p-adic Schwarzian triangle groups of Mumford type

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We define a certain p-adic analogue of classical Schwarzian triangle groups related to Mumford's uniformization of analytic curves and give a complete classification of it.

1. Introduction

1.1. *Uniformization of orbifolds and triangle groups.* The rich geometric structure of uniformized analytic varieties over non-archimedean fields has been studied by many authors, and already has a long history. Mumford [Mum72] showed that an analytic curve $\mathcal{X}$ defined over a non-archimedean field K with a split multiplicative analytic reduction can be uniformized as $\mathcal{X} \cong \Gamma\backslash(\mathbb{P}^{1,\mathrm{an}}_K - \mathcal{L}_\Gamma)$, where Γ is a finitely generated free discrete subgroup of $\mathrm{PGL}(2,K)$ and $\mathcal{L}_\Gamma$ is the set of limit points of Γ. An equally important example is the uniformization of a curve which is an étale covering of a Mumford curve, studied by van der Put [vdP83]. These are the most practical and reasonable analogues of uniformizations of complex analytic curves.

Historically, however, the theory of uniformization in complex analysis arose from interplay between geometric and function-theoretic viewpoints. This is apparent if one considers the orbifold uniformization of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}}$ with finitely many points marked by positive integers (= the ramification degrees). The link between geometry and function theory stems from the behavior of the multivalued function z, inverse to the uniformization map, which is written as a ratio $z = u_1/u_2$ of two linearly independent solutions of a Fuchsian differential equation with rational exponents (cf. [Yos87]). For example, if the orbifold is $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}}$ with precisely the three points $0, 1, \infty$ marked by $e_0, e_1, e_\infty \in \mathbb{Z}_{>0}$, then the corresponding differential equation is the Gaussian hypergeometric equation

$$x(1-x)\frac{d^2u}{dx^2} + \{c-(a+b+1)x\}\frac{du}{dx} - abu = 0 \tag{1.1.1}$$

(where $a, b, c \in \mathbb{Q}$ with $|1-c| = 1/e_0$, $|c-a-b| = 1/e_1$, and $|a-b| = 1/e_\infty$). Depending on whether $1/e_0 + 1/e_1 + 1/e_\infty > 1$, $= 1$ or < 1, the image of $z = u_1/u_2$ is isomorphic to $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{H}$, respectively, and z maps the upper-half plane onto the interior of a triangle region with the angles π/e_0, π/e_1 and π/e_∞. The corresponding orbifold fundamental group has a representation into $\mathrm{PGL}(2,\mathbb{C})$ with discrete image, the so-called *Schwarzian triangle group* $\Delta(e_0, e_1, e_\infty)$, whose action on the universal covering space is visible in terms of complex reflections (cf. [Mag74, Chap. II]).

The problems which arises from carrying out such a program in the p-adic situation are mainly topological. It is perhaps appropriate here to remind the reader of the fact that in rigid analysis an étale covering map is not necessarily a topological covering map (= the locally topologically trivial map). Consequently, in contrast to complex analysis, we have many simply connected domains, even one dimensional; for instance, the complement of finitely many points in $\mathbb{P}^{1,\mathrm{an}}_K$ is simply connected. In particular, a reasonable definition of the orbifold fundamental groups is highly non-trivial.

In [And98], Y. André studied the rather special class of étale covering maps which are composites of topological coverings followed by finite étale (not necessarily topological) coverings. He observed that the covering maps of this kind give rise to a reasonable concept of orbifold fundamental groups (denoted by π_1^{orb} in [And98]) in the p-adic situation, and discussed the relation with differential equations; one of his result phrases it as follows [And98, §6]: Consider the orbifold $\mathcal{X}$ (cf. [And98, 5.1] for the precise definition) which is supported on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ with n-points ζ_i marked by positive integers e_i $(1 \leq i \leq n)$. Then:

There exists a canonical fully faithful functor of categories

$$(1.1.2)\quad \left\{\begin{array}{l}\textit{Continuous representations } \rho\\ \textit{of } \pi_1^{\mathrm{orb}}(\mathcal{X},\overline{x}) \textit{ into } \mathrm{GL}(r,\mathbb{C}_p)\\ \textit{with discrete coimage}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\textit{Algebraic regular connections}\\ \textit{on } \mathbb{P}^1_{\mathbb{C}_p} - \{\zeta_i \,|\, 1 \leq i \leq n\}\\ \textit{of rank } r \textit{ such that the local}\\ \textit{monodromy at each } \zeta_i \textit{ is of fi-}\\ \textit{nite order dividing } e_i\end{array}\right\}.$$

Moreover, the essential image of this functor consists of the connections ∇ enjoying the following condition (Global Monodromy Condition):

(1.1.3) *There exists a connected rigid analytic curve $\mathcal{S}$ and a finite morphism $\varphi\colon \mathcal{S} \to \mathcal{X}$ ramified above precisely the points ζ_i with ramification indices dividing e_i, such that the connection $\varphi^*\nabla$ on $\mathcal{S}$ admits a full set of multivalued analytic solutions on $\mathcal{S}$ (and, moreover, on the Berkovich space associated to $\mathcal{S}$).*

In particular, if $n = 3$, $r = 2$ and the image Γ of ρ in $\mathrm{PGL}(2,\mathbb{C}_p)$ is discrete, then Γ can be regarded as a p-adic analogue of the Schwarzian triangle group $\Delta(e_0, e_1, e_\infty)$. If so, Γ gives the "projective monodromy" for the connection ∇ defined by the functor (1.1.2), which is nothing but the one associated to the Gaussian hypergeometric equation (1.1.1). In [And98, §9], André discussed such groups, called *p-adic triangle groups*, and gave a complete list of the so-called *arithmetic* p-adic triangle groups, which are constructed through the Cherednik-Drinfeld theory of uniformization of Shimura curves, starting from Takeuchi's list of arithmetic triangle groups. Notably, he deduced that *there exists no arithmetic p-adic triangle groups for $p > 5$.* The uniformizations of the orbifolds $\mathcal{X}$ corresponding to these groups are given by the Drinfeld upper-half plane or its étale coverings.

1.2. *Results of this paper.* In this paper, we will discuss (not necessarily arithmetic) p-adic triangle groups Γ as above, especially in the case that the corresponding uniformization is given by the space $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} - \mathcal{L}_\Gamma$. We call such a group Γ a p-adic triangle group *of Mumford type*; we can define it in simpler terms (without involving π_1^{orb}) as follows:

Definition. A finitely generated discrete subgroup Γ of $\mathrm{PGL}(2,\mathbb{C}_p)$ is said to be a *p-adic (Schwarzian) triangle group of Mumford type* if $\Gamma\backslash(\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}-\mathcal{L}_\Gamma)\cong\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ and the uniformization map

$$\varpi_\Gamma\colon(\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}-\mathcal{L}_\Gamma)\longrightarrow\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$$

is ramified above precisely three points.

If Γ is a finite subgroup, then $\mathcal{L}_\Gamma=\emptyset$, and the map ϖ_Γ is the one which comes from the algebraic quotient map $\mathbb{P}^1_{\mathbb{C}_p}\to\mathbb{P}^1_{\mathbb{C}_p}/\Gamma$. Hence the spherical (i.e., $1/e_0+1/e_1+1/e_\infty>1$) p-adic triangle groups of Mumford type amount to the same as the classical ones. Our main theorem gives the complete classification of the other p-adic triangle groups of Mumford type:

Theorem. (1) *A non-spherical p-adic triangle group Γ of Mumford type of index (e_0,e_1,e_∞) exists if and only if p and the unordered triple (e_0,e_1,e_∞) occur in the left two columns of Table 1, where, in the last row, the integers l,m,n obey the following condition;*

$(*)$ *$lmn\geq 4$, and at least two of l,m,n are odd.*

In particular, p-adic triangle groups of Mumford type do not exist if $p>5$.

(2) *In each of the cases in Table 1, the conjugacy class of Γ in $\mathrm{PGL}(2,\mathbb{C}_p)$ is uniquely determined. Moreover, Γ is isomorphic to the abstract group given in the right column of the table, where Z_n denotes the cyclic group of order n, D_n the dihedral group of degree n ($\cong Z_n\rtimes Z_2$), etc. In the case $(3,2l,4m)$ (resp. $(2l,2m,2n)$), the subgroups D_4 and D_2 in S_4 are chosen such that $D_2\subset D_4$ (resp. both D_2 are chosen to be equal to each other).*

TABLE 1: List of p-adic triangle groups of Mumford type

p	Index		Abstract structure
$p=5$	$(2,3,5l)$	$(l\geq 2)$	$A_5*_{D_5}D_{5l}$
$p=3$	$(2,5,3l)$	$(l\geq 2)$	$A_5*_{D_3}D_{3l}$
	$(2,4,3l)$	$(l\geq 2)$	$S_4*_{D_3}D_{3l}$
	$(2,3l,3l)$	$(l\geq 2)$	$A_4*_{Z_3}Z_{3l}$
$p=2$	$(3,5,2l)$	$(l\geq 3$ and l: odd)	$A_5*_{D_2}D_{2l}$
	$(3,2l,4m)$	$(lm\geq 2$ and l: odd)	$D_{4m}*_{D_4}S_4*_{D_2}D_{2l}$
	$(3,3,2l)$	$(l\geq 3$ and l: odd)	$A_4*_{D_2}D_{2l}$
	$(2m+1,2l,2l)$	$(m\geq 1,l\geq 2)$	$D_{2m+1}*_{Z_2}Z_{2l}$
	$(2l,2m,2n)$	(l,m,n satisfy $(*)$)	$D_{2l}*_{D_2}D_{2m}*_{D_2}D_{2n}$

Remark. (1) In each of the cases in the table, at least one of the numbers e_i $(i=0,1,\infty)$ is divisible by the residue characteristic p.

(2) The theorem shows that there are no Euclidean (i.e., $1/e_0+1/e_1+1/e_\infty=1$) p-adic triangle groups of Mumford type. The reason for this is that the elliptic curve which covers an Euclidean orbifold always has a complex multiplcation and never be a Tate curve.

(3) The theorem and the well-known fact on automorphisms of Mumford curves [GvP80, VII.§1] imply that for $p>5$ and for a Mumford curve X, if $\mathrm{Aut}(X)\backslash X\cong$

$\mathbb{P}^1_K$, then the quotient map $X \to \mathbb{P}^1_K$ ramifies above at least 4 points. Applying the classical Hurwitz formula, we therefore see that $|\mathrm{Aut}(X)| \leq 12(g-1)$ (where g is the genus of X), which partly recovers the Herrlich's result [Her80b].

(4) Our list of p-adic triangle groups has a non-empty intersection with André's list of p-adic arithmetic triangle groups, but does not include it, since André's p-adic arithmetic triangle groups are not of Mumford type in general. In other words, André's p-adic arithmetic triangle groups do not always come out with the uniformization by the spaces of form $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} - \mathcal{L}_\Gamma$, but by étale coverings of them.

1.3. *Outline of the proof.* Let us briefly view an outline of the proof.

