Transverse structures and modular Hecke algebras

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This is a report on joint work with Alain Connes that concerns the adaptation of some ideas and concepts from our approach to the transverse geometry of foliations to the study of certain noncommutative 'arithmetic spaces', described in terms of their 'holomorphic coordinates' by what we called *modular Hecke algebras*. These are algebras, associated to congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$, that arise from the fusion of the two quintessential structures on modular forms, namely the algebra structure given by the pointwise product on the one hand, and the action of the Hecke operators on the other. The Hopf algebra $\mathcal{H}_1$ that determines the 'affine' transverse geometry of codimension 1 foliations admits canonical Hopf actions on the modular Hecke algebras, that span the 'holomorphic tangent space' of the underlying noncommutative spaces. Furthermore, each of the three basic cocycles in the Hopf cyclic cohomology of $\mathcal{H}_1$, the Schwarzian 1-cocycle, the Godbillon-Vey 1-cocycle, and the transverse fundamental 2-cocycle, defines via such an action a specific geometric structure. The Schwarzian cocycle acts by an inner derivation, implemented by Eisenstein series of weight 4 that plays the role of the quadratic differential for a 1-dimensional projective structure. The image of the Hopf cyclic cocycle representing the 'transverse' fundamental class gives natural extensions of the first Rankin-Cohen bracket to the modular Hecke algebras, that confer a noncommutative 'Poisson structure' to the underlying 'spaces'. The higher Rankin-Cohen brackets also admit canonical extensions, that provide 'deformation quantizations' reflecting the integrality of the transverse fundamental class.
For lack of time, the role of the Hopf cyclic version of the Godbillon-Vey cocycle and of its transgressed avatars arising in this context has not been discussed.

1 Geometric background

1.1 Diff-invariant setting (cf. [1, Part I], [7])

In order to specify the geometry of the 'space' of leaves for a general foliation, the first problem one is confronted with is that of finding a geometric structure that is *invariant* under all diffeomorphisms of a given manifold M. Indeed, the action of the holonomy on a complete transversal M to a foliation is as wild (in general) as that of an arbitrary countable subgroup Γ of $\mathrm{Diff}(M)$, and the invariance under holonomy is an unavoidable constraint when passing to the space of leaves. While the standard geometric notions are of course equivariant with respect to $\mathrm{Diff}(M)$, they are not invariant.

By contrast, the operator theoretic spectral framework of noncommutative geometry only requires the invariance to hold at the principal symbol level (in classical pseudodifferential terms) of the operator D. When D is an elliptic operator the gain is non-existent, since in that case the principal symbol specifies the metric and the invariance condition would force Γ to be a discrete subgroup of a Lie group (of isometries). Fortunately, the theory applies with very little change when D is only *hypoelliptic*, and this allows to treat 'para-Riemannian' spaces, which admit groups of isometries as large as diffeomorphism groups.

This was the fundamental idea employed in [1] for the treatment of transverse geometry. One first replaces a given manifold M^n (with no extra structure except an orientation) by the total space of the bundle $PM = F^+M/SO(n)$, where F^+M is the $\mathrm{GL}^+(n,\mathbb{R})$–principal bundle of oriented frames on M^n. The sections of $\pi : PM \to M$ are precisely the Riemannian metrics on M but unlike the space of such metrics the space P is still a finite dimensional manifold. The total space PM itself admits a canonical, and thus $\mathrm{Diff}^+(M)$-invariant, 'para-Riemannian' structure, which can be described as follows. The vertical subbundle $\mathcal{V} \subset T(PM)$, $\mathcal{V} = \mathrm{Ker}\, \pi_*$, carries natural Euclidean structures on each of its fibers, determined solely by fixing once and for all a choice of a $GL^+(n,\mathbb{R})$-invariant Riemannian metric on the symmetric space $GL^+(n,\mathbb{R})/SO(n)$. On the other hand, the quotient bundle $\mathcal{N} = T(PM)/\mathcal{V}$ comes equipped with a tautologically defined Riemannian structure: every point $p \in PM$ is an Euclidean structure on $T_{\pi(p)}(M)$ which is identified to $\mathcal{N}_p$ via π_*.

The resulting structure on PM is *invariant* under the canonical lift of the action of $\mathrm{Diff}^+(M)$, because no non-canonical choices were involved so far. In particular any hypoelliptic operator whose principal symbol only depends upon the above 'para-Riemannian' structure will have the required invariance property.

