MAPS BETWEEN $2N$-MANIFOLDS OF NONZERO DEGREE

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§1. Introduction and Results.

In this paper all $2n$-manifolds are closed and orientable, and all matrices are integer matrices.

Let M be a $2n$-dimensional closed oriented manifold and let $\bar{H}^n(M, Z)$ be the free part of $H^n(M; Z)$. Then the cup product operator

$$\bar{H}^n(M;Z) \otimes \bar{H}^n(M;Z) \to H^{2n}(M;Z), \tag{1.1}$$

defines a bilinear unimodule form over $\bar{H}^n(M, Z)$, which is denoted by X_M, where $a \cup a' = X(a, a')[M]$ and $[M]$ is the fundamental class of $H^{2n}(M, Z)$.

Let $\alpha = (a_1, ..., a_m)$ be a basis for $\bar{H}^n(M)$. Then X_M determines an m by m matrix $A = (a_{ij})$, where a_{ij} is given by $a_i \cup a_j = a_{ij}[M]$. We can also present this formula as

$$\alpha^{\perp} \cup \alpha = A[M], \tag{1.2}$$

where $\alpha^{\perp}$ is the transpose matrix of α.

Let $\alpha' = (a'_1, ..., a'_m) \subset \bar{H}_n(M, Z)$ be the Poincare dual basis of α, then

$$\alpha'^{\perp} \cap \alpha' = A, \tag{1.3}$$

Below we will call X_M the intersection form of M, and A the intersection matrix of M (under basis α). Note that those forms (matrices) is symmetry if n is even and is anti-symmetry if n is odd. When n is odd, the Betti number of middle dimensional cohomology must be even.

Say the intersection form $X(M)$ is even if for any $x \in \bar{H}^2(M; Z)$, $x \cup x = c[M]$, where c is an even integer. Equivalently, $X(M)$ is even if under the basis $\alpha = (a_1, ..., a_m)$ of $\bar{H}^2(M; Z)$, $a_i \cup a_i = c_i[M]$, c_i is even for all i.

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Let H be a subgroup of $\bar{H}^n(M,Z)$, then X is defined on H, simply denoted as X_H. Say X_M is a direct sum of X_H and $X_{H'}$, denoted as $X_M = X_H \oplus X_{H'}$, if $\bar{H}^n(M,Z) = H \oplus H'$ and $X(a,a') = 0$ for any $a \in H$ and $a' \in H'$.

For a closed oriented $2n$-manifold L with basis $\beta = \{b_1, ..., b_l\} \subset \bar{H}^n(L;Z)$, similarly we have the intersection form X_L and the intersection matrix B.

For any map $f : M \to L$, there is an m by l integer matrix P such that $f^*(\beta) = \alpha P$.

Let $\alpha_* \in \bar{H}_n(M,Z)$ and $\beta_* \in \bar{H}_n(L,Z)$ be the algebraic dual basis of α and β respectively. Then

$$f^*(\beta) = \alpha P \text{ if and only if } f_*(\alpha_*^{\perp}) = P\beta_*^{\perp}. \tag{1.4}$$

where f^* and f_* are be the induced homomorphism on H^n and H_n respectively, and $f_*(\alpha_*^{\perp}) = P\beta_*^{\perp}$ is the abbreviation of

$$\begin{pmatrix} f_*(a_{*1}) \\ \vdots \\ f_*(b_{*n}) \end{pmatrix} = P \begin{pmatrix} b_{*1} \\ \vdots \\ b_{*l} \end{pmatrix} \tag{1.5}$$

and so on.

When $f^*(\beta) = \alpha P$ (or $f_*(\alpha_*^{\perp}) = P\beta_*^{\perp}$), we say P is realized by a map $f : M \to L$.

Below we assume that the basis α for $\bar{H}^n(M,Z)$ and β for $\bar{H}^n(L,Z)$ have been chosen, therefore the intersection matrices of M and L are given.

Our main result is Theorem 1, which gives a necessary and sufficient conditions for the existence of a degree k map between $(n-1)$-connected 2n-manifolds, $n > 1$, will be presented in Section 3.

Some part of the necessary condition of Theorem 1 is true for all maps between any 2n-manifolds. This is the following

Theorem 2. *Suppose M and L are two closed oriented $2n$-manifolds with intersection matrices A and B respectively, $n > 1$. If P is realized by a map of $f : M \to L$ of degree k, then*

$$P^{\perp}AP = kB. \tag{1.6}$$

Moreover if $k = 1$, then

$$X_M = X_H \oplus X_{H'}, \tag{1.7}$$

where X_H is isomorphic to X_N.

In dimension 4, the necessary any sufficient condition for the existence of degree k map in Theorem 1 can be extend to the maps from any 4-manifolds to simply connected 4-manifolds. The criterion for degree one maps is very simple.

Theorem 3. *Suppose M and L are two closed oriented 4-manifolds with the intersection matrix of A and B respectively. If L is simply connected, then there is a map $f : M \to L$ of degree k such that $f^*(\beta) = \alpha P$ if and only if*

$$P^{\perp} AP = kB \tag{1.8}$$

Moreover there is a map $f : M \to L$ of degree 1 if and only if

$$X_M = X_H \oplus X_{H'}, \tag{1.9}$$

where X_H is isomorphic to X_N.

Note that $-X_M = X_{-M}$. In the statements above and below, if we replace "degree 1," "X_H" and "L" by "degree -1", "$-X_H$" and "$-L$" simultaneously, then those statement are still true and the proofs are just routine replacements.

Now we will give sequences of corollaries and examples of Theorems 1, 2 and 3.

It will be interseting to compare those results with the situation of $n = 1$, the case of surfaces (see [Ed] for example).

First we have a look of maps between simply connected 4-manifolds.

Corollary 1. *Let $f : M \to L$ be a map between two simply connected oriented 4-manifolds.*

(1) If f is of degree one, then M has the homotopy type of $L\#Q$, where Q is a simply connected 4-manifold; moreover Q is unique and M is homeomrphic to $L\#Q$ when $X(M)$ is even.

(2) If f is of non-zero degree, then $\chi(M) \geq \chi(L)$; if f is of degree one and $\chi(M) = \chi(L)$, then M and L are homotopic (or homeomorphc if $X(M)$ is even).

The first part of (2) of Corollary 1 can be easily generalized to the following

Corollary 1*. *Suppose there is a map $f : M \to L$ between two closed orientable 2n-manifolds of vanishing odd dimension Betti number, $n > 1$. Then $\chi(M) \geq \chi(L)$.*

Before to state the next corollary, we introduce the following definition.

For any two closed oriented n-manifolds M and L, define

$$D(M, L) = \{degf | f : M \to L\}, \tag{1.10}$$

i.e., $D(M, L)$ is all degrees of maps from M to L. Let $Z(2) = \{0, 1, 4, 9, 16, ...\}$, the union of all perfect squares, and $2Z$ be the all even numbers.

Corollary 2. *Suppose L be a $(n-1)$-connected $2n$-manifold, $n > 1$.*

(1) $D(L, L)$ is an infinite set, in particular M admits a self map of degree larger than 1. Moreover If L is a closed oriented simply connected 4-manifold, then $Z(2) \subset D(L, L)$.

(2) If L is a closed oriented (n-1)-connected 2n-manifold and M is an closed oriented $2n$-manifold, Then $D(M, L)$ is either $\{0\}$, or an infinite set.

