Max-Planck-Institut für Mathematik Bonn

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Preprint Series 2018 (15)

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MPIM 18-15

DERIVATIVES OF EISENSTEIN SERIES OF WEIGHT 2 AND INTERSECTIONS OF MODULAR CORRESPONDENCES

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ABSTRACT. We give a formula for certain values and derivatives of Siegel series and use them to compute Fourier coefficients of derivatives of the Siegel Eisenstein series of weight $\frac{g}{2}$ and genus g. When $g = 4$, the Fourier coefficient is approximated by a certain Fourier coefficient of the central derivative of the Siegel Eisenstein series of weight 2 and genus 3, which is related to the intersection of 3 arithmetic modular correspondences. Applications include a relation between weighted averages of representation numbers of symmetric matrices.

1. INTRODUCTION

1.1. **Motivation : On the modular correspondences.** Let $j = j' = j(\tau)$ be the elliptic modular function on the upper half plane. For $m \geq 1$ let $\varphi_m \in \mathbb{Z}[j, j']$ be the classical modular polynomial defined by

$$\varphi_m(j(\tau), j(\tau')) = \prod_{A \in \mathrm{M}_2(\mathbb{Z}) \pmod{\mathrm{SL}_2(\mathbb{Z})},\ \det A = m} (j(\tau) - j(A\tau')).$$

Put $S = \operatorname{Spec} \mathbb{Z}[j, j']$ and $S_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[j, j']$. Let T_m and $T_{m,\mathbb{C}}$ be the arithmetic and geometric divisors defined by $\varphi_m = 0$. We can view S as an arithmetic threefold $\mathcal{S} = \mathcal{M} \times_{\operatorname{Spec} \mathbb{Z}} \mathcal{M}$, where $\mathcal{M}$ is the moduli stack of elliptic curves over $\mathbb{Z}$, and T_m as the moduli stack $\mathcal{T}_m$ of isogenies of elliptic curves of degree m. In the 19th century Hurwitz has computed the intersection

$$(T_{m_1,\mathbb{C}} \cdot T_{m_2,\mathbb{C}}) := \dim_{\mathbb{C}} \mathbb{C}[j, j']/(\varphi_{m_1}, \varphi_{m_2})$$

of complex curves. Gross and Keating [3] discovered that $(T_{m_1,\mathbb{C}} \cdot T_{m_2,\mathbb{C}})$ is related to the Fourier coefficients of the Siegel Eisenstein series of weight 2 for $Sp_2(\mathbb{Z})$. Moreover, they gave an explicit expression for the intersection

$$(T_{m_1} \cdot T_{m_2} \cdot T_{m_3}) := \log \sharp \mathbb{Z}[j, j']/(\varphi_{m_1}, \varphi_{m_2}, \varphi_{m_3})$$

of 3 arithmetic modular correspondences. It is already mentioned in the introduction of [3] that computations of Kudla or Zagier strongly suggest that $\deg \mathscr{Z}(B)$ equals the B-th Fourier coefficient of the derivative of the Siegel Eisenstein series of weight 2 for $Sp_3(\mathbb{Z})$, up to multiplication by a

2010 *Mathematics Subject Classification.* 11F30, 11F32.

Key words and phrases. Eisenstein series, arithmetic intersection numbers, modular correspondence.

constant which is independent of B. A complete proof of this identity has been given in [17] (cf. [11]).

The purpose of this paper is to compute the Fourier coefficients of the derivative of the Siegel Eisenstein series of weight 2 for $Sp_4(\mathbb{Z})$. One may expect that these coefficients are related to the intersection of 4 modular correspondences. However, the number

$$\log \sharp\mathbb{Z}[j,j']/(\varphi_{m_1},\varphi_{m_2},\varphi_{m_3},\varphi_{m_4}),$$

does not seem to be naturally expanded to a sum over positive semi-definite symmetric half-integral matrices of size 4 and does not seem to be a right object. The fiber product $\mathcal{T}_{m_1}\times_{\mathcal{S}}\mathcal{T}_{m_2}\times_{\mathcal{S}}\mathcal{T}_{m_3}\times_{\mathcal{S}}\mathcal{T}_{m_4}$ has a disjoint sum decomposition according to the values of the fundamental matrices:

$$\mathcal{T}_{m_1}\times_{\mathcal{S}}\mathcal{T}_{m_2}\times_{\mathcal{S}}\mathcal{T}_{m_3}\times_{\mathcal{S}}\mathcal{T}_{m_4}=\bigsqcup_T \mathscr{Z}(T),$$

where T extends over the set of positive semi-definite symmetric half-integral matrices of size 4 with diagonal entries m_1, m_2, m_3, m_4. If T is positive definite, then $\mathscr{Z}(T)$ is empty unless $\det T$ is a square and T is split except over a single prime. If T is positive definite and $\det T$ is a square, then the T-th Fourier coefficient is zero unless T is anisotropic only at a prime p, in which case the T-th Fourier coefficient is approximately equal to $\deg \mathscr{Z}(T')$, where T' is some positive semi-definite symmetric half-integral matrix of size 3 (see Theorem 1.3). Our result may imply that for each point of the intersection, where 4 surfaces intersect properly, in a small neighborhood of the point, the intersection multiplicity behaves like the intersection multiplicity of 3 surfaces of them.

In the intervening years Kudla and others have gone a long way towards proving such relations in much greater generality. In [8], he introduced a certain family of Eisenstein series of genus g and weight $\frac{g+1}{2}$. They have an odd functional equation and hence have a natural zero at their center of symmetry. The central derivatives of such series, which he refers to as incoherent Eisenstein series, have a connection with arithmetic algebraic geometry of cycles on integral models of Shimura varieties attached to orthogonal groups of signature $(2, g-1)$, at least when $g \leq 4$. We refer the reader to [14] for $g = 1$, to [8, 12, 15] for $g = 2$, to [11, 24, 17] for $g = 3$, and to [13] for $g = 4$. However, there are serious problems with the construction of arithmetic models of these Shimura varieties as soon as $g \geq 5$.

1.2. **The Fourier coefficients of derivative of Eisenstein series.** In this paper we compute the Fourier coefficients of derivatives of incoherent Eisenstein series of genus g and weight $\frac{g}{2}$. In this introductory section we will consider classical Eisenstein series of level 1. Let g be a positive integer that is divisible by 4. Let

$$E_g(Z,s)=\sum_{\{C,D\}}\det(CZ+D)^{-g/2}|\det(CZ+D)|^{-s}(\det Y)^{s/2}$$

be the Siegel Eisenstein series of genus g, where $\{C, D\}$ runs over a complete set of representatives of the equivalence classes of coprime symmetric pairs of degree g, and Z is a complex symmetric matrix of degree g with positive definite imaginary part Y. This series converges absolutely for $\Re s > \frac{g}{2} + 1$ and admits a meromorphic continuation to the whole s-plane by the general theory of Langlands.

If $\frac{g}{4}$ is even, then $E_g(Z, s)$ is holomorphic at $s = 0$ and the T-th Fourier coefficient of $E_g(Z, 0)$ is equal to

$$2\Big(\sum_i \frac{1}{N(L_i, L_i)}\Big)^{-1} \sum_i \frac{N(L_i, T)}{N(L_i, L_i)} \tag{1.1}$$

by the Siegel formula (see [23, 10, 27]), where $\{L_i\}$ is the set of isometry classes of positive definite even unimodular lattices of rank g. Here $N(L, L')$ denotes the number of isometries $L' \to L$ for two quadratic spaces L, L' over $\mathbb{Z}$. In particular, the nondegenerate Fourier coefficients are supported on a single rational equivalence class.

On the other hand, if $\frac{g}{4}$ is odd, then $E_g(Z, s)$ has a zero at $s = 0$. Our main object of study in this paper is the derivative

$$\frac{\partial}{\partial s} E_g(Z, s)|_{s=0} = \sum_{T>0} C_g(T) e^{2\pi\sqrt{-1}\mathrm{tr}(TZ)} + \sum_{\text{other } T} C_g(T, Y) e^{2\pi\sqrt{-1}\mathrm{tr}(TZ)}.$$

Fix a positive definite symmetric half-integral $n \times n$ matrix T and a rational prime p. Let $\mathbb{Q}^{(p)}$ be a subring of $\mathbb{Q}$, consisting of the numbers of the form $\frac{a}{p^n}$ with $n \in \mathbb{N}$ and $a \in \mathbb{Z}$. We define the additive character $\mathbf{e}_p$ of $\mathbb{Q}_p$ by setting $\mathbf{e}_p(x) = e^{-2\pi\sqrt{-1}y}$ with $y \in \mathbb{Q}^{(p)}$ such that $x - y \in \mathbb{Z}_p$. The Siegel series attached to T and p is defined by

$$b_p(T, s) = \sum_{z \in \mathrm{Sym}_n(\mathbb{Q}_p)/\mathrm{Sym}_n(\mathbb{Z}_p)} \mathbf{e}_p(-\mathrm{tr}(Tz)) \nu[z]^{-s},$$

where $\nu[z]$ is the product of denominators of elementary divisors of z. Put $D_T = (-4)^{[n/2]} \det T$. We denote the primitive Dirichlet character corresponding to $\mathbb{Q}(\sqrt{D_T})$ by χ_T and its conductor by $\mathfrak{d}^T$. Put $\xi_p^T = \chi_T(p)$. Let $e_p^T = \mathrm{ord}_p D_T$ or $e_p^T = \mathrm{ord}_p D_T - \mathrm{ord}_p \mathfrak{d}^T$ according as n is odd or even. There exists a polynomial $F_p^T(X) \in \mathbb{Z}[X]$ such that

$$b_p(T, s) = \gamma_p^T(p^{-s}) F_p^T(p^{-s}),$$

where

$$\gamma_p^T(X) = (1 - X) \prod_{j=1}^{[n/2]} (1 - p^{2j} X^2) \times \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \frac{1}{1 - \xi_p^T p^{n/2} X} & \text{if } n \text{ is even.} \end{cases}$$

The symbol η_p^T stands for the normalized Hasse invariant of T over $\mathbb{Q}_p$ (see Definition 2.1). We write $\mathrm{Diff}(T)$ for the finite set of prime numbers p such that $\eta_p^T = -1$. A direct calculation gives the following formula:

Proposition 5.1. *Assume that $\frac{g}{4}$ is odd. Let T be a positive definite symmetric half-integral matrix of size g.*

(1) *If $\chi_T = 1$, then $C_g(T) = 0$ unless $\mathrm{Diff}(T)$ is a singleton.*
(2) *If $\chi_T = 1$ and $\mathrm{Diff}(T) = \{p\}$, then*

$$C_g(T) = -\frac{2^{(g+2)/2}p^{-(g+e_p^T)/2}\log p}{\zeta(1-\frac{g}{2})\prod_{i=1}^{(g-2)/2}\zeta(1-2i)}\frac{\partial F_p^T}{\partial X}(p^{-g/2})\prod_{p\neq \ell|D_T}\ell^{-e_\ell^T/2}F_\ell^T(\ell^{-g/2}).$$

(3) *If $\chi_T \neq 1$, then*

$$C_g(T) = -\frac{2^{(g+2)/2}L(1,\chi_T)}{\zeta(1-\frac{g}{2})\prod_{i=1}^{(g-2)/2}\zeta(1-2i)}\prod_{\ell|D_T}p^{-e_\ell^T/2}F_\ell^T(\ell^{-g/2}).$$

Remark 1.1. If $\chi_T \neq 1$, then $L(1,\chi_T) = \frac{\sqrt{\mathfrak{d}^T}}{\log\epsilon}h$ by Dirichlet's class number formula, where h is the class number of the real quadratic field $\mathbb{Q}(\sqrt{\det T})$ and $\epsilon = \frac{t+u\sqrt{\mathfrak{d}^T}}{2}$ ($t > 0$, $u > 0$) is the solution to the Pell equation $t^2 - \mathfrak{d}^T u^2 = 4$ for which u is smallest.

