A Hilbert-Schmidt Property of Resolvent Differences of Singularly Perturbed Generalized Schrödinger Operators

M. Demuth *
J. A. van Casteren **

* Max-Planck-Research-Group
Dept. of Mathematics
University of Potsdam
Am Neuen Palais 10
O-1571 Potsdam
Germany

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

** Dept. of Mathematics and
Computer Science
University of Antwerp, UIA
Universiteitsplein 1
Antwerp 2610
Belgium

MPI / 92-48

A Hilbert-Schmidt Property of Resolvent Differences of Singularly Perturbed Generalized Schrödinger Operators

M. Demuth*
J. A. van Casteren**

March 1992

Abstract

Let K_0 be the self-adjoint generator of a Feller semi-group in $L^2(E,m)$, let V be a Kato-Feller potential and let Σ be an appropriate open subset of the locally compact second countable Hausdorff space E. Conditions are given in order that differences of (powers) of resolvents of the form $(aI+K_0\dot{+}V)^{-q}-J^*(aI+(K_0\dot{+}V)_\Sigma)^{-q}J$ are Hilbert-Schmidt operators. Here Jf is the restriction of the function f to Σ and J^*g extends the function with 0 on the complement of Σ. The operator $(K_0\dot{+}V)_\Sigma$ is the generator of the Dirichlet semigroup in $L^2(E,m)$ generated by $K_0\dot{+}V$, but killed on the complement of Σ.

Key Words

Generalized Schrödinger semigroup, Hilbert-Schmidt operator, Singular perturbation, Resolvent difference, Feynman-Kac formula, Harmonic extension operator.

AMS Classification

47D06, 47D07, 60J25

* Max-Planck-Research-Group, Dept. of Mathematics, University of Potsdam, Am Neuen Palais 10, 0-1571 Potsdam, Germany

** Dept. of Mathematics and Computer Science, University of Antwerp, UIA, Universiteitsplein 1, Antwerp 2610, Belgium

ON DIFFERENCES OF GENERALIZED SCHRÖDINGER SEMIGROUPS: HILBERT-SCHMIDT PROPERTIES

1. INTRODUCTION AND EXAMPLES

The main purpose of this paper is an exhibition of a number of conditions, guaranteeing that certain differences of generalized Schrödinger semi-groups consist of Hilbert-Schmidt operators. We will not only consider so-called regular perturbations, but also singular ones. This kind of properties has some spectral theoretical consequences like stability of the essential spectrum. In what follows we give some examples to which our results are applicable. For this reason we mention the kind of operators that generate (self-adjoint) Feller semi-groups in spaces of the form $C_\infty(E)$ or $L^2(E,m)$, where E is a locally compact second countable Hausdorff space. The formal definition of Feller semi-group will be given in BASSA, section 2.

EXAMPLE 1. Certain operators K_0 of the form

$$K_0 = -\frac{1}{2}\sum_{i,j=1}^{\nu} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{\nu} b_j(x)\frac{\partial}{\partial x_j} + c(x) \tag{1.1}$$

generate self-adjoint semi-groups in $L^2(\mathbb{R}^\nu)$; for details and conditions to be imposed see Kochubeĭ [46, Theorem 2.].

Under some appropriate assumptions the operator K_0 generator generates a Feller semi-group and hence a Markov process

$$\{(\Omega, \mathcal{F}, \mathsf{P}_x), (X(t) : t \geq 0), (\vartheta_t : t \geq 0), (E, \mathcal{E})\}$$

with state space $E = \mathbb{R}^\nu$. Moreover the one-dimensional distribution $\mathsf{P}_x(X(t) \in A)$ are given by $\mathsf{P}_x(X(t) \in A) = \int_A p_0(t,x,y)dy$. If $c \equiv 0$ and the functions a_{ij} and b_j, $1 \leq i,j \leq \nu$ are uniformly bounded then the corresponding Markov process has infinite lifetime. If the coefficients (a_{ij}) are unbounded, then the corresponding heat kernels cannot be estimated in terms of the classical Gaussian kernel, see [21, Example 2.14]. On the other hand we do have $p_0(t,x,y) \leq Ct^{-\nu/2}$ for all $t > 0$ and for all x and y in $\mathbb{R}^\nu$. In [69] Taira considers operators of the form (1.1) on (open) subsets of $\mathbb{R}^\nu$, but now with boundary conditions (Neumann, Dirichlet, Wentzell).

In the following two examples we consider relativistic Hamiltonians, which were introduced by Ichinose (see e.g. [31, 32, 33, 34, 35]). For systems without electromagnetic fields we use the notation of Carmona, Masters and Simon [9].

EXAMPLE 2. The present example is taken from Carmona, Masters and Simon [9]. Let μ be non-negative measure on $\mathbb{R}^\nu$ with the property that $\int_{\mathbb{R}^\nu} \min\left(1, |x|^2\right) d\mu(x) < \infty$, let a and b be a vector in $\mathbb{R}^\nu$ and let C be a square $\nu \times \nu$ matrix. In addition let $h : \mathbb{R}^\nu \to \mathbb{R}^\nu$ be a function of compact support with the property that $h(x) = x$ for all x in a neighborhood of the origin. Define the negative-definite function $F : \mathbb{R}^\nu \to \mathbb{R}^\nu$ by

$$F(p) = a + ib.p + p.Cp - \int_{\mathbb{R}^\nu} \left[e^{ip.x} - 1 - ip.h(x)\right] d\mu(x). \tag{1.2}$$

Then there exists a generator K_0 of a semi-group $\{\exp(-tK_0) : t \geq 0\}$ with the property that $\exp(-tK_0)$ is given by

$$\left[\exp(-tK_0)f\right](x) = \int_{\mathbb{R}^\nu} f(x+y)dm_t(y), \tag{1.3}$$

where f belongs to $C_\infty(\mathbb{R}^\nu)$. Here the family $\{m_t : t \geq 0\}$ is a vaguely continuous convolution semi-group of probability measures on $\mathbb{R}^\nu$ with the property that the Fourier transform $\widehat{m_t}$ of m_t is given by $\widehat{m_t}(p) = \exp(-tF(p))$, $p \in \mathbb{R}^\nu$. If, in addition the integrals $\int_{\mathbb{R}^\nu} \exp(-tF(p))dp$ are finite, then the function $p_0(t,x,y)$ defined by

$$p_0(t,x,y) = \frac{1}{(2\pi)^\nu} \int_{\mathbb{R}^\nu} \exp\left(-tF(p) + ip.(x-y)\right) dp \tag{1.4}$$

defines the density of the corresponding Markov process. In case $F(p) \equiv |p|^\alpha$, $0 < \alpha \leq 2$ (stable case), the following inequalities can be proved:

$$\frac{c_1}{t^{\nu/\alpha}\left|x-y\right|^{\nu+\alpha}} \leq p_0(t,x,y) \leq \frac{c_2}{t^{\nu/\alpha}\left|x-y\right|^{\nu+\alpha}}, \quad |x-y| \geq 1 \tag{1.5}$$

if $\alpha \neq 2$. In particular, if $F(p) = \sqrt{p^2+m^2} - m$, m fixed, then $K_0 = \sqrt{-\Delta + m^2} - m$. In this case the density is given by

$$\begin{aligned} &p_0(t,x,y) \\ &= \frac{1}{(2\pi)^\nu} \frac{t}{\sqrt{|x-y|^2+t^2}} \int_{\mathbb{R}^\nu} \exp\left(mt - \sqrt{\left(|x-y|^2+t^2\right)(p^2+m^2)}\right) dp. \quad (1.6) \\ &= \int_0^\infty \frac{\exp(mt - m^2u)}{(4\pi u)^{\nu/2}} \exp\left(-\frac{|x-y|^2}{4u}\right) \frac{t}{u\sqrt{u\pi}} \exp\left(-\frac{t^2}{4u}\right) du. \quad (1.7) \end{aligned}$$

The previous results can be generalized for negative definite functions F defined on a locally compact, second countable, abelian group G. In that case the variable p

varies over the dual group. It is also noticed that these results fit in the theory of Lévy processes.

EXAMPLE 3. The following example is related to the previous one and can be found in Ichinose [30]. Let the Lévy measure μ^m be defined by the equality:

$$\sqrt{p^2+m^2} = m - \int_{\{|y|>0\}} [\exp(ip.x) - 1 - ip.y 1_B(y)]\, d\mu^m(y). \tag{1.8}$$

Here $B = \{y \in \mathbb{R}^\nu : |y| < 1\}$. An explicit form of these measures is given by

$$d\mu^m(y) = 2\frac{m^{\frac{1}{2}(m+1)}}{(2\pi\,|y|)^{\frac{1}{2}(\nu+1)}} K_{\frac{1}{2}(\nu+1)}(m\,|y|)\, dy, \quad m > 0, \tag{1.9}$$

$$= \Gamma\left(\frac{\nu+1}{2}\right)\frac{1}{(\pi\,|y|)^{\nu+1}}dy, \quad m = 0. \tag{1.10}$$

Here $\Gamma(z)$ is the gamma function and $K_r(z)$ is the modified Bessel function of the third kind of order r. Let $A : \mathbb{R}^\nu \to \mathbb{R}^\nu$ be a locally bounded function with the property that the function

$$x \mapsto \int_{\{0<|y|<1\}} \frac{|A(x-y/2) - A(x)|}{|y|^\nu} dy \tag{1.11}$$

is also locally bounded. Define the *Weyl quantized relativistic Hamiltonian* H_A^m via the formula ($B_1 = \{y \in \mathbb{R}^\nu : |y| < 1\}$):

$$\begin{aligned}&[H_A^m f](x) - mf(x) \qquad (1.12)\\ &= -\int_{\{|y|>0\}} [\exp(-iyA(x+y/2))\, f(x+y) - f(x) - 1_{B_1}(y) y\, (\partial_x - iA(x))\, f(x)]\, d\mu^m(y).\end{aligned}$$

Under suitable conditions on A, the operator H_A^m is the following pseudodifferential operator:

$$\begin{aligned}&[H_A^m f](x) - mf(x)\\ &= \frac{1}{(2\pi)^\nu}\int\int_{\mathbb{R}^\nu\times\mathbb{R}^\nu} \exp(i(x-y).p)\, h_A^m\left(p, \frac{x+y}{2}\right) f(y)dydp,\end{aligned} \tag{1.13}$$

where h_A^m is the function

$$h_A^m(p,x) = \sqrt{(p - A(x))^2 + m^2}, \quad p \in \mathbb{R}^\nu,\ x \in \mathbb{R}^\nu. \tag{1.14}$$

The operator $K_0 \equiv H_A^m - mI$ is than the self-adjoint generator of a strongly continuous semi-group in $L^2(\mathbb{R}^\nu)$. Let the density function $p_0(t,x,y)$ be as Example 2. Then the integral kernel of the operator $\exp(-t(H_A^m - mI))$ is given by the imaginary path integral

$$\exp(-t(H_A^m - mI))(x,y) = \lim_{s\uparrow t} \mathsf{E}_x\left(\exp(-S(s))\, p_0(t-s, X(s), y)\right).$$

Here $\{(\Omega, \mathcal{F}, \mathsf{P}_x), (X(t): t \geq 0), (\vartheta_t : t \geq 0), (E, \mathcal{E})\}$ is the Markov process generated by $H_0^m - mI$, with state space $E = \mathbf{R}^\nu$ and with transition density $p_0(t,x,y)$. The process $S(t)$ is given by

$$\begin{aligned} S(t) &= i\int_0^t \int_{\mathbf{R}^\nu \setminus B_1} A(X(s-) + y/2)\,.y N_X(dsdy) \\ &\quad + i\int_0^t \int_{0<|y|<1} A(X(s-) + y/2)\,.y\widetilde{N}_X(dsdy) \\ &\quad + i\int_0^t \int_{0<|y|<1} [A(X(s) + y/2) - A(X(s))]\,.y\widehat{N}(dsdy) \\ &= i\int_0^t \int_{|y|>0} A(X(s-) + y/2)\,.y\widetilde{N}_X(dsdy) \\ &\quad + i\int_0^t \int_{0<|y|} [A(X(s) + y/2) - A(X(s))]\,.y\widehat{N}(dsdy) \end{aligned}$$

and here the random measures N_X, $\widetilde{N}_X$ and $\widehat{N}_X$ are respectively given by

$$\begin{aligned} &N_X((t_1,t_2] \times U) = \#\{s \in (t_1,t_2] : X(s) \neq X(s-), \quad X(s) - X(s-) \in U\}; \\ &\widetilde{N}_X(dsdy) = N_X(dsdy) - \widehat{N}(dsdy); \\ &\widehat{N}_X(dsdy) = \mathsf{E}_0(N_X(dsdy)) = dsd\mu^m, \end{aligned}$$

where $0 < t_1 < t_2$ and where U is a Borel subset of $\mathbf{R}^\nu \setminus \{0\}$. For more details see Ichinose [34].

EXAMPLE 4. This example is due to N. Jacob [38, 40]. Let $p : \mathbf{R}^\nu \times \mathbf{R}^\nu \to \mathbf{R}$ be a continuous function such that for fixed $\xi \in \mathbf{R}^\nu$ the function $x \mapsto p(x,\xi)$ is a bounded C^∞-function with bounded derivatives of all orders. Suppose that for fixed $x \in \mathbf{R}^\nu$, the function $\xi \mapsto p(x,\xi)$ is negative definite. In addition we assume that there exists a continuous negative function $a : \mathbf{R}^\nu \mapsto \mathbf{R}$ such that, for some $0 < r \leq 2$,

$$c_0 c_1 \left(1 + |\xi|^2\right)^{r/2} \leq c_1 \left(1 + a(\xi)^2\right)^{1/2} \leq p(x,\xi) \leq c_2 \left(1 + a(\xi)^2\right)^{1/2}. \tag{1.15}$$

Define the Sobolev spaces $H^{q,a}(\mathbf{R}^\nu)$, $q \geq 0$, as follows:

$$H^{q,a}(\mathbf{R}^\nu) = \left\{u \in L^2(\mathbf{R}^\nu) : \|u\|_{q,a} < \infty\right\}, \tag{1.16}$$

where $\|\cdot\|_{q,a}$ is given by:

$$\|u\|_{q,a}^2 = \int_{\mathbf{R}^\nu} \left(1 + a(\xi)^2\right)^q |\widehat{u}|^2 \, d\xi. \tag{1.17}$$

If $a(\xi) \equiv |\xi|$, then we just write $H^q(\mathbf{R}^\nu)$ instead of $H^{q,a}(\mathbf{R}^\nu)$ and $\|\cdot\|_q$ replaces $\|\cdot\|_{q,a}$. Put $H^\infty(\mathbf{R}^\nu) := \bigcap_{q\geq 0} H^q(\mathbf{R}^\nu)$ and let the pseudodifferential operator $p(x,D)$ be defined by

$$[p(x,D)u](x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^\nu} e^{ix.\xi} p(x,\xi)\widehat{u}(\xi) d\xi. \tag{1.18}$$

Again under some appropriate assumptions the operator $-p(x,D)$ generates a Feller semi-group in $C_\infty(\mathbf{R}^\nu)$. If, in addition, the inequality

$$\|u\|_{L^q}^2 \leq c\left(\mathcal{E}_0(u,u) + c_0 \|u\|_0^2\right), \quad u \in \mathrm{dom}(\mathcal{E}_0), \tag{1.19}$$

holds for some $q > 2$ and some constants c and c_0, then the associated Markov process $M = \{(\Omega, \mathcal{F}, \mathsf{P}_x), (X(t) : t \geq 0), (\vartheta_t : t \geq 0), (E, \mathcal{E})\}$, where $E = \mathbf{R}^\nu$, possesses the following property. There exists a Borel set N of capacity zero such that $\mathbf{R}^\nu \setminus N$ is M-invariant and such that:

(i) The resolvent kernel $R(\lambda)(x,\cdot)$ is absolutely continuous with respect to the Lebesgue measure for every $\lambda > 0$ and for each $x \in \mathbf{R}^\nu \setminus N$.

(ii) The transition function $\exp(tL)(x,\cdot)$ is absolutely continuous with respect to the Lebesgue measure for every $t > 0$ and for each $x \in \mathbf{R}^\nu \setminus N$.
Here $L = p(x,D)$ and $\mathcal{E}_0$ is the corresponding Dirichlet form:

$$\mathcal{E}_0(u,v) = \lim_{t\downarrow 0} \frac{\langle u,v\rangle - \langle \exp(tL)u, v\rangle}{t}, \quad u,v \in \mathrm{dom}(\mathcal{E}_0). \tag{1.20}$$

Here u belongs to $\mathrm{dom}(\mathcal{E}_0)$ if and only if the limit in the right-hand side of (1.20) exists with $v = u$.