Corollary 3. *Suppose M is a closed oriented $2n$-manifold and the Betti number $\beta_n(M) = m$ is odd, Then for any map $f : M \to M$, the degree k of f is a perfect square. In particular there is no orientation reversing homeomorphism on M.*

As a corollary of Corollaries 3 and 4, we have that if M is a simply connected 4-manifold with odd β_2, then $D(M, M)$ are exactly all perfect squares. We noticed that in [Ba], it was proved $D(M, M)$ are eaxcly all perfect cubes for certain 6-manifolds which have algebraic topology close to CP^3.

Corollary 4. *Let $f : M \to L$ be a map of degree k between closed oriented 4-manifolds. If A is even and B is not, then k must be even.*

Suppose M and L are smooth. Let w_i be the i-th Stiefel-Whitney class of M. Since M is orientable, $w_1 = 0$. Then $w_2 = 0$ implies X_M is an even form (or M is spin) (see [p. 27, K]. Then the following proposition generalizes Corollary 4.

Proposition 4^***.** *If $f : M \to L$ is a map of odd degree between smooth manifolds of dimension n. Let w_i and w_i' be the $i-$th Stiefel-Whitney classes of M and L respectively. Then $w_i = 0$, $i = 1, 2, ..., k \le n$, implies that $w_i' = 0$, $i = 1, 2, ..., k \le n$.*

The following definitions for 3-manifolds are given in [BW] and [LWZ]. They can be defined for any dimension n.

Definition: Let M, L be two n-dimensional oriented manifolds. Say M (k-)domi -nates L if there is a map $f : M \to L$ of non-zero degree (degree k), denoted by $M > N$ ($M >_k L$). Say M is minimal, if $M >_1 L$ implies that either $M = S^4$ or $M = N$.

Corollary 5. *The homotopy type of $(n-1)$-connected $2n$-manifold domimated by any given closed orientable $2n$-manifold M is finite. In particular each closed orientable 4-manifolds dominates at most finitely simply connected 4-manifolds.*

Now we will apply the results above to some simple examples.

Examples.

Let F_g, CP^2, T^4 and K be the surface of genus g, complex projective plane, 4-torus and Kummer surface respectively.

(1) Any closed orientable 4-manifold with non-trivial intersection form dominates CP^2.

(2) If $P_{m\times 1}$ is realized by a map $f : M \to CP^2$, then any map realizes P is homotopic to f. If $P_{m\times 2}$ is realized by a map $f : M \to S^2 \times S^2$ and M is simply connected, then all maps realize P have 4 homotopy classes. There are infinitely many homotopy classes of maps $S^2 \times S^2 \# S^3 \times S^1 \to S^2 \times S^2$ realizing I_2.

(3) $D(CP^2\#CP^2, S^2\times S^2) = D(S^2\times S^2, CP^2\#CP^2) = D(CP^2\#(-CP^2), CP^2\# CP^2) = D(CP^2\#CP^2, CP^2\#(-CP^2)) = \{0\}$ and $D(CP^2\#(-CP^2), S^2\times S^2) = D(S^2\times S^2, CP^2\#(-CP^2)) = 2Z$.

(4) $D(T^4, \#_3 S^2\times S^2) = Z$. That is for any k, there is a map of degree k from 4-torus to the connected sum of 3 copies of $S^2\times S^2$. In general $D(F_s\times F_r, \#_q S^2\times S^2) = Z$ if $q\le 2rs+1$. $D(F_s\times F_r, (\#_p CP^2))\#\#_q(-CP^2) = 2Z$ if $p,q\le 2rs+1$.

(5) The degree of any self-map on $K\#CP^2$ must be a perfect square. CP^2, $S^2\times S^2$, $M(E_8)$ are minimal.

The paper is organized as follow: In section 2 we will discuss related facts about Thom-Pontrjagin constructions and the prepresentations of pre-n-spaces, which will be uesd to present and to prove Theorem 1 in Section 3. Theorem 2 and Theorem 3 are proved in Section 4 and Section 5 respectively, the corollaries are proved in Section 6 and the examples are verified in Section 7.

§2. Thom-Pontrjagin Construction and presentation of pre-n-space.

Let $f: S^{2n-1}\to S^n$ be any map with $y\in S^n$ a regular value. Then $W=f^{-1}(y)\subset S^{2n-1}$ is a (n-1)-manifold and $\tau = f^*(T_yS^n)$, the pull-back of the tangent bundle of S^n at y, gives framed normal bundle of W in S^{2n-1}. Call the pair (W,τ) a manifold in S^{2n-1} with given framed normal bundle.

If both S^{2n-1} and S^n are oriented, then both W and the normal framed bundle are naturally oriented as follow: The normal space at each point of W has the orientation lifted from the tangent space at $y\in S^n$, and W is oriented so that the orientations of W it tangent bundle give the orientation of S^{2n-1}.

On the other hand, for any (W,τ), the ϵ-neighborhood W_ϵ of W in S^{2n-1} has a product structure $D^n\times W$ induced by the given framing. Let $p: D^n\times W\to D^n$ be the projection. $q: D\to D/\partial D = S^n$ be the quotient map, then we can extend $q\circ p: W_\epsilon\to S^n$ to a map $f: S^{2n-1}\to S^n$ by sending the remaining part of S^{2n-1} to the point $q(\partial D)\in S^n$.

Define

$$\mathcal{F}(n) = \{\ (n-1)\text{-manifold } W^{n-1}\subset S^{2n-1} \text{ with framed normal bundle}\}$$

and

$$\mathcal{F}^*(n) = \{\ (n-1)\text{-spheres } S^{n-1}\subset S^{2n-1} \text{ with framed normal bundle}\}$$

Note the underlying space of a element in $\mathcal{F}(n)$ or $\mathcal{F}^*(N)$ are not necessary to be connected.

Let $\Omega(n)$ and $\Omega^*(n)$ be the bordism classes of $\mathcal{F}(n)$ and $\mathcal{F}^*(n)$ respectively. For each $s \in \mathcal{F}(n)$, its bordism class is denoted by $[s]$.

Theorem 2.1. *The construction (Thom-Ponjaging) above gives an 1-1 and onto map $T : \Omega(n) \to \pi_{2n-1}(S^n)$.*

For a detailed proof of this Theorem, see [p.209 DFN].

In such a codimension, $\Omega(n)$ is not a group. For any two disjoint $s_1, s_2 \in \mathcal{F}(n)$, let $lk(s_1, s_2)$ be the linking number of those two $(n-1)$-manifolds in S^{2n-1} carrying s_1 and s_2, and $s_1 \sqcup s_2$ be the disjoint union of s_1 and s_2, which is again in $\mathcal{F}(n)$. One can verify directly $T([s_1 \sqcup s_2]) = T([s_1]) + T([s_2])$ if $lk(s_1, s_2) = 0$. In general we have

Theorem 2.2. *For $s_1, s_2 \in \mathcal{F}(n)$, we have*

$$T([s_1 \sqcup s_2]) = T([s_1]) + T([s_2]) + lk(s_1, s_2)[s^n, s^n], \tag{2.1}$$

where s^n is a generator of $\pi_n(S^n)$ and $[s^n, s^n] \subset \pi_{2n-1}(S^n)$ is the Whitehead product.