(1.3.1) The proof basically relies on the following principle, which is well-known and can be proved by using standard arguments as in [GvP80] and [vdP97]: Let K be a finite extension of $\mathbb{Q}_p$, and Γ a finitely generated discrete subgroup of $\mathrm{PGL}(2,K)$. Replacing K by a finite extension if necessary, we assume that the fixed points in $\mathbb{P}^{1,\mathrm{an}}_K$ of any element ($\neq 1$) in Γ are K-valued (cf. [GvP80, I.3.1 (1)]). Let $\mathcal{L}_\Gamma$ (resp. $\mathcal{F}_\Gamma$) be the set of all limit points (resp. fixed points) in $\mathbb{P}^{1,\mathrm{an}}_K$ of Γ. If Γ is free, then we have $\mathcal{L}_\Gamma = \overline{\mathcal{F}}_\Gamma$ (where $\bar{\ }$ stands for the topological closure), while in general we only have $\mathcal{L}_\Gamma \subseteq \overline{\mathcal{F}}_\Gamma$, and $\overline{\mathcal{F}}_\Gamma - \mathcal{L}_\Gamma$ is a discrete set consisting of the fixed points of elliptic elements in Γ. Let $\mathcal{T}_\Gamma$ (resp. $\mathcal{T}^*_\Gamma$) be the subtree of the Bruhat-Tits tree generated by the vertices determined by three pairwise distinct points in $\mathcal{L}_\Gamma$ (resp. $\overline{\mathcal{F}}_\Gamma$); cf. [Mum72, §1] and [GvP80, I.§2]. Clearly, $\mathcal{T}_\Gamma \subseteq \mathcal{T}^*_\Gamma$. It is easy to see that Γ acts on $\mathcal{T}_\Gamma$ and $\mathcal{T}^*_\Gamma$ without inversion. Set $\mathsf{T}_\Gamma = \Gamma\backslash\mathcal{T}_\Gamma$ and $\mathsf{T}^*_\Gamma = \Gamma\backslash\mathcal{T}^*_\Gamma$. Let $\Omega_\Gamma = \mathbb{P}^{1,\mathrm{an}}_K - \mathcal{L}_\Gamma$, and $\varpi_\Gamma\colon \Omega_\Gamma \to \Gamma\backslash\Omega_\Gamma$ the quotient map. Then:

There exist canonical bijections, compatible with the quotient maps,

$$\begin{array}{ccc} \left\{\begin{array}{l}\textit{Ramification points in}\\ \Omega_\Gamma \textit{ of the map } \varpi_\Gamma\end{array}\right\} & \longleftrightarrow & \{\textit{Ends of } \mathcal{T}^*_\Gamma - \mathcal{T}_\Gamma\} \\ \big\downarrow & & \big\downarrow \\ \{\textit{Branch points in } \Gamma\backslash\Omega_\Gamma\} & \longleftrightarrow & \{\textit{Ends of } \mathsf{T}^*_\Gamma\}. \end{array} \tag{1.3.2}$$

Moreover, the decomposition group of a ramification point coincides with the stabilizer of the corresponding end.

For example, the case when $\Gamma = G$ is a finite subgroup provides a nice prototype. As is well-known, the finite subgroup G is isomorphic to one of the following groups: cyclic group Z_n of order $n > 1$, dihedral group $D_n \cong Z_n \rtimes Z_2$ of degree $n > 1$, tetrahedral group A_4, octahedral group S_4, and icosahedral group A_5 (cf. [Web99, Vol. II, §67]). Since the map ϖ_G is the one associated to the algebraic map $\mathbb{P}^1_K \to G\backslash\mathbb{P}^1_K \cong \mathbb{P}^1_K$, we deduce that the quotient graph T^*_G is a tree with two ends, if G is cyclic, or with three ends, otherwise.

(1.3.3) Now we assume for a general Γ as above that $\Gamma\backslash\Omega_\Gamma \cong \mathbb{P}^{1,\mathrm{an}}_K$. Then T^*_Γ is a tree, and hence we have a lifting $\iota_\Gamma\colon \mathsf{T}^*_\Gamma \hookrightarrow \mathcal{T}^*_\Gamma$ by [Ser80, I.4.1, Prop. 17]. Let ε_i ($i = 1,\dots,n$) be the ends of $\iota_\Gamma(\mathsf{T}^*_\Gamma)$, and ζ_i the corresponding branch point in $\Gamma\backslash\Omega_\Gamma \cong \mathbb{P}^{1,\mathrm{an}}_K$. For each i the stabilizer Γ_{ε_i} of the end ε_i is a finite cyclic group; let γ_i be a generator of it, and e_i its order. Then ϖ_Γ gives the uniformization of the orbifold $\mathcal{X} = (\mathbb{P}^{1,\mathrm{an}}_K, (\zeta_i, e_i);\ i = 1,\dots,n)$. By [And98, 5.8] we have a continuous surjective homomorphism $\pi_1^{\mathrm{orb}}(\mathcal{X},\overline{x}) \to \Gamma$, from which it follows in particular that Γ is generated by γ_i ($i = 1,\ldots,n$) (cf. [And98, 5.4]).

(1.3.4) In our situation, what to do is the classification of Γ such that T^*_Γ is a tree with precisely three ends. The proof of the "only if" part of Theorem (1) goes like as follows: Suppose that T^*_Γ is a tree with precisely three ends. The first main point of the proof is that we can find a finite subgroup $G \subseteq \Gamma$ such that by the natural map

$$\mathsf{T}^*_G \longrightarrow \mathsf{T}^*_\Gamma \tag{1.3.5}$$

induced by the inclusion $\mathcal{T}^*_G \hookrightarrow \mathcal{T}^*_\Gamma$, the tree T^*_G essentially determines the shape of T^*_Γ; more precisely, the map (1.3.5) gives rise to a bijection between ends. We can actually do it by choosing G to be a subgroup whose abstract structure is "maximal" with respect to a certain order.

On the other hand, it is easily observed that the action of Γ on $\mathcal{T}^*_\Gamma$ is also "visible" like in the classical situation. In fact, Γ is generated by three elliptic elements γ_0, γ_1 and γ_∞ of finite order (by (1.3.3)) of which the action on $\mathcal{T}^*_\Gamma$ looks like a "reflection". Here, for such an elliptic element γ the apartment connecting the two fixed points of γ plays the role of reflection mirrors (cf. Definition 2.3), and the fundamental domain T^*_Γ, which is essentially identified with T^*_G, gives the counterpart of the triangle regions.

Let θ_0, θ_1 and θ_∞ be the generators of G corresponding to the three ends of T^*_G as in (1.3.3). Since there exist no two elements in Γ which have exactly one fixed point in common (because of the discreteness), each γ_i ($i = 0, 1, \infty$) has the same (up to conjugacy) fixed points as one of θ_0, θ_1 and θ_∞; hence, changing indices of γ_i's if necessary, we may assume $\gamma_i^{l_i} = \theta_i$ for some $l_i \geq 1$. Now the second main point of the proof is that such a trick with $l_i > 1$ is only possible when the mirror of θ_i runs apart enough from any vertices in $\mathcal{T}^*_G$ stablized by G; otherwise, in the tree $\mathcal{T}^*_\Gamma$ such a vertex would be stabilized by either an infinite subgroup (which contradicts the discreteness) or by a finite subgroup which is "bigger" than G with respect to the order as above.

In order to know which mirror is separated enough from the vertex stablized by G, we need a detailed study about the trees $\mathcal{T}^*_G$ and T^*_G for the finite subgroup G. This will be done in the next section, where we will find that the picture of these trees differs depending on the residue characteristic p, especially when p is small (which effects the restraint $p \leq 5$ in the theorem). Having such a knowledge of the mirror arrangement, we will be finally able to determine all the possible fundamental domains T^*_Γ and the triple (e_0, e_1, e_∞) as desired. The results in Theorem (2) are by-products of the proof of this part.

(1.3.6) The "if" part of the proof will be done by constructing Γ explicitly. We will do it, roughly speaking, along the recipe which was already indicated in the "only if" part of the proof. Comparing our construction with the abstract construction of groups as in [Ser80, I.4.5], we will show that Γ is discrete, as well as that Γ has the desired abstract structure.

1.4. *Notation and conventions.* (1) Throughout this paper K denotes a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_K$ the integer ring, and π a prime element in $\mathcal{O}_K$. We write $[K\colon\mathbb{Q}_p]=ef$, where e is the ramification degree and $q=p^f$ is the the number of elements in the residue field $k=\mathcal{O}_K/\pi\mathcal{O}_K$. We denote by $\nu\colon K^\times\to\mathbb{Z}$ the normalized (i.e., $\nu(\pi)=1$) valuation.

(2) Let $\mathfrak{T}_K^{\mathrm{BT}}$ be the Bruhat-Tits tree attached to $\mathrm{PGL}(2,K)$. The geometric realization $|\mathfrak{T}_K^{\mathrm{BT}}|$ is a metrized space with the metric denoted by $d(\cdot,\cdot)$. For two vertices Λ_0 and Λ_1, $[\Lambda_0,\Lambda_1]$ denotes the geodesic path connecting them. For $u,v\in\mathbb{P}^1(K)$ $(u\neq v)$ and a vertex $\Lambda\in\mathfrak{T}_K^{\mathrm{BT}}$, the apartment (resp. half-line) connecting u and v (resp. u and Λ) is denoted by $]u,v[$ (resp. $[\Lambda,u[$). We will use the notation $\mathcal{L}_\Gamma$, $\mathcal{F}_\Gamma$, $\mathfrak{T}_\Gamma$, $\mathfrak{T}_\Gamma^*$, T_Γ, T_Γ^* and Ω_Γ as in (1.3.1) for any finitely generated discrete subgroup $\Gamma\subset\mathrm{PGL}(2,K)$.

2. The trees associated to finite groups

2.1. The aim of this section is to describe the trees $\mathfrak{T}_G^*$ and T_G^* $(=G\backslash\mathfrak{T}_G^*)$ for G a finite subgroup in $\mathrm{PGL}(2,K)$, where K is taken so that any element $(\neq 1)$ in G has at most K-valued fixed points in $\mathbb{P}^1_K$. The procedure of the description is very elementary and straightforward, but needs a lot of computation. Here we mainly present only the results, and exhibit necessary data in Appendix.

