ON COMPOSITION SERIES AND NEW INVARIANTS OF LOCAL ALGEBRA

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§ 1 INTRODUCTION, MOTIVATION AND PRELIMINARY RESULTS

In 1965, D.A. Buchsbaum posed the problem to describe the difference between the length and multiplicity of parameter ideals of local rings, see, e.g., [SV]. 30 years later, the same problem was discussed in [Vo], [MV] for the quotient. Analyzing this approach the aim of our paper is to study two new invariants of local rings. Moreover, we describe some applications.

Let A be a local ring or a graded K-algebra, K is a field, with maximal ideal m_A. We note that a graded K-algebra is a Noetherian graded ring, say $A = A_0 \oplus A_1 \oplus \dots$ with $A_0 = K$, generated by A_1. By an A-module we always mean a unitary finitely generated module over A. Let M be a (graded) A-module. The length of M over A is denoted by $\ell_A(M)$ or $\ell(M)$. We consider a (homogeneous) ideal Q of A with $\ell(M/QM) < \infty$. Then the multiplicity of Q on M is well-defined and denoted by $e(Q; M)$, see, e.g., [Ei], [SV].

We want to study the following two invariants:

$$n_A(M) := \sup\{\ell(M/QM)/e(Q;M) | Q \text{ (homogeneous) ideal of } A \text{ with } \ell(M/QM) < \infty\} \in \mathbb{R}^+ \cup \{\infty\}.$$

$$\tilde{n}_A(M) := \sup\{\ell(M/QM)/e(Q;M) | Q \text{ (homogeneous) parameter ideal for } M\} \in \mathbb{R}^+ \cup \{\infty\}.$$

Remarks 1. (i) We only consider homogeneous ideals Q provided A is a graded K-algebra and M is a graded A-module.
(ii) Of course, $1 \le \tilde{n}_A(M) \le n_A(M)$. We get $n_A(M) = \tilde{n}_A(M)$ if A is a local ring and A/m_A is an infinite field, see our discussion about an application of the theorem of transition. This follows from well-known results of local algebra (see, e.g., [Ma], theorems 14.13 and 14.14). However, the following example shows that n_A and $\tilde{n}_A$ need not to be equal in the graded case even when the field K is infinite.

Moreover, we note that $\tilde{n}_A(M) = 1$ if and only if M is a (graded) Cohen-Macaulay module over A (see, e.g., [BH]). Using this fact and the considerations below on flat extensions we also get: $n_A(M) = 1$ if and only if M is a (graded) Cohen-Macaulay module.

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(iii) Using our notation we note that the lemma of Lech, see [Le], can be stated as follows:

$$\inf\{\ell(M/QM)/e(Q;M) | Q \text{ parameter ideal for } M\} = 1.$$

Example. Let R be the polynomial ring $K[x,y,z]$. Consider the ideal $I = xR \cap (y,z)R$ and the graded K-algebra $A := R/I$. Then we have

(1) $n_A(A) = \infty$,
(2) $\tilde{n}_A(A) = 2$.

Proof. (1) Let n be an integer ≥ 1. We set $Q_n := (x^n, y, z)R$. We note that $Q_n A$ is not a parameter ideal for A. We have $\ell(A/Q_n A) = n$. Applying the associative law for multiplicities, see, e.g., [No], Prop. 11 on page 341, we get for the multiplicity $e(Q_n; A) = 1$. Hence we get (1).
(2) We set $q := (x_0 + x_1, x_2) \cdot R$. Then we have $\ell(A/qA) = 2$. Using again the associative law for multiplicities we obtain $e(q; A) = 1$. Hence we get $\tilde{n}_A(A) \geq 2$. Our theorem 4, (i) of § 2 and example 1 of § 4 yield the desired equality. □

Remarks 2. (i) If M is a FLC A-module (i.e. the local cohomology $H^i_m(M)$ has finite length for $i = 0, \ldots, \dim M - 1$) then the corrected version of the theorem, (i) of [MV] shows that $n_A(M) < \infty$. From this point of view we want to mention that R/I in our above example is not FLC (see, e.g., [SV], Prop. 16, page 260).
(ii) Assertion (1) shows that $n_A(A) < \infty$ is not true in general. However, by considering the same ideal I we have $\tilde{n}_A(A) < \infty$. Hence we want to ask the following question.

Problem. Characterize the (graded) A-modules M with $n_A(M) < \infty$ or $\tilde{n}_A(M) < \infty$.

In order to study this problem and to describe our first applications of the new invariants we need to explain some further notations. First we require some dimension theory. Let M be an A-module. We set

$$Assh\, M := \{P \in Ass\, M | \dim A/P = \dim M\}.$$

M is said to be equidimensional if $\dim A/P = \dim M$ for all minimal prime ideals $P \in Supp\, M$, i.e., if $\min Supp\, M = Assh\, M$. Following M. Nagata [Na], page 124, an A-module M is called quasi-unmixed if $\hat{M}$ is equidimensional where $\hat{M}$ is the completion of M with respect to m_A. Moreover, M is unmixed if $\dim \hat{A}/P = \dim \hat{M}$ for all $P \in Ass\, \hat{M}$, i.e., if $Ass\, \hat{M} = Assh\, \hat{M}$ (see again [Na], page 82). It is known that there are local integral domains which are not quasi-unmixed (see our remark after Theorem 2 of § 2). However, "quasi-unmixed" and "equidimensional" coincide in the graded case. In this case the chain conditions for prime ideals are satisfied by A as A is an epimorphic image of a polynomial ring over a field (see [Na]).

