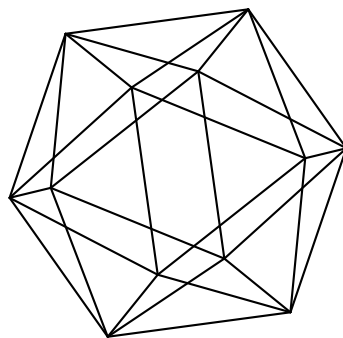# Max-Planck-Institut für Mathematik Bonn

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# EFFECTIVE GLOBAL GENERATION ON VARIETIES WITH NUMERICALLY TRIVIAL CANONICAL CLASS

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ABSTRACT. We prove a Fujita-type theorem for varieties with numerically trivial canonical bundle. We deduce our result via a combination of algebraic and analytic methods, including the Kobayashi–Hitchin correspondence and positivity of direct image bundles. As an application, we combine our results with recent work of U. Riess on generalized Kummer varieties to obtain effective global generation statements for Hilbert schemes of points on abelian surfaces.

## INTRODUCTION

Fujita's conjectures on global generation and very ampleness have long been a strong driving force in birational geometry. The conjecture says that for a polarized complex manifold $(X, \mathscr{L})$, the line bundle $\omega_X \otimes \mathscr{L}^{\otimes m}$ should be basepoint-free if $m \geq \dim(X) + 1$, and very ample whenever $m \geq \dim(X) + 2$. The statement is classical if $\dim(X) = 1$, and has been verified for $\dim(X) \leq 5$ in the case of global generation [Rei88, EL93, Kaw97, Hel01, YZ17]; there are also partial results for the case of very ampleness when $X$ is a Calabi-Yau threefold [GP98]. For arbitrary dimension, there exist strong global generation statements due to Angehrn and Siu [AS95] and Heier [Hei02], whose bounds are nevertheless not linear in $\dim(X)$.

While sharp for hyperplane bundles on projective spaces (for instance) Fujita's conjecture is very far from the truth in general. One notable class where global generation holds for much smaller powers is that of abelian varieties. Indeed, for a polarized abelian variety $(X, \mathscr{L})$ of any dimension, the line bundle $\mathscr{L}^{\otimes m}$ is globally generated for $m \geq 2$ and very ample for $m \geq 3$ by a theorem of Lefschetz. Similar statements include a result of Pareschi and Popa [PP03, Theorem 5.1] to the effect that if $(X, \mathscr{L})$ is a polarized smooth irregular variety whose Albanese morphism is finite onto its image, then $(\omega_X \otimes \mathscr{L})^{\otimes 2}$ is globally generated. We mention the recent work [Rie18] on the base locus of $\mathscr{L}^{\otimes 2}$ when $X$ is an irreducible holomorphic symplectic manifold.

Following this train of thought, we define the Fujita number $f_X$ of a given variety $X$ as

$$f_X \overset{\text{def}}{=} \min\{m \geq 1 : \omega_X \otimes \mathscr{A}^{\otimes m'} \text{ is globally generated for all } m' \geq m \text{ and all ample } \mathscr{A} \text{ on } X\} \,,$$

while for a smooth fibration $\pi \colon X \to Y$ we set $f_\pi \overset{\text{def}}{=} \max\left\{ f_{X_y} : y \in Y \right\}$.

Fujita's prediction for global generation can be phrased as $f_X \leq \dim(X) + 1$. The examples above suggest that $f_X$ may be rather strongly influenced by the Albanese dimension. Since there exist non-minimal $X$ with maximal Albanese dimension and $f_X \geq \dim(X) - 1$ (see Example 4.1), it is natural to restrict our attention to minimal varieties.

The main goal of our paper is an effective global generation result for minimal varieties of Kodaira dimension zero. Recall that these are precisely the varieties whose canonical bundle is numerically trivial, and that their Albanese maps are étale-trivial fibrations (Section 8, [Kaw85]).

**Theorem A.** *Let $X$ be a smooth projective variety for which $K_X =_{\text{num}} 0$, and let $\text{alb}_X : X \to \text{Alb}(X)$ be its Albanese fibration. If $\mathscr{L}$ is an ample line bundle on $X$, then $\mathscr{L}^{\otimes 2m}$ is globally generated for all $m \geq f_{\text{alb}_X}$.*

The closed fibers of $\mathrm{alb}_X$ have numerically trivial canonical bundles themselves (cf. Proposition 1.2), and since the Fujita numbers of such varieties can be as low as 1 or at least as high as their dimension (see Examples 4.2, 4.3 and 4.4) the shape of Theorem A is essentially optimal modulo Fujita's conjecture. Fibrations over abelian varieties have recently been studied by Cao–Păun [CP17], who proved subadditivity of Kodaira dimension in this case; their proof was recently simplified by Hacon–Popa–Schnell [HPS16].

Our starting point is a generalization of the Lefschetz theorem for global generation of line bundles to ample semihomogeneous vector bundles on abelian varieties (Proposition 2.7); it should be noted that this fails without the semihomogeneity requirement (see Remark 2.8). The most involved part of our proof of Theorem A is showing that $\mathrm{alb}_{X*}\mathscr{L}$ is a vector bundle which is ample and semihomogeneous. Here we make essential use of the Kobayashi–Hitchin correspondence. Along the way, we also rely on a positivity result of Berndtsson [Ber09] and Green–Laplace operator techniques of To-Weng [TW98].

Combining our work with the aforementioned results of [Rie18] we obtain an effective global generation result for the Hilbert scheme $\mathrm{Hilb}^n(A)$ of length-$n$ subschemes of an abelian surface $A$ which improves on the Fujita bound when $n = 2$.

**Corollary B.** *If $A$ is an abelian surface, $\mathscr{L}$ is an ample line bundle on $A^{[n]}$, and $m \geq 2$, then $\mathscr{L}^{\otimes 2m}$ is globally generated when $n = 2$ and globally generated in codimension 1 when $n \geq 3$.*

Returning to Fujita numbers, in rough analogy with the subadditivity of Kodaira dimension, several classes of examples (for instance surfaces, products of varieties with no non-trivial correspondences) including our main result point towards the following

**Conjecture.** *Let $X, Y$ be smooth projective varieties and $\pi\colon X \to Y$ a smooth fibration. Then*

$$f_X \leq f_\pi \cdot f_Y \ .$$

About the organization of the paper: the first section deals with the results we need about varieties with numerically trivial canonical bundle. Section 2 is devoted to the study of semihomogeneous bundles both from the algebraic and analytic point of view. While the relation between semihomogeneity and projective flatness has been mentioned in the literature (cf. [Yan89]), our proof uses a precise formulation coming from the Kobayashi–Hitchin correspondence, so we include all the relevant arguments here. Since many of our essential tools come from Hermitian geometry and our main result is a purely algebro-geometric statement, we go over the associated terminology in some detail. Section 3 contains the proofs of Theorem A and Corollary B, while the last section contains some examples and some discussion around Fujita numbers.