1.2 Hypoelliptic signature operator (cf. [1, Part I])

The *hypoelliptic signature operator* operator D on PM is uniquely determined by the equation $Q = D|D|$, where Q is the operator

$$Q = (d_V^* \, d_V - d_V \, d_V^*) \oplus \gamma_V \, (d_H + d_H^*) \,, \tag{1.1}$$

acting on the Hilbert space of L^2-sections

$$\mathfrak{H}_{PM} = L^2(\wedge^{\cdot}\mathcal{V}^* \otimes \wedge^{\cdot}\mathcal{N}^*, \, \varpi_{PM}) \,;$$

here d_V denotes the vertical exterior derivative, γ_V is the usual grading for the vertical signature operator, d_H stands for the horizontal exterior differentiation with respect to a fixed connection on the frame bundle, and ϖ_{PM} is the $\mathrm{Diff}^+(M)$-invariant volume form on PM associated to the connection. (When $n \equiv 1 \,\mathrm{or}\, 2 \,(\mathrm{mod}\, 4)$, for the vertical component to make sense, one has to replace PM with $PM \times S^1$ so that the dimension of the vertical fiber be even.)

While the construction of D does involve a non-canonical choice – that of a connection – this choice does not affect its principal symbol in the appropriate pseudo-differential calculus, and thus does not break the intrinsic $\mathrm{Diff}^+(M)$-invariance of the spectral triple $(\mathcal{A}_\Gamma, \mathfrak{H}_{PM}, D)$ associated to an arbitrary (countable) discrete subgroup Γ of $\mathrm{Diff}^+(M)$. Here $\mathcal{A}_\Gamma$ is the convolution algebra

$$\mathcal{A}_\Gamma \,=\, C_c^\infty(PM) \rtimes \Gamma\,, \quad fU_\varphi \cdot gU_\psi = f\,(g \circ \varphi^{-1})\, U_{\varphi\psi}\,,$$

and the resulting spectral triple represents the desired geometric structure for the 'quotient space' M/Γ.

1.3 Noncommutative local index formula (cf. [1, Part II])

Indeed, using calculus on Heisenberg manifolds, which in particular provides a noncommutative residue functional $⨍$ that extends the Dixmier trace, we proved [1, Part I] that such a spectral triple $(\mathcal{A}_\Gamma, \mathfrak{H}_{PM}, D)$ fulfills the hypotheses of the operator theoretic local index theorem of [1, Part II]. Therefore, its character-index $ch_*(D) \in HC_*(\mathcal{A}_\Gamma)$ can be expressed in terms of residues of spectrally defined zeta-functions, and is given by a cocycle $\{\phi_q\}$ in the (b, B) bi-complex of $\mathcal{A}_\Gamma$ whose components are of the following form

$$\phi_q(a^0, \ldots, a^q) = \sum_{\mathbf{k}} c_{q,\mathbf{k}} ⨍ a^0[Q, a^1]^{(k_1)} \ldots [Q, a^q]^{(k_q)}\, |Q|^{-q-2|k|}\,; \qquad (1.2)$$

we have used here the abbreviations $T^{(k)} = \nabla^k(T)$ and $\nabla(T) = D^2T - TD^2$,

$$\mathbf{k} \,=\, (k_1, \ldots, k_q)\,, \qquad |\mathbf{k}| = k_1 + \ldots + k_q\,, \quad \text{and}$$

$$c_{q,\mathbf{k}} \quad = \quad \frac{(-1)^{|\mathbf{k}|}\sqrt{2i}}{k_1! \ldots k_q!\,(k_1+1)\ldots(k_1+\ldots+k_q+q)}\,\Gamma\left(|k| + \frac{q}{2}\right).$$

Note that the operator under the $⨍$ sign in the generic term of the sum is of order $-(q + |k|)$, therefore the corresponding term vanishes whenever $q + |k|$ exceeds the 'metric dimension', i.e. summability degree $= \dfrac{n(n+1)}{2} + 2n$.

While the expression (1.2) is completely computable by symbolic calculus, its actual calculation is highly impractical, due to the very large number of terms that need to be evaluated. In any event, even if the computer can deliver the answer in symbolic terms, its understanding would still require an appropriate cohomological interpretation.

Although the curved case can also be handled (see [4]), there is no loss of generality in assuming that M is a flat affine manifold, as long as the affine structure is not required to be preserved by Γ. Secondly, one can afford to work at the level of the principal bundle F^+M, since the descent to the quotient bundle PM only involves the simple operation of taking $SO(n)$-invariants.