The following theorem is a special case of Theorem 4.3 and allows us to compute $\frac{\partial F_p^T}{\partial X}(\xi_p^T p^{-g/2})$. For simplicity we here assume p to be odd.

Theorem 1.2. *Let p be an odd rational prime and $T = \mathrm{diag}[t_1, \dots, t_g]$ with $0 \leq \mathrm{ord}_p t_1 \leq \cdots \leq \mathrm{ord}_p t_g$. Put $T' = \mathrm{diag}[t_1, \dots, t_{g-1}]$. Suppose that g is even and $p \nmid \mathfrak{d}^T$. Then*

$$F_p^T(\xi_p^T p^{-g/2}) = p^{e_p^T/2}F_p^{T'}(\xi_p^T p^{-g/2}).$$

If $\eta_p^T = -1$, then

$$\frac{\xi_p^T}{p^{g/2}}\frac{\partial F_p^T}{\partial X}\left(\frac{\xi_p^T}{p^{g/2}}\right) = \frac{F_p^{T'}(\xi_p^T p^{(2-g)/2})}{p-1} - p^{e_p^T/2}\frac{\xi_p^T}{p^{g/2}}\frac{\partial F_p^{T'}}{\partial X}\left(\frac{\xi_p^T}{p^{g/2}}\right).$$

Our key ingredient is the explicit formula for $F_p^T(X)$, given by Ikeda and Katsurada in [5], which expresses the polynomial F_p^T in terms of the (naive) extended Gross–Keating datum H of T over $\mathbb{Z}_p$. The polynomial $F_p^{T'} = F_p^{H'}$ is defined in terms of a subset $H' \subsetneq H$ for any p in a uniform way. Actually, if $g = 4$, then the values $\frac{\partial F_p^{H'}}{\partial X}(p^{-2})$ and $F_p^{H'}(p^{-1})$ depend only on (a_1, a_2, a_3) if we write (a_1, a_2, a_3, a_4) for the Gross–Keating invariant of T over $\mathbb{Z}_p$.

1.3. **Applications.**

1.3.1. *On the average of the representation numbers.* Theorem 1.2 combined with the Siegel formula will identify (1.1) with four times the average of the representation numbers of a symmetric matrix of size $g-1$ (see Conjecture 5.4 and Proposition 5.5). The following result is a special case of Proposition 5.5.

Corollary 5.6. If T is a positive definite symmetric half-integral matrix of size 4 which satisfies $\chi^T = 1$ and $\eta_\ell^T = 1$ for $\ell \neq p$, then there exists a positive definite symmetric half-integral matrix T' of size 3 such that

$$\sum_{(E',E)} \frac{N(\mathrm{Hom}(E',E),T)}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')} = 2\sum_{(E',E)} \frac{N(\mathrm{Hom}(E',E),T')}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')},$$

where (E,E') extends over all pairs of isomorphism classes of supersingular elliptic curves over $\bar{\mathbb{F}}_p$.

1.3.2. *On the Fourier coefficients and the modular correspondences.* The factor $\frac{\partial F_p^{H'}}{\partial X}(\xi_p^T p^{-g/2})$ appears in Fourier coefficients of central derivatives of incoherent Eisenstein series of genus $g-1$ and weight $\frac{g}{2}$, which have close connection with arithmetical geometry on Shimura varieties at least for $g \leq 5$ as mentioned above. We will be mostly interested in the case $g = 4$. When T_{m_1}, T_{m_2} and T_{m_3} intersect properly, the formula of Gross and Keating in [3] can be stated as follows:

$$(T_{m_1} \cdot T_{m_2} \cdot T_{m_3}) = \sum_B \deg \mathscr{Z}(B),$$

where B extends over all positive definite symmetric half-integral matrices with diagonal entries m_1, m_2, m_3. Here $\deg \mathscr{Z}(B) = 0$ unless $\mathrm{Diff}(B)$ consists of a single rational prime p, in which case

$$\deg \mathscr{Z}(B) = -\frac{(\log p)}{2p^2}\frac{\partial F_p^B}{\partial X}\left(\frac{1}{p^2}\right)\sum_{(E,E')} \frac{N(\mathrm{Hom}(E',E),B)}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')}. \tag{1.2}$$

The degree $\deg \mathscr{Z}(B)$ equals the B-th Fourier coefficient of the derivative of the Siegel Eisenstein series of weight 2 and genus 3 up to a negative constant (cf. Theorem 2.2 of [17]). We combine (1.2), Theorem 5.3 and Corollary 5.6 to obtain the following formula:

Theorem 1.3. *If T is a positive definite symmetric half-integral matrix of size 4, $\chi_T = 1$ and $\mathrm{Diff}(T)$ consists of a single prime number p, then there exists a positive definite symmetric half-integral matrix T' of size 3 such that*

$$\frac{C_4(T)}{-2^8\cdot 3^2} = \deg \mathscr{Z}(T') + \frac{F_p^{T'}(p^{-1})}{2\sqrt{p}^{e_p^T}(p-1)}\log p \sum_{(E,E')} \frac{N(\mathrm{Hom}(E',E),T')}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')},$$

where (E,E') extends over all pairs of isomorphism classes of supersingular elliptic curves over $\bar{\mathbb{F}}_p$.

Since $\mathrm{Hom}(E',E)$ is a quaternary quadratic space, if S has rank greater than 4, then $N(\mathrm{Hom}(E,E'),S) = 0$. Therefore when $g \geq 5$, the nature of Fourier coefficients of the derivative of Eisenstein series of weight 2 and genus g should be much different. The case $g = 4$ should be a boundary case. We will explicitly compute $F_p^{T'}(p^{-1})$ in Lemma 5.7 and show that

$$\left|\frac{C_4(T)}{-2^8\cdot 3^2\cdot \deg \mathscr{Z}(T')}-1\right|< \frac{20}{p\sqrt{p}}.$$

Moreover, Corollary 5.8 says that for a fixed prime number p

$$\lim_{\mathrm{ord}_p(\det T)\to\infty}\frac{C_4(T)}{-2^8\cdot 3^2\cdot \deg \mathscr{Z}(T')}=1.$$

1.4. **Organizations.** We now explain the lay-out of this paper. Section 2 extends the notion of incoherent Eisenstein series to the case where the point at which the Eisenstein series is evaluated lies within the left half-plane. We calculate the Fourier coefficients of those Eisenstein series and their derivatives. In Section 3 we derive a general formula for Fourier coefficients of derivatives of incoherent Eisenstein series. Section 4 is devoted to a local study of the Siegel series. We give the inductive expression for the special value of the derivative of the Siegel series. Section 5 is devoted to proving Theorem 5.3.

Acknowledgement. Cho is supported by JSPS KAKENHI Grant No. 16F16316. Yamana is partially supported by JSPS Grant-in-Aid for Young Scientists (B) 26800017. Yamana also thanks Max Planck Institut für Mathematik for an excellent working environment. We would like to thank Stephen Kudla for very stimulating discussions.

Notations

For a finite set A, we denote by $\sharp A$ the number of elements in A. For a ring R we denote by $\mathrm{M}_{i,j}(R)$ the set of $i\times j$-matrices with entries in R and write $\mathrm{M}_m(R)$ in place of $\mathrm{M}_{m,m}(R)$. The group of all invertible elements of $\mathrm{M}_m(R)$ and the set of symmetric matrices of size m with entries in R are denoted by $\mathrm{GL}_m(R)$ and $\mathrm{Sym}_m(R)$, respectively. Let $\mathcal{E}_m(R)$ be the set of elements $(a_{ij})\in \mathrm{Sym}_m(R)$ such that $a_{ii}\in 2R$ for every i. For matrices $B\in \mathrm{Sym}_m(R)$ and $G\in \mathrm{M}_{m,n}(R)$ we use the abbreviation $B[G]={}^tGBG$, where ${}^tG$ is the transpose of G. If $A_1,\dots,A_r$ are square matrices, then $\mathrm{diag}[A_1,\dots,A_r]$ denotes the matrix with $A_1,\dots,A_r$ in the diagonal blocks and 0 in all other blocks. Let $\mathbf{1}_m$ be the identity matrix of degree m. Put

$$Sp_g(R)=\left\{G\in \mathrm{GL}_{2g}(R)\;\middle|\; G\begin{pmatrix}0&\mathbf{1}_g\\-\mathbf{1}_g&0\end{pmatrix}{}^tG=\begin{pmatrix}0&\mathbf{1}_g\\-\mathbf{1}_g&0\end{pmatrix}\right\},$$

$$M_g(R)=\left\{\mathbf{m}(A)=\begin{pmatrix}A&0\\0&{}^tA^{-1}\end{pmatrix}\;\middle|\; A\in \mathrm{GL}_g(R)\right\},$$

$$N_g(R)=\left\{\mathbf{n}(B)=\begin{pmatrix}\mathbf{1}_g&B\\0&\mathbf{1}_g\end{pmatrix}\;\middle|\; B\in \mathrm{Sym}_g(R)\right\}.$$

Let $\mathbb{Z}$ be the set of integers and μ_n the group of n-th roots of unity. If x is a real number, then we put $[x]=\max\{m\in\mathbb{Z}\mid m\le x\}$.

2. Eisenstein series

Let k be a totally real number field with integer ring $\mathfrak{o}$. The set of real places of k is denoted by $\mathfrak{S}_\infty$. The completion of k at a place v is denoted by k_v. Let $(\ ,\ )_{k_v} : k_v^\times \times k_v^\times \to \mu_2$ denote the Hilbert symbol. We let $\mathfrak{p}$ denote a finite prime of k and do not use the letter $\mathfrak{p}$ for a real place. Let $q_\mathfrak{p} = \sharp\mathfrak{o}/\mathfrak{p}$ be the order of the residue field. We define the character $\mathbf{e}_\mathfrak{p}$ of $k_\mathfrak{p}$ by $\mathbf{e}_\mathfrak{p}(x) = \mathbf{e}(-y)$ with $y \in \mathbb{Q}^{(p)}$ such that $\mathrm{Tr}_{k_\mathfrak{p}/\mathbb{Q}_p}(x) - y \in \mathbb{Z}_p$ if p is the rational prime divisible by $\mathfrak{p}$. Put $\mathbf{e}(z) = e^{2\pi\sqrt{-1}z}$ for $z \in \mathbb{C}$ and $\mathbf{e}_\infty(z) = \prod_{v\in\mathfrak{S}_\infty} \mathbf{e}(z_v)$ for $z \in \prod_{v\in\mathfrak{S}_\infty} \mathbb{C}$.

Once and for all we fix a positive integer $g \geq 2$. Let $(V, (\ ,\ ))$ be a quadratic space of dimension m over k_v. Whenever we speak of a quadratic space, we always assume that $(\ ,\ )$ is nondegenerate, i.e., $(u, V) = 0$ implies that $u = 0$. Put $s_0 = \frac{1}{2}(m - g - 1)$. Given $u = (u_1, \dots, u_g) \in V^g$, we write (u, u) for the $g \times g$ symmetric matrix with (i, j) entry equal to (u_i, u_j). We write $\det V$ for the element in $k_v^\times/k_v^{\times 2}$ represented by the determinant of the matrix representation of the bilinear form $(\ ,\ )$ with respect to any basis for V over k_v. We define the character $\chi^V : k_v^\times \to \mu_2$ by

$$\chi^V(t) = (t, (-1)^{m(m-1)/2} \det V)_{k_v}. \tag{2.1}$$

We normalize our Hasse invariant η^V so that it depends only on the isomorphism class of an anisotropic kernel of V (cf. [2, 22]).