## 1. Minimal Varieties of Kodaira Dimension Zero

In this section we lay out the structure of our argument up to the point where we can use semihomogeneous vector bundles on complex tori. We will use the following fundamental decomposition result for compact Kähler manifolds with numerically trivial canonical class; we refer to [Bea83] for the proof. Recall that a compact Kähler manifold $Y$ is *hyperkähler* if it is simply connected and $H^0(\Omega_Y^2)$ is spanned by a symplectic 2-form, and *Calabi-Yau* if $\dim(Y) \geq 3$, the canonical bundle of $Y$ is trivial, and $H^0(\Omega_Y^p) = 0$ for $0 < p < \dim(Y)$.

**Theorem 1.1** (Beauville–Bogomolov decomposition)**.** *Let $X$ be a compact Kähler manifold with $c_1(X) = 0$. Then there exists a finite étale cover $\nu' \colon \Pi_{i=1}^d X_i \to X$ such that every factor $X_i$ is a compact complex torus, a compact hyperkähler manifold, or a Calabi–Yau manifold.*

**Proposition 1.2.** *Let $X$ be a smooth projective variety satisfying $K_X =_{\mathrm{num}} 0$. Then the Albanese fibration $\mathrm{alb}_X \colon X \to \mathrm{Alb}(X)$ fits into a commutative diagram*

(1.1)
$$
\begin{array}{ccc}
Y \times A' \times A'' & \xrightarrow{\quad\nu\quad} & X \\
{\scriptstyle \mathrm{pr}_{A' \times A''}} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{alb}_X} \\
A' \times A'' \xrightarrow{\ \mathrm{pr}_{A'}\ } A' & \xrightarrow{\ \rho\ } & \mathrm{Alb}(X)
\end{array}
$$

*where $Y$ is a smooth projective variety satisfying $K_X =_{\mathrm{num}} 0$ and $q(Y) = 0$, $A'$ and $A''$ are abelian varieties, $\nu \colon Y \times A' \times A'' \to X$ is a surjective étale map, and $\rho \colon A' \to \mathrm{Alb}(X)$ is an isogeny.*

*Proof.* By Theorem 1.1 there exists an étale covering $\nu' \colon Y \times B \to X$ where $Y$ satisfies the claimed properties and $B$ is an abelian variety. Since $q(Y) = 0$, we have that $\mathrm{Alb}(Y \times B) = B$, and the Albanese map of $Y \times B$ is the projection $\mathrm{pr}_B \colon Y \times B \to B$.

The Stein factorization of $\mathrm{alb}(\nu') \colon B \to \mathrm{Alb}(X)$ has the form $\rho' \circ \gamma$, where $\gamma \colon B \to A'$ is an epimorphism of abelian varieties with connected kernel and $\rho' \colon A' \to \mathrm{Alb}(X)$ is an isogeny. If $A'' := \ker(\gamma)$ then we have a diagram

$$
\begin{array}{ccc}
A' \times A'' & \xrightarrow{\ \theta\ } & B \\
{\scriptstyle \mathrm{pr}_{A'}} \big\downarrow & & \big\downarrow {\scriptstyle \gamma} \\
A' & \xrightarrow{\ \rho''\ } & A'
\end{array}
$$

where $\theta$ and $\rho''$ are isogenies. Letting $\rho = \rho' \circ \rho''$ and $\nu = \nu' \circ (1_Y \times \theta)$, we see that the desired diagram (1.1) exists. $\qquad\square$

*Remark* 1.3*.* If $Y$ is a single point, then $X$ is a hyperelliptic variety in the sense of [Lan01], and Theorem 1.2 of [CI14] implies that $L^{\otimes 2}$ is globally generated for any ample $L$ on $X$, i.e. that $f_X \leq 2$.

## 2. Semihomogeneous Bundles

2.1. **Algebraic Preliminaries.** The main result of this subsection, which is central to our proof of Theorem A, is Proposition 2.7, which generalizes the standard Lefschetz theorem on ample line bundles to ample semihomogeneous vector bundles. Let $A$ be a complex abelian variety.

**Definition 2.1.** A vector bundle $\mathscr{E}$ on $A$ is *semihomogeneous*, if for every $a \in A$ there exists a line bundle $\mathscr{L}$ such that

(2.1)
$$
t_a^* \mathscr{E} \simeq \mathscr{E} \otimes \mathscr{L}
$$

where $t_a \colon A \to A$ stands for translation by $a$. If $\mathscr{L} \cong \mathscr{O}_A$ in (2.1) we say that $\mathscr{E}$ is *homogeneous*.

Note that in this definition we must necessarily have $\mathscr{L} \in \mathrm{Pic}^\circ(A)$. We now collect some fundamental facts about semihomogeneous bundles from [Muk78]

**Proposition 2.2** (Propositions 6.17 and 6.18, [Muk78])**.** *Let $\mathscr{E}$ be a semihomogeneous bundle on $A$. Then*

- (1) *for any semihomogeneous bundle $\mathscr{F}$ on $A$, we have that $\mathscr{E} \otimes \mathscr{F}$ is semihomogeneous.*
- (2) *there exist simple semihomogeneous bundles $\mathscr{E}_1, \cdots \mathscr{E}_s$ and indecomposable unipotent bundles $\mathscr{U}_1, \cdots, \mathscr{U}_s$ on $A$ such that*

$$
\mathscr{E} \cong \bigoplus_{j=1}^s \mathscr{E}_j \otimes \mathscr{U}_j \ .
$$

*Moreover, for each $j \in \{1, \cdots, s\}$ there exists $\alpha_j \in \widehat{A}$ such that $\mathscr{E}_j \cong \mathscr{E}_1 \otimes \alpha_j$.*

**Proposition 2.3** (Propositions 6.13 and 6.16, [Muk78])**.** *If $\mathscr{E}$ is a semihomogeneous bundle on $A$, and $H$ is an ample divisor on $A$, then $\mathscr{E}$ is $\mu_H$-semistable, and is $\mu_H$-stable if and only if $\mathscr{E}$ is simple.*

**Proposition 2.4** (Theorem 5.8 and Proposition 7.3, [Muk78])**.** *Let $\mathscr{E}$ be a simple vector bundle on $A$. Then the following are equivalent.*