Proof. Let $\Delta(n)$ be the diffeotopy classes of $\mathcal{F}^*(n)$. For each $s \in \mathcal{F}^*(n)$, its diffeotopy class is denoted by $[s]'$. Then there is a natural surjection $S : \Delta(n) \to \Omega^*(n)$ defined by $S([s]') = [s]$.

For $s_1, s_2 \in \mathcal{F}^*(n)$, by [Lemma 2 (6), Wa]. there is a map $\alpha : \Delta(n) \to \pi_{n-1}(SO_n)$ and we have

$$\alpha([s_1 \sqcup s_2]') = \alpha([s_1]') + \alpha([s_2]') + lk(s_1, s_2)(\partial s^n). \tag{2.2}$$

where s^n is the generator of $\pi_n(S^n)$ and $\partial : \pi_n(S^n) \to \pi_{n-1}(SO_n)$ is the boundary homomorphism for the homotopy exact sequence of the fibration $S^n = SO_{n+1}/SO_n$. (Note in [Wa] the domain of α is H_n, but it is defined more directly on $\mathcal{F}^*(n)$, and the intersection number if (6) of [Wa] is precisely the linking number in (2.2). Compare with [pp 164-165, Wa]).

Now apply a classical homomorphism $J : \pi_{n-1}(SO_n) \to \pi_{2n-1}(S^{2n-1})$ to the both side of (2.2). Since $J\partial s_n = [s_n, s_n]$ ([p.167, Wa]) We have

$$J \circ \alpha([s_1 \sqcup s_2']) = J \circ \alpha([s_1]') + J \circ \alpha([s_2]') + lk(s_1, s_2)[s^n, s^n]. \tag{2.4}$$

Since $J \circ \alpha = T \circ S$ and $S([s_i]') = [s_i]$, we have

$$T([s_1 \sqcup s_2]) = T([s_1]) + T([s_2]) + lk(s_1, s_2)[s^n, s^n]. \tag{2.5}$$

and the theorem is proved for elements in $\mathcal{F}^*(n)$.

Now for any s_1 and s_2 in $\mathcal{F}(n)$, it is not hard to see that we can find $s_1', s_1", s_2', s_2" \in \mathcal{F}(n)$ such that

(2.6) $s_1", s_2" \in \mathcal{F}^*(n)$.

(2.7) $[s_1] = [s_1' \sqcup s_1"]$ and $[s_2] = [s_2' \sqcup s_2"]$.

(2.8) $[s_1 \sqcup s_2] = [s_1' \sqcup s_1" \sqcup s_2' \sqcup s_2"]$.

(2.9) $lk(s_i', s_j') = lk(s_i', s_j") = lk(s_i', s_i") = 0$ and $lk(s_i", s_j") = lk(s_i, s_j)$.

Then

$$T([s_1 \sqcup s_2]) = T([s_1' \sqcup s_1" \sqcup s_2' \sqcup s_2"]) = T([s_1']) + T([s_1" \sqcup s_2"]) + T([s_2'])$$

$$= T([s_1']) + T([s_1"]) + T([s_2"]) + T([s_2']) + lk(s_1", s_2")[s^n, s^n]$$

$$= T([s_1' \sqcup s_1"]) + T([s_2' \sqcup s_2"]) + lk(s_1, s_2)[s^n, s^n]$$

$$= T([s_1]) + T([s_2]) + lk(s_1, s_2)[s^n, s^n]$$

Those five equalities follow by (2.8), the first half of (2.9), (2.6) and (2.5), (2.9), (2.7) in order. $\square$

Call M is a pre-n-space, if M has the homotopy type of a space obtained from $V_{i=1}^m S_i^n$, the m-fold wedge of n-spheres with one point y in common, by attaching $2n$-cell D^{2n} via attaching map

$$g : S^{2n-1} = \partial D^{2n} \to V_{i=1}^m S_i^n. \tag{2.10}$$

So the homotopy type of M is determined by $g \in \pi_{2n-1}(V_{i=1}^m S_i^n)$. For short denote this space by $V_{i=1}^m S_i^n \cup_g D^{2n}$.

Suppose both S_i^n and D^{2n} are oriented. Let $y_i \in S_i^n$ be a point (other than base point y). Make g transverse to y_i for each $i = 1, ..., m$ so that $K_i = g^{-1}(y_i)$ is an oriented (n-1)-manifold with framed normal bundle in S^{2n-1}. The orientation of K_i is given by the orientation of $\partial D^{2n} = S^{2n-1}$ and S_i^n, and framed normal bundle of K_i is given by Thom-Pontragin construction, i.e., the pull back of the tangent bundle at y_i in S_i^n. The oriented framed (n-1)-manifolds $K = \{K_1, ..., K_m\}$ gives a linking matrix $A = (a_{vw})$, where $a_{vw} = lk(K_v, K_w)$, the linking number of K_v and K_w. The manifold K_i with the given framed normal bundle gives an element $t_i \in \pi_{2n-1}(S^n)$.

So we have a set of invariants $(A; t_1, ..., t_m)$ of M. Denote

$$\mathcal{I}(m, n) = \{(A_{m\times m}; t_1, ..., t_m \in \pi_{2n-1}(S^n)\}, \tag{2.11}$$

where A is a symmetric matrix if n is even and is anti-symmetric matrix if n is odd.

Note when a pre-n-space has the homotopy type of a closed orientable manifold, its linking matrix is A unimodule.

Theorem 2.3 ([Wall p. 182] and [p.26, Ki]). *There is a 1-1 correspondence between $H : \pi_{2n-1}(V_{i=1}^m S_i^n) \to \mathcal{I}(m,n)$ Hence $(A; t_1, ..., t_m)$ is a complete set of homotopy type invariant of the pre-n-space M.*

Proof. The follwoing classsical facts, will be also used later, explain that Theorem 2.3 is the same as the statement of fact in [p. 128, Wall], and coinside with fact in [p.26, Ki] when $n - 2$. □

Lemma 2.4.
(1) $\pi_{2n-1}(S^n)$ is isomorphic to $< \nu > \oplus G$ if n is even and isomorphic to G if n is odd, where G is a finite abelian group, and ν is an element of infinite order. Moreover $\pi_3(S^2) = Z$.
(2) Suppose n is even and $t \in \pi_{2n-1}(S^n)$, then $t = eH(t)\nu + \mu$, where $H : \pi_{2n-1}(S^n) \to < \nu >= Z$ is the Hopf map, which is the self-linking number defined by the map $t : S^{2n-1} \to S^n$ and $e = \frac{1}{H(\nu)}$. In particular, for each t_v in Theorem 2.3, $H(t_v) = a_{vv}$.
(3) Also $H([s^n, s^n]) = 2$ for $[s^n, s^n]$ with suitable chosen orientations in the proof of Theorem 2.2.

Proof. (1) and (2) can be found in the Chapter XI of [Hu], and (3) ha been indicated in [p.167, Wall]. □

§3. Theorem 1 and its proof.

Suppose M and L are two $(n-1)$-connected $2n$-manifolds with representation

$$M = V_{i=1}^m S_i^n \cup_g D^{2n} \qquad L = V_{j=1}^l S_j^n \cup_h D^{2n}$$

where $g \in \pi_{2n-1}(V_{i=1}^m S_i^n)$ is corresponding to $(A; t_1, ..., t_m) \in \mathcal{I}(m,n)$ and $h \in \pi_{2n-1}(V_{j=1}^l S_i^n)$ corresponding to $(B; u_1, ..., u_l) \in \mathcal{I}(l,n)$ by Theorem 2.3.