Definition 2.1. We associate to the quadratic space V over $k_\mathfrak{p}$ of dimension m an invariant $\eta^V \in \mu_2$ according to the type of V as follows:

- If m is odd, then an anisotropic kernel of V has dimension $2 - \eta^V$.
- If m is even and $\chi^V \neq 1$ and if we choose an element $c \in k_\mathfrak{p}^\times$ such that $\chi^V(c) = \eta^V$, then V is the orthogonal sum of a split form of dimension $m - 2$ with the norm form scaled by the factor c on the quadratic extension of $k_\mathfrak{p}$ corresponding to χ^V.
- If m is even and $\chi^V = 1$, then V is split or the orthogonal sum of the norm form on the quaternion algebra over $k_\mathfrak{p}$ with a split form of dimension $m - 4$ according as $\eta^V = 1$ or -1.

We denote the set of positive definite symmetric matrices over $\mathbb{R}$ of rank g by $\mathrm{Sym}_g(\mathbb{R})^+$. Let

$$\mathfrak{H}_g = \{X + \sqrt{-1}Y \in \mathrm{Sym}_g(\mathbb{C}) \mid Y \in \mathrm{Sym}_g(\mathbb{R})^+\}$$

be the Siegel upper half-space of genus g. The real symplectic group $Sp_g(\mathbb{R})$ acts transitively on $\mathfrak{H}_g$ by $GZ = (AZ + B)(CZ + D)^{-1}$ for $Z \in \mathfrak{H}_g$ and $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_g(\mathbb{R})$. We define the maximal compact subgroups by

$$K_\mathfrak{p} = Sp_g(\mathfrak{o}_\mathfrak{p}), \qquad K_v = \{G \in Sp_g(k_v) \mid G(\sqrt{-1}\mathbf{1}_g) = \sqrt{-1}\mathbf{1}_g\}$$

for $v \in \mathfrak{S}_\infty$. We have the Iwasawa decomposition

$$Sp_g(k_v) = M_g(k_v)N_g(k_v)K_v.$$

Denote the two-fold metaplectic cover of $Sp_g(k_v)$ by Mp_v. There is a canonical splitting $N_g(k_v) \to \mathrm{Mp}_v$. When $\mathfrak{p}$ does not divide 2, we have a canonical splitting $K_\mathfrak{p} \to \mathrm{Mp}_\mathfrak{p}$. We still use $N_g(k_v)$ and $K_\mathfrak{p}$ to denote the images of these splittings. Let $\tilde{K}_v$ denote the pull-back of K_v in Mp_v. Define the map $\mathrm{Mp}_v \to \mathbb{R}_+^\times$ by writing $\tilde{G} = \mathbf{n}(b)\tilde{m}\tilde{k} \in \mathrm{Mp}_v$ with $b \in \mathrm{Sym}_g(k_v)$, $a \in \mathrm{GL}_g(k_v)$, $\tilde{m} = (\mathbf{m}(a), \zeta)$ and $\tilde{k} \in \tilde{K}_v$ and setting $|a(\tilde{G})| = |\det a|_v$. We refer to Section 1.1 of [27] for additional explanation.

Let V be a quadratic space over k_v and ω_v the Weil representation of Mp_v with respect to $\mathbf{e}_v$ on the space $\mathcal{S}(V^g)$ of the Schwartz functions on V^g. We associate to $\varphi \in \mathcal{S}(V^g)$ the function on $\mathrm{Mp}_v \times \mathbb{C}$ by

$$f_\varphi^{(s)}(\tilde{G}) = (\omega_v(\tilde{G})\varphi)(0)|a(\tilde{G})|^{s-s_0}.$$

The real metaplectic group acts on the half-space $\mathfrak{H}_g$ through $Sp_g(\mathbb{R})$. There is a unique factor of automorphy $\jmath_v : \mathrm{Mp}_v \times \mathfrak{H}_g \to \mathbb{C}^\times$ whose square descends to the automorphy factor on $Sp(k_v) \times \mathfrak{H}_g$ given by $\jmath_v(G_v, Z_v)^2 = \det(C_v Z_v + D_v)$ for $G_v = \begin{pmatrix} * & * \\ C_v & D_v \end{pmatrix} \in Sp(k_v)$. We define an automorphy factor $\jmath : \prod_{v \in \mathfrak{S}_\infty} (\mathrm{Mp}_v \times \mathfrak{H}_g) \to \mathbb{C}^\times$ by $\jmath(\tilde{G}, Z) = \prod_v \jmath_v(\tilde{G}_v, Z_v)$.

Let $\mathbb{A}$ be the adele ring of k and $\mathbb{A}_\mathbf{f}$ the finite part of the adele ring. We arbitrarily fix a quadratic character χ of $\mathbb{A}^\times/k^\times$ such that $\chi_v = \mathrm{sgn}^{m(m-1)/2}$.

Definition 2.2. Let $\mathcal{C} = \{\mathcal{C}_v\}$ be a collection of local quadratic spaces of dimension m such that $\chi^{\mathcal{C}_v} = \chi_v$ for all v, such that $\mathcal{C}_v$ is positive definite for $v \in \mathfrak{S}_\infty$ and such that $\eta^{\mathcal{C}_\mathfrak{p}} = 1$ for almost all $\mathfrak{p}$. We say that $\mathcal{C}$ is coherent if it is the set of localizations of a global quadratic space. Otherwise we call $\mathcal{C}$ incoherent.

One can derive the following criterion from the theorem of Minkowski-Hasse (see Theorem 4.4 of [21]).

Lemma 2.3. *Put $d = [k : \mathbb{Q}]$. When m is odd, $\mathcal{C}$ is coherent if and only if $(-1)^{d(m^2-1)/8} \prod_\mathfrak{p} \eta^{\mathcal{C}_\mathfrak{p}} = 1$. When m is even, $\mathcal{C}$ is coherent if and only if $(-1)^{dm(m-2)/8} \prod_\mathfrak{p} \eta^{\mathcal{C}_\mathfrak{p}} = 1$.*

There is a unique splitting $Sp_g(k) \hookrightarrow \mathrm{Mp}_g$ by which we regard $Sp_g(k)$ as the subgroup of the two-fold metaplectic cover Mp_g of $Sp_g(\mathbb{A})$. Let $P_g = M_g N_g$ be the Siegel parabolic subgroup of Sp_g. Given any pure tensor $\varphi = \otimes_\mathfrak{p} \varphi_\mathfrak{p} \in \otimes'_\mathfrak{p} \mathcal{S}(\mathcal{C}_\mathfrak{p}^g)$, we consider the function

$$f_\varphi^{(s)}(\tilde{G}) = \prod_\mathfrak{p} f_{\varphi_\mathfrak{p}}^{(s)}(\tilde{G}_\mathfrak{p}), \qquad f_{\varphi_\mathfrak{p}}^{(s)}(\tilde{G}_\mathfrak{p}) = (\omega_\mathfrak{p}(\tilde{G}_\mathfrak{p})\varphi_\mathfrak{p})(0)|a(\tilde{G}_\mathfrak{p})|^{s-s_0}$$

on $\mathrm{Mp}_g \times \mathbb{C}$ and the Eisenstein series on $\prod_{v \in \mathfrak{S}_\infty} \mathfrak{H}_g$

$$E(Z, f_\varphi^{(s)}) = (\det Y)^{(s-s_0)/2} \sum_{\gamma \in P_g(k) \backslash Sp_g(k)} |\jmath(\gamma, Z)|^{s_0 - s} \jmath(\gamma, Z)^{-g} f_\varphi^{(s)}(\gamma),$$

where Y is the imaginary part of Z. The series is absolutely convergent for $\Re s > \frac{g+1}{2}$. It admits a meromorphic continuation to the whole plane and its Laurent coefficients define automorphic forms. Moreover, it is holomorphic at $s = s_0$, and if $\mathcal{C}$ is coherent, then the Siegel–Weil formula holds by [10].

From now on we require that $m \leq g+1$. Let V be a totally positive definite quadratic space of dimension m over k. We normalize the invariant measure $\mathrm{d}h$ on $\mathrm{O}(V,k)\backslash\mathrm{O}(V,\mathbb{A})$ to have total volume 1 and define the integral

$$I(Z,\varphi) = \int_{\mathrm{O}(V,k)\backslash\mathrm{O}(V,\mathbb{A})} \Theta(Z,h;\varphi)\,\mathrm{d}h$$

of the theta function

$$\Theta(Z,h;\varphi) = \sum_{u \in V(k)^g} \varphi(h^{-1}u)\mathbf{e}_\infty(\mathrm{tr}((u,u)Z)).$$

Since we are under coherent situation, the Siegel–Weil formula can now be stated as follows:

$$E(Z, f^{(s)}_\varphi)|_{s=s_0} = 2I(Z,\varphi). \tag{2.2}$$

The reader who is interested in this identity can consult Theorem 2.2(i) of [27]. On the other hand, if $\mathcal{C}$ is incoherent, then the series $E(Z, f^{(s)}_\varphi)$ has a zero at $s = s_0$ by Corollary 5.5 of [27].

Consider the Fourier expansions

$$E(Z, f^{(s)}_\varphi) = \sum_{T \in \mathrm{Sym}_g(k)} A(T,Y,\varphi,s)\mathbf{e}_\infty(\mathrm{tr}(TZ)),$$

$$\frac{\partial}{\partial s}E(Z, f^{(s)}_\varphi)|_{s=s_0} = \sum_{T \in \mathrm{Sym}_g(k)} C(T,Y,\varphi)\mathbf{e}_\infty(\mathrm{tr}(TZ)),$$

where

$$Z = X + \sqrt{-1}Y, \qquad C(T,Y,\varphi) = \frac{\partial}{\partial s}A(T,Y,\varphi,s)|_{s=s_0}.$$

Put $\mathrm{Sym}^{\mathrm{nd}}_g = \mathrm{Sym}_g(k) \cap \mathrm{GL}_g(k)$. When $T \in \mathrm{Sym}^{\mathrm{nd}}_g$, the Fourier coefficient has an explicit expression as an infinite product

$$A(T,Y,\varphi,s) = a(T,Y,s)\prod_{\mathfrak{p}} W_T\left(f^{(s)}_{\varphi_\mathfrak{p}}\right)$$

for $\Re s \gg 0$, where

$$W_T\left(f^{(s)}_{\varphi_\mathfrak{p}}\right) = \int_{\mathrm{Sym}_g(k_\mathfrak{p})} f^{(s)}_{\varphi_\mathfrak{p}}\left(\begin{pmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{pmatrix}\mathbf{n}(z_\mathfrak{p})\right)\overline{\mathbf{e}_\mathfrak{p}(\mathrm{tr}(Tz_\mathfrak{p}))}\,\mathrm{d}z_\mathfrak{p}$$

and $a(T,Y,s)\mathbf{e}_\infty(\sqrt{-1}\mathrm{tr}(TY))$ is a product of the confluent hypergeometric functions investigated in [18]. Given $T \in \mathrm{Sym}^{\mathrm{nd}}_g$, we define the quadratic form on $V^T = k^g$ by $u \mapsto T[u]$ and define the Hecke character $\chi^T = \prod_v \chi^T_v$ and the Hasse invariants $\eta^T_\mathfrak{p}$, where χ^T_v is defined in (2.1). Let $\mathrm{Diff}(T,\mathcal{C})$ denote the set of places v of k such that T is not represented by $\mathcal{C}_v$. Let

Sym_g^+ denote the set of totally positive definite symmetric $g \times g$ matrices over k.