EXAMPLE 5. It is perhaps interesting to recall Theorem 10.3. in Ikeda and Watanabe [37], stating that under appropriate conditions (boundedness of certain vector fields V_j, $0 \leq j \leq \nu$, and Hörmander's hypo-ellipticity condition) the P_x-distribution of the solution $(X(t) : t \geq 0)$ of the stochastic differential equation $dX(t) = \sigma(X(t))dB(t) + b(X(t))dt$, $X(0) = x$, defines a Markov process $\{(\Omega, \mathcal{F}, \mathsf{P}_x), (X(t) : t \geq 0), (\vartheta_t : t \geq 0), (E, \mathcal{E})\}$ with the property that, for every compact subset K, there exists a constant C_K such that, for appropriate n, $\|1_K P(t)\|_{1,\infty} \leq C_K t^{-n/2}$, $t \geq 0$. To make this precise we write $V_k(x) = \sum_{i=1}^\nu \sigma_{ik}(x)\frac{\partial}{\partial x_i}$, $k = 1, \ldots, \nu$, and

$$V_0(x) = \sum_{i=1}^\nu \left[b_i(x) - \frac{1}{2}\sum_{k=1}^\nu \sum_{j=1}^\nu \frac{\partial}{\partial x_j}\sigma_{ik}(x)\sigma_{jk}(x)\right] \frac{\partial}{\partial x_i}.$$

The above stochastic differential equation can be rewritten as follows:

$$\begin{cases} dX(t) = \sum_{k=1}^{\nu} V_k(X(t))dB_k(t) + V_0(X(t))dt, \\ X(0) = x. \end{cases}$$

Define for V a vector field on $\mathbb{R}^\nu$ the vector fields (V_k, V), $k = 1, \ldots, \nu$, by $(V_k, V) = [V_k, V]$ and define (V_0, V) by

$$(V_0, V) = [V_0, V] + \frac{1}{2}\sum_{\ell=1}^{\nu} [V_\ell, [V_\ell, V]].$$

The subsets of vector fields Σ_n, $n \in \mathbb{N}$, are given by $\Sigma_0 = \{V_1, V_2, \ldots, V_\nu\}$ and, for $n \geq 1$, Σ_n is given by

$$\Sigma_n = \{(V_k, V) : V \in \Sigma_{n-1}, k = 0, 1, \ldots, \nu\}.$$

The vector fields are V_k, $k = 0, 1, \ldots, \nu$ are said to satisfy the hypo-ellipticity condition of Hörmander at $x \in \mathbb{R}^\nu$ if there exists $m \in \mathbb{N}$ and $A_1, \ldots, A_\nu \in \bigcup_{n=0}^{n=m} \Sigma_k$ such that $A_1(x), \ldots, A_\nu(x)$ are linearly independent. Suppose that all the coefficients σ_{ij} and b_j are bounded and have bounded derivatives of all orders. Also suppose that Hörmander's condition is satisfied. Then the operator $L := \frac{1}{2}\sum_{k=1}^{\nu} V_k^2 + V_0$ generates a Feller semi-group with C^∞-density $p_0(t, x, y)$. For more details we refer the reader to Ikeda and Watanabe [37, Chapter V].

EXAMPLE 6. The previous example has its counterpart for Riemannian manifolds. In fact instead of the Laplace operator on $\mathbb{R}^\nu$ we can also consider the Laplace-Beltrami operator on a Riemannian manifold. For details we refer the reader to Elworthy [24], [23], Azencott et al [3], Bismut [6] and several others. The authors also establish existence results for and bounds on the corresponding heat kernels. A recent and very interesting paper is [17] written by Davies. It provides the reader with much insight into the behavior of heat kernels. Of course his book [16] should be consulted also.

EXAMPLE 7. In this example we consider so-called hyper-singular integrals. Define the operators Δ_h^ℓ on $C(\mathbb{R}^\nu)$ as follows:

$$\left[\Delta_h^\ell f\right](x) = \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} f(x - kh). \tag{1.21}$$

In [47] Kochubeĭ proves that operators K_0 of the form

$$\begin{aligned} [K_0 f](x) = &-\beta \sum_{i,j=1}^{\nu} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \frac{1-\beta}{d_{n,\ell}(\gamma)} \int_{\mathbb{R}^\nu} \Omega\left(x, \frac{h}{|h|}\right) \frac{\left[\Delta_h^\ell f\right](x)}{|h|^{\nu+\gamma}} dh \\ &+ \sum_{k=1}^{m} \frac{1}{d_{n,\ell}(\gamma_k)} \int_{\mathbb{R}^\nu} \Omega_k\left(x, \frac{h}{|h|}\right) \frac{\left[\Delta_h^\ell f\right](x)}{|h|^{\nu+\gamma_k}} dh \\ &+ \sum_{j=1}^{\nu} b_j(x) \frac{\partial f}{\partial x_j} \end{aligned} \tag{1.22}$$

generates a Feller semi-group in $C_\infty(\mathbb{R}^\nu)$ provided that the following conditions are satisfied:

(a) The functions Ω and Ω_k, $1 \le k \le m$, are non-negative and continuous on $\mathbb{R}^\nu \times S^{\nu-1}$. They are also even: $\Omega(x,\sigma) = \Omega(x,-\sigma)$, $x \in \mathbb{R}^\nu$, $\sigma \in S^{\nu-1}$.

(b) The orders of homogeneity γ, γ_k, $1 \le k \le m$, verify: $0 < \gamma_k < \gamma \le 2$. If $\gamma = 2$, then $\beta = 1$ and if $\gamma < 2$, then $\beta = 0$. If $\gamma = 1$, then $b_j \equiv 0$, $1 \le j \le \nu$.

(c) Some ellipticity conditions on (a_{ij}) are also required. In fact, the inequality $\mathrm{Re}\sum_{i,j=1}^{\nu} a_{ij}(x)\xi_i\overline{\xi}_j \ge a_0 |\xi|^2$, for all $\xi_j \in \mathbb{C}$, $1 \le j \le \nu$. Here a_0 is some strictly positive real number.

(d) The constants $d_{n,\ell}(\gamma)$ have to be chosen suitably. In fact they are chosen in such a way that the expression

$$\widetilde{\Omega}(x,\xi) = \frac{1}{d_{\nu,\ell}(\gamma)} \int_{\mathbb{R}^\nu} \frac{(1-\exp(-i\xi.h))^\ell}{|h|^{\nu+\gamma}} \Omega\left(x, \frac{h}{|h|}\right) dh,$$

called the *symbol* of the hyper-singular integral $D_\Omega^\gamma f$, does not depend on the particular choice of ℓ, where $\ell > \alpha$.

(e) The characteristics Ω and Ω_k, $1 \le k \le m$, are supposed to be non-negative and symmetric in the second variable, i.e. $\Omega(x,\sigma) = \Omega(x,-\sigma)$ for all $x \in \mathbb{R}^\nu$ and for all $\sigma \in S^{\nu-1}$.

Moreover the life time of the corresponding Markov process is ∞. In [47] Kochubeĭ proves that the corresponding Markov processes posses transition densities $p_0(t,x,y)$ verifying inequalities of the form

$$p_0(t,x,y) \le C \left\{ \frac{t}{\left[t^{1/\gamma} + |x-y|\right]^{\nu+\gamma}} + \sum_{k=1}^{m} \frac{t}{\left[t^{1/\gamma} + |x-y|\right]^{\nu+\gamma_k}} \right\}.$$

EXAMPLE 8. In this example we consider the generator $K_0 := -\frac{1}{2}\Delta + x.\nabla$ of the so-called Ornstein-Uhlenbeck process in $L^2\left(\mathbb{R}^\nu, \exp\left(-|y|^2\right)\pi^{-\nu/2}dy\right)$. Its integral kernel $p_0(t,x,y)$ is given by

$$\begin{aligned} &p_0(t,x,y) \\ &= \frac{1}{(1-\exp(-2t))^{\nu/2}} \exp\left(-\frac{\exp(-2t)|x|^2 + \exp(-2t)|y|^2 - 2\exp(-t)\langle x,y\rangle}{1-\exp(-2t)}\right). \end{aligned}$$

The semi-group in $L^2\left(\mathbb{R}^\nu, \exp\left(-|y|^2\right)\pi^{-\nu/2}dy\right)$ is given by

$$\begin{aligned} \left[\exp(-tK_0)f\right](x) &= \int p_0(t,x,y) f(y) \exp\left(-|y|^2\right) \frac{dy}{\pi^{\nu/2}} \\ &= \int f\left(\exp(-t)x + \sqrt{1-\exp(-2t)}\,y\right) \exp\left(-|y|^2\right) \frac{dy}{\pi^{\nu/2}}. \end{aligned}$$

For more details the reader is referred to e.g. Simon [61].

EXAMPLE 9. In this example we consider the generator $K_0 := -\frac{1}{2}\Delta + \frac{1}{2}|x|^2$ of the oscillator process. The integral kernel of the corresponding semi-group $\exp(-tK_0)(x,y)$ may be written as (again see Simon [61])

$$\exp(-tK_0)(x,y) = \exp\left(-\frac{1}{2}|x|^2\right)\frac{1}{(2\pi\sinh t)^{\nu/2}}\exp\left(-\frac{|\exp(-t)x-y|^2}{1-\exp(-2t)}\right)\exp\left(\frac{1}{2}|y|^2\right)$$

$$= \exp\left(-\frac{1}{2}|x|^2\tanh t\right)\frac{1}{(2\pi\sinh t)^{\nu/2}}\exp\left(-\frac{1}{2\sinh t}\left|\frac{x}{(\cosh t)^{1/2}} - y(\cosh t)^{1/2}\right|^2\right).$$

It follows that the corresponding semi-group $\{\exp(-tK_0) : t \geq 0\}$ is given by

$$[\exp(-tK_0)f](x)$$
$$= \frac{\exp\left(-\frac{1}{2}|x|^2\tanh t\right)}{(2\pi\cosh t)^{\nu/2}}\int \exp\left(-\frac{1}{2}|y|^2\right) f\left(\frac{x}{\cosh t} + \sqrt{\tanh t}\,y\right)dy.$$

For more details we refer again to Simon [61].

2. STOCHASTIC SPECTRAL ANALYSIS (BASSA)

There are different ways of introducing semi-groups with perturbed generators. The analytic way starts with the unperturbed semi-group and uses the Trotter-product formula to find a Feynman-Kac representation of the perturbed semi-group. The semi-analytic or semi-stochastic manner begins again with the unperturbed semi-group. Then the potentials are introduced stochastically by verifying the sensibility and the semi-group property of the Feynman-Kac formula.

In order to introduce semi-groups with perturbed generators we employ a purely stochastic approach in the sense that we begin with the process, or what is equivalent, with the transition density function. Our aim is to formulate all assumptions on the process or its generator in terms of assumptions on the density. An advantage is that we can consider a large class of generators, containing the examples in the introduction.

The objective of this paper is to present some Hilbert-Schmidt properties of differences of semi-groups generated by these operators. We start with the basic assumptions on the transition density function, which form the foundations of this theory. This theory will be called "Stochastic Spectral Analysis". The state space (or configuration space) will be a second countable locally compact Hausdorff space E with Borel field $\mathcal{E}$. A non-negative Radon measure m (reference measure) on $\mathcal{E}$ is given. Instead of $dm(x)$ or $m(dx)$ we usually write dx.

Basic Assumptions of Stochastic Spectral Analysis (BASSA).

In what follows the function $p_0(t,x,y)$ defined on $(0,\infty)\times E\times E$ will be a continuous density function with the following properties:

A1. It is non-negative and it verifies the Chapman-Kolmogorov identity, i.e. $\int p_0(s,x,z)p_0(t,z,y)dz = p_0(s+t,x,y)$, s, $t > 0$, x, $y \in E$, and its total mass is less than or equal to 1, i.e. $\int p_0(t,x,y)dy \leq 1$, $t > 0$, $x \in E$;

A2. (Feller property) For every $f \in C_\infty(E)$ the function $x \mapsto \int f(y)p_0(t,x,y)dm(y)$ belongs to $C_\infty(E)$;

A3. (continuity) For every $f \in C_\infty(E)$ and for every $x \in E$ the following identity is true: $\lim_{t\downarrow 0} \int f(y)p_0(t,x,y)dm(y) = f(x)$;

A4. The function $p_0(t,x,y)$ is symmetric: $p_0(t,x,y) = p_0(t,y,x)$ for all $t > 0$ and for all x and y in E.

Sometimes we shall need a *boundedness assumption* of the following form:

B. There exists finite constants m, b and c such that $0 \leq p_0(t,x,y) \leq ct^{-m}\exp(bt)$ for all $t > 0$ and for all x, $y \in E$.

Remark 1. It is well-known that there exists a strong Markov process

$$\{(\Omega, \mathcal{F}, \mathsf{P}_x), (X(t) : t \geq 0), (\vartheta_t : t \geq 0), (E, \mathcal{E})\}$$

(see e.g. Blumenthal and Getoor [7]) with the following properties. The one-dimensional distributions are given by $\mathsf{P}_x(X(t) \in B) = \int_B p_0(t,x,y)dy$, $t > 0$, B Borel subset of E. Its sample paths are P_x-almost surely right continuous and possess P_x-almost sure left limits in E on its life time. In other words the process $\{X(t), \mathsf{P}_x\}$ is cadlag on its life time. Moreover we may assume that the closure of the (random) set $\{X(s) : 0 \leq s < t\}$ is a compact subset of E, whenever $X(t-)$ belongs to E. In other the process does not re-enter E once it has hit δ, the point at infinity.

Remark 2. It is not necessarily true that densities are available. In principle one may formulate the basic assumptions (BASSA) in terms of the transition function $P(t,x,B) := \mathsf{P}_x(X(t) \in B)$, $t \geq 0$, $x \in E$, $B \in \mathcal{E}$, where $\mathcal{E}$ is the collection of Borel subsets of E.

This is perhaps the right place to fix some notation and insert an interesting inequality. Let K_0 be the L^2-generator of the Markov process

$$\{(\Omega, \mathcal{F}, \mathsf{P}_x), (X(t) : t \geq 0), (\vartheta_t : t \geq 0), (E, \mathcal{E})\}$$

and let a be a strictly positive real number. For any Borel function g, defined on E, we write

$$[\exp(-sK_0)g](x) = \mathsf{E}_x(g(X(s))) = \int p_0(s,x,y)g(y)dy \tag{2.1}$$

and

$$\left[(aI + K_0)^{-1}g\right](x) = \int_0^\infty e^{-as}[\exp(-sK_0)g](x)ds = \int_0^\infty e^{-as}p_0(s,x,y)g(y)dy$$

whenever these expressions make sense.

We begin with a definition and a result, due to Varopoulos, on the (spectral) dimension of a semi-group.

2.1. DEFINITION. The (kernel of the) semi-group $\{\exp(-tK_0) : t \geq 0\}$ is said to be of *spectral dimension* n, if $\|\exp(-tK_0)\|_{1,\infty} \leq Ct^{-n/2}$ for all (small) $t > 0$. In [74], Varopoulos shows that this is equivalent to saying that the generator K_0 verifies $\|f\|^2_{2n/(n-2)} \leq C\langle K_0 f, f\rangle$ for all $f \in D(K_0)$, provided $n > 2$. In [10] Coulhon gives a simple proof of this equivalence and in [11] the authors apply this result to semi-groups of operators acting on functions defined on a Lie group. In [8] the authors use, following Nash [51], the Dirichlet form associated to a semi-group, to characterize the dimension of a semi-group. Let $\mathcal{E}_0$ be the Dirichlet form associated to the semi-group $\{\exp(-tK_0) : t \geq 0\}$, i.e.

$$\mathcal{E}_0(f, f) = \lim_{t \downarrow 0} \frac{\langle f, f\rangle - \langle f, \exp(-tK_0)f\rangle}{t}, \quad f \in \mathrm{dom}(\mathcal{E}_0). \tag{2.2}$$

Then $\|\exp(-tK_0)\|_{1,\infty} \leq C_1 e^{\delta t} t^{-n/2}$, $t > 0$, if and only if

$$\|f\|_2^{2+4/n} \leq C_2 \left[\mathcal{E}_0(f, f) + \delta \|f\|_2^2\right] \|f\|_1^{4/n}$$

for all $f \in \mathrm{dom}(\mathcal{E}_0)$. The constant C_1 depends on C_2 and on n and the constant C_2 depends on C_1 and n. Another instance where the spectral dimension of a semi-group pops up is given Example 6.

Before in the next section we actually give some estimates on the norms of differences of semi-groups and resolvents, we insert a convenient inequality for the unperturbed resolvent. This inequality will among others be used in Theorem 2.5. Its proof will be omitted, but we refer to van Casteren [70, Theorem 6.4. p. 116-117] for a proof of a similar statement.

2.2. PROPOSITION. Let $g : E \to \mathbb{R}$ be a Borel measurable function and let a and η be strictly positive real numbers. The following inequalities are valid:

$$\begin{aligned}&\left(1 - e^{-a\eta}\right) \left\|(aI + K_0)^{-1} |g|\right\|_\infty \\ &\quad \leq \sup_{x \in E} \int_0^\eta \mathbb{E}_x(|g(X(s))|) ds \leq e^{a\eta} \left\|(aI + K_0)^{-1} |g|\right\|_\infty .\end{aligned} \tag{2.3}$$

For a concise formulation of our results we introduce the following definitions.

2.3. DEFINITION. Let $V : E \to [0, \infty]$ be a Borel measurable function on E.

(a) The function V is said to belong to $K(E)$ if

$$\limsup_{t \downarrow 0} \left\| \int_0^t P_0(s) V ds \right\|_{\infty,\infty} = \limsup_{t \downarrow 0} \sup_{x \in E} \int_0^t \left(\int p_0(s, x, y) V(y) dm(y) \right) ds = 0. \tag{2.4}$$

(b) The Borel measurable function $V : E \to [0, \infty]$ belongs to $K_{loc}(E) = K_{loc}(E, A_0)$ if $1_K V$ belongs to $K(E)$ for all compact subsets K of E.

(c) The Borel measurable function $V = V_+ - V_-$ is said to be a Kato-Feller potential if its positive part $V_+ = \max(V, 0)$ belongs to $K_{\text{loc}}(E)$ and if its negative part $V_- = \max(-V, 0)$ belongs to $K(E)$.

If a non-negative function W is a member of $K(E)$, then W is said to belong to *Kato's class* and if W is a member of $K_{\text{loc}}(E)$, then W is said to belong to *Kato's class locally*. The following general result can be proved. For details in the symmetric case see [70], [73], [72] and [71]. For the Gaussian semi-group the reader may consult Simon [60] and [61]. Also notice the equality

$$\|\exp(-tK_0)\|_{\infty,\infty} = \|\exp(-tK_0)\|_{1,1} = \sup_{x\in E} \int p_0(t,x,y)dm(y). \tag{2.5}$$

2.4. THEOREM. Suppose that $V = V_+ - V_-$ is a Borel measurable function defined on E such that V_- belongs to $K(E)$ and such that V_+ belongs to $K_{\text{loc}}(E)$.

(a) There exists a closed, densely defined linear operator $K_0 \dot{+} V$ in $C_\infty(E)$, extending $K_0 + V$, which generates a strongly continuous positivity preserving semi-group $\{\exp(t(K_0 \dot{+} V)) : t \geq 0\}$ in $C_\infty(E)$. Every operator $\exp(-t(K_0 \dot{+} V))$, $t > 0$, is of the form

$$\left[\exp(-t(K_0 \dot{+} V))f\right](x) = \int \exp(-t(K_0 + V))(x,y)f(y)dm(y), \quad f \in C_\infty(E), \tag{2.6}$$

where $\exp(-t(K_0 + V))(x,y)$ is a continuous function which verifies the identity of Chapman-Kolmogorov:

$$\exp(-t(K_0 + V))(x,y) = \int \exp(-s(K_0 + V))(x,z) \exp(-t(K_0 + V))(z,y)dz, \tag{2.7}$$

for $t > 0$, x, $y \in E$.