Let $a_{i*} = [S_i^n] \in H_n(M, Z)$ and $b_{j*} = [S_j^n] \in H_n(L, Z)$. Then $\alpha_* = (a_{*1}, ..., a_{*m})$ and $\beta_* = \{b_{1*}, ..., b_{l*}\}$ form basis of $H_n(M, Z)$ and $H_n(L, Z)$ respectively.

Theorem 1. *Suppose M and L are two closed oriented $(n-1)$-connected $2n$-mani -folds with detas as above and $P = (p_{ij}) \in \mathcal{M}_{m \times l}$. Then there is a map $f : M \to L$ of degree k such that $f_*(\alpha_*^\perp) = P\beta_*^\perp$ if and only if P satisfies the following equations*

$$P^\perp AP = kB \tag{3.1}$$

and

$$ku_r = \sum_v p_{vr} t_v + (\sum_v \frac{1}{2} p_{vr}(p_{vr} - 1)a_{vv} + \sum_{v<w} p_{vr} p_{wr} a_{vw})[s_n, s_n] \tag{3.2}$$

Moreover the homotopy classes of maps $f : M \to L$ of such that $f^*(\beta) = \alpha P$ is 1-1 corresponding to the elements of $\pi_{2n}(L)$. In particular there are only finitely many homotopy classes of maps realizes P when $n > 2$.

Proposition 3.1. *Let $f : V_{i=1}^m S_i^n \to V_{j=1}^n S_j^n$ be map such that $f_*\alpha_*^\perp = P\beta_*^\perp$ on $H_n = \pi_n$. Then for any element $g \in \pi_{2n-1}(V_{i=1}^m S_i^n)$ corresponding to $(A; t_1, ..., t_m) \in \mathcal{I}(m, n)$, the element $f_*(g) \in \pi_{2n-1}(V_{j=1}^n S_j^n)$ is corresponding to $(A'; t'_1, ..., t'_l) \in \mathcal{I}(l, n)$, where*

$$A' = P^\perp A P \tag{3.3}$$

and

$$t'_r = \sum_v p_{vr} t_v + (\sum_v \frac{1}{2} p_{vr}(p_{vr} - 1)a_{vv} + \sum_{v<w} p_{vr} p_{wr} a_{vw})[s_n, s_n] \tag{3.4}$$

Proof. Call a map from disc D^n to a n-sphere S^n with base point z is a boundary pinch if the interior of D is mapped onto $S^n - z$ homeomorphically and ∂D is mapped onto z. If both D and S^n are oriented, We can say a boundary pinch is degree 1 or degree -1.

Recall $P = (p_{ij})_{m\times l}$, then $P^\perp = (p_{ji})_{l\times m}$.

For each i, take $\sum_j |p_{ji}|$ disjoint n-disc in S_i^n away from the base point y_i. Each n-disk has the induced orientation of S_j^n. Then map $|p_{ji}|$ discs to S_j^n via boundary pinches of degree 1 if p_{ji} is positive and -1 otherwise. Doing this for all i, any finally map the complement of all those $\sum_{j,i} |p_{ji}|$ discs to the base point z, we get a map $f' : V_{i=1}^m S_i^n \to V_{j=1}^n S_j^n$ and clearly $f_*\alpha_*^\perp = P\beta_*^\perp$ on $H_n = \pi_n$. Up to homotopy, we may assume that $f = f'$.

Note $f^{-1}(z_j)$ consists $\sum_i |p_{ji}|$ points, and $|p_{ji}|$ points of them lie in S_i^n with sign 1 if p_{ji} is positive and sign -1 otherwise. Suppose g is tranversal to $V_i S_i^n - y$ so that the oriented (n-1)-manifolds $K_i = g^{-1}(y_i)$ and its lifted framing normal bundle provides an element $s_i \in \Omega(n)$ such that $A = (lk(K_i, K_j))$ and $t_i = T(s_i)$ provide the set of invariant $(A, t_1, ..., t_m)$. Than $f \circ g$ is transversal to $V_j S_j^n - z$. Now

$$(f \circ g)^{-1}(z_r) = g^{-1}(f^{-1}(z_r)) = \sum_{p=1}^{m} \sum_{e=1}^{p_{vr}} K_v^e, \tag{3.5}$$

where K_v^e is the preimage of a point in $f^{-1}(z_r) \cap S_v$ under g. Let $s_v^e \in \mathcal{F}(n)$ be the K_v^e with its lifted framing normal bundle.

By the definition we have

$$A' = (a_{rs}) = lk((f \circ g)^{-1}(z_r), (f \circ g)^{-1}(z_s)) \tag{3.6}$$

and

$$t'_r = T(\sqcup_{p=1}^{m} \sqcup_{e=1}^{p_{vr}} s_v^e) \tag{3.7}$$

Then

$$lk(s_v^e, s_w^{e'}) = lk(K_v^e, K_w^{e'}) = lk(K_v, K_w) = a_{vw} \,\text{and}\, T([s_v^e]) = T([s_v]) = t_v \tag{3.8}$$

For simplicity we will denote $\sum_{e=1}^{p_{vr}} s_v^e$ by $p_{vr}s_v$ and $\sum_{e=1}^{p_{vr}} K_v^e$ by $p_{vr}K_v$.
Then

$$lk((f\circ g)^{-1}(z_r), (f\circ g)^{-1}(z_s)) = lk(\sum_v p_{vr}K_v, \sum_w p_{ws}K_w)$$

$$= \sum_{p,q} p_{vr}p_{ws}lk(K_v, K_w) = \sum_{p,q} p_{vr}p_{ws}a_{vw} \tag{3.9}$$

Combine (3.6) and (3.9), we get (3.3).
Applying Theorem 2.2 to $T(p_{vr}s_v)$ we have

$$T(p_{vr}s_v) = p_{vr}T(s_v) + \frac{1}{2}p_{vr}(p_{vr}-1)lk(s_v, s_v)[s^n, s^n] \tag{3.10}$$

Apply Theorem 2.2 to $T(\sum_v p_{vr}s_v)$, and by (3.10) then
We have

$$T(\sum_v p_{vr}s_v) = \sum_v T(p_{vr}s_v) + \sum_{p<q} p_{vr}p_{qr}lk(s_v, s_w)[s^n, s^n]$$

$$= \sum_v p_{vr}T(s_v) + (\sum_v \frac{1}{2}p_{vr}(p_{vr}-1)a_{vv} + \sum_{v<w} p_{vr}p_{wr}a_{vw})[s^n, s^n] \tag{3.11}$$

(3.7) and (3.11) gives (3.4). Lemma 3.3 is proved. □

Proof of Theorem 1.
(1) The Sufficiant Part.
Let $J = (J_1, ..., J_l) \subset B \subset S^{2n-1}$ be the (n-1)-manifold provided by $h : S^{2n-1} \to V_{j=1}^{l} S_j^n$ and B is a (2n-1)-ball. Let $p_k : S^{2n-1} \to S^{2n-1}$ is a branched covering over a standard sphere $S^{2n-3} \subset S^{2n-1}$ which misses B. Then $p_k^{-1}(J) = (J^1, ..., J^k)$, each J^i lies in a homeomorphic lift B_i of B. Then

$$lk(J_v^i, J_w^i) = lk(J_v, J_w) \text{ and } lk(J_v^i, J_w^j) = 0 \text{ if } i \neq j \tag{3.12}$$

Now consider the composition

$$S^{2n-1} \xrightarrow{p_k} S^{2n-1} \xrightarrow{h} V_j S_j^n, \tag{3.13}$$

by (3.12), the discussion above and Theorem 2.2, it is easy to verify that the element in $\mathcal{I}(l, n)$ corresponding to $[h \circ p_k] = k[h] \in \pi_{2n-1}(V_{j=1}^l S_j^n)$ is $(kB; ku_1, ..., ku_l)$.