1. *$\mathscr{E}$ is semihomogeneous.*
2. *There exists an abelian variety $A'$, an isogeny $p : A' \to A$ and a line bundle $\mathscr{L}$ on $A'$ such that $\mathscr{E} \cong p_* \mathscr{L}$.*
3. *There exists an abelian variety $B$ along with an isogeny $\sigma \colon B \to A$ and a line bundle $\mathscr{M}$ on $B$ such that $\sigma^* \mathscr{E} \cong \mathscr{M}^{\oplus r}$, where $r = \operatorname{rk} \mathscr{E}$.*

**Proposition 2.5** (Proposition 5.4, [Muk78])**.** *Let $A'$ and $A''$ be abelian varieties, and let $\pi' : A' \to A$ and $\pi'' : A'' \to A'$ be isogenies. If $\mathscr{E}$ is a vector bundle on $A'$, then all three of the vector bundles $\mathscr{E}, \pi'_* \mathscr{E}, \pi''^* \mathscr{E}$ are semihomogeneous if and only if one of them is semihomogeneous.*

**Lemma 2.6** (Chern classes of semihomogeneous bundles)**.** *Let $\mathscr{E}$ be a semihomogeneous vector bundle on the abelian variety $A$. Then it has the total Chern class of a direct sum, that is,*

$$c(\mathscr{E}) = \left(1 + \frac{c_1(\mathscr{E})}{r}\right)^r$$

*with $r = \operatorname{rk} \mathscr{E}$.*

*Proof.* Assume first that $\mathscr{E}$ is simple. Then Proposition 2.4 yields the existence of an isogeny $\sigma \colon B \to A$ such that $\sigma^* \mathscr{E} \cong \mathscr{M}^{\oplus \operatorname{rk} \mathscr{E}}$ for a line bundle $\mathscr{M}$ on $B$. Then

$$c(\sigma^* \mathscr{E}) = c(\mathscr{M}^{\oplus r}) = (1 + c_1(\mathscr{M}))^r = \left(1 + \frac{c_1(\sigma^* \mathscr{E})}{r}\right)^r .$$

By the projection formula for finite flat morphisms, we then have

$$\deg \sigma \cdot c(\mathscr{E}) = \sigma_* c(\sigma^* \mathscr{E}) = \sigma_* \left(1 + \frac{c_1(\sigma^* \mathscr{E})}{r}\right)^r = \deg \sigma \cdot \left(1 + \frac{c_1(\mathscr{E})}{r}\right)^r$$

from which (2.6) follows at once for simple semihomgeneous bundles.

Dropping the simplicity assumption, let $\mathscr{E}$ now be an arbitrary semihomogeneous bundle. By Proposition 2.2, there exists a simple semihomogeneous bundle $\mathscr{E}_1$, topologically trivial line bundles $\alpha_j$, and unipotent bundles $\mathscr{U}_j$ such that

$$\mathscr{E} \cong \bigoplus_{j=1}^s \mathscr{E}_1 \otimes \alpha_j \otimes \mathscr{U}_j .$$

Since $c(\mathscr{U}_j) = c(\alpha_j) = 1$ for all $j$, we have from the previously established case of simple bundles that

$$c(\mathscr{E}) = \prod_{j=1}^s c(\mathscr{E}_1 \otimes \alpha_j \otimes \mathscr{U}_j) = \prod_{j=1}^s c(\mathscr{E}_1)^{\operatorname{rk} \mathscr{U}_j} = \prod_{j=1}^s \left(1 + \frac{c_1(\mathscr{E}_1)}{\operatorname{rk} \mathscr{E}_1}\right)^{\operatorname{rk} \mathscr{U}_j} = \left(1 + \frac{c_1(\mathscr{E}_1)}{\operatorname{rk} \mathscr{E}_1}\right)^{\operatorname{rk} \mathscr{E}} ,$$

and by $\frac{c(\mathscr{E}_1)}{\operatorname{rk} \mathscr{E}_1} = \frac{c(\mathscr{E})}{r}$ the proof is complete. □

**Proposition 2.7** (Effective global generation of ample semihomogeneous bundles)**.** *Let $\mathscr{E}$ be an ample semihomogeneous bundle on $A$ of rank $r \geq 1$, and let $m \geq 2$ be an integer. Then $\mathscr{E}^{\otimes m}$ is globally generated. In particular, $\operatorname{Sym}^m(\mathscr{E})$ is globally generated.*

*Proof.* By Proposition 2.2, the indecomposable summands of $\mathscr{E}^{\otimes m}$ are all of the form $\mathscr{U}' \otimes \mathscr{E}_{j_1} \otimes \cdots \otimes \mathscr{E}_{j_m}$ for $\mathscr{U}'$ unipotent and $\mathscr{E}_{j_1}, \cdots, \mathscr{E}_{j_m}$ simple and semihomogeneous. Given that an extension of ample globally generated semihomogeneous bundles on an abelian variety is globally generated, we are reduced to considering $\mathscr{E}_{j_1} \otimes \cdots \otimes \mathscr{E}_{j_m}$. By Proposition 2.2 (ii), we have $\mathscr{E}_{j_k} \cong \mathscr{E}_{j_1} \otimes \alpha_k$ for some $\alpha_k \in \widehat{A}$ for all $2 \le k \le m$. If $\alpha'$ is any $m$-th root of $\alpha_2 \otimes \cdots \otimes \alpha_m$, then

$$\mathscr{E}_{j_1} \otimes \cdots \otimes \mathscr{E}_{j_m} \cong (\mathscr{E}_{j_1} \otimes \alpha')^{\otimes m} .$$

We may therefore assume without loss of generality that $\mathscr{E}$ is simple.

Let $\sigma : A' \to A$ be an isogeny and let $\mathscr{L}$ be an ample line bundle on $A'$ such that $\mathscr{E} \cong \sigma_* \mathscr{L}$. We have that

$$\sigma^*(\mathscr{E}^{\otimes m}) \cong (\sigma^* \sigma_* \mathscr{L})^{\otimes m} \cong (\oplus_{x \in \ker(\sigma)} t_x^* \mathscr{L})^{\otimes m}$$

It then follows from the standard result for ample line bundles on abelian varieties that $\sigma^*(\mathscr{E}^{\otimes m})$ is globally generated. The direct image under $\sigma$ of the evaluation map of $\sigma^*(\mathscr{E}^{\otimes m})$ is then a surjective morphism

$$H^0(\sigma^*(\mathscr{E}^{\otimes m})) \otimes_{\mathscr{O}_A} \sigma_* \mathscr{O}_{A'} \to \sigma_* \sigma^*(\mathscr{E}^{\otimes m}) \cong (\mathscr{E}^{\otimes m}) \otimes_{\mathscr{O}_A} \sigma_* \mathscr{O}_{A'}$$