The strategy that led to the unwinding of the formula (1.2) started with the observation that the built-in affine invariance of the operator Q, allows to reduce the noncommutative residue functional involved in the cochains

$$\phi_{q,\mathbf{k}}(a^0,\ldots,a^q) = ⨍ a^0[Q,a^1]^{(k_1)}\ldots[Q,a^q]^{(k_q)}\,|Q|^{-(q+2|\mathbf{k}|)}\,, \tag{1.3}$$

to genuine integration, and thus replace them by sums of cochains of the form

$$\psi(a^0,\ldots,a^q) = \tau_\Gamma(a^0\,h^1(a^1)\ldots h^q(a^q))\,. \tag{1.4}$$

Here $h^1,\ldots,h^q$ are 'transverse' differential operators acting on the algebra $\mathcal{A}_\Gamma$, and τ_Γ is the canonical trace on $\mathcal{A}_\Gamma$,

$$\tau_\Gamma\,(f\,U_\varphi^*) \;=\; \begin{cases} \int_{F\mathbb{R}^n} f\,\varpi_{F\mathbb{R}^n}\,, & \text{if} \quad \varphi = Id\,, \\ \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{1.5}$$

1.4 Hopf algebraic symmetries (cf. [2, 4])

Under closer scrutiny, the above transverse differential operators turn out to arise from the action of a canonical Hopf algebra $\mathcal{H}_n$, depending only on the codimension n (cf. [2], see also [7, §2]).

Indeed, restricting for simplicity to the case $n = 1$, the operator Q, cf. (1.1), is built from the vector fields

$$Y \;=\; y\,\frac{\partial}{\partial y} \qquad \text{and} \qquad X \;=\; y\,\frac{\partial}{\partial x}\,,$$

where we used the obvious coordinates $(x,y) \in F^+(M^1) \simeq M^1 \times \mathbb{R}^+$. Therefore, the expression under the residue-integral in (1.2) involves iterated commutators of these vector fields with multiplication operators of the form

$$a = f\,U_\varphi^*\,, \quad f \in C_c^\infty(F^+M), \quad \varphi \in \Gamma\,,$$

where the diffeomorphism φ acts on $F^+(M^1)$ by

$$\varphi(x,y) \;=\; (\varphi(x),\,\varphi'(x)\cdot y)\,, \quad (x,y) \in F^+(M^1)\,.$$

The two basic vector fields become linear operators on $\mathcal{A}_\Gamma$, acting as

$$Y(f\,U_\varphi) = Y(f)\,U_\varphi\,, \qquad X(f\,U_\varphi) = X(f)\,U_\varphi\,.$$

However, while Y acts as derivation

$$Y(ab) = Y(a)\,b + a\,Y(b)\,, \qquad a,b \in \mathcal{A}_\Gamma\,.$$

X satisfies instead

$$X(ab) = X(a)\,b \,+\, a\,X(b) + \delta_1(a)\,Y(b)\,.$$

where

$$\delta_1(f\,U_{\varphi^{-1}}) \;=\; y\,\frac{d}{dx}\left(\log\frac{d\varphi}{dx}\right)\,fU_{\varphi^{-1}}\,.$$

The operator δ_1 is itself a derivation,

$$\delta_1(ab) = \delta_1(a)\, b + a\, \delta_1(b)\,,$$

but the successive commutators with X produces new operators

$$\delta_n(f\, U_{\varphi^{-1}}) \;=\; y^n\, \frac{d^n}{dx^n}\left(\log \frac{d\varphi}{dx}\right) fU_{\varphi^{-1}}\,, \qquad \forall\, n \geq 1\,,$$

that satisfy progressively more complicated Leibniz rules.

We can encode all this information in a 'universal' (for codimension 1), Hopf algebra $\mathcal{H}_1$, defined as follows. As an algebra it coincides with the universal enveloping algebra of the Lie algebra with basis $\{X, Y, \delta_n\,; n \geq 1\}$ and brackets

$$[Y, X] = X\,,\ [Y, \delta_n] = n\, \delta_n\,,\ [X, \delta_n] = \delta_{n+1}\,,\ [\delta_k, \delta_\ell] = 0\,, \quad n, k, \ell \geq 1\,.$$

As a Hopf algebra, the coproduct $\Delta : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_1$ is determined by

$$\Delta\, Y = Y \otimes 1 + 1 \otimes Y\,, \quad \begin{aligned} \Delta\, X &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y \\ \Delta\, \delta_1 &= \delta_1 \otimes 1 + 1 \otimes \delta_1 \end{aligned}$$

and the multiplicativity property

$$\Delta(h^1\, h^2) = \Delta h^1 \cdot \Delta h^2\,, \quad h^1, h^2 \in \mathcal{H}_1\ ;$$

the antipode is determined by

$$S(Y) = -Y\,,\ S(X) = -X + \delta_1 Y\,,\ S(\delta_1) = -\delta_1$$

and the anti-isomorphism property

$$S(h^1\, h^2) = S(h^2)\, S(h^1)\,, \quad h^1, h^2 \in \mathcal{H}_1\ ;$$

finally, the counit is

$$\varepsilon(h) = \text{constant term of} \quad h \in \mathcal{H}_1\,.$$

The role of $\mathcal{H}_1$ as 'transverse symmetry' comes from its natural Hopf actions on the crossed products algebras $\mathcal{A}_\Gamma = C_c^\infty(F^+(M^1)) \rtimes \Gamma$. We recall that the action of $\mathcal{H}_1$ is given on generators as follows:

$$Y(fU_\varphi) \;=\; y\, \frac{\partial f}{\partial y}\, U_\varphi\,, \qquad X(fU_\varphi) = y\, \frac{\partial f}{\partial x}\, U_\varphi\,, \tag{1.6}$$

$$\delta_n(fU_{\varphi^{-1}}) \;=\; y^n\, \frac{d^n}{dx^n}\left(\log \frac{d\varphi}{dx}\right) fU_{\varphi^{-1}}\,, \qquad \forall\, n \geq 1\,. \tag{1.7}$$