First we need to know the following fact: Let $a=(a_0\colon a_1)$, $b=(b_0\colon b_1)$, $c=(c_0\colon c_1)$ and $d=(d_0\colon d_1)$ be four pairwise distinct K-rational points of $\mathbb{P}^1_K$. We are interested in the arrangement of two apartments $]a,b[$ and $]c,d[$ in $\mathfrak{T}_K^{\mathrm{BT}}$. Let us define the cross-ratio

$$R(a,b;c,d)=\frac{(a_1c_0-a_0c_1)(b_1d_0-b_0d_1)}{(a_0b_1-a_1b_0)(c_0d_1-c_1d_0)}. \tag{2.1.1}$$

2.2. Lemma. (1) *If $|\nu(R(a,b;c,d))|=|\nu(R(b,a;c,d))|=0$, then $]a,b[$ and $]c,d[$ intersects at exactly one vertex.*

(2) *If $|\nu(R(a,b;c,d))|=|\nu(R(b,a;c,d))|\neq 0$, then $]a,b[$ and $]c,d[$ are disjoint with the distance $|\nu(R(a,b;c,d))|$.*

(3) *If $|\nu(R(a,b;c,d))|\neq|\nu(R(b,a;c,d))|$, then the intersection of $]a,b[$ and $]c,d[$ is the path $[\Lambda(a,b,c),\Lambda(b,c,d)]$ of length $\max\{|\nu(R(a,b;c,d))|,|\nu(R(b,a;c,d))|\}$, where $\Lambda(z_0,z_1,z_2)$ for pairwise distinct three points $z_0,z_1,z_2\in\mathbb{P}^1(K)$ is the unique vertex lying in the intersection of all $]z_i,z_j[$ for $i,j=0,1,2$, $i\neq j$.*

Proof. First we recall how to calculate $\Lambda(z_0,z_1,z_2)$: Let Y_i $(i=0,1,2)$ be a homogeneous coordinate of z_i, and choose $\alpha_i\in K^\times$ such that $\alpha_0Y_0+\alpha_1Y_1+\alpha_2Y_2=0$. Then $\Lambda(z_0,z_1,z_2)$ is the similarity class of $\mathcal{O}_K\alpha_iY_i+\mathcal{O}_K\alpha_jY_j$ for any $i,j=0,1,2$, $i\neq j$. By this, it is easily checked that $d(\Lambda(a,b,c),\Lambda(b,c,d))=|\nu(R(a,b;c,d))|$. Once it is checked, all the statements are clear, since the apartments $]a,b[$ and $]c,d[$ either do not intersect or intersect along a path (see Figure 1). $\square$

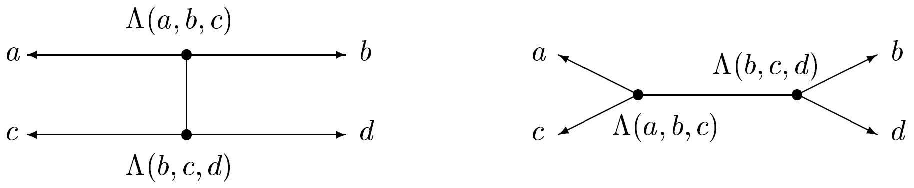


FIGURE 1: Arrangement of the apartments $]a,b[$ and $]c,d[$

2.3. Definition. Let $\gamma \in \mathrm{PGL}(2,K)$ be an elliptic element (i.e., an element having two distinct eigenvalues with the same valuation) of finite order, and suppose that γ has two K-valued fixed points $u, v \in \mathbb{P}^1(K)$. We set

$$\mathsf{M}(\gamma) =]u, v[,$$

and call it the *mirror* of γ. It is clear that for elliptic elements γ and θ of finite order the equality $\mathsf{M}(\gamma) = \mathsf{M}(\theta)$ holds if and only if $\langle\gamma, \theta\rangle$ is a cyclic group.

The element γ acts trivially on $\mathsf{M}(\gamma)$, i.e., $\mathsf{M}(\gamma)$ is contained in the fixed locus of γ (= the subtree in $\mathcal{T}_K^{\mathrm{BT}}$ generated by vertices fixed by γ); but they do not coincide with each other in general:

2.4. Lemma. *Let n be the order of γ, and set $G = \langle\gamma\rangle$.*

(1) *Let $\Lambda_0 \in \mathsf{M}(\gamma)$. If $(n,p) = 1$, then G acts freely on the $q-1$ vertices adjacent to Λ_0 not lying on $\mathsf{M}(\gamma)$.*

(2) *Suppose $n = p^r$ for $r \geq 1$, and set $s = \nu(\zeta_{p^r} - 1)$, where ζ_{p^r} is a primitive p^r-th root of unity. Then a vertex $\Lambda \in \mathcal{T}_K^{\mathrm{BT}}$ is fixed by G if and only if $0 \leq d(\Lambda, \mathsf{M}(\gamma)) \leq s$.*

PROOF. We may assume that $\gamma\colon z \mapsto \zeta_n z$, where z is the inhomogeneous coordinate. (1) follows from the fact that the adjacent vertices are in canonical one-to-one correspondence with points in $\mathbb{P}^1(k)$. (2) is due to an easy calculation collaborated with the straightforward lemma below. $\square$

2.5. Lemma. *Set $\Lambda_0 = [\mathcal{O}_K X_0 + \mathcal{O}_K X_1]$ and $\Lambda_1 = [\mathcal{O}_K(X_0 + u_0 X_1) + \mathcal{O}_K \pi X_1]$ with $u_0 \not\equiv 0 \bmod \pi$. Let Λ be a vertex such that the path $[\Lambda_0, \Lambda]$ contains Λ_1. Then we have $\Lambda = [\mathcal{O}_K(X_0 + (\sum_{i=0}^{d-1} u_i \pi^i) X_1) + \mathcal{O}_K \pi^d X_1]$, where $d = d(\Lambda, \Lambda_0)$.* $\square$

2.6. Definition. Let Γ be a finitely generated discrete subgroup of $\mathrm{PGL}(2,K)$. Let A be an apartment in $\mathcal{T}_\Gamma^*$, and $\Lambda \in A$ a vertex. Then we say $\gamma \in \Gamma$ *folds A at Λ* if $\gamma \cdot \Lambda = \Lambda$ and γ maps one of the edge in A emanating from Λ to the other such edge. The same terminology will be applied to a half-line ℓ and a vertex $\Lambda \in \ell$ which does not sit on the end point of ℓ.

Observe that, if A is mapped to a half-line (resp. a segment of finite length) by the quotient map $\mathcal{T}_\Gamma^* \to \Gamma\backslash\mathcal{T}_\Gamma^*$, then A is folded at at least one (resp. two) vertices.

2.7. *Terminology.* In the sequel the expression *order-n mirror* means the apartment A which is the mirror of an element of order n, but not that of an element of order greater than n. Clearly the number of order-n mirrors in $\mathcal{T}_G^*$ is equal to the number of maximal cyclic subgroups in G of order n.

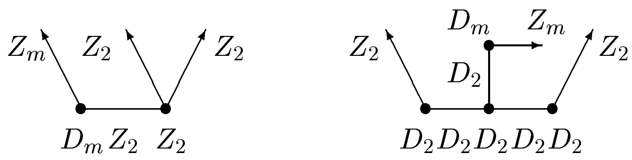
2.8. *Cyclic case:* $G \cong Z_n$. For any residue characteristic p the tree $\mathcal{T}^*_G$ consists of only one apartment which is the mirror $\mathsf{M}(\theta)$ of any element $\theta \neq 1$ in G. Since G acts on $\mathcal{T}^*_G$ trivially, the quotient tree T^*_G also consists of one straight-line whose ends corresponds to the two points above which $\mathbb{P}^1_K \to G\backslash\mathbb{P}^1_K$ ramifies.

2.9. *Dihedral case:* $G \cong D_m$. Let $N = \langle\theta\rangle$ be the normal subgroup of G of index 2, and $H_0 = \langle\varphi\rangle$ a subgroup of order 2. Then $G = \langle\theta, \varphi \mid \varphi\theta\varphi = \theta^{-1}\rangle$; G has exactly m subgroups of order 2 given by $H_i = \langle\theta^i\varphi\rangle$ for $0 \leq i \leq m-1$. The tree $\mathcal{T}^*_G$ therefore contains one order-m mirror (corresponding to N) and m order-2 mirrors (corresponding to H_i's). If m is odd, then all H_i's are conjugated among others, while they fall into two conjugacy classes if m is even. The element φ of order 2 folds the apartment $\mathsf{M}(\theta)$ to a half-line. If m is even, then $\theta^{m/2}$ folds each $\mathsf{M}(\theta^i\varphi)$ to a half-line.

(2.9.1) $p \neq 2$: All the mirrors $\mathsf{M}(\theta)$ and $\mathsf{M}(\theta^i\varphi)$ $(0 \leq i \leq m-1)$ intersect among others at precisely one vertex Λ° which is the unique vertex stabilized by G. We therefore have $\mathcal{T}^*_G = \mathsf{M}(\theta) \bigcup (\bigcup_{i=0}^{m-1} \mathsf{M}(\theta^i\varphi))$, and hence the quotient tree T^*_G simply consists of three half-lines starting at one point.

(2.9.2) $p = 2$: All the mirrors $\mathsf{M}(\theta)$ and $\mathsf{M}(\theta^i\varphi)$ $(0 \leq i \leq m-1)$ are disjoint from among others. The tree $\mathcal{T}^*_G$ is the union of all these mirrors and paths $[\Lambda^\circ, \Lambda_i]$ of length e connecting them, where $\Lambda^\circ \in \mathsf{M}(\theta)$ and $\Lambda_i \in \mathsf{M}(\theta^i\varphi)$. Each path $[\Lambda^\circ, \Lambda_i]$ is pointwise fixed by the subgroup $\theta^{-i}H\theta^i$. If m is even, it is further pointwise fixed by the subgroup $\langle\theta^{m/2}, \theta^i\varphi\rangle \cong D_2$.

The proofs of (2.9.1) and (2.9.2) are straighforward: We may assume that $\theta(z) = \zeta_m z$ and $\varphi(z) = z^{-1}$, where z is an inhomogeneous coordinate of $\mathbb{P}^1_K$. Then apply Lemma 2.2 and Lemma 2.4. The quotient tree T^*_G in $p = 2$ looks like as in Figure 2, where the arrow indicates the ends. The groups beside vertices (resp. edges, ends) indicate the stabilizers ($\subseteq G$) of the them.



$p = 2,\ m$: odd $\qquad p = 2,\ m$: even

FIGURE 2: T^*_G for $G \cong D_m$ in $p = 2$

The assertions in the following paragraphs can be proved in the same spirit: We can compute the arrangement of apartments by Lemma 2.2 using the combinatorial data presented in Appendix. The statements about fixed loci and fixers are straighforward by Lemma 2.4.

2.10. *Tetrahedral case:* $G \cong A_4$. The subtree $\mathcal{T}^*_G$ contains three order-2 mirrors and four order-3 mirrors. The order-2 mirrors are folded to half-lines, while the order-3 mirrors are not.

(2.10.1) $p \neq 2, 3$: All the $3 + 4$ mirrors intersect among others at precisely one vertex Λ°. Hence the tree $\mathcal{T}^*_G$ is simply the union of all these mirrors.

(2.10.2) $p = 3$: The three order-2 mirrors intersect at one vertex Λ°, while the order-3 mirrors are disjoint among others and do not intersect the order-2 mirrors. Let Λ be the vertex nearest to Λ° in one of the order-3 mirrors. Then $d(\Lambda^\circ, \Lambda) = e/2$, and $[\Lambda^\circ, \Lambda]$ is pointwise fixed by a subgroup isomorphic to Z_3. The tree $\mathcal{T}_G^*$ is the union of all mirrors and the four paths (of the form $[\Lambda^\circ, \Lambda]$ as above) of length $e/2$.

(2.10.3) $p = 2$: The order-3 mirrors have a path σ_0 of length e in common. Let Λ° be the middle point of σ_0. The order-2 mirrors are disjoint among others and do not intersect the order-3 mirrors. Let Λ be the vertex nearest to Λ° in one of the order-2 mirrors. Then $d(\Lambda^\circ, \Lambda) = e/2$, and $[\Lambda^\circ, \Lambda]$ is pointwise fixed by a subgroup isomorphic to D_2. The path σ_0 is fixed by the whole G. The tree $\mathcal{T}_G^*$ is the union of all mirrors and the three paths (of the form $[\Lambda^\circ, \Lambda]$ as above) of length $e/2$.