(b) The semi-group $\{\exp(-t(K_0 \dot{+} V)) : t \geq 0\}$ also acts as a strongly continuous semi-group in $L^p(E,m)$, $1 \leq p < \infty$.

(c) If $\exp(-tK_0)$ maps $L^1(E,m)$ into $L^\infty(E,m)$ for all $t > 0$ (i.e. if $\sup\{p_0(t,x,y) : x, y \in E\} < \infty$ for all $t > 0$), then $\exp(-t(K_0 \dot{+} V))$, $t > 0$, maps $L^p(E,m)$ into $L^q(E,m)$, for $1 \leq p \leq q \leq \infty$. If $t > 0$ and if $1 \leq p \leq q < \infty$, then $\exp(-t(K_0 + V))$ maps $L^p(E,m)$ into $L^q(E,m) \cap C_\infty(E)$.

(d) In $L^2(E,m)$ the family $\{\exp(-t(K_0 \dot{+} V)) : t \geq 0\}$ is a self-adjoint positivity preserving strongly continuous semi-group with a self-adjoint generator.

(e) The Feynman-Kac semi-group in $L^2(E,m)$ coincides with the semi-group corresponding to the quadratic form Q with $D(Q) = D\left(K_0^{1/2}\right) \cap D\left(V_+^{1/2}\right)$ and defined by

$$Q(f,g) = \left\langle K_0^{1/2} f, K_0^{1/2} g\right\rangle - \left\langle V_-^{1/2} f, V_-^{1/2} g\right\rangle + \left\langle V_+^{1/2} f, V_+^{1/2} g\right\rangle,$$

where f and g belong to $D(Q)$.

Remark 1. From the general assumptions it follows that, for $t > 0$, the operator $\exp(-tK_0)$ maps $L^1(E,m)$ in $C_\infty(E)$. As indicated in (c), then we may prove that, always for $t > 0$, the operator $\exp(-t(K_0\dot{+}V))$ maps $L^p(E,m)$ in $L^q(E,m)\bigcap C_\infty(E)$, provided that $1 \leq p \leq q \leq \infty$, $p \neq \infty$. This is explained in [72], in [70] and in [73]. In fact the integral kernel $\exp\left(-t(K_0+V)\right)(x,y)$ is given by

$$\exp\left(-t(K_0+V)\right)(x,y) = \lim_{\tau\uparrow t} \mathsf{E}_x\left(\exp\left(-\int_0^\tau V(X(s))ds\right)p_0(t-\tau, X(\tau), y)\right). \tag{2.8}$$

Remark 2. A proof of (e) follows from Proposition 2.13.

Remark 3. Let K be a self-adjoint generator in a Hilbert space with a lower bound. Let ω_0 be the smallest number ω with the property that $\langle Kf, f\rangle \geq -\omega\langle f,f\rangle$ for all $f \in D(K)$. Then ω_0 is called the *type* of the semi-group $\{\exp(-tK) : t \geq 0\}$ generated by K. In fact it follows that $\|\exp(-tK)\| \leq \exp(\omega_0 t)$, $t \geq 0$. The corresponding quadratic form Q with domain $D(Q) = D\left(K_0+\omega_0\right)^{1/2}$, defined by $Q(f,g) = \lim_{t\downarrow 0}\frac{\langle f,g\rangle - \langle\exp(-tK)f,g\rangle}{t}$, f and $g \in D(Q)$ possesses lower bound ω_0. In fact there exists a one-to-one correspondence between the class of self-adjoint operators with largest lower bound ω_0, the class of symmetric closed quadratic forms with largest lower ω_0 and the self-adjoint semi-groups $\{T(t) : t \geq 0\}$ with the property that $\|T(t)\| \leq \exp(\omega_0 t)$ for all $t \geq 0$. For all this the reader may for example consult Chapter 6 in [70].

Next we want to discuss the way in which the generator of the Feynman-Kac semi-group $\left\{\exp(-t(K_0\dot{+}V)) : t \geq 0\right\}$ is related to the Friedrichs' extension of K_0+V. We also are interested in "core"-type problems. Theorem 2.5. is closely related to the well-known KLMN-theorem: see Reed and Simon [55, Theorem X.17, p. 167]. The fact that $D\left(K_0^{1/2}\right)\cap D\left(V_+^{1/2}\right)$ is automatically dense implies that the Trotter-Lie product is available; see Kato [44, Theorem 1, p. 694].

2.5. THEOREM. Suppose that for every function $f \in D\left(K_0^{1/2}\right)\cap D\left(V_+^{1/2}\right)$ there exists a sequence $\{f_n : n \in \mathbb{N}\}$ in $D(K_0)\cap D(V)$ with the following properties:

(a) $\lim_{n\to\infty}\|f_n - f\|_2 = 0$;

(b) $\lim_{m,n\to\infty}\langle K_0(f_n - f_m), f_n - f_m\rangle = 0$;

(c) $\lim_{m,n\to\infty}\langle V_+(f_n - f_m), f_n - f_m\rangle = 0$.

Then the Feynman-Kac generator $K_0\dot{+}V$ is the Friedrichs' extension of K_0+V.

It is noticed that the hypotheses in Theorem 2.5. can be rephrased as "the subspace $\mathrm{dom}(K_0)\cap\mathrm{dom}(V)$ is a core for the operator $K_0^{1/2}+V_+^{1/2}$" or, equivalently, "the subspace $\mathrm{dom}(K_0)\cap\mathrm{dom}(V)$ is a form core for K_0+V_+".

PROOF. Fix a number a that is strictly larger then the type of the Feynman-Kac semi-group $\left\{\exp\left(-t(K_0\dot{+}V)\right) : t \geq 0\right\}$. The quadratic form $Q^{a,V}$ associated to the Feynman-Kac semi-group is given by (see Proposition 2.13)

$$Q^{a,V}(f,g) := \left\langle \left(aI + K_0\dot{+}V\right)^{1/2} f, \left(aI + K_0\dot{+}V\right)^{1/2} g\right\rangle \tag{2.9}$$

$$= \left\langle (aI+K_0)^{1/2} f, (aI+K_0)^{1/2} g \right\rangle + \left\langle V_+^{1/2} f, V_+^{1/2} g \right\rangle$$
$$- \left\langle V_-^{1/2} (aI+K_0)^{-1/2} (aI+K_0)^{1/2} f, V_-^{1/2} (aI+K_0)^{-1/2} (aI+K_0)^{1/2} g \right\rangle,$$

where f and g belong to the domain $D(Q^{a,V}) = D\left(K_0^{1/2}\right) \cap D\left(V_+^{1/2}\right)$. Moreover, for a large enough, we have by Proposition 2.2. together with the definition of Kato-Feller potential,

$$\left\| V_-^{1/2} (aI+K_0)^{-1/2} h \right\|_2 \leq \left\| V_-^{1/2} (aI+K_0)^{-1/2} \right\|_{2,2} \|h\|_2 \tag{2.10}$$
$$= \left\| V_-^{1/2} (aI+K_0)^{-1} V_-^{1/2} \right\|_{2,2}^{1/2} \|h\|_2 \leq \left\| (aI+K_0)^{-1} V_- \right\|_\infty^{1/2} \|h\|_2 \leq \sqrt{\varepsilon(a)}\, \|h\|_2,$$

where h belongs to $L^2(E,m)$. Since the negative part V_- of V is supposed to belong to Kato's class (because V is supposed to be a Kato-Feller potential), from Proposition 2.4. it follows that $\varepsilon(a) < 1$ for $a > 0$ large enough. From (a) it follows that the domain of $S := K_0 + V$ is dense in $L^2(E,m)$. This is so because the domain $D(Q^{a,V})$ is dense in $L^2(E,m)$. Since, in addition, the Feynman-Kac generator $K_0 \dot{+} V$ extends $K_0 + V$, we see that the operator $K_0 + V$ is closable. Let $\overline{S}$ denote this closure. We also write $\widetilde{S}$ for the Feynman-Kac generator $K_0 \dot{+} V$. Furthermore we define the operators T_1 and T_2 as follows:

$$T_1 := S^* \big|_{D\left(Q^{a,V}_{\overline{S}}\right) \cap D(S^*)}, \quad T_2 := S^* \big|_{D\left(Q^{a,V}_{\widetilde{S}}\right) \cap D(S^*)}. \tag{2.11}$$

Here $Q^{a,V}_{\overline{S}}$ is the quadratic form associated to $\overline{S}$ and $Q^{a,V}_{\widetilde{S}}$ is the quadratic form associated to $\widetilde{S}$, the so-called Feynman-Kac or Schrödinger form. Then, from Theorem 5.38 in Weidmann [81, p. 123], it follows that T_1 is the Friedrichs' extension of S. Since K_0 and V are both self-adjoint, the operator S is symmetric and so $S \subset S^*$. We also have $S \subset \widetilde{S}$ and hence $S^* \supset \widetilde{S}^* = \widetilde{S} \supset \overline{S} \supset S$. From the definition T_1 it is clear that $T_1 \subset S^*$ and hence $T_1 = T_1^* \supset \overline{S}$. We also readily see $T_2 \supset \widetilde{S}$ and thus $T_2^* \subset \widetilde{S}$. Since $D\left(Q^{a,V}_{\overline{S}}\right) \subseteq D\left(Q^{a,V}_{\widetilde{S}}\right)$, we also have $T_1 \subset T_2 \subset S^*$. A combination of these inclusions yields:

$$S \subseteq \overline{S} \subseteq T_2^* \subseteq \widetilde{S} \subseteq T_2 \subseteq S^* \tag{2.12}$$

and

$$S \subseteq \overline{S} \subseteq T_2^* \subseteq T_1 \subseteq T_2 \subseteq S^*. \tag{2.13}$$

From (a), (b) and (c) it follows that $D(S)$ forms a core for $Q^{a,V}_{\widetilde{S}}$. For let $f_0 \in D\left(Q^{a,V}_{\widetilde{S}}\right)$ be such that $\left\langle \left(aI+\widetilde{S}\right)^{1/2} f, \left(aI+\widetilde{S}\right)^{1/2} f_0 \right\rangle = 0$ for all $f \in D(S)$. Then

$\langle (aI+S)f, f_0\rangle = 0$ for all $f \in D(S)$. By properties (a), (b) and (c), there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $D(S)$ such that

$$\lim_{n\to\infty} \left\langle \left(aI+\widetilde{S}\right)^{1/2}(f_0 - f_n), \left(aI+\widetilde{S}\right)^{1/2}(f_0 - f_n)\right\rangle = 0. \tag{2.14}$$

Consequently

$$\begin{aligned} \left\| \left(aI+\widetilde{S}\right)^{1/2} f_0 \right\|_2^2 &= \lim_{n\to\infty} \left\langle \left(aI+\widetilde{S}\right)^{1/2} f_n, \left(aI+\widetilde{S}\right)^{1/2} f_0 \right\rangle \\ &= \lim_{n\to\infty} \left\langle \left(aI+\widetilde{S}\right) f_n, f_0\right\rangle = \lim_{n\to\infty} \langle (aI+S) f_n, f_0\rangle = 0. \end{aligned} \tag{2.15}$$

Hence $f_0 = 0$. Next we are going to show that

$$D(S^*) \cap D\left(Q^{a,V}_{\widetilde{S}}\right) = D\left(\widetilde{S}\right) \cap D\left(Q^{a,V}_{\widetilde{S}}\right) = D\left(\widetilde{S}\right). \tag{2.16}$$

This is true, because let $f \in L^2(E,m)$ be such that the functional $g \mapsto \left\langle (aI+S)g, \left(aI+\widetilde{S}\right)^{-1/2} f\right\rangle$, $g \in D(S)$, is continuous. Since $D(S)$ is a core for $Q^{a,V}_{\widetilde{S}}$, it follows that the functional $g \mapsto \left\langle \left(aI+\widetilde{S}\right) g, \left(aI+\widetilde{S}\right)^{-1/2} f\right\rangle$, $g \in D\left(\widetilde{S}\right)$, is continuous as well. We infer that $R\left(\left(aI+\widetilde{S}\right)^{-1/2}\right) \cap D(S^*) = R\left(\left(aI+\widetilde{S}\right)^{-1/2}\right) \cap D\left(\widetilde{S}^*\right)$, or putting it differently

$$D\left(Q^{a,V}_{\widetilde{S}}\right) \cap D(S^*) = D\left(Q^{a,V}_{\widetilde{S}}\right) \cap D\left(\widetilde{S}\right) = D\left(\widetilde{S}\right) \tag{2.17}$$

and hence

$$T_2 = S^* \restriction_{D\left(Q^{a,V}_{\widetilde{S}}\right) \cap D(S^*)} = S^* \restriction_{D(\widetilde{S})} = \widetilde{S}. \tag{2.18}$$

It follows that $T_2 = \widetilde{S}$ and hence $T_2 = T_2^* = T_1$. This shows that $\widetilde{S}$ is the Friedrichs' extension of S.

2.6. COROLLARY. If for all sufficiently large a, $a > 0$, the range of the operator $aI + K_0 + V$ is dense in $L^2(E,m)$, then the operator $K_0 + V$ is essentially self-adjoint and its closure generates the Feynman-Kac semi-group.

PROOF. Let $a > 0$ be a real number, that is strictly larger than the type of the semi-group $\{\exp(-t(K_0 \dot{+} V)) : t \geq 0\}$. Since the operator $K_0 \dot{+} V$ is bounded from below, with lower bound $-\omega$ say, it follows that $\|\exp(-t(K_0 \dot{+} V))\| \leq \exp(\omega t)$, $t > 0$ and the *type* of the Feynman-Kac semi-group is the smallest ω for which the latter inequality is valid. Let the notation be that of the proof of Theorem 2.5. Then we have $S^* \supseteq \widetilde{S} \supseteq \overline{S}$. Let f_0 belong to the domain of S^*. Since the closure of the range of $aI + K_0 + V$ is dense, it follows that the range of $aI + \overline{S}$ coincides with $L^2(E,m)$ and hence $(aI + S^*)f_0 = (aI + \overline{S})f = (aI + S^*)f$ for some f in the domain of $\overline{S}$. Consequently $(aI + S^*)(f - f_0) = 0$. Since the range of $aI + S$ is dense, it follows that $f = f_0$. Whence $D(S^*) \subseteq D(\overline{S})$. Altogether this shows that $S^* = \overline{S} = \widetilde{S}$.

2.7. PROPOSITION. Let $K_0 \dot{+} V$ be the Feynman-Kac generator of the semi-group $\left\{\exp\left(-t(K_0 \dot{+} V)\right) : t \geq 0\right\}$ and let $Q^{a,V}$ be the corresponding quadratic form. The subspace $D\left(K_0 \dot{+} V\right) \cap C_\infty(E)$ is a core for $K_0 \dot{+} V$ and the subspace $D\left(K_0^{1/2}\right) \cap D\left(V_+^{1/2}\right) \cap C_\infty(E)$ is a form core for $Q^{a,V}$.

PROOF. Let f be a member of the domain of $K_0 \dot{+} V$. Then there exists a sequence $g_n \in C_\infty(E) \cap L^2(E,m)$, $n \in \mathbb{N}$, such that $(aI + K_0 \dot{+} V)f = \lim_{n\to\infty} g_n$. Next we write g_n in the form $g_n = \left(aI + K_0 \dot{+} V\right) f_n$. Then f_n belongs to $L^2(E,m) \cap C_\infty(E)$. The fact that every f_n belongs to $C_\infty(E)$ follows because the Feynman-Kac semi-group leaves the space $C_\infty(E)$ invariant. Moreover we have $f = \lim_{n\to\infty} f_n$ and $\left(K_0 \dot{+} V\right) f = \lim_{n\to\infty}(K_0 + V)f_n$. This proves the first statement in Proposition 2.7. Upon replacing $\left(aI + K_0 \dot{+} V\right)$ with $\left(aI + K_0 \dot{+} V\right)^{1/2}$ and noticing identity (2.9) yields the second assertion.

The results in Theorem 2.5, Corollary 2.6 and Proposition 2.7 have their local counterparts. In fact, let Γ be a Borel subset of the second countable locally compact Hausdorff space E. In relation to the set Γ we shall be employing the following stopping times:

$$S = \inf\left\{s > 0 : \int_0^s 1_\Gamma(X(\sigma))d\sigma > 0\right\}, \quad T = \inf\left\{s > 0 : X(s) \in \Gamma\right\}. \tag{2.19}$$

It readily follows that $S \geq T$, P_x-almost surely, for all $x \in E$. The following proposition gives a sufficient condition on Γ, in order that, for all $x \in E$, $S = T$, P_x-almost surely. A point $x \in E$ belongs to Γ^r if $\mathsf{P}_x(T = 0) = 1$. Some authors call the time S the *penetration time*: see e.g. Herbst and Zhongxin Zhao [28].

2.8. PROPOSITION. Suppose $\Gamma^r = (\text{int}(\Gamma))^r$. Then $S = T$, P_x-almost surely for all $x \in E$.

PROOF. Since $S \geq T$, it suffices to prove that $\mathsf{P}_x(S > T) = 0$, P_x-almost surely for all $x \in E$. Since on the event $\{S > T\}$, $S = T + S \circ \vartheta_T$, P_x-almost surely, we have by the Markov property:

$$\begin{aligned}\mathsf{P}_x(S > T) &= \mathsf{P}(S > T, S \circ \vartheta_T > 0) = \mathsf{E}_x\left(\mathsf{P}_{X(T)}(S > 0), S > T\right)\\ &= \mathsf{P}_x\left(S > T, \mathsf{P}_{X(T)}(S > 0) = 1\right)\end{aligned}$$

(On the event $\{T < \infty\}$, $X(T)$ belongs to $\Gamma \bigcup \Gamma^r$ P_x-almost surely)

$$= \mathsf{P}_x\left(S > T, \mathsf{P}_{X(T)}(S > 0) = 1, X(T) \in \Gamma \bigcup \Gamma^r\right)$$

(Because of symmetry, $\Gamma \setminus \Gamma^r$ is a polar set)

$$= \mathsf{P}_x \left(S > T, \mathsf{P}_{X(T)}(S > 0) = 1, X(T) \in \Gamma^r\right)$$

$(\Gamma^r = (\mathrm{int}(\Gamma))^r)$

$$= \mathsf{P}_x \left(S > T, \mathsf{P}_{X(T)}(S > 0) = 1, X(T) \in (\mathrm{int}(\Gamma))^r\right). \tag{2.20}$$

However, if y belongs to $(\mathrm{int}(\Gamma))^r$, then $\mathsf{P}_y(S > 0) = 0$ and hence $\mathsf{P}_x(S > T) = 0$.