Then by (3.3) and (3.4) in Lemma 3.3, we have

$$[f \circ g] = k[h] = [h \circ p_k]. \tag{3.14}$$

Since and the composition

$$S^{2n-1} \xrightarrow{g} V_i S_i^n \xrightarrow{f} V_j S_j^n \xrightarrow{i} V_j S_j^n \cup_h D^{2n} \tag{3.15}$$

is homotopic to the composition

$$S^{2n-1} \xrightarrow{p_k} S^{2n-1} \xrightarrow{h} V_j S_j^n \xrightarrow{i} V_j S_j^n \cup_h D^{2n} \tag{3.16}$$

by the (3.14), where i is the inclusion. Now extending the map in (3.16) by coning $p_k : S^{2n-1} \to S^{2n-1}$ to $D^{2n} \to D^{2n}$, to a map, stiil denited by f,

$$M = V_{i=1}^m S_i^n \cup_g D^{2n} \to N = V_{j=1}^n S_j^n \cup_h D^{2n}. \tag{3.17}$$

Clearly f is degree k. and $f_*(\alpha^\perp) = P\beta^\perp$. □

(2) The 1-1 correspondence:

Suppose P is realized by a map f' from M to L, that is $f'_*(\alpha_*^\perp) = P\beta_*^\perp$. Since M and L are (n-1)-connected 2-manifolds, we may assume that up to homotopy that

$$f', f : M = V_{i=1}^m S_i^n \cup_g D^4 \to N = V_{j=1}^n S_j^n \cup_h D^4.$$

only differ only the interior of D^{2n}, where f is the map of degree k constructed in (3.17).

Now restriction of f and f' on D^{2n} give a map $S^{2n} \to L$, which is an element in $\pi_{2n}(L)$, denote as $C(f, f')$. Clearly f and f' are homotopic if and only if $C(f, f') = 0$, any for any $\tau \in \pi_{2n}(L)$, we can construct an f' such that $C(f, f') = amstexdw3\tau$

(present D^{2n} as $S^{2n}\#D^{2n}$ then present f' as $\tau\#f$). This prove that the homotopy classes of maps realizing P is 1-1 corresponding to the elements in $\pi_{2n}(L)$.

The finiteness of $\pi_{2n}(L)$, $n > 2$, is known (see [p. 45, Ka] for example) It follows from the Hilton's formular

$$\pi_{2n}(V_j S_j^n) = l\pi_{2n}(S^n) \oplus \frac{1}{2}l(l-1)\pi_{2n}(S^{2n-1}) \tag{3.18}$$

and the finiteness of groups in the rightside of (3.18). □

(3) The necessary Part.

Suppose there is a map $f' : M \to L$ such that $f'_*(\alpha_*^{\perp}) = P\beta_*^{\perp}$. Then up to homotopy, we may assume that $f'(V_i S_i^n) \subset V_j S_j^n)$ and the restriction on $V_i S_i^n$ is the same as the map f given in the proof of Lemma 3.1. Then by Lemma 3.1, the complete set of homotopy invariants $< A'; t'_1, ..., t_l >$ assocaiated with the attaching map $g \circ f'$ is given by (3.3) and (3.4).

Since the kernal of

$$i_* : \pi_{2n-1}(V_j S_j^n) \to \pi_{2n-1}(V_j S_j^n \cup_h D^{2n}) = \pi_{2n-1}(V_j S_j^n)/[h] \tag{3.19}$$

induced by the inclusion $i : V_j S_j^n \to V_j S_j^n \cup_h D^{2n}$ is the cyclic group $< [h] >$, and $[i \circ f \circ g] = 0$, we have $[f \circ g] = [dh]$

Then one can verify d must be k. □

Since Theorem 2 and 3 is stated in terms of cohomology, we need translate the sufficient part of Theorem 1 as below:

Theorem 1*. *Suppose M and L are two oriented $(n-1)$-connected (2n)-manifolds and $P \in \mathcal{M}_{m\times l}$ satisfying the equation (3.1) and (3.2) in Proposition 3.2, where A and B are the intersection matrix of M and L under some basis α for $\bar{H}^n(M, Z)$ and β for $\bar{H}^n(L, Z)$. Then there is a map $f : M \to L$ of degree k such that $f^*(\beta) = \alpha P$.*

Proof. Let $\alpha_* \in H_n(M, Z)$ and $\beta_* \in H^n(L, Z)$ be the algebraic dual basis of α and β respectively.

For $\alpha_* = (a_{1*}, ..., a_{n*}) \in H_n(M, Z)$, there is a presentation $V_{i=1}^m S_i^n \cup_g D^{2n}$ for M (contained in [p.166, Wa]), where $g \in \pi_{2n-1}(V_{i=1}^m S_i^n)$ such that $a_{i*} = [S_i^n] \in H_n(M, Z)$ and the linking matrix A' provided by g.

The following Lemma is known.

Lemma 3.2. $A' = A$, (and similarly $B' = B$).

Proof. Now we can further homotope g so that $K_i = g^{-1}(y^i) \subset S^{2n-1}$ is connected. Let $F'_i \subset D^{2n}$ be an oriented n-chain bounded by K_i. Let F_i be the cycle in $M = V_{i=1}^m S_i^n \cup_g D^{2n}$, defined by F'_i/K_i, where $g(K_i) = y_i$. Then

(a) $S_i \cap F_j$ is 1 if $i = j$ and is 0 if $i \neq j$,

(b) the intersection matrix of $H_n(M,Z)$ under $F_1,...,F_m$ is the linking matrix A' provided by g.

The fact (a) implies that α is the Poincare dual basis of $(F_1,...,F_m)$. Therefore the intersection matrix of $H^n(M,Z)$ under the basis α is the same as the intersection matrix of $H_n(M,Z)$ under the basis $(F_1,...,F_m)$, Hence $A' = A$. □

Now by Theorem 1, P can be realied by $f : M \to N$ such that $f_*(\alpha_*^\perp) = P\beta_*^\perp$. By (1.4), this is equivalent to $f^*(\beta) = \alpha P$ □

§4. Proof of Theorem 2.