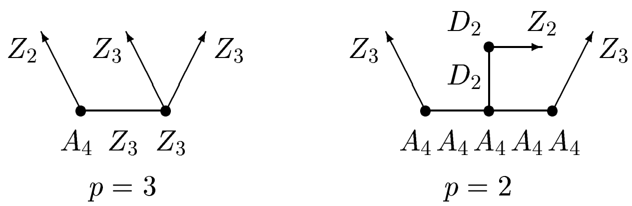


FIGURE 3: T_G^* for $G \cong A_4$

2.11. *Octahedral case:* $G \cong S_4$. The tree $\mathcal{T}_G^*$ contains six order-2 mirrors, four order-3 mirrors, and three order-4 mirrors. All these mirrors are folded to half-lines.

(2.11.1) $p \neq 2, 3$: All the $6 + 4 + 3$ mirrors intersect among others at precisely one vertex Λ°. Hence the tree $\mathcal{T}_G^*$ is simply the union of all these mirrors.

(2.11.2) $p = 3$: The order-4 mirrors intersect among others at a unique vertex Λ°. The order-2 mirrors also pass through Λ°; there are four paths σ_i $(i = 0, 1, 2, 3)$ of length e of form $\sigma_i = [\Lambda^\circ, \Lambda_i]$ such that $\sigma_i \bigcap \sigma_j = \Lambda^\circ$ for $i \neq j$, and that each order-2 mirror contains precisely two of σ_i's. The order-3 mirrors are disjoint among others, and each of them intersects one of σ_i's at exactly the middle point. Hence the tree $\mathcal{T}_G^*$ is the union of all these mirrors. Each σ_i is pointwise fixed by a subgroup isomorphic to D_3.

(2.11.3) $p = 2$: The order-3 mirrors intersect among others along a path σ_0 of length e. Let Λ° be the middle point of σ_0. The order-4 mirrors are disjoint among others and do not intersect order-3 mirrors. Let Λ be the vertex nearest to Λ° in one of the order-4 mirrors. We have $d(\Lambda^\circ, \Lambda) = e/2$. Let Λ_1 be the middle point of $[\Lambda^\circ, \Lambda]$. Then there are two order-2 mirrors to which the distance from Λ_1 is $3e/4$. The paths from Λ_1 to these order-2 mirrors share a path $[\Lambda_1, \Lambda_2]$ of length $e/4$. Moreover:

(a) The path σ_0 is pointwise fixed by the normal subgroup $\cong A_4$.
(b) The paths $[\Lambda^\circ, \Lambda]$ and $[\Lambda_1, \Lambda_2]$ are pointwise fixed by a subgroup $\cong D_4$.
(c) Each of the geodesic paths of length $e/2$ from Λ_2 to those order-2 mirrors is pointwise fixed by a subgroup $\cong D_2$.

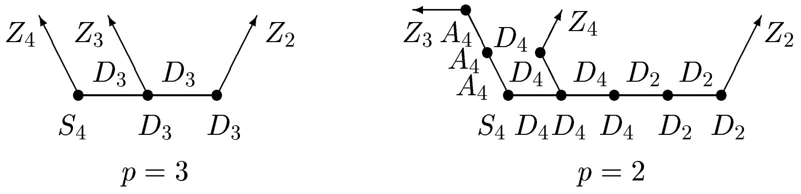


FIGURE 4: T_G^* for $G \cong S_4$

2.12. *Icosahedral case:* $G \cong A_5$. In this case, the tree $\mathcal{T}_G^*$ contains fifteen order-2 mirrors, ten order-3 mirrors, and six order-5 mirrors. All these mirrors are folded to half-lines.

(2.12.1) $p \neq 2, 3, 5$: All the $15+10+6$ mirrors intersect among others at precisely one vertex Λ°. Hence the tree $\mathcal{T}_G^*$ is simply the union of all these mirrors.

(2.11.2) $p = 5$: The order-3 mirrors intersect among others at exactly one vertex Λ°. The order-2 mirrors also pass through Λ°; there are six paths σ_i ($i = 0, 1, 2, 3, 4, 5$) of length $e/2$ of form $\sigma_i = [\Lambda^\circ, \Lambda_i]$ such that $\sigma_i \bigcap \sigma_j = \Lambda^\circ$ for $i \neq j$, and that each order-2 mirror contains two of σ_i's. The order-5 mirrors are disjoint among others, and each of them intersects one of σ_i's at exactly the middle point. Each σ_i is pointwise fixed by a subgroup isomorphic to D_5.

(2.11.3) $p = 3$: The order-5 mirrors intersect among others at exactly one vertex Λ°. The order-2 mirrors also pass through Λ°; there are ten paths $\sigma_i = [\Lambda^\circ, \Lambda_i]$ ($i = 0, \ldots, 9$) of length e such that $\sigma_i \bigcap \sigma_j = \Lambda^\circ$ for $i \neq j$, and that each order-2 mirror contains two of σ_i's. The order-3 mirrors are disjoint among others, and each of them intersects one of σ_i's at exactly the middle point. Each σ_i is pointwise fixed by a subgroup isomorphic to D_3.

(2.11.4) $p = 2$: The order-5 mirrors intersect among others at exactly one vertex Λ°. The order-3 mirrors also pass through Λ°; there are five paths $\sigma_i = [\Lambda^\circ, \Lambda_i]$ ($i = 0, 1, 2, 3, 4$) of length e such that $\sigma_i \bigcap \sigma_j = \Lambda^\circ$ for $i \neq j$, and that each order-3 mirror contains two of σ_i's. Let Λ_i' be the middle point of σ_i. Then there are precisely three order-2 mirrors to which the distance from Λ_i' is $e/2$. The geodesic paths from Λ_i' to these order-2 mirrors, which are pointwise fixed by a subgroup iromorphic to D_2, intersect among others at exactly Λ_i'. Moreover, each path σ_i is pointwise fixed by a subgroup isomorphic to A_4.

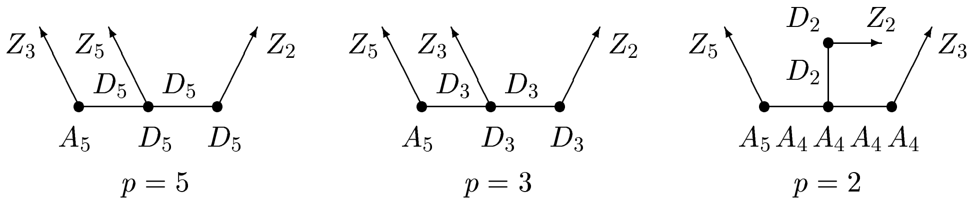


FIGURE 5: T_G^* for $G \cong A_5$

3. Proof of the theorem

3.1. In the rest of this paper we devote ourselves to prove the main theorem. We first recall the following well-known facts: Let γ be an element in $\mathrm{PGL}(2,K)$. Then γ is said to be *parabolic* (resp. *elliptic*, resp. *hyperbolic*) if it has only one eigenvalue (resp. two distinct eigenvalues with equal valuation, resp. two distinct eigenvalues with different valuations). Let Γ be a discrete subgroup of $\mathrm{PGL}(2,K)$. Then:

(3.1.1) *There exists no parabolic element other than* 1.

(3.1.2) *An element $\gamma \in \Gamma$ is of finite order if and only if it is elliptic.*

Suppose two elements θ and χ have exactly one common fixed point $\infty \in \mathbb{P}^1(K)$. We may assume $\theta = \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1}\end{smallmatrix}\right)$ and $\chi = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right)$ with $b \neq 0$. Then it is easy to see that $\theta\chi\theta^{-1}\chi^{-1}$ is a parabolic element. Hence:

(3.1.3) *No two elements in Γ have exactly one common fixed point in $\mathbb{P}^1_K$.*

In particular, for elliptic elements $\gamma, \theta \in \Gamma$, $\mathsf{M}(\gamma) \bigcap \mathsf{M}(\theta)$ contains a half-line if and only if $\langle \gamma, \theta \rangle$ is a cyclic group (cf. Definition 2.3).

Let $\Gamma \subset \mathrm{PGL}(2,K)$ be a finitely generated discrete subgroup. Replacing K by a finite extension if necessary, we may assume that every element ($\neq 1$) in Γ has at most K-valued fixed points in $\mathbb{P}^1_K$ (cf. [GvP80, I.3.1 (1)]). Then Γ acts on the Bruhat-Tits building $\mathcal{T}^{\mathrm{BT}}_K$ without inversion. We assume that the quotient graph T^* is a tree, which is certainly the case if Γ is a p-adic triangle group of Mumford type. Let $\varrho_\Gamma\colon \mathcal{T}^*_\Gamma \to \mathsf{T}^*_\Gamma$ be the quotient map of trees, and take a lifting $\iota_\Gamma\colon \mathsf{T}^*_\Gamma \hookrightarrow \mathcal{T}^*_\Gamma$ ([Ser80, I.4.1, Prop. 17]); i.e., a section of ϱ_Γ. We associate to any vertex $\Lambda \in \mathsf{T}^*_\Gamma$ (resp. edge $\sigma \subseteq \mathsf{T}^*_\Gamma$) the stabilizer of $\iota_\Gamma(\Lambda)$ (resp. $\iota_\Gamma(\sigma)$) in Γ, denoted by Γ_Λ (resp. Γ_σ), to make up the *tree of groups* $(\mathsf{T}^*_\Gamma, \Gamma_\bullet)$ ([Ser80, I.4.4, Def. 8]). Note that stabilizers of vertices and edges are finite subgroups since Γ is discrete. Since $\mathcal{T}^*_\Gamma$ is a tree, we have an isomorphism

$$\Gamma \xrightarrow{\sim} \varinjlim (\mathsf{T}^*_\Gamma, \Gamma_\bullet)$$

([Ser80, I.4.5, Thm. 10]), where the right-hand side is the associated amalgam product of the groups $\Gamma_\bullet$.

3.2. Next let us recall the following facts: Let Γ be as above. Since $\mathsf{T}^{\mathrm{BT}}_\Gamma = \Gamma\backslash\mathcal{T}^{\mathrm{BT}}_K$ is also a tree, we can take a lifting $\iota\colon \mathsf{T}^{\mathrm{BT}}_\Gamma \to \mathcal{T}^{\mathrm{BT}}_K$ and do the same as above to get the associated tree of groups $(\mathsf{T}^{\mathrm{BT}}_\Gamma, \Gamma_\bullet)$. Let Λ be a vertex in $\mathsf{T}^{\mathrm{BT}}_\Gamma$ with the stabilizer $G = \Gamma_\Lambda$. If Λ is the similarity class of a lattice $M \subset K^2$, then G acts on $\operatorname{Proj}\operatorname{Sym}_k(M/\pi M) \cong \mathbb{P}^1_k$; i.e., we have the representation (not necessarily injective) $G \to \mathrm{PGL}(2,k)$.