Furthermore, the trace τ_Γ has the invariance property:

$$\tau_\Gamma(h(a)) = \delta(h)\, \tau_\Gamma(a)\,, \qquad \forall\, h \in \mathcal{H}_1\,, a \in \mathcal{A}.$$

where $\delta \in \mathcal{H}_1^*$ is the *modular* character, defined by

$$\delta(Y) = 1, \quad \delta(X) = 0, \quad \delta(\delta_n) = 0 .$$

It should also be noted that while $S^2 \neq \mathrm{Id}$, the 'twisted' antipode

$$\widetilde{S}(h) = \delta(h_{(1)})\, S(h_{(2)}) , \qquad h \in \mathcal{H}_1 , \tag{1.8}$$

does satisfy the involution property $\widetilde{S}^2 = \mathrm{Id}$.

Finally, the cochains (1.4) can be recognized to belong to the range of a certain cohomological characteristic map. In fact, requiring the assignment

$$\begin{array}{rcl} \chi_\Gamma(h^1 \otimes \ldots \otimes h^n)(a^0, \ldots, a^n) & = & \tau_\Gamma(a^0\, h^1(a^1) \ldots h^n(a^n)) , \\ h^1, \ldots, h^n \ \in \mathcal{H}_1 & , & a^1, \ldots, a^n \ \in \mathcal{A}_\Gamma , \end{array} \tag{1.9}$$

to induce a characteristic homomorphism

$$\chi_\Gamma^* \, : \, HC^*_{\mathrm{Hopf}}\,(\mathcal{H}_1) \, \to \, HC^*(\mathcal{A}_\Gamma) , \tag{1.10}$$

practically dictates the definition of the *Hopf cyclic cohomology.*

2 Cohomological background

2.1 Hopf cyclic cohomology cf. [2, 3])

Quite generally, for an arbitrary Hopf algebra $\mathcal{H}$ over a field k containing $\mathbb{Q}$, one is led to postulate the existence of a *modular pair* (δ, σ), consisting of a character $\delta \in \mathcal{H}^*$, $\delta(ab) = \delta(a)\delta(b)$, $\forall\, a, b \in \mathcal{H}$, $\delta(1) = 1$, and a group-like element $\sigma \in \mathcal{H}$, $\Delta(\sigma) = \sigma \otimes \sigma$, $\varepsilon(\sigma) = 1$, related by the condition $\delta(\sigma) = 1$. In addition, one requires that the corresponding twisted antipode $\widetilde{S}$, defined as in (1.8), satisfy the same involution property $\widetilde{S}^2 = \mathrm{Id}$. Then (cf. [3]) the following operators define a cyclic structure on

$$\mathcal{H}^\natural_{(\delta,\sigma)} \; = \; \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes^n} \; : \tag{2.1}$$

$$\begin{array}{rcll} \delta_0(h^1 \otimes \ldots \otimes h^{n-1}) & = & 1 \otimes h^1 \otimes \ldots \otimes h^{n-1}, & \\ \delta_j(h^1 \otimes \ldots \otimes h^{n-1}) & = & h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^{n-1}, & 1 \leq j \leq n-1 \\ \delta_n(h^1 \otimes \ldots \otimes h^{n-1}) & = & h^1 \otimes \ldots \otimes h^{n-1} \otimes \sigma, & \\ \sigma_i(h^1 \otimes \ldots \otimes h^{n+1}) & = & h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{n+1}, & 0 \leq i \leq n , \\ \tau_n(h^1 \otimes \ldots \otimes h^n) & = & (\Delta^{p-1}\widetilde{S}(h^1)) \cdot h^2 \otimes \ldots \otimes h^n \otimes \sigma. & \end{array}$$

