf{(71.b)}$$

$$K_2 = -\frac{k}{q}K_1, \qquad \textbf{(71.c)}$$

$$K_1^2 = -\frac{[2p_r(k^2+q^2) + a_r]q^2}{p_r(k^2+q^2)} > 0. \qquad \textbf{(71.d)}$$

Here k and q are free parameters. For this kind of solutions, we find that it includes the case of 2D nonlinear Schroedinger equation (Eq.(61)) when $p_r = \delta_r = a_r = 0$. All the unknowns are parametrized by k, q and K_1.

$$|b|^2 = -\frac{2p_i}{\delta_i}(k^2 + q^2), p_i\delta_i < 0, \tag{72.a}$$

$$K_2 = -\frac{k}{q}K_1, \tag{72.b}$$

$$\Omega = -p_i[K_1^2 + K_2^2 + 2(k^2 + q^2)]. \tag{72.c}$$

These solutions also include the 2D RGL equation (Eq.(59)) when $p_i = \delta_i = a_i = 0$.

$$|b|^2 = -\frac{2p_r}{\delta_r}(k^2 + q^2), p_r\delta_r < 0, \tag{73.a}$$

$$K_2 = -\frac{k}{q}K_1, \tag{73.b}$$

$$\Omega = 0, \tag{73.c}$$

$$K_1^2 = -\frac{2p_r(k^2 + q^2) + a_r q^2}{p_r(k^2 + q^2)} > 0. \tag{73.d}$$

Here k and q are free parameters.

5. CONCLUSION

In summary, for the (2+1) dimensional CGL equation, there are four types of 2D localized solutions namely propagating pulses, stationary pulses, propagating hole solutions and stationary dark solutions obtaining by a combined methodology which derived from Painlevé test for integrability and Hirota's technique of multilinearization. This method is a practical one of family of complex Ginzburg-Landau equation. The approach is convenient and effective since it permits to build an ansatz in natural way in the two dimensional case. We will try in the forthcoming study to use the same method to solve the (2+1) dimensional complex cubic-quintic Ginzburg-Landau equation. By studying the degeneracies of the (2+1) dimensional cubic CGL equation and its solutions, it appears that this equation is a generalization of an integrable system (nonlinear Schroedinger equation) and of a variational equation (Real Ginzburg-Landau equation), thus exhibits rich variety of phenomena.

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