Second we need to apply the theorem of transition (see [Na], Ch. II, § 19). Let A' be a faithfully flat A-algebra. We assume that A' is again a local ring or a graded K'-algebra with a field $K' \supseteq K$. If $\ell_{A'}(A'/m_A A') < \infty$ then the length and multiplicities over A' and A differ only by the factor $\ell(A'/m_A A')$ (see [Na], (19.1)). Hence we get $n_A(M) \leqq n_{A'}(M')$ where $M' := M \otimes_A A'$. In order to obtain upper bounds for $\tilde{n}_A(M)$ and $n_A(M)$ we therefore may assume that $A/m_A (= K$ if A is a graded K-algebra) is an infinite field. We need to apply this fact in order to get our above remark 1, (ii).

Moreover, if Q' is a parameter ideal for $\hat{M}$ in $\hat{A}$ then there is a parameter ideal Q for M in A such that $\hat{M}/Q'\hat{M} \cong M/QM$. Considering the faithfully flat A-module $\hat{A}$ we therefore have $\tilde{n}_A(M) = \tilde{n}_{\hat{A}}(\hat{M})$ and $n_A(M) = n_{\hat{A}}(\hat{M})$.

Using the Cohen structure theory we therefore may assume that in the local case the ring A is a complete regular local ring with infinite residue class field or in the graded case that A is a polynomial ring in finitely many variables over an infinite field since the length and multiplicity are the same by considering epimorphic images of A.

An important approach to the proofs of § 3 is to apply filtrations: Let M be a (graded) A-module. A finite sequence $(M_i)_{0 \leq i \leq r}$ of submodules M_i of M is said to be a filtration of M if $0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$. Following [Bo], pp. 265-266 a filtration is called a composition series if for every $i = 1, \ldots, r$ we have $M_{i-1}/M_i \cong A/P_i$ for some (homogenous) prime ideal P_i of A. It follows that $Ass\, M \subseteq \{P_1, \ldots, P_r\}$. Moreover, following [Dr] and [Si] M is defined to be clean if there exists a composition series of M, say $(M_i)_{0 \leq i \leq r}$ with $M_{i-1}/M_i \cong A/P_i$, such that $\{P_1, \ldots, P_r\} = \min(Supp\, M)$.

Let us consider the following special case: $R = K[x_0, \ldots, x_n]$, I a monomial ideal of R and $M = R/I$. Then M has a composition series where the corresponding prime ideals $P_1, \ldots, P_r$ are monomial primes. This follows from the construction of composition series. Let's call such composition series of R/I a monomial composition series. Therefore our Theorem 4 of § 2 yields a first application of the new invariants. Before stating this result we need to introduce the degree of a graded module M over a graded K-algebra A. This degree denoted by $\deg_A M$ or $\deg M$ is defined as the multiplicity $e(m_A; M)$.

Corollary 1. Let I be a monomial ideal of $R := K[x_0, \ldots, x_n]$. For every monomial composition series $(M_i)_{0 \leqq i \leqq r}$ of R/I we have

$$r \geqq \deg R/I \cdot \tilde{n}_R(R/I).$$

If R/I is quasi-unmixed then we get $r \geq \deg R/I + n_R(R/I) - 1$.

Moreover, with the aid of the cleanness concept, our study on the invariants yields a new proof of the known and interesting fact: Let I be an unmixed clean monomial ideal of R then R/I is Cohen-Macaulay (see Corollary 5 of § 2).

Another application of these invariants follows from theorem 1 of § 2.

Corollary 2. Let A be a local ring and let M be an A-module. If $n_A(M) < \infty$ then M is quasi-unmixed.

However, the main result of this paper states that $\tilde{n}_A(M) < \infty$ for all graded modules M over a graded K-algebra A, see theorem 2 (2) of § 2.

§ 2 Main Results

The aim of this section is to describe our four theorems. First we study the invariant $n_A(M)$.

Theorem 1. *(1) Let A be a local ring (or a graded K-algebra) and let M be an (graded) A-module. If $n_A(M) < \infty$ then M is quasi-unmixed.*

(2) The following conditions are equivalent

- (i) *For every local ring A (or graded K-algebra) and every quasi-unmixed (graded) A-module M we have $n_A(M) < \infty$.*
- (ii) *For every complete regular local ring R with infinite residue class field (or for every polynomial ring $R = K[x_0, \dots, x_n]$ over an infinite field K) we have $n_R(R/P) < \infty$ for all (homogeneous) prime ideals P of R.*

Now we are going to examine the invariant $\tilde{n}_A(M)$.

Theorem 2. *(1) Let A be a local ring and let M be an A-module. If $\tilde{n}_A(M) < \infty$ then M is quasi-unmixed.*
(2) Let A be a graded K-algebra. Then we have $\tilde{n}_A(M) < \infty$ for all graded A-modules M.

Remarks 3. (1) By remark 1, (ii) it is clear that (1) of theorem 2 is equivalent to (1) of theorem 1 provided that the residue class field of A is infinite. Hence theorem 2, (1) gives only a new result when this residue class field is finite.
(2) M. Nagata has constructed local integral domains A which are not quasi-unmixed, see [Na], example 2 of the appendix. Theorems 1, 2, (1) show that for such local rings we have $n_A(A) = \tilde{n}_A(A) = \infty$.