Recall that $\alpha = (a_1,...,a_m)$ and $\beta = (b_1,...,b_l)$ are basis of $\bar{H}^2(M,Z)$ and $\bar{H}^2(L,Z)$ respectively. We also have

$$\alpha^\perp \cup \alpha = A[M] = (a_{ij})[M], \qquad \beta^\perp \cup \beta = B[N] = (b_{ij})[N], \qquad f^*(\beta) = \alpha P. \tag{4.1}$$

Suppose $f : M \to N$ is of degree k, then $f^*([N]) = k[M]$ and

$$kB_l[M] = (b_{ij}k[M]) = (f^*(b_{ij,l}[N])) = (f^*(b_i \cup b_{j,}))$$

$$= (f^*(b_i) \cup f^*(b_j)) = (f^*(\beta))^\perp (f^*\beta) = (\alpha P)^\perp \alpha P \tag{4.2}$$

$$= P^\perp \alpha^\perp \alpha P = P^\perp A P[M].$$

That is

$$P^\perp A P = kB. \tag{4.3}$$

Suppose $f : M \to L$ is of degree one. Then for $f^* : H^2(L,Z) \to H^2(M,Z)$ by [Theorem I.2.5, B], there is $\mu : H^2(M,Z) \to H^2(L,Z)$ such that $\mu \circ f^* : H^2(L,Z) \to H^2(L,Z)$ is the identity on $H^2(L,Z)$. Hence $\mu \circ f^*| : \bar{H}^2(L,Z) \to \bar{H}^2(L,Z)$ is the identity on $\bar{H}^2(L,Z)$. In particular $\bar{H}^2(M,Z) = f^*(\bar{H}^2(L,Z) \oplus \mathrm{ker}\mu|$.

For any $b_i, b_j \in \beta \subset \bar{H}^2(L,Z)$, Since

$$X_M(f^*(b_i), f^*(b_j))[M] = f^*(b_i) \cup f^*(b_j)$$

$$= f^*(b_i \cup b_j) = f^*(b_{ij}[N]) = b_{ij} f^*(N) = b_{ij}[M] \tag{4.4}$$

Hence the restriction of X_M on the subgroup $f^*(\bar{H}^2(L,Z)) \subset \bar{H}^2(M,Z)$ is isomorphic to X_N. In particular it is still unimodular. By orthogonal decomposition lemma [p.5 MH],

$$X_M = X_{f^*\bar{H}^2(L,Z)} \oplus X_{ker(\mu|)} = X_N \oplus X_{ker\mu|}$$

□

§5. Proof of Theorem 3.

Proposition 5.1. *Let M and L be two closed oriented 4-manifolds satisfying*

$$X_M = X_H \oplus X_{H'}, \tag{5.1}$$

where X_H is isomorphic to X_L. If L is simply connected, then there is map $g : M \to L$ of degree 1.

The proof divided into three steps:

(1) First to prove Proposition 5.1 is true in the case M is also simply connected.

This is a corollary of Freedmann's work [Fr]. Suppose $X(M) = X(L) \oplus X(H')$. Clearly $X(H')$ is a symmetric unimodule, therefore C represents the intersection form of a simply connected 4-manifold Q. That is we have $X(M) = X(L)\#X(Q) = X(L\#Q)$. By Whitehead's theorem [Theorem 2.1, K], M and $L\#Q$ are homotopic. then there is a homotopy equivalence $h : M \to L\#Q$, which is degree one. There is also an obvious degree one pinch $p : L\#Q \to L$. Then $p \circ h : M \to L$ is a map of degree one.

(2) Then to prove that if $X(M) = X(L)$ and L is simply connected, then there is a degree one map $f : M \to L$.

Let $\mathcal{F} = \{F_j,\, j = 1, ..., m\}$ be a set of oriented embedded surfaces which provides a basis of $\bar{H}_n(M, Z)$.

If M is simply connected, then M and L have the same homotopy type, hence there is a degree one map $M \to L$.

Suppose M is not simply connected. Let C_i, $i = 1, ..., l$ be a set of disjoint simple closed curves which generate $\pi_1(M)$ and

$$(\cup_i N(C_i)) \cap (\cup_j F_j) = \emptyset. \tag{5.2}$$

where $N(C_i) = C_i \times D^3$ is the regular neighborhood of C_i in M. Let M^* be the simply connected 4-manifolds obtained from M by surgery on $\{c_i\}$. Precisely

$$M^* = (M - \text{int} \sum_i C_i \times D^3) \cup_{\{h_i\}} D_i^2 \times S^2, \tag{5.3}$$

where the gluing map

$$h_i : S_i^1 \times S^2 = \partial D_i^2 \times D^2 \to \partial(C_i \times D^3) = C_i \times S^2 \tag{5.4}$$

is an orientation reversing homeomorphism.

Since $\mathcal{F}$ is linear independent in $H_2(M, Z)$, it is easy to see that $\mathcal{F}$ is still linear independent in $H_2(M^*, Z)$. Hence $\mathcal{F}$ a basis of a submodule H of $H_2(M^*, Z)$ and the restriction of the intersection form $X(M^*)$ on $\mathcal{F}$ is still $X(M)$. In particular the restriction of the form of M^* on $\mathcal{F}$ is still unimodular. By the famous orthogonal decomposition lemma [p.5 MH], We have proved that

$$X_{M^*} = X_H \oplus X_{H'} = X_L \oplus X_{H'} \tag{5.5}$$

In particular,

$$H_2(M^*, Z) = H \oplus H', \tag{5.6}$$

By the part (1) of Proposition 5.1 and its proof, we have that there is a degree one map

$$M^* \xrightarrow{h} L\#Q \xrightarrow{p} L \tag{5.7}$$

where $X(L) = X(H)$, h is a homotopy equivalence and p pinches Q. Let $f = p \circ h$, in particular

$$H' = \mathrm{Ker} f_* : H_2(M^*, Z) = \pi_2(M^*) \to H_2(L, Z) = \pi_2(L) \tag{5.8}$$

Now we will transfer the degree one map in (5.7) to a degree one from M to N.

Pick a point $*_i \in D_i^2$ for each i.

Claim: $*_i \times S^2 \in H'$ for each i:

For simplicity, let $S = *_i \times S^2$. Then we have $S = \sum x_i F_i + y$, where $y \in H'$. By (5.5),

$$< S, F_j >=< \sum x_i F_i, F_j >= a_{ij} x_i. \tag{5.9}$$

On the otherhand since S is disjoint from all F_j, we have

$$\sum a_{ij} x_i = 0, \qquad i = i, ...m, \tag{5.10}$$

Since $A = (a_{ij})$ is unimodule, which is in particular is non-singular, (5.10) has only zero sultion $x_1 = ... = x_m = 0$. This prove the claim.

By (5.8) and the Claim above, $f(*_i \times S^2)$ is null homology, therefore null-homotopy. Now we can homotopy f so that $f(D_i^2 \times S^2) = z$ for each i by first shrinking $D_i^2 \times S^2$ to its core $*_i \times S^2$, then shrinking $*_i \times S^2$ to Z.

Note $M^* - \mathrm{int} \cup_i D_i^2 \times S^2 = M - \mathrm{int} \sum_i C_i \times D^3$. Since $f : M^* \to L$ is degree one, $f(D_i^i \times S^2) = z$ for all i, we have the restriction

$$f| : (M - \text{int} \cup_i C_i \times D^3, \cup_i C_i \times S^2) \to (L, z) \tag{5.11}$$

is map between pairs of degree 1.

Since $f(D_i^2 \times S^2) = z \in L$, we have $f(C_i \times S^2) = z \in L$. Now we can extend $f|$ to $g : M \to L$ by sending $C_i \times D^3$ to z. Since f sent $C_i \times D^3$ to z, the extension $g : M \to N$ is of degree 1.

(3) Prove Proposition 5.1 from (1) and (2).