Since this map is canonical, it can be obtained by tensoring $\sigma_* \mathscr{O}_{A'}$ with the composition

$$(2.2) \qquad H^0(\sigma^*(\mathscr{E}^{\otimes m})) \otimes \mathscr{O}_A \cong H^0(\mathscr{E}^{\otimes m} \otimes \sigma_* \mathscr{O}_{A'}) \otimes \mathscr{O}_A \to H^0(\mathscr{E}^{\otimes m}) \otimes \mathscr{O}_A \to \mathscr{E}^{\otimes m}$$

where the first arrow is induced by the trace and the second map is the evaluation map of $\mathscr{E}^{\otimes m}$. Since $\sigma_* \mathscr{O}_{A'}$ is a sheaf of faithfully flat $\mathscr{O}_A$−algebras, the map (2.2) is surjective; thus $\mathscr{E}^{\otimes m}$ is globally generated as claimed. $\qquad \square$

*Remark* 2.8. The global generation of $\mathrm{Sym}^2 \mathscr{E}$ can fail when $\mathscr{E}$ is not semihomogeneous. For semihomogeneous bundles, ampleness is equivalent to being I.T. of index 0 (cf. [KM18, Proposition 2.6]). Example 3.4 of [PP08] exhibits vector bundles $E$ and $F$ on a principally polarized abelian variety $(A, \Theta)$ such that $E \otimes F$ is not globally generated and $E, F$ are both I.T. of index 0; it follows that $\mathrm{Sym}^2(E \oplus F)$—which contains $E \otimes F$ as a direct summand—is not globally generated, although $E \oplus F$ is I.T. of index 0. Since $E$ and $F$ can be shown to have different slopes with respect to $\Theta$, their direct sum $E \oplus F$ is not slope-semistable with respect to $\Theta$, and is therefore not semihomogeneous by Proposition 2.3.

*Remark* 2.9. Every abelian variety $A$ admits an ample semihomogeneous bundle which is not globally generated. Indeed, let $\rho : A' \to A$ be an isogeny for which $A'$ admits a principal polarization $\Theta$, and note that the ample semihomogeneous bundle $\rho_* \Theta$ fails to be globally generated.

## 2.2. Kobayashi–Hitchin correspondence for semihomogeneous bundles.
Our goal here is to show that on an abelian variety, semisimple semihomogeneous bundles are precisely the vector bundles which admit a projectively flat Hermitian structure.

Recall that if $X$ is a compact complex manifold and $\mathscr{E}$ is a vector bundle on $X$, a projectively flat connection $\nabla$ on $\mathscr{E}$ is a connection whose associated curvature form is $\gamma \cdot \mathrm{Id}_r$, where $\gamma$ is a 2-form on $X$. A vector bundle $\mathscr{E}$ is projectively flat precisely if it supports a projectively flat connection (see [Kob87, I. Corollary 2.7]), while $\mathscr{E}$ is said to admit a projectively flat Hermitian structure (which we call PFHS for short) if there exists a Hermitian metric $h$ on $\mathscr{E}$ whose associated Chern connection is projectively flat.

We begin with an Appell-Humbert-type characterization of bundles on complex tori which admit a PFHS. In this subsection we let $X = V/\Lambda$ be a complex torus unless otherwise mentioned.

**Proposition 2.10.** *Let $H : V \times V$ be a Hermitian form such that $\mathrm{Im}(H)$ is $\mathbb{Z}$−valued on $\Lambda \times \Lambda$, and let $G$ be a unitary semirepresentation of $\Lambda$ associated to $H$, i.e. a map $G : \Lambda \to U(r)$ satisfying*

$$G(\lambda_1 + \lambda_2) = G(\lambda_1) \cdot G(\lambda_2) \cdot \exp\left( \frac{\pi i}{r} \cdot \mathrm{Im}(H(\lambda_1, \lambda_2)) \right)$$

*for all* $\lambda_1, \lambda_2 \in \Lambda$. *Define* $J : V \times \Lambda \to \mathrm{GL}(r, \mathbb{C})$ *by*

$$(2.3) \qquad J(\mathbf{v}, \lambda) \overset{def}{=} G(\lambda) \cdot \exp\left( \frac{\pi}{r} H(\mathbf{v}, \lambda) + \frac{\pi}{2r} H(\lambda, \lambda) \right)$$

*Then there exists a PFHS bundle* $\mathscr{E}(G, H)$ *of rank* $r$ *on* $X$ *whose space of holomorphic (resp.* $\mathscr{C}^\infty$*) global sections of* $\mathscr{E}(G, H)$ *is isomorphic to that of the holomorphic (resp.* $\mathscr{C}^\infty$*) vector-valued functions* $\mathbf{f} : V \to \mathbb{C}^r$ *satisfying*

$$\mathbf{f}(\mathbf{v} + \lambda) \overset{def}{=} J(\mathbf{v}, \lambda) \cdot \mathbf{f}(\mathbf{v})$$

*Moreover, every PFHS bundle of rank* $r$ *on* $X$ *is isomorphic to* $\mathscr{E}(G, H)$ *for some* $G$ *and* $H$ *as above.*

*Proof.* See [Yan89, Section 5] or [Kob87, IV. Theorem 7.54].  □

**Lemma 2.11.** *Let* $X$ *be a complex torus,* $\mathscr{E}$ *a vector bundle on* $X$. *If* $\mathscr{E}$ *is PFHS, then it is semihomogeneous.*

*Proof.* By Proposition 2.10 we can assume that $\mathscr{E} = \mathscr{E}(G, H)$. Let $x \in X$ be arbitrary, then the expression $J_x(\mathbf{v}, \lambda) \overset{\mathrm{def}}{=} J(\mathbf{v} + \mathbf{a}, \lambda)$ for any $\mathbf{a} \in V$ mapping to $x \in X$ yields a factor of automorphy for $T_x^* \mathscr{E}$ by [Yan89, Lemma 4.2]. It follows that

$$J_x(\mathbf{v}, \lambda) = J(\mathbf{v}, \lambda) \cdot \exp\left( \pi \cdot H\left( \frac{\mathbf{a}}{r}, \lambda \right) \right) .$$

Since the second factor gives a factor of automorphy for a topologically trivial line bundle, $\mathscr{E}$ is indeed semihomogeneous, hence (3) implies (1).  □

**Definition 2.12** (cf. [LT95], Definition 2.1.1). A Hermitian metric $h$ on a holomorphic vector bundle $\mathscr{E}$ on $X$ is *Hermitian-Einstein* with respect to a Kähler metric $g$ on $X$ if its curvature form $F_h$ is of type $(1, 1)$ and satisfies the identity

$$i \Lambda_g F_h = \lambda_h \cdot \mathrm{Id}_{\mathscr{E}}$$

where $\lambda_h \in \mathbb{R}$ and $\Lambda_g$ is the contraction operator associated to $g$.