Lemma 2.4. *Let $\varphi_\mathfrak{p} \in \mathcal{S}(\mathcal{C}_\mathfrak{p}^g)$ and $T \in \mathrm{Sym}_g^{\mathrm{nd}}$.*

(1) *$a(T, Y, s)$ and $W_T\bigl(f^{(s)}_{\varphi_\mathfrak{p}}\bigr)$ are entire functions in s.*

(2) *$\lim_{s \to s_0} W_T\bigl(f^{(s)}_{\varphi_\mathfrak{p}}\bigr) = 0$ unless T is represented by $\mathcal{C}_\mathfrak{p}$.*

(3) *If $m = g$, $T \in \mathrm{Sym}_g^+$, $\chi^T = \chi$ and $\mathcal{C}$ is incoherent, then $\mathrm{Diff}(T, \mathcal{C})$ is a finite set of odd cardinality.*

Proof. The first part is well-known (see [6, 18]). Lemma on p. 73 of [16] implies (2). By assumption $\mathrm{Diff}(T, \mathcal{C}) = \{\mathfrak{p} \mid \eta^{\mathcal{C}_\mathfrak{p}} = -\eta_\mathfrak{p}^T\}$. Since $\mathcal{C}$ is incoherent, Lemma 2.3 implies $\prod_\mathfrak{p} \eta^{\mathcal{C}_\mathfrak{p}} = -\prod_\mathfrak{p} \eta_\mathfrak{p}^T$, which proves (3). □

Let $T \in \mathrm{Sym}_g^+$. Then both $a(T, Y, s_0)$ and $C(T, Y, \varphi)$ are independent of Y. Put

$$c_m(T) = a(T, Y, s_0), \qquad C(T, \varphi) = C(T, Y, \varphi), \qquad D_T = \mathrm{N}_{k/\mathbb{Q}}(\det(2T)).$$

Let $\mathfrak{d}_k$ denote the absolute value of the discriminant of k. Note that

$$c_g(T) = c_g D_T^{-1/2}, \qquad c_g = \mathfrak{d}_k^{-g(g+1)/4}\left(\mathbf{e}\left(\frac{g^2}{8}\right)\frac{2^g \pi^{g^2/2}}{\Gamma_g\left(\frac{g}{2}\right)}\right)^d \tag{2.3}$$

by (4.34K) of [18], where $\Gamma_g(s) = \pi^{g(g-1)/4} \prod_{i=0}^{g-1} \Gamma\bigl(s - \frac{i}{2}\bigr)$.

Proposition 2.5. *Let $m = g$ and $T \in \mathrm{Sym}_g^+$. Suppose that $\mathcal{C}$ is incoherent. If $\chi^T = \chi$, then $C(T, \varphi) = 0$ unless $\mathrm{Diff}(T, \mathcal{C})$ is a singleton. Moreover, if $\mathrm{Diff}(T, \mathcal{C}) = \{\mathfrak{p}\}$, then*

$$C(T, \varphi) = c_g D_T^{-1/2} \lim_{s \to -1/2} \frac{\partial W_T\bigl(f^{(s)}_{\varphi_\mathfrak{p}}\bigr)}{\partial s} \prod_{\mathfrak{l} \neq \mathfrak{p}} W_T\bigl(f^{(s)}_{\varphi_\mathfrak{l}}\bigr).$$

Proof. For given φ and T, let $\mathfrak{S}$ be a finite set of rational primes of k such that if $\mathfrak{q} \notin \mathfrak{S}$, then $\mathfrak{q}$ does not divide 2, $\chi_\mathfrak{q}$ is unramified, $\mathbf{e}_\mathfrak{q}$ is of order 0, $T \in \mathrm{GL}_g(\mathfrak{o}_\mathfrak{q})$ and the restriction of $f^{(s)}_{\varphi_\mathfrak{q}}$ to $K_\mathfrak{q}$ is 1. Since T cannot be unimodular at $\mathfrak{p} \in \mathrm{Diff}(T, \mathcal{C})$, the set $\mathfrak{S}$ necessarily contains $\mathrm{Diff}(T, \mathcal{C})$. The T-th Fourier coefficient of $E(Z, f^{(s)}_\varphi)$ is given by

$$A(T, Y, \varphi, s) = \beta^T(s) a(T, Y, s) \prod_{\mathfrak{q} \in \mathfrak{S}} \beta_\mathfrak{q}^T(s) W_T\bigl(f^{(s)}_{\varphi_\mathfrak{q}}\bigr), \tag{2.4}$$

where

$$\beta^T(s) = \frac{L\bigl(s + \frac{1}{2}, \chi^T \chi\bigr)}{\prod_{j=1}^{[(g+1)/2]} \zeta(2s + 2j - 1)} \times \begin{cases} 1 & \text{if } 2 \nmid g, \\ L\bigl(s + \frac{g+1}{2}, \chi\bigr)^{-1} & \text{if } 2 | g, \end{cases}$$

$$\beta_\mathfrak{q}^T(s) = \frac{\prod_{j=1}^{[(g+1)/2]} \zeta_\mathfrak{q}(2s + 2j - 1)}{L\bigl(s + \frac{1}{2}, \chi_\mathfrak{q}^T \chi_\mathfrak{q}\bigr)} \times \begin{cases} 1 & \text{if } 2 \nmid g, \\ L\bigl(s + \frac{g+1}{2}, \chi_\mathfrak{q}\bigr) & \text{if } 2 | g. \end{cases}$$

Notice that the product $\beta_{\mathfrak{q}}^T(s)W_T\Big(f_{\varphi_{\mathfrak{q}}}^{(s)}\Big)$ is holomorphic at $s=-\frac{1}{2}$. Indeed, if $\chi_{\mathfrak{q}}^T=\chi_{\mathfrak{q}}$, then $\beta_{\mathfrak{q}}^T(s)$ is holomorphic at $s=-\frac{1}{2}$ while if $\chi_{\mathfrak{q}}^T\neq\chi_{\mathfrak{q}}$, then $\beta_{\mathfrak{q}}^T(s)$ has a simple pole at $s=-\frac{1}{2}$, but $W_T\Big(f_{\varphi_{\mathfrak{q}}}^{(s)}\Big)$ has a zero at $s=-\frac{1}{2}$ by Lemma 2.4(2).

Assume that $\chi^T=\chi$. Then $\beta^T(s)$ is holomorphic and has no zero at $s=-\frac{1}{2}$. If $\mathfrak{q}\in\mathrm{Diff}(T,\mathcal{C})$, then $\beta_{\mathfrak{q}}^T(s)W_T\Big(f_{\varphi_{\mathfrak{q}}}^{(s)}\Big)$ has a zero at $s=-\frac{1}{2}$ by Lemma 2.4(2), which combined with (2.4) proves the first statement. We obtain the first formula by differentiating (2.4) at $s=-\frac{1}{2}$. □

Corollary 2.6. If $m=g$, $\mathcal{C}$ is incoherent and $T\in\mathrm{Sym}_g^+$ with $\chi^T\neq\chi$, then

$$C(T,\varphi)=c_gD_T^{-1/2}\lim_{s\to-1/2}\frac{\partial\beta^T}{\partial s}(s)\prod_{\mathfrak{p}}\beta_{\mathfrak{p}}^T(s)W_T\Big(f_{\varphi_{\mathfrak{p}}}^{(s)}\Big).$$

Proof. Since $\beta^T(s)$ has a zero at $s=-\frac{1}{2}$ if $\chi\neq\chi^T$, we can deduce Corollary 2.6 from (2.4). □

3. Fourier coefficients of derivatives of Eisenstein series

Let $\gamma_v(t)$ be the Weil constant associated to the character of second degree $u\mapsto\mathbf{e}_v(tu^2)$, and $\varepsilon_v(\mathcal{C}_v)$ the unnormalized Hasse invariant of $\mathcal{C}_v$. Put

$$\gamma(\mathcal{C}_v)=\varepsilon_v(\mathcal{C}_v)\gamma_v\left(\frac{1}{2}\right)^{m-1}\gamma_v\left(\frac{1}{2}\det\mathcal{C}_v\right).$$

Let $L_{\mathfrak{p}}$ be an integral lattice of $\mathcal{C}_{\mathfrak{p}}$, i.e., a finitely generated $\mathfrak{o}_{\mathfrak{p}}$-submodule of $\mathcal{C}_{\mathfrak{p}}$ which spans $\mathcal{C}_{\mathfrak{p}}$ over $k_{\mathfrak{p}}$ and such that $(u,u)\in\mathfrak{o}_{\mathfrak{p}}$ for every $u\in L_{\mathfrak{p}}$. Let

$$L_{\mathfrak{p}}^*=\{u\in\mathcal{C}_{\mathfrak{p}}\mid 2(u,w)\in\mathfrak{o}_{\mathfrak{p}}\text{ for every }w\in L_{\mathfrak{p}}\}$$

be its dual lattice. Let $\mathrm{ch}\langle L_{\mathfrak{p}}^g\rangle\in\mathcal{S}(\mathcal{C}_{\mathfrak{p}}^g)$ be the characteristic function of $L_{\mathfrak{p}}^g$. We write $S_{\mathfrak{p}}$ for the matrix for the quadratic form on $\mathcal{C}_{\mathfrak{p}}$ with respect to a fixed basis of $L_{\mathfrak{p}}$. For nondegenerate symmetric matrices $T\in\frac{1}{2}\mathcal{E}_g(\mathfrak{o}_{\mathfrak{p}})$ and $S\in\frac{1}{2}\mathcal{E}_m(\mathfrak{o}_{\mathfrak{p}})$ the local density of representing T by S is defined by

$$\alpha_{\mathfrak{p}}(S,T)=\lim_{i\to\infty}q_{\mathfrak{p}}^{ig((g+1)-2m)/2}A_i(S,T),$$

where

$$A_i(S,T)=\sharp\{X\in\mathrm{M}_{m,g}(\mathfrak{o}/\mathfrak{p}^i)\mid S[X]\equiv T\pmod{\mathfrak{p}^i}\}.$$

Proposition 3.1 (cf. [8]). *Put* $\mathcal{V}_r=\mathcal{C}_{\mathfrak{p}}\oplus\mathcal{H}(k_{\mathfrak{p}})^r$, *where* $\mathcal{H}$ *is the split binary quadratic space. We choose an integral lattice* $L_{\mathfrak{p}}^g\oplus\mathrm{M}_{2r,g}(\mathfrak{o}_{\mathfrak{p}})$ *of full rank in* $\mathcal{V}_r^g$. *Then*

$$\lim_{s\to r+s_0}W_T\Big(f_{\mathrm{ch}\langle L_{\mathfrak{p}}^g\oplus\mathrm{M}_{2r,g}(\mathfrak{o}_{\mathfrak{p}})\rangle}^{(s)}\Big)=\frac{\alpha_{\mathfrak{p}}\left(S_{\mathfrak{p}}\perp\frac{1}{2}\begin{pmatrix}&\mathbf{1}_r\\\mathbf{1}_r&\end{pmatrix},T\right)}{\gamma(\mathcal{C}_{\mathfrak{p}})^g\mathfrak{d}_k^{-g/2}[L_{\mathfrak{p}}^*:L_{\mathfrak{p}}]^{g/2}}.$$

Here, s_0 *is associated to* $\mathcal{C}_{\mathfrak{p}}$.