The normalized bi-complex

$$(CC^{*,*}(\mathcal{H}; \delta, \sigma),\, b,\, B)$$

that computes the Hopf cyclic cohomology of $\mathcal{H}$ with respect to a modular pair in involution (δ, σ) looks as follows:

$$CC^{p,q}(\mathcal{H}; \delta, \sigma) \; = \; \begin{cases} \bar{C}^{q-p}(\mathcal{H}; \delta, \sigma) , & q \geq p , \\ 0 , & q < p , \end{cases}$$

where

$$\bar{C}^n(\mathcal{H};\delta,\sigma) \;=\; \begin{cases} \bigcap_{i=0}^{n-1} \mathrm{Ker}\ \sigma_i\,, & n\geq 1\,,\\ \\ \mathbb{C}\,, & n=0\,; \end{cases}$$

the operator

$$b:\bar{C}^{n-1}(\mathcal{H};\delta,\sigma)\to\bar{C}^{n}(\mathcal{H};\delta,\sigma), \qquad b=\sum_{i=0}^{n}(-1)^i\delta_i$$

has the form $\quad b(\mathbb{C}) \;=\; 0 \;$ for $\; n=0\,,$

$$\begin{aligned} b\,(h^1\otimes\ldots\otimes h^{n-1}) \;&=\; 1\otimes h^1\otimes\ldots\otimes h^{n-1}\\ &+\; \sum_{j=1}^{n-1}(-1)^j\sum_{(h_j)} h^1\otimes\ldots\otimes h^j_{(1)}\otimes h^j_{(2)}\otimes\ldots\otimes h^{n-1}\\ &+\; (-1)^n h^1\otimes\ldots\otimes h^{n-1}\otimes 1\,, \end{aligned}$$

while the B-operator $B:\bar{C}^{n+1}(\mathcal{H};\delta,\sigma)\to\bar{C}^{n}(\mathcal{H};\delta,\sigma)$ is defined by the formula

$$B = A\circ B_0\,,\quad n\geq 0\,,$$

where

$$B_0(h^1\otimes\ldots\otimes h^{n+1}) \;=\; \begin{cases} (\Delta^{n-1}\widetilde{S}(h^1))\cdot h^2\otimes\ldots\otimes h^{n+1}\,, & n\geq 1\,,\\ \delta(h^1)\,, & \text{for}\quad n=0 \end{cases}$$

and

$$A = 1+\lambda_n+\cdots+\lambda_n^n\,, \qquad \text{with} \qquad \lambda_n=(-1)^n\tau_n\,.$$

The groups $\{HC^n_{\mathrm{Hopf}}(\mathcal{H};\delta,\sigma)\}_{n\in\mathbb{N}}$ are computed from the first quadrant total complex $(TC^*(\mathcal{H};\delta,\sigma),\, b+B)$,

$$TC^n(\mathcal{H};\delta,\sigma) \;=\; \sum_{k=0}^{n} CC^{k,n-k}(\mathcal{H};\delta,\sigma)\,,$$

and the periodic groups $\{HP^i_{\mathrm{Hopf}}(\mathcal{H};\delta,\sigma)\}_{i\in\mathbb{Z}/2}$ are computed from the full total complex $(TP^*(\mathcal{H};\delta,\sigma),\, b+B)$,

$$TP^i(\mathcal{H};\delta,\sigma) \;=\; \sum_{k\in\mathbb{Z}} CC^{k,i-k}(\mathcal{H};\delta,\sigma)\,.$$

2.2 Godbillon-Vey cocycle

In the case of $\mathcal{H}_1$, it is easily seen that $\delta_1\in\mathcal{H}_1$ is a Hopf cyclic 1-cocycle. Indeed, since δ_1 is primitive,

$$b(\delta_1) \;=\; 1\otimes\delta_1-\Delta\delta_1+\delta_1\otimes 1 \;=\; 0\,;$$

on the other hand,

$$\tau_1(\delta_1) \;=\; \widetilde{S}(\delta_1) \;=\; S(\delta_1) \;=\; -\delta_1\,.$$

The class $[\delta_1]\in HC^1_{\mathrm{Hopf}}(\mathcal{H}_1)$ is non-trivial and gives the Godbillon-Vey class for foliations (see §2.5).

2.3 Schwarzian cocycle

The element Y is also primitive, but $\widetilde{S}(Y) = -Y + 1 \neq -Y$. There is however another Hopf cyclic 1-cocycle, given by the only other primitive element

$$\delta_2' := \delta_2 - \frac{1}{2}\delta_1^2 \in \mathcal{H}_1 \,,$$

which in view of (1.7) acts on crossed product algebras $\mathcal{A}_\Gamma$ as a Schwarzian:

$$\delta_2'(fU_{\varphi^{-1}}) = y^2 \left[\frac{d^2}{dx^2}\left(\log \frac{d\varphi}{dx}\right) - \frac{1}{2}\left(\frac{d}{dx}\left(\log \frac{d\varphi}{dx}\right)\right)^2 \right] fU_{\varphi^{-1}} \,.$$

The class $[\delta_2'] \in HC^1_{\rm Hopf}(\mathcal{H}_1)$ becomes trivial in $HP^1_{\rm Hopf}(\mathcal{H}_1)$ since, for example, $\delta_2' = B(c)$, with $c = \delta_1 \otimes X + \frac{1}{2}\,\delta_1^2 \otimes Y$ and $b(c) = 0$.