Theorem 3. *We set $R := K[x_0, \dots, x_n]$. Let $M \neq 0$ be a graded R-module having the following property: There is a filtration of M, say*

$$0 = M_0 \subset M_1 \subset \dots \subset M_r = M,$$

of graded R-modules $M_0, \dots, M_r, r \geq 1$, such that the factors $N_i := M_i/M_{i-1}$ are FLC R-modules for $i = 1, \dots, r$. Then we have

$$\tilde{n}_R(M) \leqq \sum_{i=1}^{r} \frac{\deg N_i}{\deg M} \cdot \tilde{n}_R(N_i) < \infty.$$

In the special case that N_i are Cohen-Macaulay modules we get the following corollary of theorem 3.

Corollary 3. Let M and R be as in theorem 3. We assume that M has a filtration such that the factors $N_i = M_i/M_{i-1}$ are Cohen-Macaulay R-modules for $i = 1, \dots, r$. Then we have

$$\tilde{n}_R(M) \leqq \sum_{i=1}^{r} \deg N_i / \deg M.$$

Theorem 4. *Let $I \subset R := K[x_0, \dots, x_n]$ be a monomial ideal. Take a composition series $0 = M_0 \subset M_1 \subset \dots \subset M_r = R/I$ of R/I with graded R-modules $M_0, \dots, M_r$ such that for $i = 1, \dots, r$ we have $M_i/M_{i-1} = R/P_i$ for monomial prime ideals*

$P_1, \ldots, P_r$ *of* R *(a so-called monomial composition series, cf. § 1). Then we have the following bounds:*

(i) $\tilde{n}_R(R/I) \leqq r/\deg R/I$

(ii) $n_R(R/I) \leqq r - \deg R/I + 1$,

provided R/I *is equidimensional.*

Remark 4. The assertion of theorem 4, (ii) remains analogously true for complete regular local rings. If $R = W[[x_1, \ldots, x_n]]$ with a discrete valuation ring W then we may replace x_0 by a canonical generator of m_W.

With the aid of the cleanness concept, we get the following two corollaries.

Corollary 4. Let I be a square-free monomial ideal of $R := K[x_0, \ldots, x_n]$. If R/I is clean then we have:

$$\tilde{n}_R(R/I) \leqq \sharp Ass\, R/I / \sharp Assh\, R/I.$$

The next corollary shows that our approach via composition series yields sharp bounds on $n_R(R/I)$. Indeed, having additional assumptions, the bound $r - \deg R/I + 1$ of theorem 4, (ii) is equal to 1. Hence we obtain a new proof of the following known result about unmixed clean monomial ideals, see [Si], Cor. 2.2.4.

Corollary 5. Let I be a monomial ideal of $R := K[x_0, \ldots, x_n]$. Assume that R/I is unmixed, i.e., $\dim R/P = \dim R/I$ for all $P \in Ass\, R/I$. If R/I is clean then R/I is Cohen-Macaulay.

§ 3 Proofs

Proof of Theorem 1, (1) and Theorem 2, (1). Assume that M is not quasi-unmixed. We may assume w.l.o.g. that $A = R$ is a complete regular local ring or a polynomial ring in finitely many indeterminates over a field K. By assumption we can consider a minimal prime ideal $P \in Ass\, M$ such that $d' := \dim R/P < \dim M =: d$. Let N be the intersection of all P' -primary submodules of a primary decomposition of 0 in M, where $P' \in Ass\, M$ with $P \subsetneqq P'$. Let U be the intersection of the remaining primary submodules, i.e., $U = 0 :_M P^t$ for $t >> 0$. We note that $Ass\, h\, M \subseteqq Ass\, U$ and $\dim M/N = d'$. We now take (homogeneous) elements $x, y \in m_R$ such that $xM \subseteqq U, x \notin P$ and $yM \subseteqq N, y \notin p$ for all $p \in Ass\, M/U$. We note that such elements do exist since $Ann\, M/U \nsubseteq P$ and $Ann\, M/U \nsubseteq p$ for all $p \in Ass\, M/U$. We also note that $Supp\, M/(x+y)M = Supp\, M/(x,y)M = Supp\, M/(x^n, y)M = Supp\, M/(x^n + y)M$ for all $n \in \mathbb{N}^+$. Hence we have $\dim M/(x^n + y)M = \dim M/(x^n, y)M = d - 1$ for all $n \in \mathbb{N}^+$. Take (homogeneous) elements $f_1, \ldots, f_{d-d'} \in Ann\, M/N$ which form a part of a system of parameters for $M/(x,y)M$. Indeed, such elements exist since the radical of $Ann\, M/N$ is equal to P and $(x,y)R \nsubseteq P$. Consider (homogeneous) elements $f_{d-d'+1}, \ldots, f_{d-1}$ such that $f_1, \ldots, f_{d-1}$ is a system of parameters for $M/(x^n, y)M$ and for $M/(x^n + y)M$ for all $n \in \mathbb{N}^+$. We set $Q_n := (x^n, y, f_1, \ldots, f_{d-1})R$ and $Q'_n := (x^n + y, f_1, \ldots, f_{d-1})R, n \in \mathbb{N}^+$. Of course, $\ell(M/Q_nM)$ and $\ell(M/Q'_nM)$ are finite. We also get

$$\ell(M/Q'_nM) \geq \ell(M/N + Q'_nM) =$$
$$\ell(M/N + (x^n, f_{d-d'+1}, \dots, f_{d-1})M)$$

(since $(y, f_{d-d'+1}, \dots, f_{d-1})M \subseteqq N$),
$=: \ell(\bar{M}/x^n\bar{M})$ with $\bar{M} := M/N + (f_{d-d'+1}, \dots, f_{d-1})M$.
The same result is true by taking Q_n for Q'_n.