We include the following fact for lack of a suitable reference.

**Lemma 2.13.** *With notation as above, let* $\mathscr{E}$ *be a vector bundle of rank* $r$ *on a compact Kähler manifold* $(X, g)$ *equipped with a PFHS metric* $h$. *Then* $\mathscr{E}$ *admits a metric that is Hermitian–Einstein with respect to* $g$.

*Proof.* By definition the curvature form $F_h$ is a $(1, 1)$-form multiplied by the identity. Contracting the associated $(1, 1)$-form by a Kähler form yields a pure-imaginary-valued smooth function $\gamma_h$ on $X$. Since $X$ is compact, Lemma 2.1.5, (i) of [LT95] implies that $h$ can be rescaled to yield a metric on $\mathscr{E}$ which is Hermitian-Einstein.  □

**Theorem 2.14** (Kobayashi–Hitchin correspondence). *A holomorphic vector bundle* $\mathscr{E}$ *on a compact Kähler manifold* $(X, g)$ *admits a metric that is Hermitian–Einstein metric with respect to* $g$ *if and only if* $\mathscr{E}$ *is polystable with respect to* $g$.

For proofs we refer the reader to the original articles [Don83, Don87, UY86] or the monograph [LT95].

**Theorem 2.15** (Kobayashi–Hitchin correspondence for semihomogeneous vector bundles). *Let* $X$ *be an abelian variety, and let* $\mathscr{E}$ *be a vector bundle of rank* $r$ *on* $X$. *Then the following are equivalent:*

(i) $\mathscr{E}$ *is semihomogeneous and semisimple.*
(ii) $\mathscr{E}$ *admits a PFHS.*

*Proof.* (i) $\Rightarrow$ (ii): Since $\mathscr{E}$ is semihomogeneous, Proposition 2.3 implies that it is semistable with respect to an arbitrary polarization. It is also semisimple, so each direct summand of $\mathscr{E}$ is slope-stable with respect to any polarization. Fix a polarization $H$ on $X$ with associated Kähler metric $g$. By the Kobayashi–Hitchin correspondence each direct summand of $\mathscr{E}$ admits a $g$-Hermitian-Einstein metric, so $\mathscr{E}$ itself admits a $g$-Hermitian-Einstein metric $h$ as well. Lemma 2.6 gives that the total Chern class of $\mathscr{E}$ is

$$c(\mathscr{E}) = \left(1 + \frac{c_1(\mathscr{E})}{r}\right)^r .$$

In particular, the Chern classes of $\mathscr{E}$ satisfy

$$(r-1)c_1(\mathscr{E})^2 - 2rc_2(\mathscr{E}) = 0 .$$

We have just shown that $\mathscr{E}$ attains equality in Corollary 2.2.4 of [LT95], hence the Chern connection associated to $(\mathscr{E}, h)$ is projectively flat, and therefore the metric $h$ is PFHS as required.

(ii) $\Rightarrow$ (i): Given that $\mathscr{E}$ admits a PFHS, it is a vector bundle of the form $\mathscr{E}(G, H)$ as in Proposition 2.10, as a consequence $\mathscr{E}$ is semihomogeneous by Lemma 2.11. By Lemma 2.13 it admits an Hermitian–Einstein metric as well. Consequently, $\mathscr{E}$ is polystable by the Kobayashi–Hitchin correspondence, and therefore semisimple. $\square$

Combining Theorem 2.15 with the Green's operator method as used by To–Weng [TW98], we obtain a crucial technical consequence.

**Theorem 2.16.** *Let $A,B$ be complex tori, and let $\mathscr{L}$ an ample line bundle on $A \times B$. Then $(pr_B)_*\mathscr{L}$ is a semihomogeneous vector bundle.*

*Proof.* Since the restriction $\mathscr{L}|_{A_b}$ to every fibre is ample, Mumford's index theorem yields the vanishing of all higher cohomology of $\mathscr{L}_{A_b}$ for all $b \in B$. Therefore Grauert's theorem implies that $(\mathrm{pr}_B)_*\mathscr{L}$ is indeed locally free.

As $\mathscr{L}$ is an ample line bundle, it will carry a PFHS metric (for instance by 2.15), and so satisfy the criteria of [TW98, Theorem 2]. This then implies that $(\mathrm{pr}_B)_*\mathscr{L}$ will also admit a PFHS metric, and therefore will be semihomogeneous by Theorem 2.15. $\square$

2.3. **Positivity of semihomogeneous bundles.** Here we will show that for semihomogeneous vector bundles ampleness is in fact equivalent to supporting a Nakano-positive hermitian metric.

Recall that a hermitian metric $h$ on a holomorphic vector bundle $\mathscr{E}$ is *Nakano-(semi)positive* if a certain hermitian form on $T_X \otimes \mathscr{E}$ canonically associated to the curvature form $\Theta_h$ of $h$ is positive (semi)definite; we refer to Chapter VI of [SS85] or Chapter VII, §6 of [Dem] for the precise definition and further discussion. The properties of Nakano positivity that we will need are given below.

**Theorem 2.17** (Vanishing for Nakano-positive vector bundles)**.** *Let $\mathscr{E}$ be a holomorphic vector bundle on $X$ which admits a Nakano-positive hermitian metric. Then for all $i > 0$ we have that $H^i(\omega_X \otimes \mathscr{E}) = 0$.*

*Proof.* See [Dem, Chapter VII, Corollary 7.5] for instance. $\square$

We now specialize to semihomogeneous bundles. The following result plays a role in the aforementioned proof of Proposition 2.10 given in [Yan89].

**Proposition 2.18.** *Let $\mathscr{E} \cong \mathscr{E}(G, H)$ be a vector bundle of rank $r$ on an abelian variety $A = V/\Lambda$. Then*

(1) *the matrix-valued function*

$$h(\mathbf{v}) \stackrel{def}{=} \exp\left(-\frac{\pi}{r} \cdot H(\mathbf{v}, \mathbf{v})\right) \cdot \mathrm{Id}_r$$

*gives a Hermitian metric on $\mathscr{E}$,*

(2) *the curvature form $\Omega_h$ of the Chern connection associated to $(\mathscr{E}, h)$ is given by*

$$\Omega_h(\mathbf{v}) = \frac{\pi}{r}\partial\overline{\partial}H(\mathbf{v}, \mathbf{v}) \cdot \mathrm{Id}_r \ .$$

*In particular, $c_1(\mathscr{E}) = c_1(\det(\mathscr{E}))$ is represented by the $(1,1)$-form $\mathrm{tr}(\Omega_h(\mathbf{v})) = \pi\partial\overline{\partial}H(\mathbf{v}, \mathbf{v})$.*