2.4 Transverse fundamental cocycle

The class of the unit constant $[1] \in HC^0_{\rm Hopf}(\mathcal{H}_1)$ is trivial in the periodic theory, since $B(Y) = 1$.
A nontrivial element of $HP^{\rm ev}_{\rm Hopf}(\mathcal{H}_1)$ is the class of the cyclic 2-cocycle

$$\Pi = X \otimes Y - Y \otimes X - \delta_1\, Y \otimes Y \,. \tag{2.2}$$

When transported via the characteristic map (1.9), it gives an *integral* (cf. §2.5) extension to $\mathcal{A}_\Gamma$ of the fundamental class in the cyclic cohomology.

2.5 Isomorphism with Gelfand-Fuks cohomology

In [2] we have constructed an explicit isomorphism

$$\kappa_n^* \,:\, H^*(\mathfrak{a}_n, \mathbb{C}) \xrightarrow{\simeq} HP^*_{\rm Hopf}(\mathcal{H}_n)\,, \qquad * = ev \text{ or } odd$$

with the Gelfand-Fuks cohomology of the Lie algebra $\mathfrak{a}_n$ of formal vector fields on $\mathbb{R}^n$, that essentially provides the sought-for cohomological understanding of the index-class of the hypoelliptic signature operator D. In the particular case $n = 1$, $[\delta_1]$ corresponds via this isomorphism to the Godbillon-Vey class, hence is a generator for $HP^{\rm odd}_{\rm Hopf}(\mathcal{H}_1)$. The class $[\Pi]$ is a generator for $HP^{\rm ev}_{\rm Hopf}(\mathcal{H}_1)$ and gives the Connes-Chern character of the spectral triple $(\mathcal{A}_\Gamma, \mathfrak{H}_{PM}, D)$.

3 Modular Hecke algebras

3.1 Definition (cf. [5])

We abbreviate $G^+(\mathbb{Q}) = {\rm GL}^+(2,\mathbb{Q})$ and denote by $\mathcal{M}$ the algebra of (holomorphic) modular forms of all levels.
Given a congruence subgroup Γ of $\Gamma(1) = {\rm SL}(2,\mathbb{Z})$, let $\mathcal{A}(\Gamma)$ consist of all finitely supported maps

$$F : \Gamma\backslash G^+(\mathbb{Q}) \to \mathcal{M}\,, \qquad \Gamma\alpha \mapsto F_\alpha$$

such that

$$F_{\alpha\gamma} \,=\, F_\alpha|\gamma\,, \quad \forall\, \alpha \in G^+(\mathbb{Q})\,, \quad \forall\, \gamma \in \Gamma\,. \tag{3.1}$$

The twisted convolution product

$$(F^1 * F^2)_\alpha \,:=\, \sum_{\Gamma\beta\in\Gamma\backslash G^+(\mathbb{Q})} F^1_\beta \cdot F^2_{\alpha\beta^{-1}}|\beta\,.$$

is associative, and the algebra thus obtained is called the *modular Hecke algebra of level* Γ.

Any finitely supported map from $\Gamma\backslash G^+(\mathbb{Q})/\Gamma$ to the algebra $\mathcal{M}(\Gamma)$ of modular forms of level Γ trivially fulfills the equation (3.1), but such maps do not exhaust all its solutions. In fact, given $f \in \mathcal{M}$ there exists an $F \in \mathcal{A}(\Gamma)$ such that $F_\alpha = f$ if and only if $f|\gamma = f\,, \quad \forall\, \gamma \in \Gamma \cap \alpha^{-1}\Gamma\,\alpha$.

The inductive limit $\varinjlim_N \mathcal{A}(\Gamma(N))$, ordered by divisibility and with respect to imbeddings normalized by the index of the subgroup, can be interpreted as the 'adelic' crossed product

$$\mathcal{A}(G(\mathbb{A}_f)^0) \,=\, \mathcal{M} \rtimes \mathrm{GL}(2, \mathbb{A}_f)^0,$$

where $\mathrm{GL}(2, \mathbb{A}_f)^0$ is the subgroup of $\mathrm{GL}(2, \mathbb{A}_f)$ whose elements have determinant in $\mathbb{Q}^* \times \mathbb{R}^* \subset \mathrm{GL}(1, \mathbb{A})$.