Proof. This result can be deduced from the proof of [28, Lemma 8.3(2)]. □

Let $\mathcal{V}$ be a totally positive definite quadratic space of dimension g over k. Fix an integral lattice L in $\mathcal{V}$. Put

$$L_{\mathfrak{p}} = L \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}, \qquad \mathrm{ch}\langle L^g \rangle = \otimes_{\mathfrak{p}} \mathrm{ch}\langle L^g_{\mathfrak{p}} \rangle.$$

For $h \in \mathrm{O}(\mathcal{V}, \mathbb{A})$ we write hL for the lattice defined by $(hL)_{\mathfrak{p}} = h_{\mathfrak{p}} L_{\mathfrak{p}}$. Put

$$K_L = \{h \in \mathrm{SO}(\mathcal{V}, \mathbb{A}) \mid hL = L\}, \quad \mathrm{SO}(L) = \{h \in \mathrm{SO}(\mathcal{V}, k) \mid hL = L\}.$$

Definition 3.2. We mean by the genus (resp. class) of L the set of all lattices of the form hL with $h \in \mathrm{O}(\mathcal{V}, \mathbb{A})$ (resp. $h \in \mathrm{O}(\mathcal{V}, k)$). The proper class of L consists of all lattices of the form hL with $h \in \mathrm{SO}(\mathcal{V}, k)$.

We write $\Xi'(L)$ and $\Xi(L)$ for the sets of classes and proper classes in the genus of L, respectively. Define the mass of the genus of L by

$$\mathfrak{m}'(L) = \sum_{\mathscr{L} \in \Xi'(L)} \frac{1}{\sharp \mathrm{O}(\mathscr{L})}, \qquad \mathfrak{m}(L) = \sum_{\mathscr{L} \in \Xi(L)} \frac{1}{\sharp \mathrm{SO}(\mathscr{L})}.$$

Remark 3.3. For each finite prime $\mathfrak{p}$ there is $h \in \mathrm{O}(\mathcal{V}, k_{\mathfrak{p}})$ with $\det h = -1$ such that $hL_{\mathfrak{p}} = L_{\mathfrak{p}}$. The genus of L therefore consists of lattices hL with $h \in \mathrm{SO}(\mathcal{V}, \mathbb{A})$. We identify $\Xi(L)$ with double cosets for $\mathrm{SO}(\mathcal{V}, k) \backslash \mathrm{SO}(\mathcal{V}, \mathbb{A}) / K_L$ via the map $h \mapsto hL$.

Lemma 5.6(1) of [20] says that

$$\mathfrak{m}(L) = 2\mathfrak{m}'(L). \tag{3.1}$$

We consider the following sums of representation numbers of $T \in \mathrm{Sym}_g(k)$:

$$R'(L, T) = \sum_{\mathscr{L} \in \Xi'(L)} \frac{N(\mathscr{L}, T)}{\sharp \mathrm{O}(\mathscr{L})}, \qquad R(L, T) = \sum_{\mathscr{L} \in \Xi(L)} \frac{N(\mathscr{L}, T)}{\sharp \mathrm{SO}(\mathscr{L})},$$

where $N(L, T) = \sharp\{u \in L^g \mid (u, u) = T\}$.

Proposition 3.4. *Notation being as above, we have*

$$2\frac{R(L, T)}{\mathfrak{m}(L)} = c_g D_T^{-1/2} \lim_{s \to -1/2} \prod_{\mathfrak{p}} W_T\left(f^{(s)}_{\mathrm{ch}\langle L^g_{\mathfrak{p}} \rangle}\right).$$

Proof. This equality is nothing but the Siegel formula. Nevertheless we reproduce its proof here because of its importance for us. Since both sides are zero unless $V^T \simeq \mathcal{V}$ by Lemma 2.4(2), we may identify V^T with $\mathcal{V}$. As is well-known, there exists $h \in \mathrm{O}(V^T, k_{\mathfrak{p}})$ such that $hL_{\mathfrak{p}} = L_{\mathfrak{p}}$ and $\det h = -1$. Since $\mathrm{SO}(V^T, \mathbb{A}) \backslash \mathrm{O}(V^T, \mathbb{A}) = \mu_2(\mathbb{A})$, we have

$$I(Z, \mathrm{ch}\langle L^g \rangle) = \frac{1}{2} \int_{\mathrm{SO}(V^T, k) \backslash \mathrm{SO}(V^T, \mathbb{A})} \Theta(Z, h; \mathrm{ch}\langle L^g \rangle)\, \mathrm{d}h.$$

Choose a finite set of double coset representatives $h_i \in \mathrm{SO}(V^T, \mathbb{A}_{\mathbf{f}})$ so that

$$\mathrm{SO}(V^T, \mathbb{A}) = \bigsqcup_i \mathrm{SO}(V^T, k) h_i K_L.$$

Then

$$I(Z, \mathrm{ch}\langle L^g\rangle) = \frac{1}{2}\mathrm{vol}(K_L)\sum_i \frac{\Theta(Z, h_i; \mathrm{ch}\langle L^g\rangle)}{\sharp \mathrm{SO}(h_i L)}.$$

Since $\mathfrak{m}(L) = 2\mathrm{vol}(K_L)^{-1}$, the T-th Fourier coefficient of $I(Z, \mathrm{ch}\langle L^g\rangle)$ is equal to $\frac{R(L,T)}{\mathfrak{m}(L)}$. The Siegel–Weil formula (2.2) proves the declared identity. □

An examination of the proof of Proposition 3.4 confirms that

$$\frac{R(L,T)}{\mathfrak{m}(L)} = \frac{R'(L,T)}{\mathfrak{m}'(L)}. \tag{3.2}$$

We can prove the following result by combining Propositions 2.5 and 3.4.

Proposition 3.5. *We assume that* $\mathrm{Diff}(T, \mathcal{C}) = \{\mathfrak{p}\}$*, notation and assumption being as in Proposition 2.5. Take an integral lattice* L *in* V^T *such that*

$$\lim_{s=-1/2} W_T\left(f^{(s)}_{\mathrm{ch}\langle L^g_{\mathfrak{p}}\rangle}\right) \neq 0.$$

If $\varphi_{\mathfrak{l}} = \mathrm{ch}\langle L^g_{\mathfrak{l}}\rangle$ *for every prime ideal* $\mathfrak{l}$ *distinct from* $\mathfrak{p}$*, then*

$$C(T,\varphi) = 2\frac{R(L,T)}{\mathfrak{m}(L)} \lim_{s\to -1/2} W_T\left(f^{(s)}_{\mathrm{ch}\langle L^g_{\mathfrak{p}}\rangle}\right)^{-1} \frac{\partial W_T\left(f^{(s)}_{\varphi_{\mathfrak{p}}}\right)}{\partial s}.$$

4. Siegel series

In this section we drop the subscript $_{\mathfrak{p}}$. Thus k is a nonarchimedean local field of characteristic zero with integer ring $\mathfrak{o}$. We denote the maximal ideal of $\mathfrak{o}$ by $\mathfrak{p}$ and the order of the residue field $\mathfrak{o}/\mathfrak{p}$ by q. Fix a prime element ϖ of $\mathfrak{o}$. We define the additive order $\mathrm{ord} : k^\times \to \mathbb{Z}$ by $\mathrm{ord}(\varpi^i \mathfrak{o}^\times) = i$.

Let $T \in \frac{1}{2}\mathcal{E}_g(\mathfrak{o})$ with $\det T \neq 0$. Denote the conductor of χ^T by $\mathfrak{d}^T$. Put

$$D_T = (-4)^{[g/2]}\det T,$$

$$e^T = \begin{cases} \mathrm{ord}\, D_T & \text{if } g \text{ is odd,} \\ \mathrm{ord}\, D_T - \mathrm{ord}\, \mathfrak{d}^T & \text{if } g \text{ is even,} \end{cases}$$

$$\xi^T = \begin{cases} 1 & \text{if } D_T \in k^{\times 2}, \\ -1 & \text{if } D_T \notin k^{\times 2} \text{ and } \mathfrak{d}^T = \mathfrak{o}, \\ 0 & \text{if } D_T \notin k^{\times 2} \text{ and } \mathfrak{d}^T \neq \mathfrak{o}. \end{cases}$$

The Siegel series associated to T is defined by

$$b(T,s) = \sum_{z\in \mathrm{Sym}_g(k)/\mathrm{Sym}_g(\mathfrak{o})} \psi(-\mathrm{tr}(Tz))\nu[z]^{-s},$$

where $\nu[z] = [z\mathfrak{o}^g + \mathfrak{o}^g : \mathfrak{o}^g]$ and ψ is an arbitrarily fixed additive character on k which is trivial on $\mathfrak{o}$ but nontrivial on $\mathfrak{p}^{-1}$. As is well-known, there exists a polynomial $\beta(T,X)\in\mathbb{Z}[X]$ such that $\beta(T,q^{-s})=b(T,s)$. Moreover, this polynomial $\beta(T,X)$ is divisible by the following polynomial

$$\gamma^T(X)=(1-X)\prod_{j=1}^{[g/2]}(1-q^{2j}X^2)\times\begin{cases}1 & \text{if } g \text{ is odd,}\\ \frac{1}{1-\xi^T q^{g/2}X} & \text{if } g \text{ is even.}\end{cases}$$

Put

$$\beta(T,X)=\gamma^T(X)F^T(X), \qquad \mathcal{F}^T(X)=X^{-e^T/2}F^T(q^{-(g+1)/2}X).$$

If g is even, then $\mathcal{F}^T\in\mathbb{Q}[\sqrt{q}][X+X^{-1}]$. If g is odd, then $\mathcal{F}^T\in\mathbb{Q}\big[\sqrt{X},\frac{1}{\sqrt{X}}\big]$.

Let $\mathcal{C}$ be a g-dimensional quadratic space over k. Recall that S is the matrix for the quadratic form on $\mathcal{C}$ with respect to a fixed basis of L, where L is an integral lattice of $\mathcal{C}$ as explained at the beginning of Section 3. If g is even, $\chi=\chi^{\mathcal{C}}$ is unramified and $\det(2S)\in\mathfrak{o}^\times$, then Lemma 14.8 combined with Proposition 14.3 of [19] gives

$$\alpha\left(S\perp\frac{1}{2}\begin{pmatrix} & \mathbf{1}_r\\ \mathbf{1}_r & \end{pmatrix},T\right)=\beta(T,\chi(\varpi)q^{-(g+2r)/2}). \tag{4.1}$$

For the rest of this paper we require g to be even.