Since $\dim \bar{M} \geqq \dim M/N - (d'-1) = 1$ and $\ell(\bar{M}/x^n\bar{M}) < \infty$ we have $\dim \bar{M} = 1$ and x^n is a parameter element for $\bar{M}$. We now choose $m \in \mathbb{N}^+$ such that $0 :_{\bar{M}} x^m = 0 :_{\bar{M}} x^{m+1} = \dots$. Hence we may consider the following exact sequence:

$$0 \to \bar{M}/0 :_{\bar{M}} x^m + x\bar{M} \stackrel{x^n}{\to} \bar{M}/x^{n+1}\bar{M} \to \bar{M}/x^n\bar{M} \to 0$$

(up to a shift of degrees in the graded case) for $n \geq m$. Then it follows by induction on n that $\ell(\bar{M}/x^n\bar{M}) = (n-m)\ell(\bar{M}/0 :_{\bar{M}} x^m + x\bar{M}) + \ell(\bar{M}/x^m\bar{M}) > n - m$. On the other hand, we get for the multiplicity $e(Q'_n; M) = e(Q'_n; M/U) = e(y, f_1, \dots, f_{d-1}; M/U)$ since $x\,M/U = 0$. The same result is again true by taking Q_n for Q'_n. Hence we have that $e(Q'_n; M)$ and $e(Q_n; M)$ do not depend on n by construction of $y, f_1, \dots, f_{d-1}$ and by using the above facts. Therefore we get

$$\lim_{n\to\infty} \frac{\ell(M/Q'_nM)}{e(Q'_n; M)} \geq \lim_{n\to\infty} \frac{n-m}{e(y, f_1, \dots, f_{d-1}; M/U)} = \infty$$

The same is true for Q_n. Finally we obtain that $n_R(M) = \infty$ for the local and graded case, and $\tilde{n}_R(M) = \infty$ for the local case since in this situation $(x^n + y, f_1, \dots, f_{d-1})R$ is a parameter ideal for M for all $n \in N^+$. This gives the desired contradiction and completes the proof. □

Proof of Theorem 1, (2). (i) $\Rightarrow$ (ii) is trivial.
(ii) $\Rightarrow$ (i): We assume again w.l.o.g. that $R = A$ is either a complete regular local ring or a polynomial ring in finitely many indeterminates over a field K. Choose a composition series $0 = M_0 \subset M_1 \subset \dots \subset M_r = M$ of M consisting of (graded) submodules $M_0, M_1, \dots, M_r$ of M such that $M_i/M_{i-1} \cong R/P_i$ (up to shifts of degree) for some (homogeneous) prime ideal P_i of $R, i = 1, \dots, r$. Since M is quasi-unmixed and $\{P_1, \dots, P_r\} \subseteqq Supp\, M$, for each $i = 1, \dots, r$, there is a prime ideal $P \in Assh\, M$ such that $P \subseteqq P_i$. Therefore we get for any (homogeneous) m_R -primary ideal Q:

$$\ell(M/QM) \leqq \sum_{i=1}^{r} \ell(R/P_i + Q) \leq \sum_{P \in Assh\, M} \left(\sum_{\substack{1 \leq i \leqq r \\ P \subseteq P_i}} \ell(R/P_i + Q) \right)$$
$$\leqq \sum_{P \in Assh\, M} \beta_P \ell(R/P + Q), \quad \text{where} \quad \beta_P := \sharp\{i | 1 \leq i \leq r, P \subseteq P_i\}$$

for all $P \in Assh\, M$. Moreover, we define $\alpha_P := \ell_{R_P}(M_P)$ for all $P \in Assh\, M$. Then it follows by [Bo], Remark 1 on page 275 that $1 \leq \alpha_P \leq \beta_P$ for all $P \in Assh\, M$. We set $\rho := \max\{\beta_P/\alpha_P | P \in Assh\, M\} \geqq 1$. Then we get

$$\ell(M/QM) \leqq \rho \sum_{P \in Assh\, M} \alpha_P \ell(R/P+Q).$$

Applying the associative law for multiplicities we obtain

$$e(Q;M) = \sum_{P \in Assh\, M} \alpha_P e(Q;R/P).$$

Hence we have

$$\frac{\ell(M/QM)}{e(Q;M)} \leq \frac{\rho \Sigma \alpha_P \ell(R/P+Q)}{\Sigma \alpha_P e(Q;R/P)} \leq \rho \max\{\frac{\ell(R/P+Q)}{e(Q;R/P)} \mid$$

$$P \in Assh\, M\} \leqq \rho \max\{n_R(R/P) | P \in Assh\, M\} < \infty$$

by our assumption (ii). This completes the proof. □

Remarks 5. Our proof of Theorem 1 (2) remains true if we take the following refined definition of β_P: We are going to count each $P_i, 1 \leq i \leq r$, precisely once, i.e., if $P \subseteq P_i$ and $P' \subseteq P_i$ with $P, P' \in Assh\, M$ then we count P_i, say for β_P, but not for $\beta_{P'}$. Counting the primes P_i in this sense we get $\sum_{P \in Assh\, M} \beta_P = r$. Hence we have