Let Q be a simply connected 4-manifold with $X(Q) = X(M) = X(L) + C$. Then there is a degree one map from M to Q by (2) and a degree one map from Q to L by (1), hence there is a degree one map form M to L. □

Proof of Theorem 3 from Propositions 5.1 and Theorem 1.* Let M and L given as in Theorem 3 and let Q be a simply connected 4-manifold with $X(Q) = X(M)$. By proposition 5.1, there is a degree one map $f' : M \to L$ which induces isomorphism $f_* : \bar{H}_2(M, Z) \to H_2(Q, Z)$.

Under the basis $f'_*(\alpha_*)$, the intersection matrix of $X(Q)$ is A the matrix for f'_* is the unit matrix of order n.

For the case of simply connected 4-manifolds, since $\pi_3(S^2) = Z$, the equation (3.2) is covered by (3.1). Then by Theorem 1^*, there is a map $f" : Q \to L$ of degree k realizes P. Finally $f = f" \circ f' : M \to L$ is a map of degree k realizes P. □

§6. Proof of the corollaries and examples.

Proof of Corollary 1.

For (1). Suppose there is a degree one map $f : M \to L$. By Theorem 2,

$$X_M = X_L \oplus X_{H'}.$$

By Freedmann's Classification Theorem [F], $X_{H'}$ represents the intersection form of a simply connected 4-manifold L. That is we have

$$X_M = X_L \# X_Q = X_{L\#Q}.$$

By Whitehead's theorem [Theorem 2.1, K], M and $L\#Q$ are homotopic. If the intersection form $X(M)$ is even, then both B and C are even, and therefore the simply connected 4-manifold L with intersection form C is unique up to homeomorphism, and M and $L\#L$ are homeomorphic by Freedmann's Classification Theorem.

For (2). If there is a map $f : M \to L$ of non-zero degree, then $\beta_2(M) \geq \beta_2(L)$ by Theorem 2. Since $\beta_0(M) = \beta_0(L) = \beta_4(M) = \beta_4(L) = 0$, and the odd dimension cohomology groups vanish for simply connected 4-manifolds, we have immediately that $\chi(M) > 0$ and $\chi(L) > 0$ and $\chi(M) \geq \chi(L)$.

In the later case, $\chi(M) = \chi(L)$ implies that $\beta_2(M) = \beta_2(L)$. Since f is of degree one, by (1.6) in Theorem 1, the intersection form of M and L must be isomorphic. Then (2) follows by applying Freedmann's work, as we did in the proof of Corollary 1. □

Proof of Corollary 1^*. By the proof of Corollary 2, we need only to show $\beta_i(M) \geq \beta_i(L)$ for all i. A proof of this fact is as follow.

Recall that Poincare duality $P_M : H^p(M, Z) \to H_{n-p}(M, Z)$ is given by $z^p \to z^p \cap [M]$, where $[M]$ be the fundamental class class of $H_n(M, Z)$ and we also have the formula

$$f_\#(f^\# z^p \cap z_w) = z^p \cap f_\# Z_w$$

for any map $f : M \to L$.

To show $\beta_i(M) \geq \beta_i(L)$, we need only to show that the kernal of $f^\# : H^\#(L, Z) \to H^\#(M, Z)$ must be in the torsion part. Suppose that $f^\#(z^q) = 0$ for some $0 \neq z^q \in H^q(L, Z)$. Then by (1) we have

$$0 = f_\#(f^\# z^p \cap [M]) = z^p \cap d[N] = dz^p \cap [N],$$

where $d \neq 0$ is the degree of f. That is $dz^p = 0$. □

Proof of Corollary 2. (1) Let $(A; t_1, ..., t_m)$ be the complete homotopy invariant of M. Recall that $\pi_{2n-1}(S^n) =< \nu > \oplus G$, where ν is infinite order if n is even and is 0 if n is odd, and G is a finite group. Now $t_v = a_{vv}\nu + \mu_v$, $[s^n, s^n] = 2\nu + \mu$ and where $\mu_v, \mu \in Tor\pi_{2n-1}(S^n)$, and $H : \pi_{2n-1}(S^n) \to Z$ is the Hopf invariant.

Let I_m be the m by m unit matrix. Let $P = kI_m$ for some integer k, we first have

$$P^\perp AP = k^2 A. \tag{6.1}$$

Now the leftside of (3.1) in Theorem 1 can be deduced to

$$\sum_v p_{vr}t_v + (\sum_v \frac{1}{2}p_{vr}(p_{vr} - 1)a_{vv} + \sum_{v<w} p_{vr}p_{wr}a_{vw})[s_n, s_n]$$

$$= kt_v + \frac{1}{2}k(k-1)a_{vv}[s_n, s_n]$$

$$= k(a_{vv}\lambda\nu + \mu_v) + \frac{1}{2}k(k-1)a_{vv}(2\lambda\nu + \mu). \tag{6.2}$$

The first equality of (6.2) is due to the fact $p_{vv} = k$ and $p_{vw} = 0$ if $v \neq w$. The second equality is by Lemma 2.4.

When k be a multiple of $|Tor\pi_{2n-1}(S^n)|$, the last term of (6.2) can be further reduced into

$$k(a_{vv}\lambda\nu + \mu_v) + \frac{1}{2}k(k-1)a_{vv}(2\lambda\nu + \mu)$$

$$= ka_{vv}\lambda\nu + k(k-1)a_{vv}\lambda\nu = k^2 a_{vv}\lambda\nu \tag{6.3}$$

On the otherhand clearly

$$k^2 t_v = k^2(a_{vv}\lambda\nu + \mu_v) = k^2 a_{vv}\lambda\nu. \tag{6.4}$$

(6.1) implies that (3.1) of Theorem 1 hold, and (6.2), (6.3) and (6.4) implies that (3.2) of Theorem 1 hold. By Theorem 1 there is a map of degree k^2 from M to itself realizing P. In the case $n = 2$, we have $\pi_3(S^2) =< \nu >$, and $|Tor\pi_{2n-1}(S^n)| = 1$. We have proved (1).

(2) If $D(M, N)$ is not the $\{0\}$, then there is a map $f : M \to L$ of degree $d \neq 0$. By (1), for some integer k, there is a map $g_k : L \to L$ of degree k^2, then $g_k \circ f : M \to L$ is of degree dk^2. That is $D(M, N)$ is infinite set. □

Proof of Corollary 3. Let P be the matrix realized by f, by (1.6) in Theorem 2 we have

$$P^{\perp}AP = kA, \tag{6.5}$$

where both A, P are square matrices of order $\beta_2(M)$. By taking the determinants we have $|P|^2|A| = k^{\beta_2(M)}|A|$. That is $|P|^2 = k^{\beta_2(M)}$. Since $\beta_2(M)$ is odd, k itself must be a perfect square. In particular $k \neq -1$. □

Proof of Corollary 4. From the equation $PAP^{\perp} = kB$ (1.6) in Theorem 2 we have

$$kb_{ii} = \sum a_{rs}p_{ir}p_{is} = \sum a_{rr}p_{ir}^2 + \sum_{i<s}(a_{rs}p_{ir}p_{is} + a_{sr}p_{is}p_{ir}) \tag{6.6}$$

Since $X(M)$ is even, each a_{rr} is even. Since and $a_{rs} = a_{sr}$, the right side of the equation is even. Since B is not even, b_{ii} is odd for some i. It follows k is even. □

Proof of Proposition 4*. Let v_i and v_i' be the i-th Wu class of M and L respectively, then v_i is characterized by [p. 132, MS]

$$< v_i \cup x, [M] >=< sq^i x, [M] > \tag{6.7}$$

Suppose $v_i' \neq 0$. Then there is $x' \in H_*(M, Z_2)$ such that $< v_i' \cup x', [N] > \neq 0$. Let the degree be $2k+1$ Then we have

$$0 \neq < v_i' \cup x', [N] > = < sq^r x', [N] > = < sq^r x', f_{\#}[M] >$$

$$= < f^{\#} sq^r x', (2k+1)[M] > = < f^{\#} sq^r x', [M] >$$

$$= < sq^r f^{\#}(x'), [M] > = < v_i \cup f^{\#}(x'), [M] > \tag{6.8}$$

It follows $v_i \neq 0$. That is if $v_i = 0$ then $v_i' = 0$.