2.9. DEFINITION. Let S be the penetration time of Γ. The integral kernel $\exp\left(-\lambda K_\Sigma\right)(x, y)$ is given by

$$\exp(-\lambda K_\Sigma)(x, y) \tag{2.21}$$
$$= \lim_{\lambda' \uparrow \lambda} \mathsf{E}_x \left(\exp\left(- \int_0^{\lambda'} V(X(\sigma))d\sigma \right) p_0(\lambda - \lambda', X(\lambda'), y) : S > \lambda' \right).$$

In the results below we let $\Sigma = E^\triangle \setminus \Gamma$ be an open subset of E and $\left(K_0 \dot{+} V\right)_\Sigma$ denotes the Feynman-Kac generator of the semi-group killed in the complement of Σ, i.e. the semi-group $\left\{\exp\left(-t(K_0 \dot{+} V)_\Sigma\right) : t \geq 0\right\}$, defined by

$$\left[\exp(-t(K_0 \dot{+} V)_\Sigma) f\right](x) = \mathsf{E}_x \left(\exp\left(- \int_0^t V(X(s))ds \right) f(X(t)) : S > t \right)$$
$$= \int \exp\left(-t(K_0 + V)_\Sigma\right)(x, y) f(y) dy. \tag{2.22}$$

If $\Gamma = (\mathrm{int}(\Gamma))^r$, then the penetration time S may be replaced with the exit time T. In Proposition 2.12. the complement of Σ is supposed to be regular, which yields the fact that the Feynman-Kac semi-group, killed on the complement of Σ, leaves $C_\infty(\Sigma)$ invariant. Proofs are not given; they follow the same lines as the ones given above.

2.10. THEOREM. Suppose that for every function $f \in D\left((K_0)_\Sigma^{1/2}\right) \cap D\left(\left(V_+^{1/2}\right)_\Sigma\right)$ there exists a sequence $\{f_n : n \in \mathbb{N}\}$ in $D((K_0)_\Sigma) \cap D((V)_\Sigma)$ with the following properties:

(a) $\lim_{n \to \infty} \|f_n - f\|_2 = 0$;

(b) $\lim_{m,n \to \infty} \langle (K_0)_\Sigma (f_n - f_m), f_n - f_m \rangle = 0$;

(c) $\lim_{m,n \to \infty} \langle V_+(f_n - f_m), f_n - f_m \rangle = 0$.

Then the Feynman-Kac generator $\left(K_0 \dot{+} V\right)_\Sigma$ is the Friedrichs' extension of $(K_0)_\Sigma + (V)_\Sigma$.

2.11. COROLLARY. If for all sufficiently large a, $a > 0$, the range of the operator $aI + (K_0)_\Sigma + (V)_\Sigma$ is dense in $L^2(\Sigma, m)$, then the operator $(K_0)_\Sigma + (V)_\Sigma$ is essentially self-adjoint and its closure generates the Feynman-Kac semigroup, killed on the complement of Σ.

2.12. PROPOSITION. Let $\left(K_0 \dot{+} V\right)_\Sigma$ be the Feynman-Kac generator of the semigroup $\left\{\exp\left(-t\left(K_0 \dot{+} V\right)_\Sigma\right) : t \geq 0\right\}$ and let $Q_\Sigma^{a,V}$ be the corresponding quadratic form. The subspace $D\left(K_0\dot{+}V\right)_\Sigma \cap C_\infty(E)$ is a core for $\left(K_0\dot{+}V\right)_\Sigma$ and the subspace $D\left(\left(K_0\right)_\Sigma^{1/2}\right) \cap D\left(\left(V_+^{1/2}\right)_\Sigma\right) \cap C_\infty(E)$ is form core for $Q_\Sigma^{a,V}$, provided that Γ, defined by $\Gamma = E^\Delta \setminus \Sigma$, is regular in the sense that $\Gamma = \Gamma^r$.

In (2.8) the following result was employed.

2.13. PROPOSITION. Let V be a Kato-Feller potential and let $Q^{a,V}$ be the corresponding Schrödinger form ($a > 0$ is larger than the type of the corresponding Feynman-Kac semi-group). Then $D\left(Q^{a,V}\right) = D\left(K_0^{1/2}\right) \cap D\left(V_+^{1/2}\right)$ and

$$Q^{a,V}(f,g) = \left\langle (aI + K_0)^{1/2} f, (aI + K_0)^{1/2} g\right\rangle + \int V(x) f(x)\overline{g(x)}dm(x), \qquad (2.23)$$

where f and g belong to $D(Q^{a,V})$.

PROOF. Put $V_{m,n} = \max(\min(V,n), -m)$, m, n in $\mathbb{N}$ and fix $f \in D\left(Q^{a,V}\right)$. Define the functions $f_{m,n} \in D\left(K_0^{1/2}\right)$ by $\left(aI + K_0\dot{+}V\right)^{1/2} f = \left(aI + K_0 + V_{m,n}\right)^{1/2} f_{m,n}$. By the Feynman-Kac formula it follows that, in L^2-sense, $f = \lim_{m\to\infty}\lim_{n\to\infty} f_{m,n}$. In addition we have

$$\begin{aligned}
&Q^{a,V}(f,f) = \left\langle \left(aI + K_0\dot{+}V\right)^{1/2} f, \left(aI + K_0\dot{+}V\right)^{1/2} f\right\rangle \\
&= \left\langle \left(aI + K_0\dot{+}V_{m,n}\right)^{1/2} f_{m,n}, \left(aI + K_0\dot{+}V_{m,n}\right)^{1/2} f_{m,n}\right\rangle \\
&= \left\langle (aI + K_0)^{1/2} f_{m,n}, (aI + K_0)^{1/2} f_{m,n}\right\rangle \\
&\qquad - \int (V_{m,n})_- \left|f_{m,n}\right|^2 dx + \int (V_{m,n})_+ \left|f_{m,n}\right|^2 dx \\
&= \left\langle (aI + K_0)^{1/2} f, (aI + K_0)^{1/2} f\right\rangle - \int V_- |f|^2 dx + \int V_+ |f|^2 dx. \qquad (2.24)
\end{aligned}$$

The ultimate equality in (2.24) follows upon letting, first n and then m, tend to infinity and by observing the following general argument for closed linear operators in Hilbert space. Let B be closed linear operator in a Hilbert space (in our situation we may take $B = (aI + K_0)^{1/2}$ or $B = V_+^{1/2}$ or $B = V_-^{1/2}$ in $L^2(E,m)$). Let (g_n) be a sequence in the domain $D(B)$ with the following properties:

(a) $g = \lim_{n\to\infty} g_n$ exists in the weak sense;

(b) $\sup_{n\in\mathbb{N}} \|Bg_n\| < \infty$. Then the vector g belongs to $D(B)$ and some sequence $(\tilde{g}_n)$ in the convex hull of the sequence (Bg_n) converges in Hilbert space sense to Bg.

This shows Proposition 2.13.

Next we want to compare the operators $\left(K_0\dot{+}V\right)_\Sigma$ and $K_0\dot{+}V$. The operator H_Σ^{a+V} is defined as follows. Its domain is $D(K_0\dot{+}V)$ and its action is given by

$$\left[H_\Sigma^{a+V} f\right](x) = \mathbb{E}_x\left(\exp\left(-\int_0^S (a + V(X(s)))\,ds\right) f(X(S)) : S < \infty\right), \qquad (2.25)$$

where f belongs to $D(K_0\dot{+}V)$. Intuitively, the function $H_\Sigma^{a+V}f$ is a function, that on $\Gamma = E\setminus\Sigma$ coincides with f and that on Σ is "$a+V$-harmonic". The operator J_Σ restricts functions defined on E to Σ and its dual J_Σ^* extends functions, defined on Σ, with 0 in Γ. The operator H_Σ^{a+V} as defined in (2.25) is a priori an operator defined on bounded continuous functions. It is not clear at all that it is defined on $L^2(E,m)$. In fact the latter does not seem to be true. However, in a natural way it is defined on the domain $D(K_0\dot{+}V)$ and a little more thought will show that it has a well-defined meaning on the domain of the corresponding quadratic form given by $D\left(aI+K_0\dot{+}V\right)^{1/2}$. For details see Proposition 2.14. and Corollary 2.15. below.

2.14. PROPOSITION. (a) Let $a>0$ be large enough. The following identity holds:

$$\left(aI+\left(K_0\dot{+}V\right)_\Sigma\right)J_\Sigma\left(I-H_\Sigma^{a+V}\right)=J_\Sigma\left(aI+K_0\dot{+}V\right), \tag{2.26}$$

in the sense of domains and of equality of operators.

(b) The identity

$$\left(K_0\dot{+}V\right)_\Sigma=J_\Sigma\left(K_0\dot{+}V\right)J_\Sigma^* \tag{2.27}$$

is valid in the sense of domains and of equality of operators.

PROOF. (a) First let f belong to $D\left(\left(K_0\dot{+}V\right)_\Sigma\right)$. Then define the function $g\in D\left(K_0\dot{+}V\right)$ by the equality $J_\Sigma^*\left(aI+\left(K_0\dot{+}V\right)_\Sigma\right)f=\left(aI+K_0\dot{+}V\right)g$. For $x\in\Sigma$ we have

$$\begin{aligned}
&g(x)-f(x)\\
&=\left[\left\{\left(aI+K_0\dot{+}V\right)^{-1}-\left(aI+\left(K_0\dot{+}V\right)_\Sigma\right)^{-1}J_\Sigma\right\}\left(aI+K_0\dot{+}V\right)g\right](x)\\
&=\int_0^\infty ds\mathsf{E}_x\left(\exp\left(-\int_0^s\left(a+V(X(u))\right)du\right)\left(aI+K_0\dot{+}V\right)g(X(s)):S\le s\right)\\
&=\mathsf{E}_x\left(\int_S^\infty ds\exp\left(-\int_0^s\left(a+V(X(u))\right)du\right)\left(aI+K_0\dot{+}V\right)g(X(s)):S<\infty\right)\\
&=\mathsf{E}_x\left(\exp\left(-\int_0^S\left(a+V(X(u))\right)du\right)\right.\\
&\quad\left.\times\int_0^\infty ds\exp\left(-\int_0^s\left(a+V(X(u+S))\right)du\right)\left(aI+K_0\dot{+}V\right)g(X(s+S)):S<\infty\right)
\end{aligned}$$

(Markov property)

$$\begin{aligned}
&=\mathsf{E}_x\left(\exp\left(-\int_0^S\left(a+V(X(u))\right)du\right)\right.\\
&\quad\left.\times\int_0^\infty ds\mathsf{E}_{X(S)}\left(\exp\left(-\int_0^s\left(a+V(X(u))\right)du\right)\left(aI+K_0\dot{+}V\right)g(X(s))\right):S<\infty\right)\\
&=\mathsf{E}_x\left(\exp\left(-\int_0^S\left(a+V(X(u))\right)du\right)g(X(S)):S<\infty\right).
\end{aligned} \tag{2.28}$$

Consequently $g(x) - f(x) = \left[H_{\Sigma}^{a+V} g\right](x)$, $x \in \Sigma$. Conversely, let g belong to $D(K_0 \dot{+} V)$ and define $f \in D\left(\left(K_0 \dot{+} V\right)_{\Sigma}\right)$ by the identity:

$$\left(aI + \left(K_0 \dot{+} V\right)_{\Sigma}\right) f = J_{\Sigma}\left(aI + K_0 \dot{+} V\right) g. \tag{2.29}$$

For $x \in \Sigma$ we have as above $g(x) - f(x) = \left[H_{\Sigma}^{a+V} g\right](x)$. This proves Proposition 2.14(a).

(b) From formula (2.26) in Proposition 2.14(a) we infer ($a > 0$ large enough)

$$\begin{aligned}\left(K_0 \dot{+} V\right)_{\Sigma} &= \left(K_0 \dot{+} V\right)_{\Sigma} J_{\Sigma} J_{\Sigma}^{*} = \left(aI + \left(K_0 \dot{+} V\right)_{\Sigma}\right) J_{\Sigma} J_{\Sigma}^{*} - aJ_{\Sigma} J_{\Sigma}^{*} \\ &= \left(aI + \left(K_0 \dot{+} V\right)_{\Sigma}\right) J_{\Sigma}\left(I - H_{\Sigma}^{a+V}\right) J_{\Sigma}^{*} - aJ_{\Sigma} J_{\Sigma}^{*} \\ &= J_{\Sigma}\left(K_0 \dot{+} V\right) J_{\Sigma}^{*}.\end{aligned} \tag{2.30}$$

As a corollary we have the following. The result should be compared to the fundamental identity for so-called λ-potentials in Port and Stone [53, p. 41].

2.15. COROLLARY. Let $a > 0$ be sufficiently large. The following identity holds in $L^2(E, m)$:

$$\left(aI + K_0 \dot{+} V\right)^{-1} - J^{*}\left(aI + \left(K_0 \dot{+} V\right)_{\Sigma}\right)^{-1} J = H_{\Sigma}^{a+V}\left(aI + K_0 \dot{+} V\right)^{-1}. \tag{2.31}$$

In addition the operator $H_{\Sigma}^{a+V}\left(aI + K_0 \dot{+} V\right)^{-1}$ is self-adjoint and form positive. In fact the following identities are true:

$$\begin{aligned}&H_{\Sigma}^{a+V}\left(aI + K_0 \dot{+} V\right)^{-1} \\ &= \left(\left(aI + K_0 \dot{+} V\right)^{1/2} H_{\Sigma}^{a+V}\left(aI + K_0 \dot{+} V\right)^{-1}\right)^{*}\left(aI + K_0 \dot{+} V\right)^{1/2} H_{\Sigma}^{a+V}\left(aI + K_0 \dot{+} V\right)^{-1} \\ &= H_{\Sigma}^{a+V}\left(aI + K_0 \dot{+} V\right)^{-1/2}\left(H_{\Sigma}^{a+V}\left(aI + K_0 \dot{+} V\right)^{-1/2}\right)^{*}.\end{aligned} \tag{2.32}$$

Hence the operators

$$\begin{aligned}&H_{\Sigma}^{a+V}\left(aI + K_0 \dot{+} V\right)^{-1}, \quad H_{\Sigma}^{a+V}\left(aI + K_0 \dot{+} V\right)^{-1/2} \\ &\text{and also} \quad \left(aI + K_0 \dot{+} V\right)^{1/2} H_{\Sigma}^{a+V}\left(aI + K_0 \dot{+} V\right)^{-1}\end{aligned}$$

are bounded operators in $L^2(E, m)$. Moreover the operator H_{Σ}^{a+V} is self-adjoint in the space $D(Q^{a+V})$ equipped with the inner-product

$$Q^{a+V}(f, g) = \left\langle\left(aI + K_0 \dot{+} V\right)^{1/2} f, \left(aI + K_0 \dot{+} V\right)^{1/2} g\right\rangle, \tag{2.33}$$

where f and g belong to $D(Q^{a+V}) = D\left(\left(aI + K_0 \dot{+} V\right)^{1/2}\right)$. In particular the operator $\left(aI + K_0 \dot{+} V\right)^{1/2} H_{\Sigma}^{a+V}\left(aI + K_0 \dot{+} V\right)^{-1/2}$ is a self-adjoint projection in $L^2(E, m)$.

PROOF. Notice the identity $\left[H_{\Sigma}^{a+V}f\right](x) = f(x)$, for $x \in \Gamma^r$ and for $f \in D\left(K_0\dot{+}V\right)$. Also notice the fact that the operator H_{Σ}^{a+V} is a projection in the sense that its square $H_{\Sigma}^{a+V} \circ H_{\Sigma}^{a+V}$ equals H_{Σ}^{a+V}. In fact (2.31) is a reformulation of (2.26). The identities in (2.32) follow because the operator $H_{\Sigma}^{a+V}\left(aI + K_0\dot{+}V\right)^{-1}$ is self-adjoint. The same argument applies for the proof of the the self-adjointness of the operator H_{Σ}^{a+V} with respect to the inner-product in (2.33). The latter also implies the final statement in Corollary 2.15.

3. HILBERT-SCHMIDT PROPERTIES OF RESOLVENT AND SEMIGROUP DIFFERENCES

3.1. NOTATION. We denote by $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_\infty$ the collection of Hilbert-Schmidt, the collection of trace class operators and the collection of compact operators respectively.

3.2. HYPOTHESES. As in section 2 we place ourselves in the surroundings of the basic assumptions on stochastic spectral analysis (BASSA). In fact, let K_0 be the generator of a self-adjoint semi-group $\{\exp(-tK_0) : t \geq 0\}$ in $L^2(E,m)$ of the form: $\exp[(-tK_0)f](x) = \int p_0(t,x,y)f(y)dm(y)$, where $p_0(t,x,y)$ is symmetric and continuous on $(0,\infty) \times E \times E$ and where m is some non-negative Borel measure on E. Briefly, assumptions A1-A4 are verified. Usually we write dy instead of $dm(y)$. As in section 2 the generator K_0 will be perturbed in two ways. First there will be a "regular" perturbation, being a multiplication operator V and secondly, there will be a potential barrier on Γ. In principle Γ will be a closed subset of E. The singularity projection operator is defined by: $[Pf](x) = 1_\Gamma(x)f(x)$, $\quad f \in L^2(E,m)$. Put $\Sigma := E \setminus \Gamma$ and introduce the restriction operator J as follows: $Jf = f\,|_\Sigma$. Hence $J^* = Id_{L^2(\Sigma)\to L^2(E)}$, $J^*J = I - P$ and $JJ^* = I_{L^2(\Sigma)}$. Let $K := K_0\dot{+}V$ be the Feynman-Kac generator, $K_M = K + MP$ with domain $\text{dom}(K) = \text{dom}(K_M)$ and denote with $K_\Sigma := \left(K_0\dot{+}V\right)_\Sigma$ the Feynman-Kac generator of the semi-group $\left\{\exp\left(-t(K_0\dot{+}V)_\Sigma\right) : t \geq 0\right\}$, killed on Γ. From formula (2.26) in Proposition 2.14(a) we infer $\left(K_0\dot{+}V\right)_\Sigma = J\left(K_0\dot{+}V\right)J^*$. Also notice that most of the time we write J instead of J_Σ. Also see the remarks preceding Proposition 2.14.