The 'rational' version is the (discrete) crossed product

$$\mathcal{A}(G^+(\mathbb{Q})) \,=\, \mathcal{M} \rtimes G^+(\mathbb{Q})\,.$$

whose elements are finite sums of symbols of the form $\sum f\, U^*_\gamma$, with $f \in \mathcal{M}$, $\gamma \in G^+(\mathbb{Q})$, and with the product given by the rule

$$f\, U^*_\alpha \cdot g\, U^*_\beta = \,(f \cdot g|\alpha)\, U^*_{\beta\,\alpha}\,,$$

3.2 Hopf actions of $\mathcal{H}_1$ on modular Hecke algebras (cf. [5])

The Hopf algebra $\mathcal{H}_1$ admits canonical Hopf actions on the algebras $\mathcal{A}(G^+(\mathbb{Q}))$ and $\mathcal{A}(G(\mathbb{A}_f)^0)$. Y acts as a grading operator

$$Y(f) \,=\, \frac{k}{2} \cdot f\,, \quad \forall\, f \in \mathcal{M}_k\,.$$

while X can be let to act as the classical derivation operator that corrects the ordinary derivative on modular forms,

$$X \,=\, \frac{1}{2\pi i}\frac{d}{dz} - \frac{1}{12\pi i}\,\frac{d}{dz}(\log \Delta)\, Y = \frac{1}{2\pi i}\,\frac{d}{dz} - \frac{1}{2\pi i}\,\frac{d}{dz}(\log \eta^4)\, Y\,.$$

Theorem 1 *There is a unique action of* $\mathcal{H}_1$ *on* $\mathcal{A}(G(\mathbb{A}_f)^0)$ *defined on the generators of* $\mathcal{H}_1$ *by*

$$Y(F)_\alpha \,=\, Y(F_\alpha)\,, \quad X(F)_\alpha \,=\, X(F_\alpha)\,, \quad \delta_n(F)_\alpha \,=\, \mu_{n,\alpha}\, F_\alpha\,,$$

where

$$\mu_{n,\alpha} = X^{n-1}(\mu_\alpha), \quad \forall\, n \geq 1\,, \qquad \mu_\alpha\,(z) = \frac{1}{2\pi i}\,\frac{d}{dz}\log\frac{\eta^4|\alpha}{\eta^4}\,.$$

• *Transported to* $\mathcal{A}(G^+(\mathbb{Q}))$*, this action commutes with the natural coaction of* $G^+(\mathbb{Q})$ *on* $\mathcal{A}(G^+(\mathbb{Q}))$*. All its conjugates by* 1*-cocycles of* $\mathcal{H}_1$ *with values in* $\mathcal{A}(G^+(\mathbb{Q}))$ *that preserve this property are of the form*

$$\begin{aligned} \tilde{Y} = Y\,, \quad \tilde{X}(a) &= X(a) + [(\xi - t\mu), a] - t\,\delta_1(a) + \mu Y(a)\,, \\ \tilde{\delta_1}(a) &= \delta_1(a) + [\mu, a]\,, \qquad a \in \mathcal{A}(G^+(\mathbb{Q}))\,. \end{aligned}$$

where the parameters

$$\xi \in \mathcal{M}_2, \quad t \in \mathbb{C} \quad \text{and} \quad \mu \in \mathcal{M}_2$$

can take arbitrary values.

• *The Schwarzian always acts on* $\mathcal{A}(G(\mathbb{A}_f)^0)$ *by inner derivations,*

$$\tilde{\delta_2'}(F) = [X(\mu) + \frac{\mu^2}{2} + \frac{E_4}{72},\ F]\,, \quad E_4(q) = 1 + 240\sum_1^\infty n^3 \frac{q^n}{1-q^n}\,,$$

and there is no choice for the above parameters such that $\tilde{\delta_2'} \equiv 0$*.*

• *The tranverse fundamental cocycle* Π *acts as a Hochschild* 2*-cocycle*

$$\begin{aligned} [\Pi] &\in H^2(\mathcal{A}(G(\mathbb{A}_f)^0), \mathcal{A}(G(\mathbb{A}_f)^0))\,, \\ \tilde{\Pi}(F_1, F_2) &= \tilde{X}(F_1)Y(F_2) - Y(F_1)\tilde{X}(F_2) - \delta_1(Y(F_1))Y(F_2)\,, \quad F_1, F_2 \in \mathcal{A}(\Gamma) \end{aligned}$$

that defines a noncommutative Poisson structure, i.e. the Gerstenhaber bracket $[\Pi, \Pi]$ *is a Hochschild boundary.*

3.3 Rankin-Cohen deformations (cf. [6])

The above formulae for $\tilde{X}$ and $\tilde{\delta_2'}$ are strikingly similar to those occuring in Zagier's definition [8] of 'canonical' Rankin-Cohen algebras, suggesting a close relationship between such brackets and the Hopf actions of $\mathcal{H}_1$.

Assume that $\mathcal{H}_1$ acts on an algebra $\mathcal{A}$ such that the derivation δ_2' is inner and implemented by an element $\Omega \in \mathcal{A}$ satisfying

$$\delta_2'(a) = \Omega\, a - a\,\Omega\,, \qquad \delta_k(\Omega) = 0\,, \qquad \forall\, k \in \mathbb{N}, \qquad \forall\, a \in \mathcal{A}\,.$$

Such a datum, called an 1-*dimensional projective structure on* $\mathcal{A}$ *with quadratic differential* Ω, allows to construct canonical bilinear operators RC_n, $n \geq 1$, that generalize the classical Rankin-Cohen brackets. Their construction, briefly sketched below, is inspired by Zagier's approach in [8].