*Proof.* Consider

$$J(\lambda, v) = G(\lambda) \cdot \exp\left(\frac{\pi}{r}H(v, \lambda) + \frac{\pi}{2r}H(\lambda, \lambda)\right)$$

where $G : \Lambda \to U(r)$ is a unitary semi-representation for $H$. (1) Consider the pairing on vector-valued $\mathscr{C}^\infty$ global sections of $\mathscr{E}$ given by

(2.4) $$\langle \mathbf{f}, \mathbf{g}\rangle(\mathbf{v}) = \mathbf{f}(\mathbf{v})^\star h(\mathbf{v})\mathbf{g}(\mathbf{v})$$

To show that (1) yields a hermitian metric on $\mathscr{E}$ as claimed, it suffices to show that this pairing is $\Lambda$-periodic. Let $\mathbf{v} \in V$ and $\lambda \in \Lambda$ be given. By (2.3) we have

$$J(\lambda, \mathbf{v})^\star J(\lambda, \mathbf{v}) = \exp\left(\frac{\pi}{r} \cdot (2 \cdot \mathrm{Re}(H(\mathbf{v}, \lambda)) + H(\lambda, \lambda))\right) \cdot \mathrm{Id}_r$$

We then have

$$\langle \mathbf{f}, \mathbf{g}\rangle(\mathbf{v} + \lambda) = \exp\left(-\frac{\pi}{r} \cdot H(\mathbf{v} + \lambda, \mathbf{v} + \lambda)\right) \cdot \mathbf{f}(\mathbf{v} + \lambda)^\star \mathbf{g}(\mathbf{v} + \lambda)$$

$$= \exp\left(-\frac{\pi}{r} \cdot (H(\mathbf{v}, \mathbf{v}) + 2 \cdot \mathrm{Re}(H(\mathbf{v}, \lambda)) + H(\lambda, \lambda))\right) \cdot \mathbf{f}(\mathbf{v})^\star J(\lambda, \mathbf{v})^\star J(\lambda, \mathbf{v})\mathbf{g}(\mathbf{v})$$

$$= \exp\left(-\frac{\pi}{r} \cdot H(\mathbf{v}, \mathbf{v})\right) \cdot \mathbf{f}(\mathbf{v})^\star \mathbf{g}(\mathbf{v}) = \langle \mathbf{f}, \mathbf{g}\rangle(\mathbf{v}).$$

The curvature computation in (2) and the formula for the first Chern class follow immediately from (1) since $\Omega_h(\mathbf{v}) = -\partial\overline{\partial}\log H(\mathbf{v}, \mathbf{v})$ in our case. $\square$

**Proposition 2.19.** *Let $\mathscr{E} = \mathscr{E}(G, H)$ be a simple semihomogeneous vector bundle of rank $r$ on an abelian variety $A$ of dimension $g \geq 1$. Then the following are equivalent:*

(i) *The Hermitian form $H$ is positive definite.*

(ii) *The Hermitian metric $h$ on $\mathscr{E}$ given by (1) is Nakano-positive.*

(iii) *$\mathscr{E}$ is ample.*

The proof of the special case $r = g = 2$ of this result is essentially contained in the proof of [Ume73, Theorem 3.2].

*Proof.* (i)$\Rightarrow$(ii): Since $H$ is positive definite and ampleness coincides with Nakano-positivity in the line bundle case, we have from Proposition 2.18 (ii) that the hermitian metric given by (1) is Nakano-positive as claimed.

(ii)$\Rightarrow$(iii): This follows from Theorem 6.30 in [SS85].

(iii)$\Rightarrow$(i): If $\mathscr{E}$ is ample, then $\det(\mathscr{E})$ is ample, and $c_1(\det(\mathscr{E}))$ is represented by $\pi\partial\overline{\partial}H(-, -)$ according to (ii) of Proposition 2.18. But then [BL04, Proposition 4.5.2] implies that $H$ is positive definite. $\square$

**Corollary 2.20.** *If $\mathscr{E}$ is a semihomogeneous bundle, then $\mathscr{E}$ is ample if and only if $\mathscr{E}$ admits a Nakano-positive hermitian metric.*

*Proof.* It suffices to show that ampleness of our semihomogeneous bundle $\mathscr{E}$ implies the existence of a Nakano-positive hermitian metric. Since the property of having a Nakano-positive hermitian metric is preserved under extension [Ume73, Lemma 2.2] and the case when $\mathscr{E}$ is simple follows from Proposition 2.19, the general case follows from Proposition 2.2. $\square$

In studying direct image bundles we will make essential use of the following result of Berndtsson [Ber09] (see also the main result of [MT08] for a generalization).

**Theorem 2.21** ([Ber09], Theorem 1.2)**.** *Let $f\colon X \to Y$ be a smooth surjective morphism of compact Kähler manifolds, $\mathscr{L}$ an ample line bundle on $X$. Then $f_*(\omega_{X/Y} \otimes \mathscr{L})$ is locally free and admits a Nakano-positive Hermitian metric.*

## 3. Proofs of Theorem A and Corollary B

Before proceeding to the proof of our main result, we make a brief detour to discuss the use of fibrations in ensuring global generation. The statements below are surely known to experts, we include them for the sake of completeness and the lack of a precise reference.

**Lemma 3.1.** *Let $f\colon Y \to Z$ be a morphism of projective varieties, $\mathscr{E}$ a vector bundle on $Y$. Assume furthermore that the adjunction map $f^*f_*\mathscr{E} \to \mathscr{E}$ is surjective.*
   (1) *If $\mathrm{Sym}^m(f_*\mathscr{E})$ is globally generated for some positive integer $m$ then so is $\mathrm{Sym}^m \mathscr{E}$.*
   (2) *If $\omega_Z \otimes f_*(\omega_{Y/Z} \otimes \mathscr{E})$ is globally generated then so is $\omega_Y \otimes \mathscr{E}$.*