It is useful to observe that weighted estimates are important for at least two reasons. A first reason is the fact that the trace norm of the product of two operators T and S is dominated by the product of the Hilbert-Schmidt norms of $T\varphi$ and of $\varphi^{-1}S$ or, in a formula, $\|TS\|_{\text{trace}} \leq \|T\varphi\|_{\text{HS}}\left\|\varphi^{-1}S\right\|_{\text{HS}}$. Here φ is a nowhere vanishing Borel function. The second reason is that for certain points λ_0 in the absolutely continuous part of the spectrum of a self-adjoint operator K, the limit absorption principle is valid. This means that, for certain functions φ and ψ and for a certain interval I containing λ_0 in its interior, the quantity $\sup_{\lambda\in\mathbb{C},\text{Re}\lambda\in I,\text{Im}\lambda\neq 0}\left\|\varphi\left(\lambda I + K\right)^{-1}\psi\right\|$ is finite. For more details we refer the reader to results based on Mourre estimates as exhibited in Chapter 4 of Cycon et al [14], Perry [52] and Mourre [50]. Closely related results and applications can be found in Agmon [1], Lavine [48], Ben-Artzi [5], Robert and Tamura [58].

A. Regular perturbations.

We want to discuss some Hilbert-Schmidt and trace class properties. First we do this for regular perturbations. The following results improve some corresponding results in [21]. The first result improves Theorem 5.5. in [21].

3.3. THEOREM. Let V and W be Kato-Feller potentials. Suppose that

$$\int_{\{|V-W|\geq 1\}} \exp\left(-2t(K_0+\min(V,W))\right)(x,x)\,|V(x)-W(x)|\,dx<\infty \tag{3.1}$$

and that

$$\int_{\{|V-W|\leq 1\}} \exp\left(-2t(K_0+\min(V,W))\right)(x,x)\,|V(x)-W(x)|^2\,dx<\infty. \tag{3.2}$$

Then the operator $D(t)$, defined by

$$D(t)=\exp\left(-t(K_0\dot{+}V)\right)-\exp\left(-t(K_0\dot{+}W)\right),$$

belongs to $\mathcal{I}_2$ and

$$\begin{aligned}\|D(t)\|_{\mathrm{HS}}\leq \sqrt{2t}&\left(\int_{\{|V-W|\geq 1\}} \exp\left(-2t(K_0+\min(V,W))\right)(x,x)\,|V(x)-W(x)|\,dx\right)^{1/2}\\ &+t\left(\int_{\{|V-W|\leq 1\}} \exp\left(-2t(K_0+\min(V,W))\right)(x,x)\,|V(x)-W(x)|^2\,dx\right)^{1/2}.\end{aligned} \tag{3.3}$$

PROOF. Write $D(t)=D_1(t)-D_2(t)$ and $V_1=(V-W)1_{\{|V-W|<1\}}$, where

$$D_1(t)=\exp\left(-t\left(K_0\dot{+}V\right)\right)-\exp\left(-t\left(K_0\dot{+}W+V_1\right)\right) \tag{3.4}$$

and with

$$D_2(t)=\exp\left(-t\left(K_0\dot{+}W\right)\right)-\exp\left(-t\left(K_0\dot{+}W+V_1\right)\right). \tag{3.5}$$

Then $\|D(t)\|_{\mathrm{HS}}\leq\|D_1(t)\|_{\mathrm{HS}}+\|D_2(t)\|_{\mathrm{HS}}$. A more or less straightforward calculation will show the following identities:

$$\|D_1(t)\|_{\mathrm{HS}}^2=-\int_0^{2t} ds\frac{\partial}{\partial s}\min(s,2t-s)\int dx\int dz \tag{3.6}$$
$$\exp\left(-s\left(K_0+V\right)\right)(x,z)\exp\left(-(2t-s)\left(K_0+W+V_1\right)\right)(z,x)\left(W(x)+V_1(x)-V(x)\right)$$

and

$$\|D_2(t)\|_{\mathrm{HS}}^2=\int_0^{2t} ds\,\min(s,2t-s)\int dx\int dz \tag{3.7}$$
$$\exp\left(-s\left(K_0+W\right)\right)(x,z)\exp\left(-(2t-s)\left(K_0+W+V_1\right)\right)(z,x)V_1(x)V_1(z).$$

Since $W + V_1 \geq \min(W, V)$, it follows from (3.6) that

$$\|D_1(t)\|_{\mathrm{HS}}^2 \leq \int_0^{2t} ds \int dx \int dz \exp\left(-s\left(K_0 + \min(V,W)\right)\right)(x,z) \tag{3.8}$$
$$\exp\left(-(2t-s)\left(K_0 + \min(V,W)\right)\right)(z,x)\left|W(x) + V_1(x) - V(x)\right|$$
$$= 2t \int dx \exp\left(-2t\left(K_0 + \min(V,W)\right)\right)\left|W(x) + V_1(x) - V(x)\right|.$$

From Cauchy-Schwarz' inequality it follows that

$$\|D_2(t)\|_{\mathrm{HS}}^2 \leq \int_0^{2t} ds \min(s, 2t-s) \int dx \int dz$$
$$\exp\left(-s\left(K_0 + W\right)\right)(x,z) \exp\left(-(2t-s)\left(K_0 + W + V_1\right)\right)(z,x) V_1(x)^2$$

and again using $W + V_1 \geq \min(W, V)$, it follows from (3.8) that

$$\|D_2(t)\|_{\mathrm{HS}}^2 \leq \int_0^{2t} ds \min(s, 2t-s) \int dx \int dz \tag{3.9}$$
$$\exp\left(-s\left(K_0 + \min(V,W)\right)\right)(x,z) \exp\left(-(2t-s)\left(K_0 + \min(V,W)\right)\right)(z,x) V_1(x)^2$$
$$= t^2 \int dx \exp\left(-2t\left(K_0 + \min(V,W)\right)\right)(x,x) V_1(x)^2.$$

The inequalities (3.8) and (3.9) yield the desired conclusion.

The following result gives conditions on the Kato-Feller potentials V and W in order that the operator $D(t)$ in (3.10) is compact.

3.4. THEOREM. Let V and W be Kato-Feller potentials with the property that, for $a > 0$ sufficiently large, the functions $(aI + K_0)^{-1}|V|$ and $(aI + K_0)^{-1}|W|$ are functions in $C_\infty(E)$. The following operators are compact in $L^2(E, m)$:

$$\exp\left(-t\left(K_0 \dot{+} W\right)\right) - \exp\left(-t\left(K_0 \dot{+} V\right)\right), \quad t \geq 0, \tag{3.10}$$
$$\left(aI + K_0 \dot{+} W\right)^{-1} - \left(aI + K_0 \dot{+} V\right)^{-1}, \quad a > 0 \quad \text{sufficiently large.}$$

Remark. The condition that the function $(aI + K_0)^{-1}|V|$ belongs to $C_\infty(E)$ should be compared to Weder [80, Lemma III.4] and to [78, Theorem II.9]. More results on relative compactness can be found in Smits [64, Chapter 5], in Weder [79] and in Reed and Simon [57, p. 117, Example 5].

PROOF. Put $V_{k,\ell,m} = V 1_{\{-k \leq V \leq \ell\}} 1_{K_m}$ and put $W_{k,\ell,m} = W 1_{\{-k \leq W \leq \ell\}} 1_{K_m}$, where $(K_m : m \in \mathbb{N})$ is a sequence of compact subsets of E with the following properties: $E = \bigcup_{m \in \mathbb{N}} K_m$, $K_m \subseteq \mathrm{int}(K_{m+1})$. From Theorem 3.3 it follows that, for k, ℓ and $m \in \mathbb{N}$, the operators $\left(aI + K_0 \dot{+} V_{k,\ell,m}\right)^{-1} - \left(aI + K_0 \dot{+} W_{k,\ell,m}\right)^{-1}$, for $a > 0$

large enough, and $\exp\left(-t\left(K_0\dot{+}V_{k,\ell,m}\right)\right)-\exp\left(-t\left(K_0\dot{+}W_{k,\ell,m}\right)\right)$, $t\geq 0$, are compact indeed. It suffices to prove that the differences

$$\left\|\exp\left(-t\left(K_0\dot{+}V\right)\right)-\exp\left(-t\left(K_0\dot{+}V_{k,\ell,m}\right)\right)\right\|_{2,2}$$

and

$$\left\|\exp\left(-t\left(K_0\dot{+}W\right)\right)-\exp\left(-t\left(K_0\dot{+}W_{k,\ell,m}\right)\right)\right\|_{2,2}$$

tend to zero, if k, ℓ and m tend to ∞. Therefore we estimate

$$\left\|\exp\left(-t\left(K_0\dot{+}V\right)\right)-\exp\left(-t\left(K_0\dot{+}V_{k,\ell,m}\right)\right)\right\|_{2,2}$$

(Riesz-Thorin interpolation together with symmetry)

$$\begin{aligned}
&\leq \left\|\exp\left(-t\left(K_0\dot{+}V\right)\right)-\exp\left(-t\left(K_0\dot{+}V_{k,\ell,m}\right)\right)\right\|_{\infty,\infty}\\
&= \left\|\int_0^t \exp\left(-s(K_0+V_{k,\ell,m})\right)\left(V-V_{k,\ell,m}\right)\exp\left(-(t-s)(K_0+V)\right)ds\right\|_{\infty,\infty}\\
&\leq \sup_{x\in E}\int_0^t \mathsf{E}_x\left(\exp\left(-\int_0^s V_{k,\ell,m}(X(u))du\right)\left|V(X(s))-V_{k,\ell,m}(X(s))\right|\right.\\
&\qquad\left.\times\mathsf{E}_{X(s)}\left(\exp\left(-\int_0^{t-s}V(X(v))dv\right)\right)\right)ds
\end{aligned}$$

(Markov property together with Schwarz' inequality a couple of times)

$$\begin{aligned}
&\leq 2^{1/4}\sup_{x\in E}\left(\mathsf{E}_x\left(\exp\left(8\int_0^t V_-(X(u))du\right)\right)\right)^{1/4}\\
&\quad\times\sup_{x\in E}\mathsf{E}_x\left(\int_0^t\left|V(X(s))-V_{k,\ell,m}(X(s))\right|ds\right).
\end{aligned}\tag{3.11}$$

Since the function $(aI+K_0)^{-1}|V|$ belongs to $C_\infty(E)$ we infer that

$$\lim_{k,\ell,m\to\infty}\sup_{x\in E}\mathsf{E}_x\left(\int_0^t\left|V(X(s))-V_{k,\ell,m}(X(s))\right|ds\right)=0. \tag{3.12}$$

From (3.12) it follows that the right-hand side of (3.11) tends to zero and hence the claim in the theorem follows.

The following theorem seems more practical then the corresponding result in [21, Theorem 5.7].

3.5. THEOREM. Let V and W be Kato-Feller potentials. Suppose that the quantity $M(t)$, defined by

$$\begin{aligned}
M(t)=\int_0^t ds\int dz\,|V(z)-W(z)| \qquad\qquad (3.13)\\
\times\left\{\left(\exp\left(-s(K_0+V)\right)(z,z)\right)^{1/2}\sup_{t\leq s\leq 2t}\exp\left(-s\left(K_0+W\right)\right)^{1/2}(z,z)\right.\\
\left.+\left(\exp\left(-s(K_0+W)\right)(z,z)\right)^{1/2}\sup_{t\leq s\leq 2t}\exp\left(-s\left(K_0+V\right)\right)^{1/2}(z,z)\right\}
\end{aligned}$$

is finite. Then $D(t) := \exp(-t(K_0+V)) - \exp(-t(K_0+W))$ belongs to $\mathcal{I}_1$ and $\|D(t)\|_1 \leq \frac{1}{2}M(t)$.

Remark. For semi-groups of dimension n (see Definition 2.1.), the quantity in (3.13) is finite if $V-W$ belongs to $L^1(E,m)$ and if $n \leq 3$. The result is applicable in scattering theory: see Baumgärtel and Wollenberg [4] and also Reed and Simon [56].

PROOF. Suppose that the unitary operator U verifies $D(t) = |D(t)|\,U^*$. This is the so-called polar decomposition of the operator $D(t)$. Then the following (in-)equalities are more or less self-explanatory:

$$\begin{aligned}
&\|D(t)\|_1 = \operatorname{trace}(D(t)U^*)\\
&= \int dx \int_0^t ds \int dz \exp(-s(K_0+V))(x,z)(V(z)-W(z))\\
&\quad \times \overline{[U\exp(-(t-s)(K_0+W))(z,\cdot)](x)}\\
&= \int_0^t ds \int dz\,(V(z)-W(z))\,\langle \exp(-s(K_0+V))(\cdot,z),\\
&\qquad U\exp(-(t-s)(K_0+W))(z,\cdot)\rangle
\end{aligned}$$

(Cauchy-Schwarz and Chapman-Kolmogorov)

$$\begin{aligned}
&\leq \frac{1}{2}\int_0^{2t} ds \int dz\,|V(z)-W(z)|\,(\exp(-s(K_0+V))(z,z))^{1/2}\\
&\quad \times (\exp(-(2t-s)(K_0+W))(z,z))^{1/2}. \qquad (3.14)
\end{aligned}$$

From (3.14) the claim in Theorem 3.5. readily follows.

B. Singular perturbations: Hilbert-Schmidt properties.

We wish to establish a number of Hilbert-Schmidt properties of resolvent and semi-group differences. We begin with a proposition on differences of powers of resolvents. The main ingredient of the proof is the observation that the process $\{M_V^t(\tau) : 0 \leq \tau < t\}$ is a $\mathbf{P}_z$-martingale on the interval $(0,t)$ for all $z \in E$ and for all $t > 0$. Here $M_V^t(\tau)$ is defined by

$$M_V^t(\tau) = \exp\left(-\int_0^\tau V(X(u))du\right)\exp(-(t-\tau)(K_0+V))(X(\tau),y).$$

3.6. PROPOSITION. Suppose that $a > 0$ and $q \geq 1$ are chosen in such a way that the integral $\int_\Sigma dx \left[H_\Sigma^{a+V}(aI+K_0+V)^{-2q}(\cdot,x)\right](x)$ is finite. Then the operator

$$J\,(aI+K_0\dot{+}V)^{-q} - (aI+(K_0\dot{+}V)_\Sigma)^{-q}\,J \qquad (3.15)$$

is a Hilbert-Schmidt operator and

$$\begin{aligned}
&\left\|J\,(aI+K_0\dot{+}V)^{-q} - (aI+(K_0\dot{+}V)_\Sigma)^{-q}\,J\right\|_{\mathrm{HS}}^2\\
&\leq \frac{\Gamma(2q-1)}{\Gamma(q)^2}\int_\Sigma dx\left[H_\Sigma^{a+V}(aI+K_0+V)^{-2q}(\cdot,x)\right](x). \qquad (3.16)
\end{aligned}$$

Here $S = \inf\left\{s > 0 : \int_0^s 1_\Gamma(X(\sigma))d\sigma > 0\right\}$. The operator H_Σ^{a+V} is discussed in equality (2.31) of Corollary 2.15.