(3.2.1) *We have a canonical bijection equivariant with the action of G,*

$$\big\{ \textit{Edges } \sigma \textit{ in } \mathcal{T}^{\mathrm{BT}}_K \textit{ emanating from } \iota(\Lambda) \big\} \longleftrightarrow \mathbb{P}^1(k).$$

(3.2.2) *Consider the quotient map $\mathbb{P}^1_k \to G\backslash\mathbb{P}^1_k$. Let $H \subseteq G$ be a subgroup. Then* *we have a canonical bijection,*

$$\left\{\begin{array}{l}\textit{Edges } \sigma \textit{ in } \mathsf{T}_\Gamma^{\mathrm{BT}} \textit{ emanating}\\ \textit{from } \Lambda \textit{ with the stabilizer}\\ \Gamma_\sigma \textit{ a conjugate of } H \textit{ in } \Gamma\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{Points of } G\backslash\mathbb{P}^1(k) \textit{ above}\\ \textit{which points in } \mathbb{P}^1(k) \textit{ are}\\ \textit{fixed by a conjugate of } H\end{array}\right\}.$$

3.3. *Notation.* In the sequel the notation $\Lambda \vdash \sigma$ for Λ a vertex and σ an edge in a tree means that the edge σ emanates from Λ.

3.4. Definition. Let $\mathsf{T} \subseteq \mathcal{T}_\Gamma^*$ be a subtree, and $\Lambda \in \mathsf{T}$ a vertex. A half-line $\ell \subset \mathcal{T}_\Gamma^*$ starting at Λ is said to be *perpendicular to* T if for any edge $\sigma \in \mathsf{T}$ with $\Lambda \vdash \sigma$ and for the edge $\varsigma \in \ell$ with $\Lambda \vdash \varsigma$ we have $\varrho_\Gamma(\sigma) \neq \varrho_\Gamma(\varsigma)$.

Since $\sigma \dashv \Lambda \vdash \varsigma$, the condition $\varrho_\Gamma(\sigma) \neq \varrho_\Gamma(\varsigma)$ is equivalent to: There exists no $\eta \in \Gamma_\Lambda$ such that $\sigma = \eta \cdot \varsigma$.

Now we quote here a few necessary facts which are proved in [CKK99]:

3.5. Proposition ([CKK99], lemma 3.4). *Let $\Lambda \in \mathcal{T}_\Gamma^*$ be a vertex and $\sigma \subset \mathcal{T}_\Gamma^*$ an edge emanating from Λ. Suppose there exists an element $\gamma \in \Gamma_\Lambda$ such that $|\langle\gamma\rangle \cdot \sigma| > 1$ (i.e. σ is not fixed by all powers of γ). Then there exists $\chi \in \Gamma_\Lambda$ ($\chi \neq 1$) such that any half-line ℓ starting at Λ converging to an end of $\mathsf{M}(\chi)$ is perpendicular to $\langle\gamma\rangle \cdot \sigma$.* □

3.6. Proposition ([CKK99], lemma 3.5). *Let $\mathcal{T} \subseteq \mathcal{T}_\Gamma^*$ be a subtree such that*

$$\{\textit{Ends of } \mathsf{T}\} = \{\textit{Ends of } \mathsf{T}_\Gamma^*\}, \tag{3.6.1}$$

where $\mathsf{T} = \varrho_\Gamma(\mathcal{T})$. Then every half-line in $\mathcal{T}_\Gamma^*$ starting from a vertex of $\mathcal{T}$ is not perpendicular to $\mathcal{T}$. □

3.7. Definition. Let $(\mathsf{T}, G_\bullet)$ be a tree of groups, and $\mathsf{T}' \subseteq \mathsf{T}$ a subtree. Then the induced tree of groups $(\mathsf{T}', G_\bullet)$ is said to be a *contraction* of $(\mathsf{T}, G_\bullet)$ if (i) $\{\text{Ends of } \mathsf{T}'\} = \{\text{Ends of } \mathsf{T}\}$ and (ii) for every vertex Λ in $\mathsf{T} - \mathsf{T}'$ the stabilizers of vertices on the path from Λ to Λ' are ordered increasingly with respect to inclusion upon approaching T', where Λ' is the vertex in T' nearest to Λ.

It is clear that if $(\mathsf{T}', G_\bullet)$ is a contraction of $(\mathsf{T}, G_\bullet)$ then we have a canonical isomorphism

$$\varinjlim(\mathsf{T}', G_\bullet) \xrightarrow{\sim} \varinjlim(\mathsf{T}, G_\bullet). \tag{3.7.1}$$

3.8. Proposition ([CKK99], proposition 1). *Let $\mathsf{T} \subseteq \mathsf{T}_\Gamma^*$ be a subtree enjoying the condition (3.6.1). Then $(\mathsf{T}, \Gamma_\bullet)$ is a contraction of $(\mathsf{T}_\Gamma^*, \Gamma_\bullet)$.* □

3.9. In what follows the subgroup Γ is assumed to be a non-spherical p-adic triangle group of Mumford type of index (e_0, e_1, e_∞). By (1.3.2) the quotient graph T_Γ^* is a tree with three ends, corresponding to 0, 1 and ∞, and the stabilizer of the end corresponding to 0 (resp. 1, ∞) is a cyclic subgroup of order e_0 (resp. e_1, e_∞).

3.10. Lemma. *Under the hypothesis above there exists a finite non-cyclic subgroup G in Γ.*

Proof. Otherwise, the group Γ_Λ is a finite cyclic group for any $\Lambda \in \mathsf{T}^*_\Gamma$. Then the mirror of any element ($\neq 1$) of any Γ_Λ is not folded, and is mapped to a straight-line (isometric to $\mathbb{R}$). Hence, T^*_Γ would have even number of ends. $\square$

3.11. We choose a non-cyclic finite subgroup $G \subseteq \Gamma$ which is maximal with respect to the following ordering: $D_{2m+1} \prec D_{2n} \prec A_4 \prec S_4 \prec A_5$ for any positive integers m and n, and $D_{2m+1} \prec D_{2n+1}$ and $D_{2m} \prec D_{2n}$ for $m \leq n$. We can actually find a maximal one since elements of finite order in Γ comprise a finite number of conjugacy classes ([GvP80, I.§3]). Let us denote by $\Phi\colon \mathcal{T}^*_G \hookrightarrow \mathcal{T}^*_\Gamma$ the natural inclusion of trees, and by $\phi\colon \mathsf{T}^*_G \to \mathsf{T}^*_\Gamma$ the induced map. As we have seen in (2.9)$\sim$(2.12) the tree T^*_G is the union of three half-lines ℓ_i ($i = 0, 1, \infty$) starting at one vertex v such that $\ell_i \bigcap \ell_j = \{v\}$ for $i \neq j$. Take a section $\iota_G\colon \mathsf{T}^*_G \hookrightarrow \mathcal{T}^*_G$ of the quotient map $\varrho_G\colon \mathcal{T}^*_G \to \mathsf{T}^*_G$. Set $\Lambda = \iota_G(v)$. Clearly we have $G_\Lambda \subseteq \Gamma_{\Phi(\Lambda)}$.

3.12. Proposition. (1) *The map ϕ induces an injective map,*

$$\{\textit{Edges } \sigma \dashv v \textit{ in } \mathsf{T}^*_G\ \} \longhookrightarrow \{\textit{Edges } \sigma \dashv \phi(v) \textit{ in } \mathsf{T}^*_\Gamma\ \}. \tag{3.12.1}$$

(2) *The mirror of any elliptic element in Γ is mapped by ϱ_Γ to either a stright-line or a half-line.*

(3) *The map ϕ induces an isometry between T^*_G and $\phi(\mathsf{T}^*_G)$. Moreover, the subtree $\phi(\mathsf{T}^*_G)$ in T^*_Γ with the induced groups is a contraction of $(\mathsf{T}^*_\Gamma, \Gamma_\bullet)$.*

Proof. The proof goes step by step.

Step 1. First we prove (1) except the case $G \cong A_4$ with $p = 3$. If $G_\Lambda = G$, then we have $\Gamma_{\Phi(\Lambda)} = G$ by our choice of G. In this case, the claim is clear by (3.2.2). Suppose $G_\Lambda \neq G$; this occurs if $G \cong A_5$ ($p = 2, 3, 5$), $G \cong S_4$ ($p = 2, 3$), $G \cong A_4$ ($p = 3$), or $G \cong D_l$ ($p = 2$). If the map (3.12.1) is not injective, then there exists an element $\chi \in \Gamma_{\Phi(\Lambda)}$ of order 2 or 3 which permutes the edges in $\Phi(\iota_G(\mathsf{T}^*_G))$ emanating from $\Phi(\Lambda)$. We clearly have $G_\Lambda \bigcap \langle\chi\rangle = \{1\}$ in $\Gamma_{\Phi(\Lambda)}$. In case $G \cong D_{2m+1}$ and $p = 2$ ($G_\Lambda \cong Z_2$), $\Gamma_{\Phi(\Lambda)}$ is dihedral of even degree or non-dihedral; in case $G \cong D_{2n}$ and $p = 2$ ($G_\Lambda \cong D_2$), $\Gamma_{\Phi(\Lambda)}$ would be non-dihedral; both contradict our choice of G. For the other cases, we consider the local representation $\rho\colon \Gamma_{\Phi(\Lambda)} \to \mathrm{PGL}(2, k)$ around $\Phi(\Lambda)$ as in (3.2). As we see in Figure 3$\sim$5, there exists a non-trivial subgroup H in G_Λ which fixes three points in $\mathbb{P}^1_k$, and hence acts on $\mathbb{P}^1_k$ by the identity. Hence $\Gamma_{\Phi(\Lambda)}$ is a subgroup such that (a) $G_\Lambda \subset \Gamma_{\Phi(\Lambda)}$, (b) $\Gamma_{\Phi(\Lambda)} \prec G$ and (c) there exists a normal subgroup $N \subset \Gamma_{\Phi(\Lambda)}$ with $H \subseteq N$ and $N \bigcap \langle\chi\rangle = \{1\}$. But, except $G \cong A_4$ with $p = 3$, we easily find that such a group does not exist, thereby a contradiction. If $G \cong A_4$ and $p = 3$, then $\Gamma_{\Phi(\Lambda)} \cong D_3$ would occur; we postpone this case to Step 3 below.

Step 2. Next we prove (2). Let $\gamma \in \Gamma$ be an elliptic element, and suppose on the contrary that $\mathsf{M}(\gamma)$ is mapped to a segment of finite length. Then it is folded at at least two vertices Λ and Λ'. We can take such vertices so that the path $[\Lambda, \Lambda']$ is mapped by ϱ_Γ bijectively onto a path. By Proposition 3.5 we have non-trivial $\chi \in \Gamma_\Lambda$ (resp. $\chi' \in \Gamma_{\Lambda'}$) such that each half-line starting at Λ (resp. Λ') and converging to an end of $\mathsf{M}(\chi)$ (resp. $\mathsf{M}(\chi')$) is perpendicular to $\mathsf{M}(\gamma)$. We have therefore four half-lines; since T^*_Γ has only three ends, at least one of them should be folded somewhere. Let $[\Lambda'', \Lambda''']$ be the minimal path which connects $\mathsf{M}(\chi)$ and $\mathsf{M}(\chi')$. We define a subtree $\mathcal{T}_0$ of $\mathcal{T}^*_\Gamma$ as follows:

— If these mirrors are not folded at Λ'' or Λ''', define $\mathcal{T}_0$ to be the union of $[\Lambda'', \Lambda''']$, $\mathsf{M}(\chi)$ and $\mathsf{M}(\chi')$.