$$\rho \leqq r + 1 - \sum_{P \in Assh\, M} \ell_{R_P}(M_P).$$

Before embarking in the proof of the main result of this paper given by theorem 2, (2) we need some investigations on local cohomology. Let A be a graded K-algebra and let $M, N \neq 0$ be graded A-modules, where M is assumed to be finitely generated with $\dim M =: d \geqq 0$ but where N need not be finitely generated. We set $a(M) := \inf\{i \in \mathbb{Z} \mid [M]_i \neq 0\}$, and $N\, |_p := \bigoplus_{i \geq p} [N]_i$ for any $p \in \mathbb{Z}$. Let $H^i_m(M)$ be the local cohomology module for $i \geqq 0$. We introduce the following nonnegative integer:

$$I_t(M) := \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H^i_m(M)|_{a(M)-it})$$

for an integer $t \in \mathbb{Z}$.

Lemma 1. *Assume K is an infinite field. Let $x_1, \ldots, x_d$ be a homogeneous system of parameters for M with $\deg x_i \leqq t$ for some $t \in \mathbb{Z}$ and for all $i = 1, \ldots, d$. Then we have*

$$\ell(M/(x_1, \ldots, x_d)M) - e((x_1, \ldots, x_d); M) \leqq I_t(M).$$

Proof. For $d = 0$ there is nothing to prove. For $d = 1$ we get

$$\ell(M/xM) - e(x;M) = \ell(0 :_M x) \leq \ell(H_m^0(M)) =$$
$$\ell(H_m^0(M)|_{a(M)}) = I_t(M) \quad \text{for all} \quad t \in \mathbb{Z}.$$

Let $d \geqq 2$. We can assume w.l.o.g. that $x_1 \notin p$ for all $p \in Ass\, M \setminus \{m\}$. We set $\delta := \deg x_1 \leq t$. Then we have an exact sequence

$$0 \to M/0 :_M x_1(-\delta) \xrightarrow{x_1} M \to M/x_1M \to 0$$

which gives rise to exact sequences for all $i \geq 0$

$$H_m^i(M) \to H_m^i(M/x_1M) \to H^{i+1}(M/0 :_M x_1)(-\delta).$$

By assumption on the element x_1 we have $H_m^{i+1}(M/0 :_M x_1) \cong H_m^{i+1}(M)$. Hence we get

$$\ell(H_m^i(M/x_1M)\,|_{a(M/x_1M)-it}) \leq \ell(H_m^i(M)|_{a(M)-it}) + \ell(H_m^{i+1}(M)(-\delta)\,|_{a(M)-it}) \leq$$
$$\ell(H_m^i(M)|_{a(M)-it}) + \ell(H_m^{i+1}(M)|_{a(M)-(i+1)t}) \quad \text{since} \quad a(M/x_1M) = a(M)$$

(note that $[x_1M]_{a(M)} = 0$ because $\deg x_1 > 0$) and $\ell(H_m^{i+1}(M)(-\delta)|_p) = \ell(H_m^{i+1}(M)|_{p-\delta}) \leq \ell(H_m^{i+1}(M)|_{p-t})$ for all $p \in \mathbb{Z}$. By induction hypothesis we therefore get:

$$\ell(M/(x_1, \ldots, x_d)M) - e((x_1, \ldots, x_d); M) =$$
$$\ell(M/x_1M/(x_2, \ldots, x_d)M/x_1M) - e((x_2, \ldots, x_d); M/x_1M) \leq$$
$$\sum_{i=0}^{d-2} \binom{d-2}{i} \ell(H_m^i(M/x_1M)|_{a(M/x_1M)-it}) \leq$$
$$\sum_{i=0}^{d-2} \binom{d-2}{i} (\ell(H_m^i(M))|_{a(M)-it}) + \ell(H_m^{i+1}(M)|_{a(M)-(i+1)t})$$
$$= \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H_m^i(M)|_{a(M)-it}) = I_t(M).$$

□

In order to state the next lemma we need to recall the definition of the Castelnuovo-Mumford regularity in terms of local cohomology: Let N be an Artinian graded A-module. Then we set $e(N) := \max\{i \mid [N]_i \neq 0\} \in \mathbb{Z}$. The Castelnuovo-Mumford regularity of a graded A-module M, denoted by $reg\, M$, is defined as follows:

$$reg\, M := \; max\{i + e(H_m^i(M)) | 0 \leq i \leq \dim M\}.$$

Now we are going to examine $reg\, M$.