PROOF. Observe that the integral kernel of the operator in (3.15) is given by

$$\frac{1}{\Gamma(q)}\int_0^\infty dte^{-at}t^{q-1} \tag{3.17}$$
$$\times \mathsf{E}_x\left(\exp\left(-\int_0^S V(X(u))du\right)\exp\left(-(t-S)\left(K_0\dot{+}V\right)\right)(X(S),y) : S < t\right),$$

where x belongs to Σ and where y is in E. Hence from Chapman-Kolmogorov's identity it follows that

$$\int dy\left(J\left(aI + K_0\dot{+}V\right)^{-q}(x,y) - \left(aI + \left(K_0\dot{+}V\right)_\Sigma\right)^{-q}J(x,y)\right)^2$$
$$= \frac{1}{\Gamma(q)^2}\int_0^\infty dt_1\int_0^\infty dt_2 e^{-a(t_1+t_2)}(t_1t_2)^{q-1}$$
$$\mathsf{E}_x\otimes\mathsf{E}_x\left((\omega,\omega')\mapsto\exp\left(-\int_0^{S(\omega)}V(X(u))(\omega)du\right)1_{[0,t_1)}(S(\omega))\right.$$
$$\times\exp\left(-\int_0^{S(\omega')}V(X(u))(\omega')du\right)1_{[0,t_2)}(S(\omega'))$$
$$\left.\times\exp\left(-\left(t_1+t_2-S(\omega)-S(\omega')\right)(K_0+V)\right)(X(S)(\omega),X(S)(\omega'))\right)$$

(apply Fubini, substitute $t_1 - S(\omega) = \tau_1$ and $t_2 - S(\omega') = \tau_2$ and apply Fubini again)

$$= \frac{1}{\Gamma(q)^2}\int_0^\infty d\tau_1\int_0^\infty d\tau_2$$
$$\mathsf{E}_x\otimes\mathsf{E}_x\left((\omega,\omega')\mapsto\exp\left(-\int_0^{S(\omega)}(a+V(X(u))(\omega))\,du\right)1_{[0,\infty)}(S(\omega))\right.$$
$$\times\exp\left(-\int_0^{S(\omega')}(a+V(X(u))(\omega'))\,du\right)1_{[0,\infty)}(S(\omega'))$$
$$\times\exp\left(-(\tau_1+\tau_2)(a+K_0+V)\right)(X(S)(\omega),X(S)(\omega'))$$
$$\left.(\tau_1+S(\omega))^{q-1}(\tau_2+S(\omega'))^{q-1}\right)$$

(substitute $\tau_1+\tau_2 = \tau$ and $\tau_1 = \sigma$)

$$= \frac{1}{\Gamma(q)^2}\int_0^\infty d\tau$$
$$\mathsf{E}_x\otimes\mathsf{E}_x\left((\omega,\omega')\mapsto\exp\left(-\int_0^{S(\omega)}(a+V(X(u))(\omega))\,du\right)1_{[0,\infty)}(S(\omega))\right.$$

$$\times \exp\left(-\int_0^{S(\omega')} (a+V(X(u))(\omega'))\,du\right) 1_{[0,\infty)}(S(\omega'))$$

$$\times \exp\left(-\tau(a+K_0+V)\right)(X(S)(\omega),X(S)(\omega'))$$

$$\left.\int_0^{\tau} d\sigma\,(\sigma+S(\omega))^{q-1}\,(\tau-\sigma+S(\omega'))^{q-1}\right)$$

($2ab \le a^2+b^2$ with $a=(\sigma+S(\omega))^{q-1}$ and with $b=(\tau-\sigma+S(\omega'))^{q-1}$)

$$\le \frac{1}{2(2q-1)}\frac{1}{\Gamma(q)^2}\int_0^{\infty} d\tau$$

$$\mathsf{E}_x\otimes\mathsf{E}_x\left((\omega,\omega')\mapsto \exp\left(-\int_0^{S(\omega)}(a+V(X(u))(\omega))\,du\right)1_{[0,\infty)}(S(\omega))\right.$$

$$\times \exp\left(-\int_0^{S(\omega')} (a+V(X(u))(\omega'))\,du\right) 1_{[0,\infty)}(S(\omega'))$$

$$\times \exp\left(-\tau(a+K_0+V)\right)(X(S)(\omega),X(S)(\omega'))$$

$$\left.\left((\tau+S(\omega))^{2q-1}+(\tau+S(\omega'))^{2q-1}\right)\right)$$

(the roles of ω and ω' are interchangeble,
Fubini's theorem is applicable and $t=\tau+S(\omega')$ is substituted)

$$=\frac{1}{(2q-1)\Gamma(q)^2}\mathsf{E}_x\left(\omega\mapsto \exp\left(-\int_0^{S(\omega)}(a+V(X(u))(\omega))\,du\right)1_{[0,\infty)}(S(\omega))\right.$$

$$\times\int_0^{\infty} dt\,t^{2q-1}\mathsf{E}_x\left(\exp\left(-\int_0^{S(\omega')}(a+V(X(u))(\omega'))\,du\right)1_{[0,t)}(S(\omega'))\right.$$

$$\left.\left.\exp\left(-(t-S(\omega'))(a+K_0+V)\right)(X(S)(\omega),X(S)(\omega'))\right)\right)$$

(the process $\exp\left(-\int_0^{\tau}V(X(u))du\right)\exp\left(-(t-\tau)(K_0+V)\right)(y,X(\tau))$
is a martingale on the interval $(0,t)$)

$$=\frac{1}{(2q-1)\Gamma(q)^2}\int_0^{\infty}dt\,t^{2q-1}\mathsf{E}_x\left(\exp\left(-\int_0^{S}(a+V(X(u)))\,du\right)\right.$$

$$\left.\times\exp\left(-t\,(a+K_0+V)\right)(X(S),x) : S<\infty\right)$$

(definition of H_Σ^{a+V})

$$=\frac{\Gamma(2q)}{(2q-1)\Gamma(q)^2}\left[H_\Sigma^{a+V}\,(aI+K_0+V)^{-2q}\,(\cdot,x)\right](x). \tag{3.18}$$

Inequality (3.16) in the proposition follow upon integrating (3.18) with respect to x.

In what follows we write

$$C_\Sigma(t) = \sup_{z,w\in E} p_0(t/2,z,w)\int_\Sigma \left(\mathsf{E}_x\left(p_0(t,X(S),x) : S < \infty\right)\right)^{1/2} dx.$$

3.7. PROPOSITION. Suppose that there is a $q \in \mathbb{N}$, $q \geq 1$, such that for some constant $a_0 > 0$ and for some $q > 0$ the expression

$$\int_0^\infty t^{2q-1} e^{-a_0 t} C_\Sigma(t) dt \tag{3.19}$$

is finite. The following assertions hold true:

(i) $J(K_M - zI)^{-p} - (K_\Sigma - zI)^{-p} J \in \mathcal{C}_\infty\left(L^2(E,m), L^2(\Sigma,m)\right)$ for all $p \in \mathbb{N}$, for all $M \geq 0$ and for all $z \in \operatorname{res}(K_M)\bigcap \operatorname{res}(K_\Sigma)$.

(ii) Moreover $J(K_M - zI)^{-p} - (K_\Sigma - zI)^{-p} J$ belongs to $\mathcal{C}_2\left(L^2(E,m), L^2(\Sigma,m)\right)$, for all $p \geq q$, for all $M \geq 0$ and for all $z \in \operatorname{res}(K_M)\bigcap\operatorname{res}(K_\Sigma)$.

(iii) $\lim_{M\to\infty}\left\|J(K_M - zI)^{-p} - (K_\Sigma - zI)^{-p} J\right\|_r = 0$, for all $z \in \operatorname{res}(K_\Sigma)$ for $r = 2$ if $p \geq q$ and for $r = \infty$ if $p = 1$.

(iv) The rate of convergence in (iii) is the same for all $z \in \operatorname{res}(K_\Sigma)$.

(v) Suppose that the dimension of the semi-group $\{\exp(-tK_0) : t \geq 0\}$ is m, i.e. suppose $p_0(t,x,y) \leq c_1 t^{-m/2} e^{b_1 t}$. Also suppose that the inequality $\int dx \left(\mathsf{E}_x\left(p_0(t,X(S),x) : S < \infty\right)\right)^{1/2} \leq c_2 t^{-m/2} e^{b_2 t}$ is valid. Then, for $\operatorname{Re} a > 0$ large,
$$\left\|J\left(aI + K_0 \dot{+} V\right)^{-q} - \left(aI + (K_0\dot{+}V)_\Sigma\right)^{-q} J\right\|_{\mathrm{HS}} \leq \text{Constant} \times (\operatorname{Re}\, a)^{-q+m/4}.$$

Suppose $m < 4$. If $-\operatorname{Re} z_0 = a$ is sufficiently large, then the following representation in the sense of Hilbert-Schmidt norm is valid:

$$\begin{aligned} &J(K_M - z_0 I)^{-1} - (K_\Sigma - z_0 I)^{-1} J \\ &= (q-1)\int_0^\infty t^{q-2}\left[J(K_M - (z_0 - t)I)^{-q} - (K_\Sigma - (z_0-t)I)^{-q} J\right] dt. \end{aligned} \tag{3.20}$$

Remark 1. Representation (3.20) says that if the Hilbert-Schmidt property is true for some $q \in \mathbb{N}$, then it is true for all $q \in \mathbb{N}$.

Remark 2. It follows that the semi-group difference $\exp(-tK_M) - J^*\exp(-tK_\Sigma) J$, $t > 0$, consists of Hilbert-Schmidt operators, whenever the integral (see (3.41) below): $\int dx \sup_{\frac{1}{2}t<s<t}\sup_{z\in\Gamma} p_0(s,z,x)^{1/2}$ is finite.

Remark 3. Suppose that the penetration time S and the hitting T of Γ are equal P_x-almost surely. Since V is a Kato-Feller potential (i.e. $V_- \in K(E)$ and $V_+ \in K_{\mathrm{loc}}(E)$) the operator K is a well-defined self-adjoint operator in $L^2(E,m)$. The spectrum of K is contained in $[-\gamma,\infty)$, $\gamma > 0$, or $\mathbb{C}\setminus[-\gamma,\infty) \subseteq \operatorname{res}(K)$, the resolvent set of K. If the operator norm convergence $\text{u-}\lim_{M\to\infty} J(K_M - zI)^{-1}J^* = (K_\Sigma - zI)^{-1}$ can be established for some $z \in \text{res}(K_\Sigma)$, then this convergence is true for all $z \in \text{res}(K_\Sigma)$: see Kato [42, p. 211-212]. Because $\text{res}(K_\Sigma) = \bigcap_{M>M_0} \text{res}(K_M)$, for M_0 large enough, we also have $\mathbb{C} \setminus [-\gamma, \infty) \subseteq \text{res}(K_\Sigma)$.

PROOF of Proposition 3.15. Choose M_0 and C_0 in such a way that

$$\exp\left(-t(K_0+V)\right)(x,y) \leq M_0 e^{C_0 t} p_0(t,x,y)^{1/2} \sup_{z,w\in E} p_0(t/2,z,w)^{1/2}.$$

Such constants M_0 and C_0 exist: see [72, p. 301]. For the proof we need the following property of functions V, that belong to the Kato-Feller class. For $a > 0$ sufficiently the following supremum $\sup_{x\in E} \mathsf{E}_x \left(\exp\left(-2aS + 2\int_0^S V_-(X(u))du\right) : S < \infty\right)$ is finite. A proof of this fact runs as follows. Fix $t_0 > 0$. From Khas'minskii's lemma it follows that $\sup_{y\in E} \mathsf{E}_y \left(\exp\left(2\int_0^{t_0} V_-(X(u))du\right)\right) < \infty$. Choose $a > 0$ so large that $e^{-at_0} \sup_{y\in E} \mathsf{E}_y \left(\exp\left(2\int_0^{t_0} V_-(X(u))du\right)\right) < 1$. From the Markov property it then follows that:

$$\begin{aligned}
&\mathsf{E}_x\left(\exp\left(-2aS + 2\int_0^S V_-(X(u))du\right) : S < \infty\right) \\
&\leq \sum_{k=1}^{\infty} \mathsf{E}_x\left(\exp\left(-2a(k-1)t_0 + 2\int_0^{kt_0} V_-(X(u))du\right) : (k-1)t_0 < S \leq kt_0\right) \\
&\leq \sum_{k=1}^{\infty} \mathsf{E}_x\left(\exp\left(-2a(k-1)t_0 + 2\int_0^{(k-1)t_0} V_-(X(u))du\right)\right. \\
&\qquad \left.\mathsf{E}_{X((k-1)t_0)}\left(\exp\left(2\int_0^{t_0} V_-(X(u))du\right)\right) : (k-1)t_0 < S\right) \\
&\leq \sum_{k=1}^{\infty} e^{-2a(k-1)t_0}\left(\sup_{y\in E} \mathsf{E}_y\left(\exp\left(2\int_0^{t_0} V_-(X(u))du\right)\right)\right)^k < \infty.
\end{aligned}$$

Henceforth we pick $a > C_0 + a_0$ so large that

$$\sup_{x\in E} \mathsf{E}_x\left(\exp\left(-2aS + 2\int_0^S V_-(X(u))du\right) : S < \infty\right) < \infty.$$

For such a it follows that

$$\begin{aligned}
&\mathsf{E}_x\left(\exp\left(-\int_0^S (a + V(X(u)))\,du\right)\exp\left(-t(K_0+V)\right)(X(S),x) : S < \infty\right) \\
&\leq \left(\mathsf{E}_x\left(\exp\left(-2\int_0^S (a + V(X(u)))\,du\right) : S < \infty\right)\right)^{1/2}
\end{aligned}$$

$$\times \left(\mathsf{E}_x\left(\left(\exp\left(-t(K_0+V)\right)(X(S),x)\right)^2 : S<\infty\right)\right)^{1/2}$$
$$\leq M_0(a)\exp(C_0 t)\sup_{z,w\in E} p_0\left(t/2,z,w\right)^{1/2}\left(\mathsf{E}_x\left(p_0(t,X(S),x) : S<\infty\right)\right)^{1/2},$$

where

$$M_0(a) = M_0\sup_{x\in E}\left(\mathsf{E}_x\left(\exp\left(-2\int_0^S (a+V(X(u)))\,du\right) : S<\infty\right)\right)^{1/2}.$$

Hence from this together with (3.19) it follows that

$$\left\|J\left(aI+K_0\dot{+}V\right)^{-q} - \left(aI+(K_0\dot{+}V)_\Sigma\right)^{-q}J\right\|_{\mathrm{HS}}^2$$
$$\leq \frac{1}{(2q-1)\Gamma(q)^2}\int_0^\infty dt e^{-at}t^{2q-1}\int_\Sigma dx\mathsf{E}_x\left(\exp\left(-\int_0^S (a+V(X(u)))\,du\right)\right.$$
$$\left.\exp\left(-t(K_0+V)\right)(X(S),x) : S<\infty\right)$$
$$\leq \frac{M_0(a)}{(2q-1)\Gamma(q)^2}\int_0^\infty dt t^{2q-1}e^{-(a-C_0)t}C_\Sigma(t)<\infty. \tag{3.21}$$

The proof of Proposition 3.15. begins with establishing the Hilbert-Schmidt property in (ii). Let z_0 be such that $\mathrm{Re} z_0 = -a < -2A$. Suppose $p\geq q$. Then, as above, the Hilbert-Schmidt norm of the operator in (ii) can be estimated by

$$\left\|\left[J\left(K_M - z_0 I\right)^{-p} - \left(K_\Sigma - z_0 I\right)^{-p}J\right]\right\|_{\mathrm{HS}}^2$$
$$\leq \frac{M_0(a)}{(2p-1)\Gamma(p)^2}\int_0^\infty dt t^{2p-1}e^{-(a-C_0)t}C_\Sigma(t)$$
$$\leq \frac{M_0(a)}{(2p-1)\Gamma(p)^2}\int_0^\infty dt t^{2q-1}\max(1,t)^{2p-2q}e^{-(a-C_0)t}C_\Sigma(t)$$
$$\leq M_0'(a)\int_0^\infty dt t^{2q-1}e^{-a_0 t}C_\Sigma(t),$$

where $M_0'(a) = \sup_{t>0}\left\{t^{2p-2q}e^{-(a-C_0-a_0)t}\right\}\dfrac{M_0(a)}{(2p-1)\Gamma(p)^2}$. So from (3.19) it follows that the operator in (ii) is a Hilbert-Schmidt operator for $-\mathrm{Re} z > C_0 + a_0$ and $p\geq q$.
(i) Representation (3.20) always holds in the sense of the usual operator norm. Hence the compactness of $J\left(K_M - z_0 I\right)^{-1} - \left(K_\Sigma - z_0 I\right)^{-1}J$ follows for z_0 as in (i). But (i) and (ii) also hold for all other $z\in \mathrm{res}(K_M)\bigcap \mathrm{res}(K_\Sigma)$. Let d be the distance between

z_0 and $\sigma(K)$. For $|z - z_0| < d$ one gets from the Neumann series:

$$\begin{aligned}
&\left\| J\left(K_M - zI\right)^{-1} - \left(K_\Sigma - zI\right)^{-1} J \right\|_r \\
&\leq \sum_{k=0}^{\infty} (k+1) \frac{|z - z_0|^k}{d^k} \left\| J\left(K_M - z_0 I\right)^{-1} - \left(K_\Sigma - z_0 I\right)^{-1} J \right\|_r
\end{aligned} \tag{3.22}$$

where $r = \infty$ or $r = 2$. Consequently (i) and (ii) now follow.

From (3.22) together with the definition of the operator H_Σ^V it also follows that in the inequality (we always suppose $p \geq q$):

$$\begin{aligned}
&\left\| \left[J\left(K_M - z_0 I\right)^{-p} - \left(K_\Sigma - z_0 I\right)^{-p} J \right] \right\|_{\mathrm{HS}}^2 \\
&\leq \frac{\Gamma(2p-1)}{\Gamma(p)^2} \int_\Sigma dx \left[H_\Sigma^{a+V} \left(aI + K_0 + V + M 1_\Gamma\right)^{-2p} (\cdot, x) \right](x)
\end{aligned} \tag{3.23}$$

we may apply Lebesgue's theorem on dominated convergence. In fact fix x and $y \in E$ and let $\mathsf{E}_{y,0}^{x,t}$ be the conditional expectation that pins the process $\{X(s) : s \geq 0\}$ in y at time 0 and in x at time t. More precisely $\mathsf{E}_{y,0}^{x,t}$ is determined by the property that for all $0 < s < t$ and for all $A \in \mathcal{F}_s$ $\mathsf{E}_{y,0}^{x,t}(1_A) p_0(t, y, x) = \mu_{y,0}^{x,t}(A)$, where $\mu_{y,0}^{x,t}(A) = \mathsf{E}_y\left(p_0(t - s_1, X(s_1), x), A\right)$, with $s \leq s_1 < t$. Since the process $\{p_0(t - s, X(s), x) : 0 < s < t\}$ is a martingale on $(0, t)$, the measure $\mu_{y,0}^{x,t}$ is well-defined. It has the property that

$$\begin{aligned}
&\exp\left(-t(K_0 + V + M 1_\Gamma)\right)(y, x) - \exp\left(-t(K_0 + V)_\Sigma\right)(y, x) \\
&= \lim_{\tau \uparrow t} \mathsf{E}_y \left(\exp\left(- \int_0^\tau \left(V(X(u)) + M 1_\Gamma(X(u))\right) du \right) p_0(t - \tau, X(\tau), x) \right) \\
&\quad - \lim_{\tau \uparrow t} \mathsf{E}_y \left(\exp\left(- \int_0^\tau \left(V(X(u)) + M 1_\Gamma(X(u))\right) du \right) p_0(t - \tau, X(\tau), x), S > \tau \right) \\
&= \lim_{\tau \uparrow t} \mathsf{E}_y \left(\exp\left(- \int_0^\tau \left(V(X(u)) + M 1_\Gamma(X(u))\right) du \right) p_0(t - \tau, X(\tau), x), S \leq \tau \right) \\
&= \int \exp\left(- \int_0^t \left(V(X(u)) + M 1_\Gamma(X(u))\right) du \right) 1_{\{S < t\}} d\mu_{y,0}^{x,t}.
\end{aligned} \tag{3.24}$$