— If, for instance, $\mathsf{M}(\chi)$ is folded at Λ'', then $\Gamma_{\Lambda''}$ is isomorphic either to D_{2m}, S_4 or A_5, since it interchanges two pairs of ends. Then there exists one more mirror; we accordingly define $\mathcal{T}_0$ to be the union of $[\Lambda'', \Lambda''']$ and all such mirrors.

In any case the tree $\varrho_\Gamma(\mathcal{T}_0)$ has at least one tip (= a vertex from which only one edge emanates); this is clear in the first case, and is easy in the second case by what we have proved in Step 1. The folding above the tip of $\varrho_\Gamma(\mathcal{T}_0)$ yields another pair of half-lines perpendicular to $\mathcal{T}_0$. Repeating this procedure, we can define inductively a sequences $\{\mathcal{T}_n\}_{n=0}^\infty$ of subtrees of $\mathcal{T}^*_\Gamma$ with $\mathcal{T}_i \subset \mathcal{T}_{i+1}$ such that

(a) the number of tips of $\varrho_\Gamma(\mathcal{T}_{i+1})$ is twice as large as that of $\varrho_\Gamma(\mathcal{T}_i)$,

(b) any tips of $\varrho_\Gamma(\mathcal{T}_{i+1})$ is not in $\varrho_\Gamma(\mathcal{T}_i)$.

But this immediately yields a contradiction, since $\mathsf{T} = \bigcup_{n=0}^\infty \varrho_\Gamma(\mathcal{T}_n)$ is a subtree of T^*_Γ with infinitely many ends. We therefore have proved (2).

Step 3. Suppose $G \cong A_4$ and $p = 3$ ($G_\Lambda \cong Z_3$). If the map (3.12.1) is not injective, then $\Gamma_{\Phi(\Lambda)}$ would be D_3, as we already deduced in Step 1. Let χ be as in Step 1; it is easy to see that χ is of order 2. We have three mirrors (two order-2 and one order-3) which are distinct in $\mathcal{T}^*_\Gamma$ up to the action of Γ. The order-2 mirror other than $\mathsf{M}(\chi)$ is folded at the vertex fixed by G (see Figure 3). Since T^*_Γ has only three ends, the mirror $\mathsf{M}(\chi)$ also should be folded, but not at $\Phi(\Lambda)$ (because of (2.9)). Because the folding yields another half-line, we have at least four half-lines which is mapped by ϱ_Γ to pairwise distinct half-lines (by (2)), which is a contradiction. Hence the proof of (1) is finished.

Step 4. Finally we prove (3). It suffices by Proposition 3.8 to show that the map ϕ induces the bijection between ends of T^*_G and of T^*_Γ. If not, at least one half-line in $\Phi(\iota_G(\mathsf{T}^*_G))$ is folded; by (1) these foldings do not occur at $\Phi(\Lambda)$. Then, by an argument similar to that in Step 2, we can find a sequences $\{\mathcal{T}_n\}_{n=0}^\infty$ of subtrees of $\mathcal{T}^*_\Gamma$ satisfying (a) and (b), which is a contradiction. Now the proof of (3), and hence of the proposition is finished. $\square$

3.13. By Proposition 3.12 we particularly deduce the following: For any vertex $\Lambda \in \mathcal{T}^*_G$ we have $G_\Lambda \subset \Gamma_{\Phi(\Lambda)}$, $\Gamma_{\Phi(\Lambda)} \prec G$, and that there exists no cyclic subgroup of $\Gamma_{\Phi(\Lambda)}$ of which every conjugate has only the trivial intersection with G_Λ (otherwise, $\mathcal{T}^*_\Gamma$ has a half-line which is perpendicular to $\mathcal{T}^*_G$, and contradicts Proposition 3.6). We therefore find that

(a) $\Gamma_{\Phi(\Lambda)} = G_\Lambda$ if $G_\Lambda = G$,

(b) $\Gamma_{\Phi(\Lambda)} = G_\Lambda$ if $G_\Lambda \cong A_4$, S_4, A_5,

(c) $\Gamma_{\Phi(\Lambda)} \cong D_{ml}$ for $l \geq 1$ if $G_\Lambda \cong D_m$,

(d) $\Gamma_{\Phi(\Lambda)} \cong Z_{ml}$ for $l \geq 1$ if $G_\Lambda \cong Z_m$.

In particular, for any $\theta \in G$ ($\theta \neq 1$) $\Phi(\mathsf{M}(\theta))$ is the mirror of an element $\gamma \in \Gamma$ such that $\gamma^l = \theta$ for some $l \geq 1$ (by (3.1.3)). Let θ_0, θ_1 and θ_∞ be generators of the stabilizers of ends of $\mathcal{T}^*_G$ which are mapped by ϱ_G to the ends different among others.

For each $i = 0, 1, \infty$ let γ_i be a generator of an end of $\phi(\mathsf{M}(\theta_i))$; by Proposition 3.12 (3) and (1.3.3) Γ is generated by γ_0, γ_1 and γ_∞, and moreover, changing the indices if necessary, we may assume $e_i =$ the order of γ_i. We also deduce that there exists positive integers l_0, l_1 and l_∞ such that $\gamma_i^{l_i} = \theta_i$. Here l_i (for each $i = 0, 1, \infty$) is allowed to be > 1 only if the mirror $\mathsf{M}(\theta_i)$ does not pass through a vertex Λ in $\mathcal{T}^*_G$ with $G_\Lambda = G$, since $\Gamma_{\Phi(\Lambda)} = G_\Lambda$. Hence:

3.14. Corollary. $p \leq 5$.

PROOF. As we saw in (2.9)∼(2.12) all the mirrors $\mathsf{M}(\theta_i)$ pass through the unique vertex Λ° which is fixed by G if $p > 5$. □

3.15. Now the rest of the proof of "only if" part of Theorem (1) goes as follows: Suppose $G \cong A_5$. Then we have three possibilities:

(1) If $p = 5$, then only the order-5 mirrors are separated from the vertex fixed by G (cf. (2.11.2)). By Lemma 2.4 we find that we can replace the index 5 by $5l$ ($l \geq 2$) keeping the rules (a)∼(d). Hence we have $(e_0, e_1, e_\infty) = (2, 3, 5l)$ for $l \geq 2$.

(2) If $p = 3$, then only the order-3 mirrors are separated from the vertex fixed by G (cf. (2.11.3)). By Lemma 2.4 we find that we can replace the index 3 by $3l$ ($l \geq 2$) keeping the rules (a)∼(d). Hence we have $(e_0, e_1, e_\infty) = (2, 5, 3l)$ for $l \geq 2$.

(3) If $p = 2$, then only the order-2 mirrors are separated from the vertex fixed by G (cf. (2.11.4)). We have $(e_0, e_1, e_\infty) = (3, 5, 2l)$ ($l \geq 2$). But, if l is even, then the vertices Λ'_i (in the notation as in (2.11.4)) are fixed by the group generated by a subgroup $H \cong A_4$ and a subgroup $Z \cong Z_4$ with $H \bigcap Z \cong Z_2$ by Lemma 2.4. But this contradicts (b) above. Hence we have $(e_0, e_1, e_\infty) = (3, 5, 2l)$ for $l \geq 3$ and l: odd.

All the rest cases are similar except the following two cases:

(3.15.1) Suppose $G \cong S_4$ and $p = 2$. Then by (2.11.3) order-4 mirrors and order-2 are separated from the vertex fixed by S_4. Hence we have $(e_0, e_1, e_\infty) = (3, 2l, 4m)$. But, if l is even, then the vertex Λ_2 (in the notation as in (2.11.3)) is fixed by the group generated by a subgroup $H \cong D_4$ and a subgroup $Z \cong Z_4$ with $H \bigcap Z \cong Z_2$ due to Lemma 2.4. As we see immediately this group must be isomorphic to S_4. But this contradicts (c) above. Hence l must be odd.

(3.15.2) Suppose $G \cong D_{2n}$ and $p = 2$. Similarly as above, we get $(e_0, e_1, e_\infty) = (2l, 2m, 2n)$. But, if two of l, m, n are even, then by Lemma 2.4 the vertex Λ° (cf. (2.9.2)) is fixed by a group isomorphic to S_4, which is a contradiction. Hence we deduce at least two of l, m, n are odd.

Now the "only if" part of Theorem (1) is proved. The above argument also shows that by (3.7.1), Lemma 2.4 and Proposition 3.8 the subgroup Γ is (if it exists) isomorphic to the amalgam product given in Table 1. Moreover, since the conjugacy class of G in $\mathrm{PGL}(2, K)$ is unique (cf. [Web99, Vol. II, §67]), it follows that the conjugacy class of Γ is also unique.

Hence it remains only to show the "if" part of Theorem (1). This part involves a criterion of discreteness:

3.16. Lemma. *Let Γ be a subgroup in $\mathrm{PGL}(2, K)$ which acts on a subtree $\mathcal{T}$ of*

$\mathcal{T}_K$. *Suppose that the stabilizer of at least one vertex is finite. Then* Γ *is discrete. (Hence, in particular, the stabilizer of any vertex is finite.)*

PROOF. If $\{\gamma_i\} \subset \Gamma$ is a sequence converging to 1, then, except finitely many γ_i's, they are contained in the stabilizer of Λ, since the stabilizer of a vertex in $\mathrm{PGL}(2,K)$ is an open neighborhood of 1. $\square$

3.17. The rest of the proof goes as follows: For each of the above cases as in Table 1 we choose three elements $\gamma_i \in \mathrm{PGL}(2,K)$ $(i = 0,1,2)$ having K-valued fixed points such that at least two of $\mathsf{M}(\gamma_i)$'s are disjoint. Then we construct a tree $\mathcal{T}$ as follows: Let T be the smallest connected subtree of $\mathcal{T}_K$ which contains all $\mathsf{M}(\gamma_i)$. Let Γ be the subgroup in $\mathrm{PGL}(2,K)$ generated by γ_i's. Then $\mathcal{T}$ is defined to be the union of all $\gamma \cdot \mathsf{T}$ for $\gamma \in \Gamma$ (it is easy to see that $\mathcal{T}$ is connected (cf. [Ser80, I.4.5 Theorem 9])). Clearly Γ acts on Σ. Note that our construction of the tree $\mathcal{T}$ is the same as that in [Ser80, I.4.5 Theorem 9]; the only difference is that the tree in [loc. cit.] are abstract one, while we constructed it inside $\mathcal{T}_K$. If Γ and $\mathcal{T}$ satisfies the hypothesis of Lemma 3.16, and if a fundamental domain (in T) has exactly three half-lines with groups of expected order, we have done.