Lemma 2. *Let A, M and $x_1, \dots, x_d$ be as in lemma 1. Then we have*

$$reg\, M/(x_1, \dots, x_d)M \leqq \sum_{i=1}^{d} (\deg x_i - 1) + reg\, M.$$

Proof. We can assume that $x_i \notin p$ for all $p \in Ass\, M/(x_1, \dots, x_{i-1})M \setminus \{m\}$ and $i = 1, \dots, d$. Under this assumption we will prove by induction on $j, 1 \leq j \leq d$, that

$$reg\, M/(x_1, \dots, x_j)M \leq \sum_{i=1}^{j} (\deg x_i - 1) + reg\, M.$$

It is therefore enough to consider just the case $j = 1$. Set $x := x_1$ and $\delta := \deg x \geqq 1$. The proof of lemma 1 has established the following exact sequence for all $i \geq 1$:

$$H^i_m(M) \to H^i_m(M/xM) \to H^{i+1}_m(M)(-\delta).$$

Hence we get

$$\begin{aligned} reg\, M/xM &= \max\{i + e(H^i_m(M/xM)|0 \leq i \leq \dim M - 1) \\ &\leq \max\{\max\{i + e(H^i_m(M)), i + e(H^{i+1}_m(M)(-\delta)\}| \\ 0 \leq i \leq \dim M - 1\} &= \max\{e(H^0_m(M)), \delta + e(H^1_m(M)), \\ &\dots, d - 1 + \delta + e(H^d_m(M))\} \leq \delta - 1 + \max\{i + \\ &e(H^i_m(M))|0 \leq i \leq d\} = \delta - 1 + reg\, M. \end{aligned}$$

□

In order to prove theorem 2, (2) we need the following application of lemma 2.

Corollary 2.1. Take an integer $t \geq 1 + \sum_{i=1}^{d} (\deg\, x_i - 1) + reg\, M - a(M)$ then we have

$$m^t M \subseteqq (x_1, \dots, x_d)M.$$

Proof. Since $reg\, M/(x_1, \dots, x_d)M = e(M/(x_1, \dots, x_d)M)$ we have $m^s M \subseteqq (x_1, \dots, x_d)M$ for all integers $s \geq 1 + e(M/(x_1, \dots, x_d)M) - a(M/(x_1, \dots, x_d)M) = 1 + e(M/(x_1, \dots, x_d)M) - a(M)$. Applying lemma 2 we therefore get corollary 2.1. □

Proof of theorem 2, (2). Let $x_1, \dots, x_d$ be a homogeneous system of parameters for M. Assume w.l.o.g. that $\delta_1 := \deg x_1 \geq \deg x_2 \geq \dots \geq \deg x_d$. We set $t_1 := 1 + \sum_{i=1}^{d} (\deg x_i - 1) + reg\, M - a(M)$. Then corollary 2.1 shows that $m^{t_1} M \subseteq (x_1, \dots, x_d)M$. Take a linear form $\ell_1 \in [A]_1$ such that $\ell_1, x_2, \dots, x_d$ is again a system of parameters for M. Then we get $(\ell_1^{t_1}, x_2, \dots, x_d)M \subseteq (x_1, \dots, x_d)M$. Hence we have

$$\ell(M/(x_1, \dots, x_d)M) \leq \ell(M/(\ell_1^{t_1}, x_2, \dots, x_d)M) \leqq t_1 \cdot \ell(M/(\ell_1, x_2, \dots, x_d)M)$$

by applying the following exact sequence for all $t \in \mathbb{N}^+$:

$$M/(\ell_1, x_2, \dots, x_d)M(-t) \xrightarrow{\ell_1^t} M/(\ell_1^{t+1}, x_2, \dots, x_d)M \to M/(\ell_1^t, x_2, \dots, x_d)M \to 0.$$

Now we need to apply the following Bezout-type theorem, see, e.g. [Se]:
Let $f_1, \dots, f_d$ be a homogeneous system of parameters for M generating a parameter ideal, say Q. Then we have for the multiplicity

$$e(Q; M) = (\deg A/Q) \deg M = \deg f_1 \cdot \ldots \cdot \deg f_d \cdot \deg M.$$

An application to $x_1, \dots, x_d$, and $\ell_1, x_2, \dots, x_d$ gives the equality $e((x_1, \dots, x_d); M) = \delta_1 e((\ell_1, x_2, \dots, x_d); M)$. Therefore we get with $Q := (x_1, \dots, x_d)A$ and $Q' := (\ell_1, x_2, \dots, x_d)A$:

$$\frac{\ell(M/QM)}{e(Q, M)} \leqq \frac{t_1}{\delta_1} \frac{\ell(M/Q'M)}{e(Q'; M)}.$$

Now we have

$$\frac{t_1}{\delta_1} = 1 + \sum_{i=2}^{d} \frac{\deg x_i - 1}{\delta_1} + \frac{reg\, M - a(M)}{\delta_1} \leqq d + reg\, M - a(M).$$

Therefore we obtain

$$\frac{\ell(M/QM)}{e(Q; M)} \leq (d + reg\, M - a(M)) \cdot \frac{\ell(M/Q'M)}{e(Q'; M)}.$$

Repeating this process we get

$$\frac{\ell(M/QM)}{e(Q; M)} \leq (d + reg\, M - a(M))^d \cdot \frac{\ell(M/Q^*M)}{e(Q^*; M)},$$

where Q^* is a parameter ideal for M generated by elements of degree one. Applying lemma 1 we obtain

$$\frac{\ell(M/Q^*M)}{e(Q^*; M)} \leqq 1 + \frac{I_1(M)}{e(Q^*; M)} = 1 + \frac{I_1(M)}{\deg M}$$

by Bezout's theorem. Finally we get

$$\tilde{n}_A(M) \leqq (d + reg\, M - a(M))^d (1 + \frac{I_1(M)}{\deg M}) < \infty.$$