According to Wu's formula

$$w_i = \sum_{j+l=i} sq^l v_j \qquad i = 1, ..., n.$$

it is easy to see that $w_1 = w_2 = ... = w_k = 0$ if and only if $v_1 = v_2 = ... = v_k = 0$, $k \leq n$. Then the conclusion follows. □

Proof of Corollary 5. If M dominates L, then the rank of the form of L is at most the rank of the form of M by Theorem 2. For the given positive integer m, there are only finitely many distinct isomorphic forms of rank m [p.18 and p.25, MH]. Since the torsion part of $\pi_{2n-1}(S^n)$ is finite, so the homotopy type of L is finite. In the case of 4-manifolds, there are at most two simply connected 4-manifolds with the given form X [Fr]. So the conclusion follows. □

§. 7. Verification of Examples.

Let

$$A_l = \oplus_l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad P_{l,k} = \oplus_l \begin{pmatrix} 0 & k \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad I_l = \oplus_l (1)$$

(1) Suppose M has intersection matrix $A = (a_{ij})_{m \times m}$. For any $a_{ij} \neq 0$, let $c_i = c_j = 1$ and $c_r = 0$ for $r \neq i, j$. Let $C = (c_i)_{m \times 1}$. Then

$$C^{\perp} A C = k(1)$$

Then $k = 2a_{ij}$ if $i \neq j$ and $k = a_{ii}$ otherwise. By Theorem 1, there is a map $f : M \to CP^2$ of degree $k \neq 0$.

(2) By exact sequence of fibration $S^5 \to CP^2$, we have $\pi_4(CP^2) = \pi_4(S^5) = 0$. Moreover $\pi_4(S^2 \times S^2) = \pi_4(S^2) \oplus \pi_4(S^2) = Z_2 \oplus Z_2$. Then the conclusion follows from Theorem 1.

For the remaining Consider the composition

$$S^2 \times S^2 \# S^3 \times S^1 \xrightarrow{p_1} S^2 \times S^2 V S^3 \times S^1 \xrightarrow{p_2} S^2 \times S^2 V S^3 \xrightarrow{g_k} S^2 \times S^2$$

where p_1 pinches the 3-sphere separating $S^2 \times S^2$ and $S^3 \times S^1$ to a point to get the the one point union of $S^2 \times S^2$ and $S^3 \times S^1$, $p_2| : S_3 \times S^1 \to S^3$ is the projection; and $p_2|S^2 \times S^2$ is the identity; $g_k|S^3$ sends S^3 to the first factor of $S^2 \times S^2$ via with Hopf invariant k and $g|S^2 \times S^2 \to S^2 \times S^2$ realizes I_2. Let $f_k = g_k \circ p_2 \circ p_1$, then $\{f_k\}$ forms infinitely many homotopy classes realizing I_2.

(3) Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since

$$P^\perp I_2 P = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}$$

Hence there is no solution P for $P^\perp I_2 P = A_1$ or $P^\perp I_2 P = B$. Since

$$P^\perp B P = \begin{pmatrix} a^2 - b^2 & ac - bd \\ ac - bd & c^2 - d^2 \end{pmatrix}$$

Hence there is no solution P for $P^\perp B P = k I_2$ and $P = \begin{pmatrix} 1 & 1 \\ k & -k \end{pmatrix}$ is a solution for $P^\perp B P = 2k A_1$. Since

$$P^\perp A_1 P = \begin{pmatrix} 2ab & ad + bc \\ ad + bc & 2cd \end{pmatrix}$$

Hence there is no solution P such that $P^\perp A_1 P = k I_2$ and $P = \begin{pmatrix} 1 & k \\ -1 & k \end{pmatrix}$ is a solution $P^\perp A_1 P = 2kB$

Since I_2, B and A_1 are the intersection form of $CP^2 \# CP^2$, $CP^2 \# (-CP^2)$ and $S^2 \times S^2$ respectively, the conclusion for in Example (2) follows from Theorem 3.

(4) Note $T^4 = S^1_1 \times S^1_2 \times S^1_3 \times S^1_4$, Let $a_i \in H^1(T^4, Z)$ be the Poincare dual of S^1_i, Let $c_{ij} = a_i \cup a_j$. Then under the basis $(c_{12}, c_{34}, c_{13}, c_{24}, c_{14}, c_{23})$ of $H^2(T^4, Z)$, the intersection matrix is A_3, which is the same as the matrix of $\#_3 S^2 \times S^2$ under the obvious basis. Then it is easy to verify

$$P^\perp_{l,k} A_3 P_{,k} = k A_3.$$

There is a map $f_k : T^4 \to \#_3 S^2 \times S^2$ of degree k by Theorem 3.

Let $a_1, b_1, ..., a_s, b_s$ be the standard basis for $H_1(F_s, Z)$ and $c_1, d_1, ..., c_r, d_r$ be the standard basis for $H_1(F_r, Z)$. Then we use the same symbol to denote their Poincare dual in $H^1(F_s \times F_g, Z)$. Let under the basis $(a_1 \cup b_1, c_1 \cup d_1, a_1 \cup c_1, b_1 \cup d_1, a_1 \cup d_1, b_1 \cup c_1, ..., a_s \cup c_r, b_s \cup d_r, a_s \cup d_r, b_s \cup c_r, ..., a_s \cup c_1, b_s \cup d_1, a_s \cup d_1, b_s \cup c_1, ..., a_s \cup c_r, b_s \cup d_r, a_s \cup d_r, b_s \cup c_r)$. The matrix of $F_s \times F_g$ is A_{2sr_1}. So for any integer k there is a map $f_k : F_s \times F_r \to \#_{2sr+1} S^2 \times S^2$ of degree k. Since there is degree one pinch $g : \#_{2sr+1} S^2 \times S^2 \to \#_q S^2 \times S^2$ for any $q \leq 2rs + 1$. The conclusion follows.

By last paragraph there is a map $f_k : F_r \times F_s \to \#_{2rs+1} CP^2 \#\#_{2rs+1}(-CP^2)$ of $2k$ for any k, and clearly there is a degree one pinch $\#_{2rs+1} CP^2 \#\#_{2rs+1}(-CP^2) \to (\#_v CP^2)\#\#_w(-CP^2)$ for any $p, q \leq 2rs + 1$.

(5) The first claim follows from Corollary 3 and its proof. The second claim follows that those forms do not split as direct sum [p.26, Ki].

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