Hence

$$\begin{aligned}
&\lim_{M \to \infty} \exp\left(-t(K_0 + V + M 1_\Gamma)\right)(y, x) - \exp\left(-t(K_0 + V)_\Sigma\right)(y, x) \\
&= \lim_{M \to \infty} \int \exp\left(- \int_0^t \left(V(X(u)) + M 1_\Gamma(X(u))\right) du \right) 1_{\{S < t\}} d\mu_{y,0}^{x,t} \\
&= \int \exp\left(- \int_0^t V(X(u)) du \right) 1_{\left\{\int_0^t 1_\Gamma(X(u)) du = 0\right\}} 1_{\{S < t\}} d\mu_{y,0}^{x,t} \\
&= \int \exp\left(- \int_0^t V(X(u)) du \right) 1_{\{S \geq t\}} 1_{\{S < t\}} d\mu_{y,0}^{x,t} = 0.
\end{aligned} \tag{3.25}$$

Inserting the result of (3.25) in (3.24) yields

$$\lim_{M\to\infty}\left\|\left[J\left(K_M - z_0 I\right)^{-p} - \left(K_\Sigma - z_0 I\right)^{-p} J\right]\right\|_{\mathrm{HS}}^2$$
$$\leq \frac{1}{(2p-1)\Gamma(p)^2}\int_0^\infty dt e^{-at} t^{2p-1}\int_\Sigma dx \mathsf{E}_x\left(\exp\left(-\int_0^S (a+V(X(u)))\,du\right)\right.$$
$$\left.\lim_{M\to\infty}\exp\left(-t(K_0+V+M1_\Gamma)\right)(X(S),x) : S<\infty\right)$$
$$= \frac{1}{(2p-1)\Gamma(p)^2}\int_0^\infty dt e^{-at} t^{2p-1}\int_\Sigma dx \mathsf{E}_x\left(\exp\left(-\int_0^S (a+V(X(u)))\,du\right)\right.$$
$$\left.\exp\left(-t(K_0+V)_\Sigma\right)(X(S),x) : S<\infty\right) = 0.$$

because, on $\{S<\infty\}$, $X(S)$ is P_x-almost surely in Γ and for $y\in\Gamma$ the expression $\exp\left(-t(K_0+V)_\Sigma\right)(y,x)$ vanishes. All this is true provided $\mathrm{Re}z_0 = -a < -2A$, i.e. for $-\mathrm{Re}z_0$ positive and large enough. An argument as in (3.22) yields the same result, not only for $-\mathrm{Re}z_0$ large, but for all $z\in \mathrm{res}(K_\Sigma)$. Instead of the Neumann series we write ($|z-z_0|<d$)

$$\left(K_M - zI\right)^{-p} = \sum_{k=0}^\infty \binom{k+p-1}{k}(z-z_0)^k\left(K_M - z_0 I\right)^{-k-1}.$$

This shows (iii) and also (iv), except for the convergence in operator norm for $p=1$. For this we again take $-\mathrm{Re}z_0$ large enough and we use representation (3.20) for the operator norm. In fact we have

$$\left\|J\left(K_M - z_0 I\right)^{-1} - \left(K_\Sigma - z_0 I\right)^{-1} J\right\|$$
$$\leq (q-1)\int_0^\infty t^{q-2}\left\|J\left(K_M - (z_0-t)I\right)^{-q} - \left(K_\Sigma - (z_0-t)I\right)^{-q} J\right\| dt$$

and, since the Hilbert-Schmidt norm dominates the operator norm, we know that $\lim_{M\to\infty}\left\|J\left(K_M - (z_0-t)I\right)^{-q} - \left(K_\Sigma - (z_0-t)I\right)^{-q} J\right\| = 0$. This proves (iii) for $p=1$ and for the operator norm replacing the Hilbert-Schmidt norm.

Next we shall prove (v). It suffices to take for a a large real positive number. From (3.21) we infer

$$\left\|J\left(aI + K_0\dot{+}V\right)^{-p} - \left(aI + (K_0\dot{+}V)_\Sigma\right)^{-p} J\right\|_{\mathrm{HS}}$$
$$\leq \frac{M_0(a)}{(2p-1)\Gamma(p)^2}\int_0^\infty dt t^{2p-1} e^{-(a-C_0)t} C_\Sigma(t)$$

$$\leq \frac{M_0(a)}{(2p-1)\Gamma(p)^2} \int_0^\infty dt t^{2p-1} e^{-(a-C_0)t} c_1^{1/2} 2^{m/4} t^{-m/4} \exp\left(\frac{1}{2} b_1 t\right) c_2^{1/2} t^{-m/4} \exp(b_2 t)$$
$$\leq \frac{M_0(a) c_1^{1/2} c_2 2^{m/4}}{(2p-1)\Gamma(p)^2} \int_0^\infty dt t^{2p-1-m/2} \exp\left(-\left(a - C_0 - \frac{1}{2} b_1 - b_2\right) t\right)$$
$$= \frac{M_0(a) c_1^{1/2} c_2 2^{m/4}}{(2p-1)\Gamma(p)^2} \frac{\Gamma(2p - m/2)}{\left(a - C_0 - \frac{1}{2} b_1 - b_2\right)^{2p-m/2}}.$$

This proves the first part of (v). For $-\mathrm{Re} z_0 = a$ sufficiently large and for $t > 0$, the Hilbert-Schmidt norm of $J(K_M - (z_0 - t)I)^{-q} - (K_\Sigma - (z_0 - t)I)^{-q} J$ is dominated by a constant times $(a + t)^{-q+m/4}$. Since the mapping $t \mapsto \left\| J(K_M - (z_0 - t)I)^{-q} - (K_\Sigma - (z_0 - t)I)^{-q} J \right\|_{\mathrm{HS}}$ and since, upon employing the first claim of (v), the Hilbert-Schmidt norm of the right hand side of (3.20) can be estimated by $(q > 1 > m/4)$

$$\leq \text{Constant} \times (q-1) \int_0^\infty \frac{t^{q-2}}{(a+t)^{-q+m/4}} dt = \text{Constant} \times \frac{\Gamma(q)\Gamma(1-m/4)}{a^{1-m/4}\Gamma(q-m/4)},$$

the assertion in (v) follows.

Next we turn our attention to resolvent differences on the whole space.

3.8. PROPOSITION. Suppose that the BASSA hypotheses A1-A4 are satisfied. Also suppose that $m(\mathrm{bdr}(\Gamma)) = 0$ and assume that the boundedness condition B is verified. Then the following assertions are valid.

(i) For every $q \in \mathbb{N}$, for every $M \geq 0$ and for every $z \in \mathrm{res}(K_M)$, the operator $P(K_M - zI)^{-q}$ is compact.
(ii) In fact, for every $q \in \mathbb{N}$, $q > m/2$, for every $M \geq 0$ and for every $z \in \mathrm{res}(K_M)$ the operator $P(K_M - zI)^{-q}$ is a Hilbert-Schmidt operator.
(iii) $\lim_{M\to\infty} \left\| P(K_M - zI)^{-1} \right\| = 0$ for every $z \in \mathrm{res}(K_\Sigma)$.

PROOF. It suffices to prove these assertions for $z_0 \in \mathrm{res}(K)$, $\mathrm{Re} z_0 = -a < -2A$. Assertion (ii) follows because

$$\left\| P(K_M - z_0 I)^{-q} \right\|_2^2$$
$$= \frac{1}{((q-1)!)^2} \int_\Gamma dx \int_E dy \left| \int_0^\infty d\lambda e^{z_0\lambda} \lambda^{q-1} \exp(-\lambda K_M)(x,y) \right|^2 \tag{3.26}$$
$$= \frac{1}{((q-1)!)^2} \int_\Gamma \int_0^\infty \int_0^\infty d\lambda d\mu e^{z_0(\lambda+\mu)} \lambda^{q-1} \mu^{q-1} \exp(-(\lambda+\mu)K_M)(x,x)$$
$$\leq C m(\Gamma) \int_0^\infty d\lambda \int_0^\infty d\mu e^{-(a-A)(\lambda+\mu)} \lambda^{q-1} \mu^{q-1} \sup_{x,y\in E} p_0\left(\frac{\lambda+\mu}{2}, x, y\right)$$
$$\leq C_1 m(\Gamma). \tag{3.27}$$

Here, the constant C_1 is finite, provided the integral kernel is of dimension $2m$, where $2q > m$. For the notion of dimension see Definition 2.1. Then (i) is a consequence of

$$P(K_M + aI)^{-1} = (q-1)\int_0^\infty dt t^{q-2} P(K_M + aI + tI)^{-q}. \tag{3.28}$$

The next step shows that the equality in (3.26) yields:

$$\lim_{M\to\infty} \left\|P(K_M + aI)^{-q}\right\|_2^2 = 0, \quad \text{if} \quad q > \frac{m}{2}. \tag{3.29}$$

Therefore we estimate:

$$\begin{aligned}
&((q-1)!)^2 \left\|P(K_M + a)^{-q}\right\|_2^2 \\
&= \int_\Gamma dx \int_E dy \left|\int_0^\infty d\lambda e^{-a\lambda}\lambda^{q-1} \exp\left(-\lambda K_M\right)(x,y)\right|^2 \\
&= \int_0^\infty d\lambda \int_0^\infty d\mu e^{-a\lambda - a\mu}\lambda^{q-1}\mu^{q-1} \int_\Gamma \exp\left(-(\lambda+\mu)K_M\right)(x,x)dx \\
&= \int_0^\infty d\lambda \int_0^\infty d\mu e^{-a\lambda - a\mu}(\lambda\mu)^{q-1} \int_\Gamma \Big(\exp\left(-(\lambda+\mu)K_M\right)(x,x) - \exp\left(-(\lambda+\mu)K_\Sigma\right)(x,x)\Big)dx \\
&\quad + \int_0^\infty d\lambda \int_0^\infty d\mu e^{-a\lambda - a\mu}(\lambda\mu)^{q-1} \int_\Gamma \exp\left(-(\lambda+\mu)K_\Sigma\right)(x,x)dx.
\end{aligned} \tag{3.30}$$

Here

$$\exp\left(-\lambda K_\Sigma\right)(x,y) = \lim_{M\to\infty} \exp(-\lambda K_M)(x,y) \tag{3.31}$$

$$= \lim_{\lambda'\uparrow\lambda} \mathsf{E}_x\left(\exp\left(-\int_0^{\lambda'} V(X(\sigma))d\sigma\right) p_0(\lambda-\lambda', X(\lambda'), y) : S > \lambda'\right), \tag{3.32}$$

where $S = \inf\left\{s > 0 : \int_0^s 1_\Gamma(X(\sigma))d\sigma > 0\right\}$. The equality in (3.29) will follow from (3.30) together with (3.32), as soon as we have shown that the integral $\int_\Gamma \exp\left(-\lambda K_\Sigma\right)(x,x)dm(x)$ vanishes. Put $\widetilde{\Gamma}^r = \{x \in E : \mathsf{P}_x(S=0) = 1\}$. Then $\Gamma \setminus \widetilde{\Gamma}^r$ is contained in the boundary of Γ. Consequently $m(\Gamma \setminus \widetilde{\Gamma}^r) = 0$ and hence $\int_\Gamma \exp\left(-\lambda K_\Sigma\right)(x,x)dm(x) = 0$.

In the same way we have, for $t \geq 0$, $\lim_{M\to\infty} \|P(K_M + aI + tI)^{-q}\|_{\mathrm{HS}} = 0$. Hence (iii) follows by means of (3.28) and the dominated convergence theorem.

3.9. COROLLARY. Let the hypotheses be as in Proposition 3.7. The following assertions hold:

(i) For $M \geq 0$ and for $z \in \mathrm{res}(K_M) \cap \mathrm{res}(K_\Sigma)$ the resolvent difference $(K_M - zI)^{-1} - J^*\left(K_\Sigma - zI\right)^{-1} J$ is a compact operator.

(ii) The equality $\lim_{M\to\infty} \left\|(K_M - zI)^{-1} - J^*\left(K_\Sigma - zI\right)^{-1} J\right\| = 0$ is valid for $z \in \mathrm{res}\left(K_\Sigma\right)$.

PROOF. These assertions follow from the propositions 3.7 and 3.8 together with the identities:

$$J(K_M - zI)^{-1} - (K_\Sigma - zI)^{-1} J = J\left[(K_M - zI)^{-1} - J^* (K_\Sigma - zI)^{-1} J\right]$$

and

$$(K_M - zI)^{-1} - J^* (K_\Sigma - zI)^{-1} J = P(K_M - zI)^{-1} - J^*\left[J(K_M - zI)^{-1} - (K_\Sigma - zI)^{-1} J\right]. \tag{3.33}$$

We conclude this paper with a weighted semi-group difference consisting of Hilbert-Schmidt operators, where one of the semi-groups is singularly perturbed.

3.10. PROPOSITION. Let φ and ψ be real functions defined on E with the property that the expression

$$\int dx \left(|\varphi(x)|^4 + |\psi(x)|^4\right) \left(\mathsf{E}_x \left(p_0(2t - S, X(S), x) : S < t\right)\right)^{1/2} \tag{3.34}$$

is finite. Then, for $t > 0$, the operator $\varphi(\exp(-tK_M) - J^* \exp(-tK_\Sigma) J)\psi$ is a Hilbert-Schmidt operator and moreover

$$\begin{aligned} &\|\varphi(\exp(-tK_M) - J^* \exp(-tK_\Sigma) J)\psi\|_2^2 \\ &\le \frac{1}{2}\int dx \left(|\varphi(x)|^4 + |\psi(x)|^4\right) \left(\mathsf{E}_x \left(p_0(2t - S, X(S), x) : S < t\right)\right)^{1/2} \\ &\quad \times \left\|\exp\left(-t\left(K_0 \dot{+} (-2V_-)\right)\right)\right\|_{\infty,\infty} \sup_{\frac{1}{2}t < s < t} \|\exp(-sK_0)\|_{1,\infty}^{\frac{1}{2}}. \end{aligned} \tag{3.35}$$

Remark. Suppose that $\psi \equiv 1$. An inspection of the proof below will show that the conclusion (3.35) still holds if in (3.34) as well as in (3.35) the quantity $|\varphi(x)|^4 + |\psi(x)|^4$ is replaced by $2|\varphi(x)|^2$.

PROOF. From Theorem 4.6. in [21] we obtain the following:

$$\begin{aligned} &\|\varphi(\exp(-tK_M) - J^* \exp(-tK_\Sigma) J)\psi\|_2^2 \\ &= \int\int dx dy\, |\varphi(x)|^2 |\psi(y)|^2 \\ &\quad \times \left(\mathsf{E}_x\left(\exp\left(-\int_0^S V(X(\sigma))d\sigma\right) \exp(-(t - S)K_M)(X(S), y) : S < t\right)\right)^2 \end{aligned}$$

(write $u(x, y) = u(y, x)$

$$= \mathsf{E}_x\left(\exp\left(-\int_0^S V(X(\sigma))d\sigma\right) \exp(-(t - S)K_M)(X(S), y) : S < t\right))$$

$$= \int\int dx dy\, |\varphi(x)|^2 |\psi(y)|^2 u(x, y) u(y, x)$$

$$= \int dx\, |\varphi(x)|^2\, \mathsf{E}_x\left(\exp\left(-\int_0^S V(X(\sigma))d\sigma\right) \int u(x,y)\, |\psi(y)|^2 \right.$$

$$\left. \times \exp\left(-(t-S)K_M\right)(X(S),y)dy : S < t\right)$$

(Feynman-Kac formula)

$$= \int dx\, |\varphi(x)|^2\, \mathsf{E}_x\left[\exp\left(-\int_0^S V(X(\sigma))d\sigma\right)\right.$$

$$\left. \times \mathsf{E}_{X(S)}\left(\exp\left(-\int_0^{t-S} [V + M1_\Gamma](X(\sigma))d\sigma\right) u\left(x, X(t-S)\right) |\psi(X(t-S))|^2\right) : S < t\right]$$

(time dependent strong Markov property)

$$= \int dx\, |\varphi(x)|^2\, \mathsf{E}_x\left(\exp\left(-\int_0^t [V + M1_\Gamma](X(\sigma))d\sigma\right) u\left(X(t), x\right) |\psi(X(t))|^2 : S < t\right)$$

(definition $u(x,y)$)

$$= \int dx\, |\varphi(x)|^2\, \mathsf{E}_x\left(\exp\left(-\int_0^t [V + M1_\Gamma](X(\sigma))d\sigma\right) |\psi(X(t))|^2\right.$$

$$\left. \times \mathsf{E}_{X(t)}\left(\exp\left(-\int_0^S V(X(\sigma))d\sigma\right) \exp\left(-(t-S)K_M\right)(X(S),x) : S < t\right) : S < t\right)$$

$$\le \int dx\, |\varphi(x)|^2 \int dy\, |\psi(y)|^2 \exp(-tK_M)(x,y)$$

$$\times \mathsf{E}_y\left(\exp\left(-\int_0^S V(X(\sigma))d\sigma\right) \exp\left(-(t-S)K_M\right)(X(S),x) : S < t\right)$$

($2ab \le a^2 + b^2$ for a, $b \in \mathbf{R}$ together with symmetry and the identity of Chapman-Kolmogorov: $a = |\varphi(x)|^2$ and $b = |\psi(y)|^2$)

$$\le \frac{1}{2}\int dx\, |\varphi(x)|^4\, \mathsf{E}_x\left(\exp\left(-\int_0^S V(X(\sigma))d\sigma\right) \exp\left(-(2t-S)K_M\right)(X(S),x) : S < t\right)$$

$$+ \frac{1}{2}\int dy\, |\psi(y)|^4\, \mathsf{E}_y\left(\exp\left(-\int_0^S V(X(\sigma))d\sigma\right) \exp\left(-(2t-S)K_M\right)(X(S),y) : S < t\right). \tag{3.36}$$

Write $V_M = V + M1_\Gamma$. Since

$$\left(\exp(-\lambda K_M)(x,y)\right)^2 = \lim_{\lambda'\uparrow\lambda} \mathsf{E}_x\left(\exp\left(-\int_0^{\lambda'} V_M(X(\sigma))d\sigma\right) p_0\left(\lambda - \lambda', X(\lambda'), y\right)\right)^2$$

$$
\begin{aligned}
&\le \lim_{\lambda'\uparrow\lambda} \mathsf{E}_x \left(\exp\left(-2\int_0^{\lambda'} V_M(X(\sigma))d\sigma\right) p_0\left(\lambda-\lambda', X(\lambda'), y\right)\right) p_0(\lambda, x, y)\\
&\le \left\|\exp\left(-\lambda\left(K_0+2V_M\right)\right)\right\|_{1,\infty} p_0(\lambda, x, y)\\
&\le \left\|\exp\left(-\frac{1}{2}\lambda\left(K_0+2V_M\right)\right)\right\|_{2,\infty}^2 p_0(\lambda, x, y)\\
&\le \left\|\exp\left(-\frac{1}{2}\lambda\left(K_0+4V_M\right)\right)\right\|_{\infty,\infty} \left\|\exp\left(-\frac{1}{2}\lambda K_0\right)\right\|_{1,\infty} p_0(\lambda, x, y),
\end{aligned}
\tag{3.37}
$$

we infer the following inequalities:

$$
\begin{aligned}
&\left[\mathsf{E}_x\left(\exp\left(-\int_0^S V(X(\sigma))d\sigma\right)\exp\left(-(2t-S)K_M\right)(X(S), x) : S < t\right)\right]^2\\
&\le \mathsf{E}_x\left(\exp\left(-2\int_0^S V(X(\sigma))d\sigma\right) : S < t\right)\\
&\quad \times \mathsf{E}_x\left(\exp\left(-(2t-S)K_M\right)(X(S), x)^2 : S < t\right)\\
&\le \mathsf{E}_x\left(\exp\left(-2\int_0^S V(X(\sigma))d\sigma\right) : S < t\right)\\
&\quad \times \mathsf{E}_x\Bigg(\left\|\exp\left(-(t-\frac{1}{2}S)K_M\right)\right\|_{\infty,\infty}\left\|\exp\left(-\left(t-\frac{1}{2}S\right)K_0\right)\right\|_{1,\infty}\\
&\quad \times p_0(2t-S, X(S), x) : S < t\Bigg)\\
&\le \left\|\exp\left(-t\left(K_0\dot{+}(-2V_-)\right)\right)\right\|_{\infty,\infty}^2 \sup_{\frac{1}{2}t<s<t}\left\|\exp\left(-sK_0\right)\right\|_{1,\infty}\mathsf{E}_x\left(p_0(2t-S, X(S), x) : S < t\right)\\
&\le \left\|\exp\left(-t\left(K_0\dot{+}(-2V_-)\right)\right)\right\|_{\infty,\infty}^2 \left\|\exp\left(-\frac{t}{2}K_0\right)\right\|_{1,\infty}\mathsf{E}_x\left(p_0(2t-S, X(S), x) : S < t\right).
\end{aligned}
\tag{3.38}
$$

Consequently the result in Proposition 3.10. follows.