The constructions of Γ with given (e_0, e_1, e_∞) of all cases are quite similar to among others; here we only indicate the construction in $(2,3,5l)$-case. This is the case in $p = 5$. Let G be a finite subgroup in $\mathrm{PGL}(2,K)$ isomorphic to A_5. It has three generators γ_0, γ_1, and θ_∞ of order 3, 2, and 5, respectively. Let γ_∞ be an element in $\mathrm{PGL}(2,K)$ of order $5l$ $(l > 1)$ such that $\gamma_\infty^l = \theta_\infty$. Then $\mathsf{M}(\gamma_1)$ and $\mathsf{M}(\gamma_\infty)$ intersects at exactly one vertex Λ_1, and $\mathsf{M}(\gamma_0)$ is disjoint from $\mathsf{M}(\gamma_\infty)$ (cf. (2.11.2)). Hence T as above is the union of these three mirrors. Moreover, we find by Lemma 2.4 that the subgroup Γ generated by γ_0, γ_1 and γ_∞ is isomorphic to $A_5 *_{D_5} D_{5l}$ as desired. Since the stabilizer of Λ_0 is G (cf. [Ser80, I.4.5 Theorem 9]), Γ is discrete. Now by (2.11) those three mirrors are folded to three half-lines; the order of the groups attached to them are exactly $(3,2,5l)$ as desired.

Now all the statements in the theorem have been proved.

A. Appendix: Combinatorial data

A.1. This appendix gives the tables of combinatorial data by which one can see how the mirrors of elements in finite subgroup in $\mathrm{PGL}(2,K)$ are arranged in $\mathcal{T}_K^{\mathrm{BT}}$ so that one can draw the picture of $\mathcal{T}_G^*$ (cf. §2). The basic principle is as follows: Suppose G is a finite subgroup in $\mathrm{PGL}(2,K)$ and $\gamma, \theta \in \mathrm{PGL}(2,K)$ $(\gamma \neq 1,\ \theta \neq 1)$. We assume that fixed points a, b (resp. c, d) of γ (resp. θ) are in $\mathbb{P}^1(K)$. Then the mirror of γ (resp. θ) is given by $]a,b[$ (resp. $]c,d[$). As we saw in Lemma 2.2 the correlation between $\mathsf{M}(g)$ and $\mathsf{M}(h)$ can be calculated by the cross-ratios $R(a,b;c,d)$ and $R(b,a;c,d)$. If we have the complete list of these values for every pair of generators of maximal cyclic subgroups in G, we therefore can perfectly describe $\mathcal{T}_G^*$.

A.2. *Convention.* For the subgroups $G \cong A_4, S_4, A_5$, which are given in the standard forms as presented below, we will give a complete list of maximal cyclic subgroups by choosing generators. To two of these generators, say γ and θ, we associate an expression $P(\gamma,\theta)=P(\theta,\gamma)$, which is either a number or an unordered pair of numbers. The meaning of $P(\gamma,\theta)$ is:

— If it is simply a number, then both $|\nu(R(a,b;c,d))|$ and $|\nu(R(b,a;c,d))|$ are equal to $|\nu(P(\gamma,\theta))|$.

— If it is a pair of numbers, say $P(\gamma,\theta)=\{s,t\}$, then, as a set, $\{|\nu(s)|,|\nu(t)|\}$ coincides with $\{|\nu(R(a,b;c,d))|,|\nu(R(b,a;c,d))|\}$.

A.3. Here, for the careful reader, we briefly mention how to calculate $P(\gamma,\theta)$: First, calculate the algebraic numbers $R=R(a,b;c,d)$ and $S=R(b,a;c,d)$ (faster by computer), which are sometimes fairly complicated. Then, look at $R\cdot S$ and $R+S$. If they are both p-adic integers, then so are R and S; if not, replace R and S by the inverse of them. Then, in our case, $R\cdot S$ and $R+S$ will turn out to be p-adic integers. Now the two numbers $R\cdot S$ and $R+S$ are always very simple, and can be easily valuated, from which the valuations of R and S can be easily calculated (actually, only the following two cases occur; either $\nu(R\cdot S)=\nu(R+S)$ or $\nu(R+S)=0$; see Figure 1 in §2).

Drawing $\mathcal{T}^*_G$ in a picture is a tedious work; but as a hint for the interested reader we give it for the case $G\cong A_5$ and $p=2$ in (A.6) below.

A.4. *Case* $G\cong A_4$ *or* S_4. Since A_4 is a subgroup of S_4, the calculation can be mixed up. Let G be a subgroup isomorphic to S_4. By [Web99, §73] we may assume that G is generated by θ and χ with

$$\theta=\begin{bmatrix}\sqrt{i} & 0\\ 0 & \frac{1}{\sqrt{i}}\end{bmatrix},\qquad \chi=\begin{bmatrix}\frac{1-i}{2} & \frac{1-i}{2}\\ -\frac{1+i}{2} & \frac{1+i}{2}\end{bmatrix},$$

where i denotes a primitive 4-th root of unity. We set $\omega=\chi\theta\chi\theta^2$. The group G has three cyclic groups of order 4 generated by each of $\theta,\chi^2\theta^3,\chi\omega\theta^2$, four cyclic groups of order 3 generated by each of $\chi,\theta\chi\theta^3,\theta^2\chi\theta^2,\theta^3\chi\theta$, and six cyclic groups of order 2, not coming from those of order 4, generated by each of $\omega,\omega\theta^2,\omega\chi,\chi^2\theta,\omega\chi^2,\theta\chi^2$. Table 2 presents the $P(g,h)$ for any distinct two of these elements.

The group G has a subgroup isomorphic to A_4 generated by θ^2 and χ. It has four cyclic groups of order 3, $\langle\chi\rangle$, $\langle\theta^2\chi\theta^2\rangle$, $\langle\theta\chi\theta^3\rangle$, and $\langle\theta^3\chi\theta\rangle$, and three cyclic groups of order 2, $\langle\theta^2\rangle$, $\langle\chi\theta^2\chi^2\rangle$, and $\langle\chi^2\theta^2\chi\rangle$.

TABLE 2: $P(g,h)$ for g and h generators of cyclic subgroups in $G \cong S_4$. Here $\heartsuit = \{3,6\}$ and $\diamondsuit = \{4,12\}$.

	$\chi^2\theta^3$	$\chi\omega\theta^2$	χ	$\theta\chi\theta^3$	$\theta^2\chi\theta^2$	$\theta^3\chi\theta$	ω	$\omega\theta^2$	$\omega\chi$	$\chi^2\theta$	$\omega\chi^2$	$\theta\chi^2$
θ	2	2	$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	2	2	$2\sqrt{2}$	$2\sqrt{2}$	$2\sqrt{2}$	$2\sqrt{2}$
$\chi\theta\chi^2$		2	$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	$2\sqrt{2}$	$2\sqrt{2}$	2	2	$2\sqrt{2}$	$2\sqrt{2}$
$\chi^2\theta\chi$			$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	$2\sqrt{2}$	$2\sqrt{2}$	$2\sqrt{2}$	$2\sqrt{2}$	2	2
χ				$\heartsuit$	$\heartsuit$	$\heartsuit$	2	$2\sqrt{3}$	2	$2\sqrt{3}$	2	$2\sqrt{3}$
$\theta\chi\theta^3$					$\heartsuit$	$\heartsuit$	$2\sqrt{3}$	2	$2\sqrt{3}$	2	2	$2\sqrt{3}$
$\theta^2\chi\theta^2$						$\heartsuit$	2	$2\sqrt{3}$	$2\sqrt{3}$	2	$2\sqrt{3}$	2
$\theta^3\chi\theta$							$2\sqrt{3}$	2	2	$2\sqrt{3}$	$2\sqrt{3}$	2
ω								2	$\diamondsuit$	$\diamondsuit$	$\diamondsuit$	$\diamondsuit$
$\omega\theta^2$									$\diamondsuit$	$\diamondsuit$	$\diamondsuit$	$\diamondsuit$
$\omega\chi$										2	$\diamondsuit$	$\diamondsuit$
$\chi^2\theta$											$\diamondsuit$	$\diamondsuit$
$\omega\chi^2$												2

A.5. *Case* $G \cong A_5$. By [loc. cit. §74] G is generated by θ and χ with

$$\theta = \begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix}, \quad \chi = \begin{bmatrix} \zeta+\zeta^4 & 1 \\ 1 & -(\zeta+\zeta^4) \end{bmatrix},$$

where ζ is a primitive 5-th root of unity. We set $\varphi = \chi\theta^{-1}\chi\theta\chi\theta^{-1}$, which sends $z \mapsto -1/z$. The group G has

— six cyclic subgroups of order 5

$$\langle\theta\rangle,\ \langle\theta\chi\rangle,\ \langle\theta\chi\theta^3\rangle,\ \langle\theta^2\chi\theta^2\rangle,\ \langle\theta^3\chi\theta\rangle,\ \langle\theta^4\chi\rangle,$$

— ten cyclic subgroups of order 3

$$\langle\theta\chi\theta\rangle,\ \langle\theta^2\chi\rangle,\ \langle\theta^3\chi\rangle,\ \langle\theta^2\chi\theta\rangle,\ \langle\theta^3\chi\theta^4\rangle,$$

$$\langle\theta\chi\varphi\rangle,\ \langle\theta^4\chi\varphi\rangle,\ \langle\theta^2\chi\varphi\theta^4\rangle,\ \langle\theta^2\chi\varphi\theta^2\rangle,\ \langle\theta^3\chi\varphi\theta\rangle,$$

— and fifteen cyclic subgroups of order 2

$$\langle\varphi\theta^i\rangle,\ \langle\theta^{-i}\chi\theta^i\rangle,\ \langle\theta^{-i}\chi\varphi\theta^i\rangle \quad (i = 0,1,2,3,4).$$

Under these notation we have:

— $P(g,h) = \sqrt{5}$ if g and h are generators of cyclic groups of order 5 ($g \neq h$).

— $P(g,h) = \sqrt{3}\cdot 5^{1/4}$ if g (resp. h) is one of the generators of cyclic groups of order 5 (resp. 3).

All the rest are shown in the tables below:

Table 3: $P(g,h)$ for g and h generators of cyclic subgroups in $G \cong A_5$ of order 3. Here $\heartsuit = \{3, 6\}$.

	$\theta^2\chi$	$\theta^3\chi$	$\theta^2\chi\theta$	$\theta^3\chi\theta^4$	$\theta\chi\varphi$	$\theta^4\chi\varphi$	$\theta^2\chi\varphi\theta^4$	$\theta^2\chi\varphi\theta^2$	$\theta^3\chi\varphi\theta$
$\theta\chi\theta$	$\heartsuit$	$\heartsuit$	3	3	$\heartsuit$	$\heartsuit$	$\heartsuit$	3	$\heartsuit$
$\theta^2\chi$		3	3	$\heartsuit$	$\heartsuit$	$\heartsuit$	$\heartsuit$	$\heartsuit$	3
$\theta^3\chi$			$\heartsuit$	3	$\heartsuit$	$\heartsuit$	3	$\heartsuit$	$\heartsuit$
$\theta^2\chi\theta$				$\heartsuit$	3	$\heartsuit$	$\heartsuit$	$\heartsuit$	$\heartsuit$
$\theta^3\chi\theta^4$					$\heartsuit$	3	$\heartsuit$	$\heartsuit$	$\heartsuit$
$\theta\chi\varphi$						3	3	$\heartsuit$	$\heartsuit$
$\theta^4\chi\varphi$							$\heartsuit$	$\heartsuit$	3
$\theta^2\chi\varphi\theta^4$								3	$\heartsuit$
$\theta^2\chi\varphi\theta^2$									3

Table 4: $P(g,h)$ for g (resp. h) a generator of cyclic group subgroups in $G \cong A_5$ of order 5 (resp. 2). Here $* = 2 \cdot 5^{1/4}$.