Proposition 4.1. *If g is even, χ is unramified, $\chi^T=\chi$, $\eta^T=-1$, $\eta^{\mathcal{C}}=1$ and L is a self-dual lattice of $\mathcal{C}$, then*

$$\frac{\partial}{\partial s}W_T\Big(f^{(s)}_{\mathrm{ch}\langle L^g\rangle}\Big)\Big|_{s=-1/2}=-\frac{\sqrt{\mathfrak{d}_k}^g\log q}{\gamma(\mathcal{C})^g}\frac{\xi^T}{\sqrt{q^g}}\gamma^T\left(\frac{\xi^T}{\sqrt{q^g}}\right)\frac{\partial F^T}{\partial X}\left(\frac{\xi^T}{\sqrt{q^g}}\right).$$

Proof. By assumption $\lim_{s\to-1/2}W_T(f^{(s)}_\varphi)=0$ in view of Lemma 2.4(2). We combine Proposition 3.1 and (4.1) with Lemmas A.2-A.3 of [8] to see that

$$\begin{aligned}W_T\Big(f^{(s)}_\varphi\Big)&=\gamma(\mathcal{C})^{-g}\sqrt{\mathfrak{d}_k}^g\beta\Big(T,\xi^Tq^{-(g+1+2s)/2}\Big)\\&=\gamma(\mathcal{C})^{-g}\sqrt{\mathfrak{d}_k}^g\gamma^T\Big(\xi^Tq^{-(g+1+2s)/2}\Big)F^T\Big(\xi^Tq^{-(g+1+2s)/2}\Big).\end{aligned}$$

Since $\chi^T=\chi$, we see that $F^T(\xi^Tq^{-g/2})=0$. We can obtain the stated identity by differentiating this equality at $s=-\frac{1}{2}$. □

Definition 4.2. Let $T=(t_{ij})\in\frac{1}{2}\mathcal{E}_g(\mathfrak{o})\cap\mathrm{GL}_g(k)$. We denote by $S(T)$ the set of all nondecreasing sequences $(a_1,\dots,a_g)$ of nonnegative integers such that $\mathrm{ord}\,t_{ii}\geq a_i$ and $\mathrm{ord}(2t_{ij})\geq\frac{a_i+a_j}{2}$ for $1\leq i,j\leq g$. The Gross–Keating invariant $\mathrm{GK}(T)$ of T is the greatest element of $\bigcup_{U\in\mathrm{GL}_g(\mathfrak{o})}S(T[U])$ with respect to the lexicographic order.

Here, the lexicographic order is defined as follows: $(y_1,\dots,y_g)$ is greater than $(z_1,\dots,z_g)$ if there is an integer $1\leq j\leq g$ such that $y_i=z_i$ for $i<j$ and $y_j>z_j$. Ikeda and Katsurada [5] define a set $\mathrm{EGK}(T)$ of invariants of T attached to $\mathrm{GK}(T)$, which they call the extended Gross–Keating datum of

T. They associated to an extended Gross–Keating datum H a polynomial $\mathcal{F}^H(Y,X) \in \mathbb{Z}[Y^{1/2}, Y^{-1/2}, X, X^{-1}]$ and show that

$$\mathcal{F}^{\mathrm{EGK}(T)}(\sqrt{q}, X) = \mathcal{F}^T(X).$$

When g is even and $\mathfrak{d}^T = \mathfrak{o}$, one can associate to $\mathrm{EGK}(T)$ truncated extended Gross–Keating datum $\mathrm{EGK}(T)'$ of length $g-1$ by Proposition 4.4 of [5]. By Definitions 4.2-4.4 of [5]

$$\begin{aligned}\mathcal{F}^{\mathrm{EGK}(T)}(Y,X) =& Y^{\mathfrak{e}'/2}X^{-(\mathfrak{e}-\mathfrak{e}'+2)/2}\frac{1-\xi^T Y^{-1}X}{X^{-1}-X}\mathcal{F}^{\mathrm{EGK}(T)'}(Y,YX)\\ &+ Y^{\mathfrak{e}'/2}X^{(\mathfrak{e}-\mathfrak{e}'+2)/2}\frac{1-\xi^T Y^{-1}X^{-1}}{X-X^{-1}}\mathcal{F}^{\mathrm{EGK}(T)'}(Y,YX^{-1}),\end{aligned}$$

where $\mathrm{GK}(T) = (a_1, \cdots, a_g)$, $\mathfrak{e} = 2\left[\frac{a_1+\cdots+a_g}{2}\right]$ and $\mathfrak{e}' = a_1 + \cdots + a_{g-1}$. It is worth noting that since $\mathfrak{d}^T = \mathfrak{o}$, we have $\mathfrak{e} = a_1 + \cdots + a_g = e^T$. We put

$$F^H(X) = (q^{(g+1)/2}X)^{\mathfrak{e}/2}\mathcal{F}^H(\sqrt{q}, q^{(g+1)/2}X).$$

If q is odd, then T is equivalent to a diagonal matrix $\mathrm{diag}[t_1, \cdots, t_g]$ with $\mathrm{ord}\, t_1 \le \cdots \le \mathrm{ord}\, t_g$ and the (naive) extended Gross–Keating datum $\mathrm{EGK}(T) = (a_1, \cdots, a_g; \varepsilon_1, \ldots, \varepsilon_g)$ is given by

$$a_i = \mathrm{ord}\, t_i, \qquad T^{(i)} = \mathrm{diag}[t_1, \cdots, t_i], \qquad \varepsilon_i = \begin{cases}\eta^{T^{(i)}} & \text{if } i \text{ is odd,}\\ \xi^{T^{(i)}} & \text{if } i \text{ is even}\end{cases}$$

and $\mathrm{EGK}(T)' = (a_1, \cdots, a_{g-1}; \varepsilon_1, \ldots, \varepsilon_{g-1})$.

Theorem 4.3. *Assume that g is even and that $\mathfrak{d}^T = \mathfrak{o}$. Then*

$$F^H(\xi^T q^{-g/2}) = q^{e^T/2}F^{H'}(\xi^T q^{-g/2}),$$

where we put $H = \mathrm{EGK}(T)$ and $H' = \mathrm{EGK}(T)'$. If $\eta^T = -1$, then

$$\frac{\xi^T}{\sqrt{q}^g}\frac{\partial F^H}{\partial X}\left(\frac{\xi^T}{\sqrt{q}^g}\right) = \frac{F^{H'}(\xi^T q^{(2-g)/2})}{q-1} - \sqrt{q}^{e^T}\frac{\xi^T}{\sqrt{q}^g}\frac{\partial F^{H'}}{\partial X}\left(\frac{\xi^T}{\sqrt{q}^g}\right).$$

Proof. Substituting $Y = \sqrt{q}$ into $\mathcal{F}^H(Y,X)$, we get

$$\begin{aligned}\mathcal{F}^H(\sqrt{q},X) =& X^{-(\mathfrak{e}+2)/2}\frac{1-\xi^T q^{-1/2}X}{X^{-1}-X}(\sqrt{q}X)^{\mathfrak{e}'/2}\mathcal{F}^{H'}(\sqrt{q},\sqrt{q}X)\\ &+ X^{(\mathfrak{e}+2)/2}\frac{1-\xi^T q^{-1/2}X^{-1}}{X-X^{-1}}(\sqrt{q}X^{-1})^{\mathfrak{e}'/2}\mathcal{F}^{H'}(\sqrt{q},\sqrt{q}X^{-1})\\ =& X^{-(e^T+2)/2}\frac{1-\xi^T q^{-1/2}X}{X^{-1}-X}F^{H'}(q^{(1-g)/2}X)\\ &+ X^{(e^T+2)/2}\frac{1-\xi^T q^{-1/2}X^{-1}}{X-X^{-1}}F^{H'}(q^{(1-g)/2}X^{-1}).\end{aligned}$$

By letting $X = \xi^T\sqrt{q}$, we get

$$(\xi^T\sqrt{q})^{-e^T/2}F^H(\xi^T q^{-g/2}) = \mathcal{F}^H(\sqrt{q}, \xi^T\sqrt{q}) = (\xi^T\sqrt{q})^{e^T/2}F^{H'}(\xi^T q^{-g/2}).$$

In the proof of Proposition 4.1 we have seen that if $\eta^T = -1$, then

$$\mathcal{F}^H(\sqrt{q}, \xi^T\sqrt{q}) = \mathcal{F}^T(\xi^T\sqrt{q}) = (\xi^T\sqrt{q})^{-e^T/2}F^T(\xi^T q^{-g/2}) = 0,$$

and hence $F^{H'}(\xi^T q^{-g/2}) = 0$. We can prove the stated identity by differentiating the equality above at $X = \xi^T\sqrt{q}$. □

We will use the following result in the next section.

Lemma 4.4. *If T is a split symmetric half-integral matrix of size 4 over $\mathbb{Z}_p$, then there exists a nondegenerate isotropic symmetric half-integral matrix B of size 3 over $\mathbb{Z}_p$ such that $F_p^B = F_p^{\mathrm{EGK}_p(T)'}$.*

Proof. If $p = 2$, then the existence of such B follows from Proposition 6.4 of [4] and Theorem 1.1 of [5]. If p is odd, then T is equivalent to a diagonal matrix $\mathrm{diag}[t_1, \cdots, t_4]$ with $\mathrm{ord}\, t_1 \le \cdots \le \mathrm{ord}\, t_4$. Then we may choose B as $\mathrm{diag}[t_1, \cdots, t_3]$ by using the argument explained in the paragraph just before Theorem 4.3. □

5. The case $g = 4$

We discuss the classical Eisenstein series of Siegel. For this it is simplest to work over $k = \mathbb{Q}$. Provided that g is a multiple of 4, we consider the series

$$E_g(Z, s) = \sum_{\{C,D\}} \det(CZ + D)^{-g/2}|\det(CZ + D)|^{-s}(\det Y)^{s/2}.$$

Here the sum extends over all symmetric coprime pairs modulo $\mathrm{GL}_g(\mathbb{Z})$. Let $\mathcal{C}_p = \mathcal{H}(\mathbb{Q}_p)^{g/2}$ be the split quadratic space of dimension g over $\mathbb{Q}_p$. Define $\varphi = \otimes_p\varphi_p$ by taking $\varphi_p = \mathrm{ch}\langle \mathrm{M}_{g,g}(\mathbb{Z}_p)\rangle \in \mathcal{S}(\mathcal{C}_p^g)$. It is known that $E_g\big(Z, s + \frac{1}{2}\big) = E(Z, f_\varphi^{(s)})$ (see §IV.2 of [9]). The series is incoherent if and only if $\frac{g}{4}$ is odd due to Lemma 2.3.

Fix a positive definite symmetric half-integral matrix T of size g. Recall that χ_T stands for the primitive Dirichlet character corresponding to χ^T. The T-th Fourier coefficient of $E_g(Z, s)$ is given by

$$A(T, Y, s) = \frac{a\big(T, Y, s - \frac{1}{2}\big)L(s, \chi_T)}{\zeta\big(s + \frac{g}{2}\big)\prod_{i=1}^{g/2}\zeta(2s + 2i - 2)}\prod_{p|D_T} F_p^T(p^{-(2s+g)/2}).$$

The T-th Fourier coefficient of $\frac{\partial}{\partial s}E_g(Z, s)|_{s=0}$ is given by

$$C_g(T) = \frac{\partial}{\partial s}A(T, Y, s)|_{s=0}.$$

Recall that $\mathrm{Diff}(T) = \{p \mid \eta_p^T = -1\}$.