*Proof.* For (1), the adjunction morphism is surjective, so the same is true of the induced map $\mathrm{Sym}^2(f^*f_*\mathscr{E}) \to \mathscr{E}^{\otimes 2}$. It follows that the composition

$$f^*(\mathrm{Sym}^2 f_*\mathscr{E}) \simeq \mathrm{Sym}^2(f^*f_*\mathscr{E}) \longrightarrow \mathscr{E}^{\otimes 2}$$

is surjective as well. As $f^*(\mathrm{Sym}^2 f_*\mathscr{E})$ is globally generated by assumption, we have that $\mathscr{E}^{\otimes 2}$ is globally generated as claimed. Part (2) follows by a similar argument using the projection formula

$$f_*(\omega_Y \otimes \mathscr{E}) \cong f_*(\omega_{Y/Z} \otimes f^*\omega_Z \otimes \mathscr{E}) \cong \omega_Z \otimes f_*(\omega_{Y/Z} \otimes \mathscr{E}) \ ,$$

and the surjectivity of the adjunction map. $\square$

*Remark* 3.2 (Variant for continuous global generation)*.* The above statements have natural counterparts in the setting of continuous global generation (cf. [PP03]). Here is an example. Let $Y, Z$ be projective varieties, $Z$ irregular, $f\colon Y \to Z$ be an arbitrary morphism of varieties. Let $\mathscr{F}$ be a coherent sheaf, $\mathscr{L}$ a line bundle on $Y$. Assume that
   (1) $f_*\mathscr{F}$ and $f_*\mathscr{L}$ are continuously globally generated;
   (2) the adjunction morphisms $f^*f_*\mathscr{F} \to \mathscr{F}$ and $f^*f_*\mathscr{L} \to \mathscr{L}$ are surjective.
Then $\mathscr{F} \otimes \mathscr{L}$ is globally generated.
   Indeed, this is a slight variation of [PP03, Proposition 2.12]. Due to *loc. cit.* applied to the identity morphism of $Z$, the sheaf $f_*\mathscr{F} \otimes f_*\mathscr{L}$ is globally generated. This implies that

$$f^*f_*\mathscr{F} \otimes f^*f_*\mathscr{L} \cong f^*(f_*\mathscr{F} \otimes f_*\mathscr{L})$$

is globally generated as well. By the surjectivity of the adjunction morphisms in (2), the tensor product

$$f^*f_*\mathscr{F} \otimes f^*f_*\mathscr{L} \xrightarrow{\otimes} \mathscr{F} \otimes \mathscr{L}$$

is surjective, too. Therefore $\mathscr{F} \otimes \mathscr{L}$ is globally generated as promised.

**Theorem 3.3.** *Let $X$ be a minimal variety of Kodaira dimension zero, and let $\mathscr{L}$ be an ample line bundle on $X$ such that $\mathrm{Bs}\,|\mathscr{L}_a|$ is of codimension $k$ in $X_a$ for all $a \in \mathrm{Alb}(X)$. Then $\mathrm{Bs}\,|\mathscr{L}^{\otimes 2}|$ is of codimension $k$ in $X$.*

*Proof.* Consider the diagram (1.1) from Proposition 1.2

(3.1)

$$
\begin{array}{ccc}
Y \times A' \times A'' & \xrightarrow{\ \ \nu\ \ } & X \\
\downarrow{\scriptstyle \mathrm{pr}_{A' \times A''}} & & \downarrow{\scriptstyle \mathrm{alb}_X} \\
A' \times A'' \ \xrightarrow{\ \mathrm{pr}_{A'}\ } \ A' & \xrightarrow{\ \rho\ } & \mathrm{Alb}(X) \ .
\end{array}
$$

Define $\mathscr{F}_1 \overset{\text{def}}{=} (\mathrm{pr}_{A' \times A''})_* \nu^* \mathscr{L}$. Since $\nu^* \mathscr{L}$ is ample and there are no non-trivial correspondences between $Y$ and $A' \times A''$, $\nu^* \mathscr{L} = \mathscr{M}_1 \boxtimes \mathscr{M}_2$ for ample line bundles $\mathscr{M}_1$ on $Y$ and $\mathscr{M}_2$ on $A' \times A''$. Pushing forward via the projection to $A' \times A''$ yields

$$\mathscr{F}_1 \cong (\mathrm{pr}_{A' \times A''})_*(\mathscr{M}_1 \boxtimes \mathscr{M}_2) \cong ((\mathrm{pr}_{A' \times A''})_*(\mathrm{pr}_Y)^* \mathscr{M}_1) \otimes \mathscr{M}_2 \cong H^0(Y, \mathscr{M}_1) \otimes \mathscr{M}_2 \ ,$$

Since $\mathscr{M}_2$ is ample, $\mathscr{F}_2 \overset{\text{def}}{=} (\mathrm{pr}_{A'})_* \mathscr{F}_1$ is an ample semihomogeneous bundle on $A'$ thanks to Theorem 2.16 and Theorem 2.21. The morphism $\rho \colon A' \to \mathrm{Alb}(X)$ is an isogeny, so $\rho_* \mathscr{F}_2$ is an ample semihomogeneous vector bundle on $\mathrm{Alb}(X)$. By the commutativity of the diagram (3.1),

$$\rho_* \mathscr{F}_2 \cong (\mathrm{alb}_X)_*(\nu_* \nu^* \mathscr{L}) \cong (\mathrm{alb}_X)_*(\mathscr{L} \otimes \nu_* \mathscr{O}_{Y \times A' \times A''}) \ .$$

As $\nu$ is finite and étale, $\nu_* \mathscr{O}_{Y \times A' \times A''} \cong \mathscr{O}_X \oplus \mathscr{G}$ for a vector bundle $\mathscr{G}$ on $X$, therefore

$$\rho_* \mathscr{F}_2 \cong (\mathrm{alb}_X)_* \mathscr{L} \oplus (\mathrm{alb}_X)_*(\mathscr{L} \otimes \mathscr{G}) \ .$$

Being a direct summand of an ample semihomogeneous vector bundle, $(\mathrm{alb}_X)_* \mathscr{L}$ is ample and semihomogeneous itself. Consequently, $\mathrm{Sym}^2(\mathrm{alb}_X)_* \mathscr{L}$ is globally generated thanks to Proposition 2.7.

Our assumption on the base loci of the $\mathscr{L}_a$ implies that the map $(\mathrm{alb}_X)^*(\mathrm{alb}_X)_* \mathscr{L} \to \mathscr{L}$ is surjective away from a subset of codimension $k$, so the same is true of its symmetric square

$$\mathrm{Sym}^2((\mathrm{alb}_X)^*(\mathrm{alb}_X)_* \mathscr{L}) \to \mathscr{L}^{\otimes 2}$$

It follows that $\mathrm{Bs}|\mathscr{L}^{\otimes 2}|$ is of codimension $k$ in $X$ as desired. $\qquad \square$

Finally, we prove our main result.