This completes the proof of theorem 2. □

Proof of Theorem 3. Since $Supp\, N_i \subseteq Supp\, M$ for all $i = 1, \ldots, r$ we have $d_i := \dim N_i \leq \dim M =: d$. Let $Q := (f_1, \ldots, f_d)R$ be a homogeneous ideal of R generated by a homogeneous system of parameters for M. Then $\ell(N_i/QN_i) < \infty$. Hence there are d_i elements of a suitable minimal generating set of Q consisting of homogeneous elements which generate a parameter ideal, say F_i, for $N_i, i = 1, \ldots, r$. Then we have: $\ell(N_i/QN_i) \leq \ell(N_i/F_iN_i)$, and $e(F_i; N_i) \leqq \prod_{j=1}^{d} \deg f_i \cdot \deg N_i$ by Bezout's theorem (see the proof of theorem 2, (2)). Moreover, it follows by induction on r: $\ell(M/QM) \leqq \sum_{i=1}^{r} \ell(N_i/QN_i)$. Using again Bezout's theorem we therefore get:

$$\frac{\ell(M/QM)}{e(Q;M)} \leqq \sum_{i=1}^{r} \frac{\ell(N_i/F_iN_i)}{(\prod_{j=1}^{d} \deg f_j) \deg M} \leq$$

$$\leq \sum_{i=1}^{r} \frac{\deg N_i}{\deg M} \cdot \frac{\ell(N_i/F_iN_i)}{e(F_iR; N_i)} \leqq \sum_{i=1}^{r} \frac{\deg N_i}{\deg M} \cdot \tilde{n}_R(N_i).$$

Since N_i are FLC modules we get $\tilde{n}_R(N_i) \leq n_R(N_i) < \infty$ by a corrected version of [MV]. This shows theorem 3. □

Proof of theorem 4. (i) Since R/P_i are Cohen-Macaulay modules for $i = 1, \ldots, r$ we have from Corollary 3 of theorem 3:

$$\tilde{n}_R(R/I) \leqq \sum_{i=1}^{r} \frac{\deg R/P_i}{\deg R/I} = \frac{r}{\deg R/I}.$$

(ii) Let Q be a homogeneous m-primary ideal of R. Using the notation and method of the proof of theorem 1, (2) we get:

$$\frac{\ell(R/IR+Q)}{e(Q;R/I)} \leqq \rho \cdot \frac{\sum_{P \in Assh\, R/I} \alpha_P \ell(R/P+Q)}{\sum_{P \in Assh\, R/I} e(Q; R/P)}.$$

Since R/P are Cohen-Macaulay modules with $\dim R/P = \dim R/I$ for all $P \in Assh\, R/I$ we have $\ell(R/I+Q) \leq e(Q; R/P)$, i.e., we get

$$\frac{\ell(R/I+Q)}{e(Q;R/I)} \leqq \rho \leqq r + 1 - \sum_{P \in Assh\, R/I} \ell_{R_P}(R_P/IR_P)$$

by applying remark 5

$$= r + 1 - \deg R/I.$$

□

Proof of Corollary 4 and 5. With the aid of the cleannes concept, we get both corollaries from theorem 4.

§ 4 Examples and Problems

Example 1. We consider the example discussed in § 1: $R = K[x_0, x_1, x_2]$, and $I = (x_0)R \cap (x_1, x_2)R$. In order to complete the proof of the assertation of this example of § 1 we need to show that $\tilde{n}_R(R/I) \leqq 2$. Applying theorem 4, (i) we describe a filtration of R/I as follows:

$$0 \subset x_0R/I \subset R/I$$

having the factors $x_0R/I \cong R/(x_1, x_2)R$, and $R/I/x_0R/I \cong R/x_0R$ with the corresponding monomial prime ideals $P_1 = (x_1, x_2)R$ and $P_2 = x_0R$. Hence theorem 4, (i) gives $\tilde{n}_R(R/I) \leq 2$.
With a view to theorem 3 and Corollary 3 we want to describe the following class of examples.

Example 2. We set $R = K[x_0, \dots, x_n]$. Let I and J be homogeneous ideals of R with the following two properties:

(i) R/I and R/J are locally Cohen-Macaulay
(ii) There is a form, say f of R such that

$$I + J = J + fR \quad \text{and} \quad J :_R f = J.$$

Then we have

$$\tilde{n}_R(R/I \cap J) \leqq \frac{\deg R/I}{\deg R/I \cap J} \cdot \tilde{n}_R(R/I) + \frac{\deg R/J}{\deg R/I \cap J} \cdot \tilde{n}_R(R/J) < \infty$$

Proof. We may assume that $f \in I$. Consider the following filtration of $R/I \cap J$:

$$0 \subset (I \cap J) + fR/I \cap J \subset R/I \cap J$$

having the two factors $(I \cap J) + fR/I \cap J \cong R/J(-\deg f)$ and $R/I \cap J/(I \cap J) + fR/I \cap J \cong R/(I \cap J) + fR = R/I$. Hence assumption (i) and theorem 3 provide our assertion of example 2.

We note that $R/I \cap J$ is not a FLC module provided that $ht\, I < ht\, J \leqq n$, or $ht\, J < ht\, I \leqq n$. Moreover, we note that example 1 is a special case of example 2.

In the light of theorem 1 we want to state the following problem. A positive solution of this problem proves our conjecture at least in the graded case below.