Proposition 5.1. *Assume that $\frac{g}{4}$ is odd. Let $T \in \frac{1}{2}\mathcal{E}_g(\mathbb{Z}) \cap \mathrm{Sym}_g^+$.*

(1) *If $\chi_T = 1$, then $C_g(T) = 0$ unless $\mathrm{Diff}(T)$ is a singleton.*

(2) *If* $\chi_T = 1$ *and* $\mathrm{Diff}(T) = \{p\}$, *then*

$$C_g(T) = -\frac{2^{(g+2)/2} p^{-(g+e_p^T)/2} \log p}{\zeta\left(1-\frac{g}{2}\right)\prod_{i=1}^{(g-2)/2}\zeta(1-2i)} \frac{\partial F_p^T}{\partial X}(p^{-g/2}) \prod_{p\neq \ell | D_T} \ell^{-e_\ell^T/2} F_\ell^T(\ell^{-g/2}).$$

(3) *If* $\chi_T \neq 1$, *then*

$$C_g(T) = -\frac{2^{(g+2)/2} L(1,\chi_T)}{\zeta\left(1-\frac{g}{2}\right)\prod_{i=1}^{(g-2)/2}\zeta(1-2i)} \prod_{p|D_T} p^{-e_p^T/2} F_p^T(p^{-g/2}).$$

Proof. We have already proved (1) in Proposition 2.5. Taking

$$\zeta(2i) = (-1)^i \frac{(2\pi)^{2i}}{2(2i-1)!}\zeta(1-2i)$$

into account, we have

$$\zeta\left(\frac{g}{2}\right)\prod_{i=1}^{(g-2)/2}\zeta(2i) = \frac{(2\pi)^{g^2/4}\zeta\left(1-\frac{g}{2}\right)}{2^{g/2}\left(\frac{g}{2}-1\right)!}\prod_{i=1}^{(g-2)/2}\frac{\zeta(1-2i)}{(2i-1)!}$$

Recall that $a\left(T,Y,-\frac{1}{2}\right)= \frac{2^g \pi^{g^2/2}}{\Gamma_g(\frac{g}{2}) D_T^{1/2}}$ by (2.3). Since

$$\Gamma_g\left(\frac{g}{2}\right) = \frac{\pi^{g^2/4}}{2^{(g^2-2g)/4}}\prod_{i=1}^{(g-2)/2}(2i)!, \quad \zeta(0) = -\frac{1}{2}, \quad L'(0,\chi_T) = \frac{\sqrt{\mathfrak{d}^T}}{2}L(1,\chi_T),$$

we get (2) and (3). □

Hereafter we let $g = 4$. By a quaternion algebra over a field k we mean a central simple algebra over k of dimension 4. Let $\mathbb{B}_p$ denote the definite quaternion algebra over $k = \mathbb{Q}$ that ramifies only at a prime number p. The reduced norm Nrd on $\mathbb{B}_p$ defines a positive definite quadratic space $\mathcal{V}_p$. Fix a maximal order $\mathcal{O}_p$ of $\mathbb{B}_p$. Let $\varphi_\ell \in \mathcal{S}(\mathcal{C}_\ell^g)$ be the characteristic function of $\mathrm{M}_2(\mathbb{Z}_\ell)^g$ and $\varphi'_p \in \mathcal{S}(\mathcal{V}_p^g(\mathbb{Q}_p))$ the characteristic function of $\mathcal{O}_p^g \otimes \mathbb{Z}_p$. We regard $\varphi' = \varphi'_p \otimes (\otimes_{\ell\neq p}\varphi_\ell)$ as the characteristic function of $\mathcal{O}_p^g \otimes \hat{\mathbb{Z}}$. We write S_p for the matrix representation of $\mathcal{V}_p$ with respect to a $\mathbb{Z}$-basis of $\mathcal{O}_p$. Put

$$S_0 = \mathrm{diag}\left[\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0\end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0\end{pmatrix}\right].$$

Lemma 5.2. *Let* $T \in \mathrm{Sym}_g(\mathbb{Q}_p)$.

(1) *If* $T \notin \frac{1}{2}\mathcal{E}_4(\mathbb{Z}_p)$, *then* $W_T\left(f_{\varphi_p}^{(s)}\right)$ *is identically zero.*

(2) *If* $T \in \frac{1}{2}\mathcal{E}_4(\mathbb{Z}_p)$ *with* $\det T \neq 0$, $\chi^T = 1$ *and* $\eta_p^T = -1$, *then*

$$\lim_{s\to -1/2}\frac{W_{S_p}\left(f_{\varphi'_p}^{(s)}\right)}{W_T\left(f_{\varphi'_p}^{(s)}\right)}\frac{\frac{\partial}{\partial s}W_T\left(f_{\varphi_p}^{(s)}\right)}{pW_{S_0}\left(f_{\varphi_p}^{(s)}\right)} = \left(p^{-2}\frac{\partial F_p^{H'}}{\partial X}(p^{-2}) - \frac{p^{-e_p^T/2}}{p-1}F_p^{H'}(p^{-1})\right)\log p,$$

where we put $H' = \mathrm{EGK}_p(T)'$.

Proof. The first part is trivial. Since

$$\alpha_p(S_p,T)=p^{(e_p^T-2)/2}\alpha_p(S_p,S_p)$$

by Hilfssatz 17 of [23], it follows from Proposition 3.1 that

$$\lim_{s\to-1/2}\frac{W_{S_p}\left(f^{(s)}_{\varphi'_p}\right)}{W_T\left(f^{(s)}_{\varphi'_p}\right)}=p^{-(e_p^T-2)/2}.$$

On the other hand, Proposition 4.1 and Theorem 4.3 give

$$\lim_{s\to-1/2}\frac{\frac{\partial}{\partial s}W_T\left(f^{(s)}_{\varphi_p}\right)}{W_{S_0}\left(f^{(s)}_{\varphi_p}\right)}=\left(p^{(e_p^T-4)/2}\frac{\partial F_p^{H'}}{\partial X}(p^{-2})-\frac{F_p^{H'}(p^{-1})}{p-1}\right)\log p.$$

These complete our proof. □

Let $\bar{\mathbb{F}}_p$ be an algebraic closure of a finite field $\mathbb{F}_p$ with p elements. For two supersingular elliptic curves E, E' over $\bar{\mathbb{F}}_p$ we consider the free $\mathbb{Z}$-module $\mathrm{Hom}(E',E)$ of homomorphisms $E'\to E$ over $\bar{\mathbb{F}}_p$ together with the quadratic form given by the degree. As E and E' are supersingular, $\mathrm{Hom}(E',E)$ has rank 4 as a $\mathbb{Z}$-module. For two quadratic spaces over $\mathbb{Z}$ we write $N(L,L')$ for the number of isometries $L'\to L$.

We are now ready to prove our main result.

Theorem 5.3. *If $T\in\frac{1}{2}\mathcal{E}_4(\mathbb{Z})$ is positive definite, $\chi_T=1$ and $\mathrm{Diff}(T)$ consists of a single prime p, then*

$$C_4(T)=2^6{\cdot}3^2\left(p^{-2}\frac{\partial F_p^{H'}}{\partial X}(p^{-2})-\frac{F_p^{H'}(p^{-1})}{\sqrt{p^{e_p^T}}(p-1)}\right)\log p\sum_{(E',E)}\frac{N(\mathrm{Hom}(E',E),T)}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')},$$

where we put $H'=\mathrm{EGK}_p(T)'$ and where (E',E) extends over all pairs of isomorphism classes of supersingular elliptic curves over $\bar{\mathbb{F}}_p$.

Proof. Proposition 3.5 and (3.2) applied to $L=\mathcal{O}_p$ gives

$$C_4(T)=R'(\mathcal{O}_p,T)c\lim_{s\to-1/2}\frac{W_{S_p}\left(f^{(s)}_{\varphi'_p}\right)}{W_T\left(f^{(s)}_{\varphi'_p}\right)}\frac{\frac{\partial}{\partial s}W_T\left(f^{(s)}_{\varphi_p}\right)}{pW_{S_0}\left(f^{(s)}_{\varphi_p}\right)},$$

where

$$c=\frac{2p}{\mathfrak{m}'(\mathcal{O}_p)}\lim_{s\to-1/2}\frac{W_{S_0}\left(f^{(s)}_{\varphi_p}\right)}{W_{S_p}\left(f^{(s)}_{\varphi'_p}\right)}.$$

If $T=S_p$, then we claim that $R'(\mathcal{O}_p,S_p)=1$. To prove this, it suffices to show that $N(\mathscr{L},S_p)=0$ if $\mathscr{L}$ is not isometric to $\mathcal{O}_p$ and $N(\mathcal{O}_p,S_p)=\sharp\mathrm{O}(\mathcal{O}_p)$, where $\mathscr{L}\in\Xi'(\mathcal{O}_p)$. If $N(\mathscr{L},S_p)\neq 0$, then there is an injection $f:\mathcal{O}_p\to\mathscr{L}$ as a lattice preserving the associated quadratic forms. Thus we only need to show that f is surjective. If it is not surjective, then $\mathscr{L}$ and

$\mathcal{O}_p$ have different discriminant, which is a contradiction to the assumption that $\mathscr{L}$ and $\mathcal{O}_p$ are in the same genus.

Applying Proposition 3.4 and (3.2) to $T = S_p$, we get

$$\frac{2}{\mathfrak{m}'(\mathcal{O}_p)} = c_4 D_{S_p}^{-1/2} \lim_{s\to -1/2} W_{S_p}\left(f^{(s)}_{\varphi'_p}\right) \prod_{\ell\neq p} W_{S_p}\left(f^{(s)}_{\varphi_\ell}\right).$$

It follows that

$$\begin{aligned} c &= p c_4 D_{S_p}^{-1/2} \lim_{s\to -1/2} \prod_{\ell} W_{S_0}\left(f^{(s)}_{\varphi_\ell}\right) \\ &= c_4 \lim_{s\to -1/2} \prod_{\ell} \gamma_\ell^S(\ell^{-(5+2s)/2}) = \frac{c_4}{\zeta(2)^2} \lim_{s\to -1/2} \frac{\zeta\left(s+\frac{1}{2}\right)}{\zeta(2s+1)} = 2^7\cdot 3^2. \end{aligned}$$

Since $R(\mathcal{O}_p, T) = 2R'(\mathcal{O}_p, T)$ by (3.1) and (3.2), and

$$R(\mathcal{O}_p, T) = \sum_{\mathscr{L}\in\Xi(\mathcal{O}_p)} \frac{N(\mathscr{L}, T)}{\sharp \mathrm{SO}(\mathscr{L})} = \sum_{(E',E)} \frac{N(\mathrm{Hom}(E',E),T)}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')} \tag{5.1}$$

by Proposition 4.1 of [25], our statement follows from Lemma 5.2(2). □

Conjecture 5.4. Let $\mathcal{V}$ be a totally positive definite quadratic space over a totally real number field k of dimension g. Fix a maximal integral lattice L of $\mathcal{V}$. Let $T \in \frac{1}{2}\mathcal{E}_g(\mathfrak{o})$ be totally positive definite. If g is even and $\chi^{\mathcal{V}} = 1$, then there is a totally positive definite matrix $T' \in \frac{1}{2}\mathcal{E}_{g-1}(\mathfrak{o})$ such that

$$R(L, T) = 2R(L, T').$$

Proposition 5.5. *If $k = \mathbb{Q}$ and $g = 4$, then Conjecture 5.4 is true.*

Proof. Since $R(L, T) = 0$ unless $\mathrm{Diff}(T) = \mathrm{Diff}(\mathcal{V})$, we may assume that

$$\mathrm{Diff}(T) = \mathrm{Diff}(\mathcal{V}).$$

Lemma 4.4 gives $T'_p \in \frac{1}{2}\mathcal{E}_3(\mathbb{Z}_p)$ such that $F_p^{T'_p} = F_p^{\mathrm{EGK}_p(T)'}$ for every rational prime p. In addition, the proof of Lemma 4.4 yields that T'_p is unimodular for almost all primes p. Thus we can find a positive rational number $0 < \delta \in \mathbb{Q}^\times$ such that $\delta^{-1}\det T'_p \in \mathbb{Z}_p^\times$ for every $p \notin \mathrm{Diff}(\mathcal{V})$. For $p \in \mathrm{Diff}(\mathcal{V})$ we fix an arbitrary anisotropic ternary quadratic form T'_p over $\mathbb{Z}_p$. Recall that $\alpha_p(S_p, T'_p)$ is independent of the choice of T'_p.