	φ	$\varphi\theta$	$\varphi\theta^2$	$\varphi\theta^3$	$\varphi\theta^4$	χ	$\theta\chi\theta^4$	$\theta^2\chi\theta^3$	$\theta^3\chi\theta^2$
θ	2	2	2	2	2	$*$	$*$	$*$	$*$
$\theta\chi$	$*$	$*$	$*$	$*$	2	$*$	$*$	2	$*$
$\theta\chi\theta^3$	$*$	$*$	2	$*$	$*$	2	$*$	$*$	2
$\theta^2\chi\theta^2$	2	$*$	$*$	$*$	$*$	$*$	2	$*$	$*$
$\theta^3\chi\theta$	$*$	$*$	$*$	2	$*$	2	$*$	2	$*$
$\theta^4\chi$	$*$	2	$*$	$*$	$*$	$*$	2	$*$	2

$\theta^4\chi\theta$	$\chi\varphi$	$\theta\chi\varphi\theta^4$	$\theta^2\chi\varphi\theta^3$	$\theta^3\chi\varphi\theta^2$	$\theta^4\chi\varphi\theta$	
$*$	$*$	$*$	$*$	$*$	$*$	θ
2	2	2	$*$	$*$	$*$	$\theta\chi$
$*$	$*$	2	2	$*$	$*$	$\theta\chi\theta^3$
2	$*$	$*$	2	2	$*$	$\theta^2\chi\theta^2$
$*$	$*$	$*$	$*$	2	2	$\theta^3\chi\theta$
$*$	2	$*$	$*$	$*$	2	$\theta^4\chi$

Table 5: $P(g,h)$ for g (resp. h) a generator of cyclic group subgroups in $G \cong A_5$ of order 3 (resp. 2).

	φ	$\varphi\theta$	$\varphi\theta^2$	$\varphi\theta^3$	$\varphi\theta^4$	χ	$\theta\chi\theta^4$	$\theta^2\chi\theta^3$	$\theta^3\chi\theta^2$
$\theta\chi\theta$	2	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$
$\theta^2\chi$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	2	$\sqrt{6}$	$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$	
$\theta^3\chi$	$2\sqrt{3}$	$\sqrt{6}$	2	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$	$2\sqrt{3}$
$\theta^2\chi\theta$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	2	$\sqrt{6}$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$
$\theta^3\chi\theta^4$	$\sqrt{6}$	2	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$2\sqrt{3}$
$\theta\chi\varphi$	$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$	$2\sqrt{3}$	2	2	2	$\sqrt{6}$	$2\sqrt{3}$
$\theta^4\chi\varphi$	$2\sqrt{3}$	2	$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$	2	$\sqrt{6}$	$2\sqrt{3}$	$\sqrt{6}$
$\theta^2\chi\varphi\theta^4$	$\sqrt{6}$	$2\sqrt{3}$	2	$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$	2	2	$\sqrt{6}$
$\theta^2\chi\varphi\theta^2$	2	$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	2	2
$\theta^3\chi\varphi\theta$	$\sqrt{6}$	$\sqrt{6}$	$2\sqrt{3}$	2	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$\sqrt{6}$	2

$\theta^4\chi\theta$	$\chi\varphi$	$\theta\chi\varphi\theta^4$	$\theta^2\chi\varphi\theta^3$	$\theta^3\chi\varphi\theta^2$	$\theta^4\chi\varphi\theta$	
$2\sqrt{3}$	$2\sqrt{3}$	2	$\sqrt{6}$	$\sqrt{6}$	2	$\theta\chi\theta$
$\sqrt{6}$	2	$\sqrt{6}$	2	$\sqrt{6}$	$\sqrt{6}$	$\theta^2\chi$
$2\sqrt{3}$	2	$\sqrt{6}$	$\sqrt{6}$	2	$2\sqrt{3}$	$\theta^3\chi$
$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$	2	$2\sqrt{3}$	2	$\theta^2\chi\theta$
$\sqrt{6}$	$\sqrt{6}$	2	$2\sqrt{3}$	2	$\sqrt{6}$	$\theta^3\chi\theta^4$
$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$\sqrt{6}$	$\theta\chi\varphi$
2	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$\theta^4\chi\varphi$
$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$\theta^2\chi\varphi\theta^4$
$\sqrt{6}$	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	$\theta^2\chi\varphi\theta^2$
2	$\sqrt{6}$	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\theta^3\chi\varphi\theta$

TABLE 6: $P(g,h)$ for g and h generators of cyclic group subgroups in $G \cong A_5$ of order 2. Here $\bigstar = \{4, 4\sqrt{5}\}$ and $\Diamond = \{4, 12\}$.

	$\varphi\theta$	$\varphi\theta^2$	$\varphi\theta^3$	$\varphi\theta^4$	χ	$\theta\chi\theta^4$	$\theta^2\chi\theta^3$	$\theta^3\chi\theta^2$
φ	$\bigstar$	$\bigstar$	$\bigstar$	$\bigstar$	2	$\bigstar$	$\Diamond$	$\Diamond$
$\varphi\theta$		$\bigstar$	$\bigstar$	$\bigstar$	$\Diamond$	$\bigstar$	2	$\bigstar$
$\varphi\theta^2$			$\bigstar$	$\bigstar$	$\bigstar$	$\Diamond$	$\Diamond$	$\bigstar$
$\varphi\theta^3$				$\bigstar$	$\bigstar$	2	$\bigstar$	$\Diamond$
$\varphi\theta^4$					$\Diamond$	$\Diamond$	$\bigstar$	2
χ						$\Diamond$	$\bigstar$	$\bigstar$
$\theta\chi\theta^4$							$\Diamond$	$\bigstar$
$\theta^2\chi\theta^3$								$\Diamond$
$\theta^3\chi\theta^2$								
$\theta^4\chi\theta$								
$\chi\varphi$								
$\theta\chi\varphi\theta^4$								
$\theta^2\chi\varphi\theta^3$								
$\theta^3\chi\varphi\theta^2$								

$\theta^4\chi\theta$	$\chi\varphi$	$\theta\chi\varphi\theta^4$	$\theta^2\chi\varphi\theta^3$	$\theta^3\chi\varphi\theta^2$	$\theta^4\chi\varphi\theta$	
$\bigstar$	2	$\Diamond$	$\bigstar$	$\bigstar$	$\Diamond$	φ
$\Diamond$	$\bigstar$	$\Diamond$	2	$\Diamond$	$\bigstar$	$\varphi\theta$
2	$\Diamond$	$\bigstar$	$\bigstar$	$\Diamond$	2	$\varphi\theta^2$
$\Diamond$	$\Diamond$	2	$\Diamond$	$\bigstar$	$\bigstar$	$\varphi\theta^3$
$\bigstar$	$\bigstar$	$\bigstar$	$\Diamond$	2	$\Diamond$	$\varphi\theta^4$
$\Diamond$	2	$\bigstar$	$\bigstar$	$\bigstar$	$\bigstar$	χ
$\bigstar$	$\bigstar$	2	$\bigstar$	$\bigstar$	$\bigstar$	$\theta\chi\theta^4$
$\bigstar$	$\bigstar$	$\bigstar$	2	$\bigstar$	$\bigstar$	$\theta^2\chi\theta^3$
$\Diamond$	$\bigstar$	$\bigstar$	$\bigstar$	2	$\bigstar$	$\theta^3\chi\theta^2$
	$\bigstar$	$\bigstar$	$\bigstar$	$\bigstar$	2	$\theta^4\chi\theta$
		$\bigstar$	$\Diamond$	$\Diamond$	$\bigstar$	$\chi\varphi$
			$\bigstar$	$\Diamond$	$\Diamond$	$\theta\chi\varphi\theta^4$
				$\bigstar$	$\Diamond$	$\theta^2\chi\varphi\theta^3$
					$\bigstar$	$\theta^3\chi\varphi\theta^2$

A.6. *Picture of $\mathcal{T}^*_G$ with $G \cong A_5$ and $p = 2$.* Mirrors are denoted by piecewise linear lines; both ends of them are indicated by arrows. Each generator of cyclic subgroups of order 3 is denoted twice at both ends of its mirror, while that of cyclic subgroups of order 2 is written once beside its mirror. The mirrors of generators of cyclic subgroups of order 5 are not drawn; they simply pass through the center of the picture. Every unit-segment is of length $e/2$.

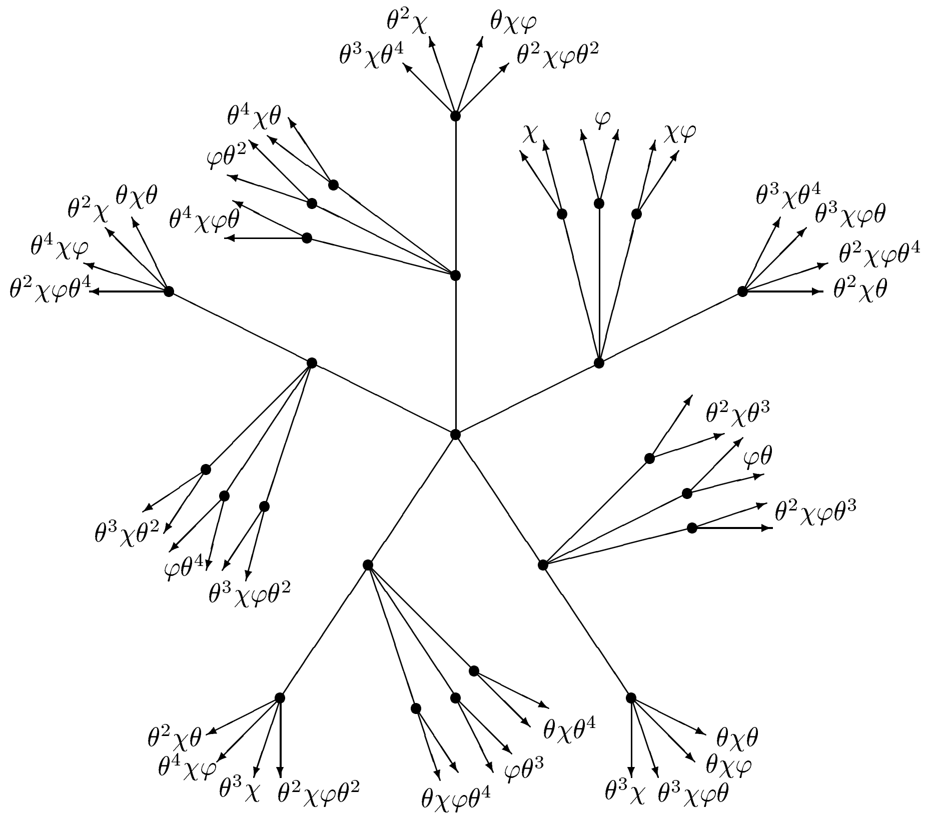


FIGURE 6: $\mathcal{T}^*_G$ for $G \cong A_5$ and $p = 2$

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