Problem 1. We set $R = K[x_0, \dots, x_n]$. Let P be a homogeneous prime ideal of R. Is then $n_R(R/P) < \infty$?
The first open case of this problem is given by $n = 4$ and $\dim R/P = 3$. To show this we want to study the following example.

Example 3. Let S be the toric surface of $\mathbb{P}^4_K$ given parametrically by

$$\{s^{10}, t^{10}, u^{10}, s^6tu^3, st^6u^3\}.$$

Let P be the defining prime ideal of S in $K[x_0, \dots, x_4]$. Considering the system of parameters x_0, x_1, x_2 for R/P we get $n_R(R/P) \geq \tilde{n}_R(R/P) \geqq 3$. We believe that $n_R(R/P) = \tilde{n}_R(R/P) = 3$. A possible way to prove such results is given by the following problem stated in terms of the theory of Gröbner bases.

Problem 2. We set $R = K[x_0, \dots, x_n]$. Let P be a homogeneous prime ideal of R. Let $\mathrm{in}(P)$ be the initial ideal of P for some term order. Describe a relationship between $\tilde{n}_R(R/P)$ and $\tilde{n}_R(R/\mathrm{in}\,(P))$. For example, is $\tilde{n}_R(R/P) \leq \tilde{n}_R(R/in(P)$? The same problem is given in terms of $n_R(\dots)$.

Continuation of example 3. Taking any term order with $x_4 > x_3$ we get the following initial ideal $\mathrm{in}(P) = (x_0x_4^2, x_3^{10}, x_3^8x_4^2, \dots, x_4^{10})R =: Q_1 \cap Q_2$, where

$$Q_1 := (x_3^{10}, x_4^2)R,\ Q_2 := (x_0, x_3^{10}, x_3^8x_4^2, \dots, x_4^{10})R.$$

Moreover, we set $Q_3 := (x_0, x_3^8, x_3^6x_4^2, \dots, x_4^8)R$. Then we consider the following filtration of $R/\ \mathrm{in}P$:
$0 \subset \mathrm{in}P + x_4^2R/\mathrm{in}P \subset R/\mathrm{in}P$ having the factors $\mathrm{in}P + x_4^2R/\mathrm{in}P \cong R/Q_3$, and

$$R/\mathrm{in}\,P/\mathrm{in}P + x_4^2R/\mathrm{in}\,P \cong R/\mathrm{in}\,P + x_4^2R = R/Q_1 \cap (Q_2 + x_4^2R) = R/Q_1.$$

Both factors are Cohen-Macaulay modules. Hence Corollary 3 shows that

$$\tilde{n}_R(R/\mathrm{in}P) \leqq \frac{\deg R/Q_1}{\deg R/\mathrm{in}P} + \frac{\deg R/Q_3}{\deg R/\mathrm{in}P} = \frac{20+40}{20} = 3.$$

Taking the parameter ideal $Q := (x_0, x_1, x_2)R$ for $R/\mathrm{in}P$ we get

$$\tilde{n}_R(R/\mathrm{in}P) \geqq \frac{\ell(R/\mathrm{in}P + Q)}{e(Q; R/\mathrm{in}\,P)} = \frac{\ell(R/Q_2 + Q)}{\ell(R/Q_1 + Q)} = \frac{60}{20} = 3$$

Hence $\tilde{n}_R(R/\mathrm{in}P) = 3$. It is not too difficult to show that $n_R(R/\mathrm{in}P) \leqq 3$. This gives $n_R(R/\mathrm{in}P) = \tilde{n}_R(R/\mathrm{in}P) = 3$.

Example 4. We set $R := K[x_0, \dots, x_4]$. Considering theorem 4 we want to give a square-free monomial ideal I of R such that

$$\tilde{n}_R(R/I) < n_R(R/I) < \infty.$$

Take $I = (x_0, x_1)R \cap (x_2, x_3)R \cap (x_3, x_4)R$.

Claim. $n_R(R/I) = 3/2 > \tilde{n}_R(R/I) = 4/3$.

Proof. Taking the following filtration of R/I:

$$0 \subset I + x_3R/I \subset (x_2x_4, x_3)R/I \subset (x_3, x_4)R/I \subset R/I$$

having the factors $R/(x_0, x_1)R, R/(x_0, x_1, x_3)R, R/(x_2, x_3)R$ and $R/(x_3, x_4)R$ (up to shifts in gradings). Hence theorem 4, (i) gives the upper bound $\tilde{n}_R(R/I) \leqq 4/3$. Considering the homogeneous system of parameters $f := x_0 - x_2, g = x_2 - x_4, h = x_1 - x_3$ for R/I we get $\tilde{n}_R(R/I) \geqq 4/3$. Hence we have $\tilde{n}_R(R/I) = 4/3$.

We note that theorem 4, (ii) just gives $n_R(R/I) \leqq 2$. However, taking the ideal $Q' := (x_0, x_2^2, g, h)R$ we obtain $n_R(R/I) \geqq 3/2$. A more careful study shows that $n_S(S/IS) \leqq 3/2$, where $S = K[[x_0, \dots, x_4]]$. Since $n_R(R/I) \leqq n_S(S/IS)$ we have $4/3 = \tilde{n}_R(R/I) < n_R(R/I) = 3/2$. This completes the proof of our claim.

We want to conclude with one of possible conjectures.

Conjecture. Let A be a local ring or a graded K-algebra. Then $n_A(M) < \infty$ for every quasi-unmixed (graded) A-module M.

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