Since $F_p^{uT'_p} = F_p^{T'_p}$ for $u \in \mathbb{Z}_p^\times$, there is no harm in assuming that $\delta = \det T'_p$. Since $\eta_p^{T'_p} = 1$ for $p \notin \mathrm{Diff}(\mathcal{V})$, the Minkowski-Hasse theorem gives $z \in \mathrm{Sym}_3(\mathbb{Q})$ which is positive definite and such that $z \in T'_p[\mathrm{GL}_3(\mathbb{Q}_p)]$ for every p. Take $A \in \mathrm{GL}_3(\mathbb{A}_\mathbf{f})$ so that $z = T'_p[A_p]$ for every p. We can take $D \in \mathrm{GL}_3(\mathbb{Q})$ in such a way that $AD^{-1} \in \mathrm{GL}_3(\mathbb{Z}_p)$ for every p. Put $T' = z[D^{-1}]$. Then $T' \in T'_p[\mathrm{GL}_3(\mathbb{Z}_p)]$ for every p. In particular, $T' \in \frac{1}{2}\mathcal{E}_3(\mathbb{Z})$.

In view of (3.2) it suffices to show that

$$\frac{R'(L,T)}{\mathfrak{m}'(L)} = 2\frac{R'(L,T')}{\mathfrak{m}'(L)}.$$

We see by the Siegel formula that

$$\frac{R'(L,T)}{\mathfrak{m}'(L)}=2^{-1}d_\infty(L,T)2^4\prod_{p\in\mathrm{Diff}(\mathcal{V})}\frac{\alpha_p(S_p,T)}{2}\prod_{q\notin\mathrm{Diff}(\mathcal{V})}(1-q^{-2})^2F_q^T(q^{-2}).$$

Recall that the archimedean densities are given by

$$d_\infty(L,T)=\frac{\prod_{i=1}^4\frac{\pi^{i/2}}{\Gamma(\frac{i}{2})}}{\det(2T)^{1/2}[L^*:L]^2},\qquad d_\infty(L,T')=\frac{\prod_{i=2}^4\frac{\pi^{i/2}}{\Gamma(\frac{i}{2})}}{[L^*:L]^{3/2}}.$$

Since

$$\alpha_p(S_p,T')=2(p+1)(1+p^{-1}),\qquad \alpha_p(S_p,T)=4p^{e_p^T/2}(p+1)^2.$$

by [26, Theorem 1.1] and Proposition 6.5 of [1]. The latter result can be derived more generally from Shimura's exact mass formula. Since $[L^*:L]=\prod_{p\in\mathrm{Diff}(\mathcal{V})}p^2$ by assumption, we have

$$d_\infty(L,T)=[L^*:L]^{-2}\det(2T)^{-1/2}\prod_{i=1}^4\frac{\pi^{i/2}}{\Gamma\left(\frac{i}{2}\right)}=\frac{d_\infty(L,T')}{\det(2T)^{1/2}}\prod_{p\in\mathrm{Diff}(\mathcal{V})}p^{-1}.$$

We combine these with Theorem 4.3 to obtain

$$\frac{R'(L,T)}{\mathfrak{m}'(L)}=d_\infty(L,T')2^3\prod_{p\in\mathrm{Diff}(\mathcal{V})}\alpha_p(S_p,T')\prod_{q\notin\mathrm{Diff}(\mathcal{V})}(1-q^{-2})^2F_q^{T'}(q^{-2}).$$

The final expression equals $2\frac{R'(L,T')}{\mathfrak{m}'(L)}$ by the Siegel formula. □

Corollary 5.6. If T is a positive definite symmetric half-integral matrix of size 4 which satisfies $\chi^T=1$ and $\eta_\ell^T=1$ for $\ell\neq p$, then there exists a positive definite symmetric half-integral matrix T' of size 3 such that

$$\sum_{(E',E)}\frac{N(\mathrm{Hom}(E',E),T)}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')}=2\sum_{(E',E)}\frac{N(\mathrm{Hom}(E',E),T')}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')},$$

where (E,E') extends over all pairs of isomorphism classes of supersingular elliptic curves over $\bar{\mathbb{F}}_p$.

Proof. Proposition 4.1 of [25] gives

$$R(\mathcal{O}_p,T')=\sum_{L\in\Xi(\mathcal{O}_p)}\frac{N(L,T')}{\sharp\mathrm{SO}(L)}=\sum_{(E',E)}\frac{N(\mathrm{Hom}(E',E),T')}{\sharp\mathrm{Aut}(E)\sharp\mathrm{Aut}(E')}.$$

We can derive Corollary 5.6 from (5.1) and Proposition 5.5. □

Let $T\in\frac{1}{2}\mathcal{E}_4(\mathbb{Z}_p)$ be an anisotropic symmetric matrix with (naive) extended Gross-Keating invariant $(a_1,a_2,a_3,a_4;\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$. Note that $\varepsilon_1=\varepsilon_4=1$ by definition. One can easily see that $\varepsilon_2\neq 1$ and $\varepsilon_3=-1$. Proposition 5.3 of [1] gives a partition $\{1,2,3,4\}=\{i,j\}\cup\{k,l\}$ such that

$$a_i\equiv a_j\not\equiv a_k\equiv a_l \pmod 2.$$

Lemma 5.7. (1) *If* $a_1 \not\equiv a_2 \pmod 2$*, then*

$$F_p^{T'}(p^{-1}) = \frac{p^{a_1+1}-1}{(p-1)(p^3-1)}\left(p^{\{a_1+3(a_2+1)\}/2} - \frac{p^{a_1+1}+1}{p+1}\right) - \frac{p^{(a_1+a_2+2a_3+1)/2}}{p-1}\left\{(a_1+1)p^{(a_1+a_2+1)/2} - \frac{p^{a_1+1}-1}{p-1}\right\}.$$

(2) *If* $a_1 \equiv a_2 \pmod 2$*, then*

$$F_p^{T'}(p^{-1}) = \frac{p^{a_1+1}-1}{(p-1)(p^3-1)}\left(p^{(a_1+3a_2)/2} - \frac{p^{a_1+1}+1}{p+1}\right) - \frac{p^{(a_1+a_2+2a_3+2)/2}}{p-1}\left\{(a_1+1)p^{(a_1+a_2)/2} - \frac{p^{a_1+1}-1}{p-1}\right\} + p^{(a_1+3a_2)/2}\frac{p^{a_1+1}-1}{p^2-1}(p^{a_1-a_2+1}+1).$$

Proof. We write the naive extended Gross-Keating invariant of T as

$$\mathrm{EGK}_p(T) = (a_1, a_2, a_3, a_4; 1, \varepsilon_2, \varepsilon_3, 1).$$

Let σ be either 1 or 2 according as $a_1 - a_2$ is odd or even. Section 8 of [5] expresses $F_p^{\mathrm{EGK}_p(T)'}(X)$ in terms of $\mathrm{EGK}_p(T)' = (a_1, a_2, a_3; 1, \varepsilon_2, \varepsilon_3)$:

$$F_p^{\mathrm{EGK}_p(T)'}(p^{-2}X) = \sum_{i=0}^{a_1}\sum_{j=0}^{(a_1+a_2-\sigma)/2-i} p^{i+j}X^{i+2j}$$
$$\varepsilon_3\sum_{i=0}^{a_1}\sum_{j=0}^{(a_1+a_2-\sigma)/2-i} p^{(a_1+a_2-\sigma)/2-j}X^{a_3+\sigma+i+2j}$$
$$+ \varepsilon_2^2 p^{(a_1+a_2-\sigma+2)/2}\sum_{i=0}^{a_1}\sum_{j=0}^{a_3-a_2+2\sigma-4}\varepsilon_2^j X^{a_2-\sigma+2+i+j}.$$

We now specialize the formula to $X = p$ and $\varepsilon_3 = -1$. Then

$$F_p^{T'}(p^{-1}) = \frac{p^{a_1+1}-1}{(p-1)(p^3-1)}\left(p^{\{a_1+3(a_2-\sigma+2)\}/2} - \frac{p^{a_1+1}+1}{p+1}\right) - \frac{p^{(a_1+a_2+2a_3+\sigma)/2}}{p-1}\left((a_1+1)p^{(a_1+a_2-\sigma+2)/2} - \frac{p^{a_1+1}-1}{p-1}\right) + \varepsilon_2^2 p^{\{a_1+3(a_2-\sigma+2)\}/2}\frac{(p^{a_1+1}-1)(1-(\varepsilon_2 p)^{a_1-a_2+2\sigma-3})}{(p-1)(1-\varepsilon_2 p)}.$$

Since $\varepsilon_2 = 0$ or -1 according as $a_1 - a_2$ is odd or even by Proposition 2.2 of [4] and Proposition 5.4 of [1], we obtain the stated formulas. □

The degree $\deg \mathscr{Z}(B)$ is defined in (1.2) for positive definite symmetric half-integral 3×3 matrices B such that $\mathrm{Diff}(B)$ is a singleton.

Corollary 5.8. Let T be a positive definite symmetric half-integral 4×4 matrix such that $\chi_T=1$ and $\mathrm{Diff}(T)=\{p\}$. Let σ be either 1 or 2 according as a_1-a_2 is odd or even. If $\deg\mathscr{Z}(T')\neq 0$, then

$$\left|\frac{C_4(T)}{-2^8\cdot 3^2\cdot\deg\mathscr{Z}(T')}-1\right|<\frac{4}{p\sqrt{p}}\left(p^{-(a_4-3+\sigma)/2}+\frac{4p^{-(a_4-a_1)/2}}{a_1+1}\right),$$

where $\mathrm{GK}_p(T)=(a_1,a_2,a_3,a_4)$. In particular,

$$\left|\frac{C_4(T)}{-2^8\cdot 3^2\cdot\deg\mathscr{Z}(T')}-1\right|<\frac{20}{p\sqrt{p}},\qquad \lim_{e_p^T\to\infty}\frac{C_4(T)}{-2^9\cdot 3^2\cdot\deg\mathscr{Z}(T')}=1.$$

Proof. By (2.12) and (2.13) of [26]

$$\begin{aligned}-p^{-2}\frac{\partial F_p^{H'}}{\partial X}(p^{-2})&\geq (a_1+1)p^{(a_1+a_2)/2}\left(\frac{a_3-a_2+2\sigma}{\sqrt{p}^{\sigma}}+\varepsilon_2^2\frac{a_3-a_2+1}{2}\right)\\ &\geq (a_1+1)p^{(a_1+a_2-(2-\sigma))/2}.\end{aligned}$$

Recall that if $\sigma=1$, then $a_1<a_2\leq a_3\leq a_4$ while if $\sigma=2$, then $a_1\leq a_2<a_3\leq a_4$. An examination of the proof of Lemma 5.7 confirms that

$$\begin{aligned}\left|\frac{F_p^{H'}(p^{-1})}{\sqrt{p}^{e_p^T}(p-1)}\right|&\leq\frac{a_1+1}{(p-1)^2}p^{(a_1+a_2-a_4+2)/2}+\frac{p^{a_1+a_2-(a_3+a_4+3\sigma)/2+4}}{(p-1)^2(p^3-1)}\\ &\quad+\varepsilon_2^2\frac{p^{2a_1+2-(a_3+a_4)/2}}{(p-1)^2(p+1)}+\varepsilon_2^2\frac{p^{a_1+a_2+1-(a_3+a_4)/2}}{(p-1)^2(p+1)}\\ &<4p^{(a_1+a_2)/2-1}\{(a_1+1)p^{-a_4/2}+2p^{-(a_4-a_1+3\sigma)/2}+2\varepsilon_2^2p^{-(a_4-a_1+1)/2}\}.\end{aligned}$$

Now our proof is completed by Theorem 1.3. □

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