COROLLARY. Let φ and ψ be real functions defined on E with the property that the expression

$$
\int dx \left(|\varphi(x)|^4 + |\psi(x)|^4\right)\left(\mathsf{E}_x\left(p_0(2t-S, X(S), x) : S < t\right)\right)^{1/2} \tag{3.39}
$$

is finite. Then, for $z \in \operatorname{res}(K_M) \cap \operatorname{res}(K_\Sigma)$, the operator

$$
\varphi\left[\left(K_M - zI\right)^{-1} - J^*\left(K_\Sigma - zI\right)^{-1} J\right]\psi \tag{3.40}
$$

is compact.

Remark. Notice that the expression in (3.39) is finite whenever

$$\int dx \left(|\varphi(x)|^4 + |\psi(x)|^4\right) \sup_{\frac{1}{2}t<s<t} \sup_{z\in\Gamma} p_0(s,z,x)^{1/2} < \infty. \tag{3.41}$$

This is so because, on $\{S < \infty\}$, $X(S)$ belongs to the closure of Γ P_x-almost surely.

Acknowledgement. The authors are grateful to R. Seiler, Technische Universität Berlin, for the support, which the second author received during his stay in Berlin (September 1990). The first author is grateful for the support given by the DFG for the project "Schrödinger operators" that he enjoyed together with Prof. W. Kirsch from Bochum. The second author is obliged to the University of Antwerp (UIA) and to the National Fund for Scientific Research (NFWO) for their material support. He is also indebted to the European Science Project: Evolutionary Systems, Deterministic and Stochastic Evolution Equations, Control Theory and Mathematical Biology.

REFERENCES

[1] S. Agmon, Spectral properties of Schrödinger operators and scattering theory, Annali della Scuola Normale Superiore di Pisa, no. II, 2, 1975, 151-218.

[2] M. Aizenman and B. Simon, Brownian motion and Harnack inequality for Schrödinger operators, Comm. Pure and Applied Math., Vol. XXXV, 1982, 209-273.

[3] R. Azencott, P. Baldi, A. Bellaiche, C. Bellaiche, P. Bougerol, M. Chaleyat-Maurel, L. Elie, J. Granara, Géodésiques et diffusions en temps petit, séminaire de probabilités, Université de Paris VII, Astérisque 84-85, Soc. Math. de France 1981.

[4] K. Baumgärtel and M. Wollenberg, Mathematical scattering theory, Birkhäuser, Basel 1983.

[5] M. Ben-Artzi and A. Devinatz, The limiting absorption principle for partial differential operators, Memoirs of the Amer. Math. Soc. 1987, Vol. 66, Number 364.

[6] J.-M. Bismut, Large deviations and the Malliavin Calculus, Birkhäuser 1984.

[7] R.M. Blumenthal and R.K. Getoor, Markov processes and potential theory, Academic Press 1986.

[8] E.A. Carlen, S. Kusuoka and D.W. Stroock, Upper bounds for symmetric Markov transition functions, *Ann. Inst. Henri Poincaré* (Probabilités et Statistiques), Sup. au no. 2, 1987, 245-287.

[9] R. Carmona, W.C. Masters and B. Simon, Relativistic Schrödinger operators: Asymptotic behaviour of eigenfunctions, *J. Funct. Anal.* 91, 1990, 117-142.

[10] T. Coulhon, Dimensions of continuous and discrete semigroups on the L^p-spaces, preprint.

[11] T. Coulhon et L. Saloff-Coste, Semi-groupes d'opérateurs et espaces fonctionnels sur les groupes de Lie, preprint.

[12] T. Coulhon et L. Saloff-Coste, Thérèmes de Sobolev pour les semi-groupes d'opérateurs et applications aux groupes de Lie unimodulaires, *C.R. Acad. Sci. Paris*, t. 309, Série I, 1989, 289-294.

[13] T. Coulhon, Dimension à l'infini d'un semi-groupe analytique, preprint 1990.

[14] H.L. Cycon, R.G. Froese, W. Kirsch and B. Simon, Schrödinger operators with applications to Quantum Mechanics and global geometry, Springer Verlag, Berlin 1987.

[15] E.B. Davies, Pointwise bounds on the space and time derivatives of heat kernels, *J. Operator Theory*, no. 21, 1989, 367-378.

[18] E.B. Davies, Gaussian upper bounds for the heat kernels of some second-order operators on Riemannian manifolds, *J. of Funct. Analysis* **80**, 1988, 16-32.

[16] E.B. Davies, Heat kernels and spectral theory, Cambridge University Press, Cambridge 1989.

[17] E.B. Davies, Heat kernel bounds, conservation of probability and the Feller property, preprint London, 1991.

[19] M. Demuth, Report Institut für Mathematik, 1985, 04-85.

[20] M. Demuth, On large coupling operator norm convergences of resolvent differences, SFB 237 - Preprint Nr. 83, Institut für Mathematik, Ruhr-Universität, Bochum, to be published in J. of Math. Physics.

[21] M. Demuth and J.A. van Casteren, On spectral theory for selfadjoint Feller generators, *Reviews in Mathematical Physics*, Vol 1, no. 4, 1989, 325-414.

[22] M. Demuth and J.A. van Casteren, On potential continuity in stochastic spectral analysis, to appear in the Proceedings of the International Conference "Rigorous results in quantum dynamics", Liblice, CSSR, 11-15 June 1990, World Scientific.

[23] K.D. Elworthy, Geometric aspects of diffusions on manifolds, in École d'été de Probabilités de Saint-Flour XV-XVII, 1985-1987, editor P.L. Hennequin, Lecture Notes in Math. 1362, Springer-Verlag, Berlin 1988.

[24] K.D. Elworthy, Stochastic differential equations on manifolds, London Math. Soc. Lecture Notes in Mathematics **70**, Cambridge University Press 1982.

[25] K.D. Elworthy and A. Truman, Classical Mechanics, the diffusion (heat) equation, and the Schrödinger equation on a Riemannian manifold, *J. of Math. Physics* **22**, 1981, 2144-2166.

[26] The semi-classical expansion for a charged particle on a curved space background, preprint Univ. of Warwick. 1985.

[27] I.W. Herbst and A.D. Sloan, Perturbations of translation invariant positivity preserving semigroups in $L^2(\mathrm{R}^N)$, *Transactions Amer. Math. Soc.* **236**, 1978, 325-360.

[28] I.W. Herbst and Zhongxin Zhao, Sobolov spaces, Kac regularity and the Feynman-Kac formula, in Seminar on Stochastic Processes 1987, edited by E. Cinlar, K.L. Chung, R.K. Getoor and J. Glover, Birkhäuser Basel 1988.

[29] W. Hoh and N. Jacob, Some Dirichlet forms generated by pseudo differential operators, preprint Universität Erlangen-Nürnberg 1991.

[30] T. Ichinose, Essential selfadjointness of the Weyl quantized relativistic Hamiltonian, *Ann. Inst. Henri Poincaré*, Vol. 51, no. 3, 1889, 265-298.

[31] T. Ichinose, The nonrelativistic limit problem for a relativistic spinless particle in an electromagnetic field, *J. Functional Analysis* **73**, 233-257, 1987.

[32] T. Ichinose, Path integral for a Weyl quantized relativistic Hamiltonian and the nonrelativistic limit problem, in Differential Equations and Mathematical Physics, Lecture Notes in Mathematics **1285**, 205-210, 1988.

[33] T. Ichinose, Kato's inequality and essential self-adjointness for Weyl quantized relativistic Hamiltonian, *Proc. Japan Acad.* **64A**, 367-369, 1988.

[34] T. Ichinose and H. Tamura, Imaginary-time path integral for a relativistic spinless particle in a electromagnetic field, *Comm. in Math. Physics Vol.* **105**, 239-257, 1986.

[35] T. Ichinose and H. Tamura, Path integral for Weyl quantized relativistic Hamiltonian, *Proc. Japan Acad.* **62A**, 91-93, 1986.

[36] N. Ikeda and H. Matsumoto, Short time asymptotics of the traces of heat kernels of Schrödinger operators with magnetic fields, to appear in *Bull. Sci. Math.*

[37] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes, second edition, North-Holland 1989.

[38] N. Jacob, Feller semigroups, Dirichlet forms, and pseudo differential operators, preprint Universität Erlangen, 1990, to appear in *Forum Math.*.

[39] N. Jacob, A class of elliptic differential operators generating symmetric Dirichlet forms, preprint Universität Erlangen, 1990.

[40] N. Jacob, Pseudo differential operators with negative definite functions as symbol: Apllications in probability theory and Mathematical physics, Preprint University of Erlangen-Nürnberg.

[41] A. Jensen, High energy resolvent estimates for Schrödinger operators in Besov spaces, preprint Aalborg University, Aalborg Denmark 1990.

[42] T. Kato, Perturbation theory for linear operators, Springer Verlag, Berlin 1976.

[43] T. Kato, Monotonicity theorems in scattering theory, *Hadronic Journal* **1**, 1978, 134-154.

[44] T. Kato, On the Trotter-Lie product formula, *Proc. Japan Acad.*, **50**, 1974, 694-698.

[45] R.Z. Khas'minskii, On positive solutions of the equation $Au + Vu = 0$, *Theory of Probability and its applications*, Vol. IV, no. 3, 1959, 309-318.

[46] A.N. Kochubeĭ, Singular parabolic equations and Markov processes, *Math. USSR Izvestiya*, Vol. **24**, no. 1, 1985, 73-97.

[47] A.N. Kochubeĭ, Parabolic pseudodifferential equations, hypersingular integrals and Markov processes, *Math. USSR Izvestiya*, Vol. **33**, no. 2, 1989, 233-259.

[48] R. Lavine, Absolute continuity of positive spectrum for Schrödinger operators with long-range potentials, *J. of Funct. Analysis*, no. **12**, 1973, 30-54.

[49] R. Léandre, Minoration en temps petit de la densité d'une diffusion déférée, *J. of Funct. Analysis*, **74**, 1987, 399-414.

[50] E. Mourre, Absence of singular continuous spectrum for certain operators, *Commun. Math. Phys.* no. **78**, 1981, 391-408.

[51] J. Nash, Continuity of solutions of parabolic and elliptic equations, *Amer. J. Math.* Vol. **80**, 1958, 931-954.

[52] P. Perry, I.M. Sigal and B. Simon, Spectral analysis of N-body Schrödinger operators, *Annals of Mathematics*, Vol. **114**, 1981, 519-567.

[53] S.C. Port and C.J. Stone, Brownian motion and classical potential theory, Academic Press, New York, 1978.

[54] N.I. Portenko, Generalized diffusion processes, in *Proceedings of the third Japan-USSR Symposium on Probability Theory* , edited by G. Maruyama and J.V. Prokhorov, Lecture Notes in Mathematics **550**, 500-523, Springer Verlag, Berlin 1976.

[55] M. Reed and B. Simon, Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness, Academic Press, New York 1975.

[56] M. Reed and B. Simon, Methods of Modern Mathematical Physics III: Scattering Theory, Academic Press, New York 1979.

[57] M. Reed and B. Simon, Methods of Modern Mathematical Physics IV: Analysis of operators, Academic Press, New York 1978.

[58] D. Robert and H. Tamura, Semi-classical estimates for resolvents and asymptotics for total scattering cross-sections, *Ann. Inst. Henri Poincaré*, Vol. 46, no. 4, 1987, p. 415-442.

[58] D. Robert and H. Tamura, Asymptotic behavior of scattering amplitudes in semi-classical and low energy limits, Annales Inst. Fourier (Grenoble) 39, 1989, no. 1, 155-192.

[60] B. Simon, Schrödinger semigroups, *Bulletin (New Series) of the Amer. Math. Soc.*, Vol. **7**, no. 3, 1982, p. 447-526.

[61] B. Simon, Functional integration and quantum physics, Academic Press, New York 1979.

[62] B. Simon, Semiclassical analysis of low lying eigenvalues, I. Non-degenerate minima: Asymptotic expansions, *Ann. Inst. H. Poincaré* **38**, 1983, 295-307.

[63] B. Simon, Semiclassical analysis of low lying eigenvalues, II. Tunneling, *Annals of Math.* **120**, 1984, 89-118.

[64] L. Smits, Perturbations of Markov semigroups by additive functionals, Ph.D.-thesis, Antwerp 1991.

[65] L. Smits and J.A. van Casteren, Semigroups defined by additive processes, in Semigroup Theory and Evolution Equations, "Proceedings of the second International Conference on Trends in Semigroups and Evolution Equations", Delft 1989, Ph. Clément, E. Mitidieri, B. de Pagter editors, Marcel Dekker Lecture Notes in Pure and Applied Mathematics **135**, 463-482, Basel 1991.

[67] K.-T. Sturm, Heat kernel bounds on manifolds, preprint Universität Erlangen 1990.

[68] K.-Th. Sturm, Schrödinger semigroups on manifolds, Preprint, Univ. Erlangen-Nürnberg, 1991.

[69] K. Taira, On the existence of Feller semigroups with boundary conditions, Memoirs of the Am. Math. Soc. 267, Providence 1991.
[70] J.A. van Casteren, Generators of strongly continuous semigroups, Research Notes in Math. 115, Pitman, London 1985.
[71] J.A. van Casteren, Integral kernels and the Feynman-Kac formalism, in Aspects of Positivity in Functional Analysis, R. Nagel, U.Schlotterbeck and M.P.H. Wolff (editors), North-Holland, 1986, 179-185.
[72] J.A. van Casteren, A pointwise inequality for generalized Schrödinger semigroups, in the Proceedings of the Conference to be published by Teubner, Leipzig, in the Proceedings of the conference "Symposium Partial Differential Operators Holzhau 1988" held from April 24 until April 30, 1988, (B.W. Schulze and H. Triebel, editors), Teubner Texte zur Mathematik, Band 112, 298-312, Teubner Verlagungsgesellschaft, DDR-7010 Leipzig 1989.
[73] J.A. van Casteren, On generalized Schrödinger semigroups, in Markov Processes and Control Theory (H. Langer and V. Nollau, editors), Proceedings of the conference ISAM 88, "Markovsche Prozessen und Steuerungstheorie" held at Gaussig, DDR, 11-15 January 1988 under auspices of Technical University of Dresden, Mathematical Research Vol. 54, Akademie Verlag Berlin 1989, DDR, 16-39.
[74] N. Varopoulos, Hardy-Littlewood theory for semigroups, *J. Functional Analysis*, Vol. **63**, no. 2, 1985, p. 240-260.
[75] N. Varopoulos, Analysis on Lie groups, *J. of Funct. Analysis* **76**, 1988, 346-410.
[76] N. Varopoulos, Small time Gaussian estimates of heat diffusion kernels I, the semigroup technique, *Bull. Sci. Math. Série* II **113**, 1989, 263-277.
[77] N. Varopoulos, Small time Gaussian estimates of heat diffusion kernels II, the theory of large deviations, *J. of Funct. Analysis* **93**, 1990, 1-33.
[78] R. Weder, The unified approach to spectral analysis, *Commun. Math. Physics* **60**, 1978, 291-299.
[79] R. Weder, The unified approach to spectral analysis II, *Proceedings of the Amer. Math. Soc.* **75**, no. 1, 1979, 81-84.
[80] R. Weder, Second order operators in the uniform norm, *Comm. in Partial Differential Equations* **3**, no. 5, 1978, 381-406.
[81] J. Weidmann, Linear operators in Hilbert space, Graduate texts in Mathematics **68**, Springer-Verlag, Berlin 1980.

M. Demuth, Max-Planck Institut
Arbeitsgruppe Part. Differentialgleich. und kompl. Analysis
FB Mathematik, Universität Potsdam
Am neuen Palais 10, O-1571 Potsdam

J.A. van Casteren, University of Antwerp (UIA)
Department of Mathematics and Computer Science
Universiteitsplein 1, 2610 Wilrijk/Antwerp, Belgium