*Proof of Theorem A:* Upon replacing $\mathscr{L}$ by $\mathscr{L}^{\otimes m}$ for a positive integer $m \geq f_\pi$ we can assume without loss of generality that $\mathscr{L}_a$ is globally generated for all $a \in A$; the adjunction map associated to the fibration $\pi$ is then surjective. Applying Theorem 3.3 to $\mathscr{L}$ and $\pi$, we obtain from Lemma 3.1 that $\mathscr{L}^{\otimes 2m}$ is indeed globally generated. $\qquad \square$

We now turn to Corollary B. If $A$ is an abelian surface, the Hilbert scheme $\mathrm{Hilb}^n(A)$ parametrizing length-$n$ subschemes of $A$ is smooth, projective, and has trivial canonical bundle (see [Bea83] for instance); in particular, it satisfies the hypothesis of Theorem A.

*Proof of Corollary B:* Consider the natural map $\kappa_n \colon \mathrm{Hilb}^n(A) \longrightarrow A$ defined by composing the Hilbert-Chow morphism to the symmetric product $A^{(n)}$ with the addition map from $A^{(n)}$ to $A$. The fiber of $\kappa_n$ over each closed point of $A$ is isomorphic to the generalized Kummer variety $\mathrm{Kum}^{n-1}(A)$, which is a compact hyperkähler manifold (cf. [Bea83]) and thus simply connected. It follows that $\kappa_n$ is the Albanese map of $\mathrm{Hilb}^n(A)$.

By [Rie18, Corollary 4.9] we have that for any ample line bundle $\mathscr{L}$ on $\mathrm{Kum}^{n-1}(A)$, the tensor square $\mathscr{L}^{\otimes 2}$ is globally generated in codimension 1. Corollary B then follows from Theorem 3.3 and the fact that $\mathrm{Kum}^1(A)$ is a K3-surface, so that $f_{\mathrm{Kum}^1(A)} \leq 2$. $\qquad \square$

## 4. Examples and Complements

The material in this section will hopefully illuminate, and point towards extensions of, the effective global generation phenomenon discussed earlier in the paper. First we highlight the importance of our earlier minimality hypothesis on $X$.

*Example* 4.1. Let $A$ be an abelian variety of dimension $n \geq 2$, let $\phi : \widetilde{A} \to A$ be the blow-up of a point $p \in A$ with exceptional divisor $E \cong \mathbb{P}^{g-1}$, and let $L$ be an ample line bundle on $\widetilde{A}$ such that $L|_E \cong \mathscr{O}_E(1)$. To see that such an $L$ exists, first consider that if $\phi' : \widetilde{\mathbb{P}^N} \to \mathbb{P}^N$ is the blow-up at $p' \in \mathbb{P}^N$ with exceptional divisor $E' \cong \mathbb{P}^{N-1}$, and $H'$ is the pullback of the hyperplane class on $\mathbb{P}^N$ via $\phi'$, then there exists $m' > 0$ such that $L' := m'H' - E'$ is ample and $L'|_{E'} \cong \mathscr{O}_{E'}(1)$. It follows that if we take an embedding of $A$ in $\mathbb{P}^N$ which maps $p$ to $p'$, the desired line bundle $L$ on $\widetilde{A}$ can be taken as the restriction of $L'$ to $\widetilde{A}$ in the induced embedding $\widetilde{A} \subset \widetilde{\mathbb{P}^N}$.

Since $K_{\widetilde{A}} = (n-1)E$ and $K_{\widetilde{A}}|_E \cong \mathscr{O}_E(1-n)$, we have for all $m \geq 1$ that

$$(4.1) \qquad (K_{\widetilde{A}} + mL)|_E \cong \mathscr{O}_E(m+1-n)$$

hence for $K_X + mL$ to be globally generated one must have $m \geq n-1$. In conclusion, $f_{\widetilde{A}} \geq n-1$ even though $\widetilde{A}$ has maximal Albanese dimension.

*Example* 4.2. There exists for each $g \geq 1$ an abelian variety $A$ of dimension $g$ such that $f_A = 1$. By [Gar06] or [DHS94], there exists a primitive polarization type $\mathbf{d}$ such that for a very general member $(A, L)$ of the moduli space $\mathscr{A}_g(\mathbf{d})$, any line bundle on $A$ algebraically equivalent to $L$ is globally generated. We can also take $(A, L)$ to be sufficiently general so that $\mathrm{NS}(A)$ is generated by $c_1(L)$. For all such $A$, we have that $f_A = 1$ as desired.

*Example* 4.3. There exists for each odd integer $n \geq 3$ a Calabi-Yau manifold $X$ of dimension $n$ with $f_X \geq n+1$. One such $X$ is a general smooth hypersurface of degree $2n+2$ in the weighted projective space $\mathbb{P}(1^n, 2, n+2)$; see Example 3.2 in [Kaw00] for details.

*Example* 4.4. If $X \subseteq \mathbb{P}^N$ is a smooth hypersurface of degree $d \geq 2$ when $N \geq 4$, or a very general hypersurface of degree $d \geq 4$ when $N = 3$, then $\mathrm{Pic}(X)$ is generated by $\mathscr{O}_X(1)$ and $\omega_X \cong \mathscr{O}_X(d-N-1)$; therefore $f_X = \max\{1, N+1-d\}$. In particular, $f_X = 1$ for the Calabi-Yau case $d = N+1$.

We now estimate $f_X$ in some low-dimensional cases.

*Example* 4.5. If $\dim X = 1$, then $\omega_X \otimes \mathscr{A}^{\otimes 2}$ is globally generated for every ample line bundle $\mathscr{A}$, so that $f_X \leq 2$. At the same time, if $P \in X$ is a point and $\mathscr{A} = \mathscr{O}_X(P)$ the divisor $\omega_X \otimes \mathscr{A}$ is never globally generated, hence $f_X = 2$.

*Example* 4.6. When $\dim X = 2$, Reider's theorem implies $f_X \leq 3$. However, if we assume further that $X$ is a minimal surface of Kodaira dimension 0, Reider's theorem implies $f_X \leq 2$.

*Example* 4.7. If $X$ is a surface which admits a smooth fibration $f\colon X \to C$ over a curve $C$, the preceding examples imply that $f_X \leq 3 < f_\pi \cdot f_C$.

Based on these examples and Theorem A, we offer the following conjecture.

**Conjecture 4.8** (Fujita numbers in fibrations). *Let $X, Y$ be smooth projective varieties and $\pi\colon X \to Y$ a smooth fibration. Then $f_X \leq f_\pi \cdot f_Y$.*

If this holds, then $\mathscr{L}^{\otimes 2m}$ can be replaced by $\mathscr{L}^{\otimes f_{\mathrm{Alb}(X)} \cdot m}$ in the statement of Theorem A. Comparing Remark 2.9 with the proof of Theorem 3.3, it seems unlikely that the case of Conjecture 4.8 where $K_X =_{\mathrm{num}} 0$, $\pi = \mathrm{alb}_X$ and $f_{\mathrm{Alb}(X)} = 1$ can be addressed with the methods of this paper alone.

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