Given two operators Z and Θ acting linearly on $\mathcal{A}$ and fulfilling

$$[Y, Z] = Z\,, \quad [Y, \Theta] = 2\Theta\,,$$

define a sequence of operators as follows:

$$C_0 = 1, \quad C_1 := Z, \quad C_{n+1} := Z\,C_n - n\,\Theta\,(Y - \frac{n-1}{2})C_{n-1}\,.$$

Lemma 2 *The series*

$$\Phi(Z,\Theta)(s) := \sum \frac{s^n\, C_n}{n!}\, \Gamma(2Y+n)^{-1}$$

is the unique solution of the operator-valued differential equation

$$s(\frac{d}{ds})^2\Phi - 2(Y-1)\,\frac{d}{ds}\Phi + Z\,\Phi - \frac{s}{2}\,\Theta\,\Phi = 0$$

that satisfies the initial conditions

$$\Phi(0) = \Gamma(2Y)^{-1}, \quad \frac{d}{ds}\Phi(0) = Z\,\Gamma(2Y+1)^{-1}$$

Lemma 3 *Let μ be an operator in $\mathcal{A}$ such that*

$$[\Theta,\, \mu] = 0\,, \quad [Y,\, \mu] = \mu\,, \quad [[Z,\, \mu],\, \mu] = 0\,. \tag{3.2}$$

Then

$$\Phi(Z + \,\mu\, Y,\; \Theta + \,[Z,\, \mu]\, + \,\frac{\mu^2}{2})(s) \,=\, e^{\frac{s\,\mu}{2}}\, \Phi(Z,\Theta)(s)\,.$$

The formulas for the bilinear operators RC_n generalizing the higher Rankin-Cohen brackets are obtained by expanding in powers of s the products

$$\Phi(-X+\delta_1 Y, R(\Omega)(s)(a)\,\Phi(X, L(\Omega))(s)(b)\,, \quad a\,, b \in \mathcal{A}\,,$$

where $L(\Omega)$ and $R(\Omega)$ stand for the left and right multiplication by Ω. This gives bilinear operators of the form

$$RC_n(a,\, b) := \sum_{k=0}^{n} \frac{A_k}{k!}\,(2Y+k)_{n-k})(a)\,\frac{B_{n-k}}{(n-k)!}\,(2Y+n-k)_k)(b), \tag{3.3}$$

with $(C)_k := C\,(C+1)....(C+k-1)$.

Lemma 4 *The bilinear operators $RC_n\,:\,\mathcal{A}\otimes\mathcal{A}\,\to\,\mathcal{A}$ satisfy the following covariance property: if $u \in \mathcal{A}$ is invertible and such that*

$$X(u) = 0,\; Y(u) = 0,\; \delta_k(u^{-1}\delta_1(u)) = 0, \quad \forall\, k \geq 1,$$

then

1^0 *for all $n \geq 1$, $RC_n(a\,u,\, b) = \,RC_n(a,\, u\,b), \quad \forall\, a,\, b \in \mathcal{A}$;*

2^0 $RC_n(u\,a,\, b) = \,u\,RC_n(a,\;b)\,, \quad RC_n(a,\, b\,u) = \,RC_n(a,\;b)\,u$*;*

3^0 *with $\alpha(u)(x) = uxu^{-1}$, one has $RC_n(\alpha(a),\, \alpha(b)) = \,\alpha(RC_n(a,\, b))$.*

Theorem 5 *The product formula*

$$a * b := \sum t^n\, RC_n(a,b)$$

gives an associative deformation of any modular Hecke algebra $\mathcal{A}(\Gamma)$.

Relying on the fact that the above result applies to all Hopf actions described in §3.2, one can 'lift' the associativity property to the Hopf algebra level and then transfer it to any action defining a projective structure on an algebra.

Theorem 6 *Let $\mathcal{A}$ be any algebra endowed with an $\mathcal{H}_1$ action confering it a projective structure. Then*

1^0 *the transverse fundamental class $[RC_1] \in H^2(\mathcal{A},\mathcal{A})$ defines a noncommutative Poisson structure;*

2^0 *the bilinear operators $RC_n : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $n \geq 1$, depend solely on the action of the constituents Y and $S(X) = -X + \delta_1 Y$ of the transverse fundamental cocycle Π on $\mathcal{A}$, and on the quadratic differential Ω;*

3^0 *the product formula*

$$a * b := \sum t^n \, RC_n(a,b)$$

gives an associative deformation that 'quantizes' the Poisson structure.

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