Geometrization Conjecture and the Ricci Flow

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Contents

Abstract

The goal of this talk is to state Thurston's Geometrization Conjecture for 3-manifolds and outline the Ricci flow approach to this conjecture following Hamilton and Perelman.

1 Geometrization Conjecture for 3-manifolds.

In dimension 3 TOP=PL=DIFF (Moise), i.e. each topological 3-manifold admits a unique PL/smooth structure. Hence throughout I will be working in the category of differentiable manifolds, assuming for simplicity that all 2- and 3-manifolds are orientable.

Loosely speaking, the goal of the Geometrization Conjecture (GC) is to generalize the classification of surfaces by their genus.

Definition 1. *A* **geometry** *is a simply-connected homogeneous unimodular Riemannian manifold X. Unimodularity means that X admits a discrete group of isometries with compact quotient.*

A lá Felix Klein we will be identifying geometry with its group of isometries.

Definition 2. *A compact manifold M is called* **geometric** *if $int(M) = X/\Gamma$ has finite volume, where X is a geometry and Γ is a discrete group of isometries of X acting freely.*

3-dimensional geometries (the first 5 are symetric spaces):

- $S^3, \mathbb{E}^3, \mathbb{H}^3$, are the constant (sectional) curvature geometries.
- $S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$ are the product geometries.
- $Nil, Sol, \widetilde{SL_2}(\mathbb{R})$ are the twisted product geometries.

Note that only the spherical geometry is compact. The hyperbolic geometry is the most interesting one. See [16] for a detailed discussion of these geometries.

Decomposition of 3-manifolds:

Assume that M is closed (compact, no boundary).

Step 1: Connected sum decomposition of M into *prime* pieces (closed manifolds which cannot be decomposed further).

Step 2. If M is prime, consider a *toral* decomposition of M along *incompressible* [1] tori into *simple* pieces (the ones which cannot be decomposed further). Note that simple pieces typically have nonempty toral boundary.

Both decomposition processes terminate (Kneser, Haken: theory of normal surfaces).

Uniqueness of the decompositions: (1) Components of the connected sum decomposition are uniquely determined by M (Milnor). (2) The toral decomposition is unique up to isotopy if we consolidate simple pieces into maximal geometric pieces (Jaco, Shalen; Johannson).

Similar decompositions exist for compact manifolds with boundary.

Thurston's Geometrization Conjecture (GC): *Each prime closed 3-manifold M is either geometric or its simple pieces are geometric.*

A similar conjecture can be stated (and is proven by Thurston!) if M has nonempty boundary.

A restatement of the GC: Each closed prime 3-manifold is either geometric or it splits along disjoint incompressible tori as $M_{thick} \cup M_{thin}$, where M_{thick} is a disjoint union of hyperbolic manifolds, and M_{thin} is a *graph-manifold*, i.e. a manifold obtained by gluing along boundary tori of geometric 3-manifolds which are **not** modelled on $\mathbb{H}^3$.

Graph-manifolds are interesting and well-understood objects, they appear for instance in theory of complex surface singularities. Example of a graph-manifold: let Σ be a surface of genus ≥ 1 with one boundary circle, M_1, M_2 are copies of $\Sigma \times S^1$. Now glue M_1, M_2 along their boundary tori.

Omnibus Theorem (Thurston et al.):

1. GC is equivalent to the conjunction of PC (Poincare conjecture), SSFC (spherical space form conjecture) and HC (Hyperbolization conjecture).

2. (Thurston) GC holds if M is prime but not simple.

3. (Thurston) GC holds for *Haken manifolds*[2]. (For proofs of this theorem, which includes (2) as a special case, see [13], [11].)

4. If M is (prime) aspherical then GC holds for $M \iff$ GC holds for all manifolds finitely covered by $M \iff$ GC holds for all (prime) manifolds which are homotopy-equivalent to M. (See [7] for the proof in the most difficult case.)

5. GC holds if $\pi_1(M)$ contains $\mathbb{Z} \times \mathbb{Z}$ or has infinite center. (See [18], [6, 2] for the key parts of the proof of this.)

Explanation:

PC: If M is homotopy-equivalent to the sphere then it is diffeomorphic to the sphere. Equivalently, if M is (closed) simply-connected, then $M = S^3$.

SSFC: If the universal cover of M is the 3-sphere then M admits a metric of (positive) constant curvature, i.e. it is geometric, modelled on S^3.

[1] I.e. π_1-injective.

[2] I.e. M is prime and contains an incompressible surface: a π_1-injective surface which is not S^2.

HC: If M is prime, aspherical (i.e. its universal cover is contractible) and $\pi_1(M)$ does not contain $\mathbb{Z} \times \mathbb{Z}$ then M is hyperbolic.

HC is the most interesting (although, not the most famous) of the 3 parts of the geometrization conjecture.

A confidence-building exercise: GC implies PC. Indeed, suppose that M is closed and simply-connected. Consider connected sum decomposition of M into prime components $M_1, ..., M_k$. Then each M_i is also closed and simply-connected. Since $\pi_1(M_i)$ is trivial, M_i contain no incompressible tori, hence, by GC, M_i is geometric. Since the only compact 3-dimensional geometry is spherical, we conclude that $M_i = S^3$ for each i. Hence $M = S^3$ as well.

2 Ricci Flow

The previous section described the status of the GC until November of 2002. In November of 2002 Perelman had posted a preprint [14] which is the first part of the proof of the entire GC. The second Perelman's paper [15] appeared in March of 2003 and the third paper (concerning collapse) should appear some time in the future. The goal of this section is to outline the approach (Ricci flow) to GC used in Perelman's papers.

Ricci flow was introduced by Hamilton in 1982 as a possible approach to GC. I refer to [9], [3], [1], [4] for surveys of the Ricci flow.

We consider a closed 3-manifold M and a smooth family of Riemannian metrics $g(t), g \in [0, T)$, $T \leq \infty$, on M. This family is said to be *Ricci flow* if it satisfies the Ricci Flow Equation (RF):

$$g'(t) = -2Ric(g(t)),$$

where $Ric(g(t))$ is the *Ricci tensor* of $g(t)$.

Ricci flow decreases the volume; by rescaling both space and time one gets *Normalized* Ricci Flow Equation (NRF):

$$\widehat{g}'(t) = -2Ric(\widehat{g}(t)) + \frac{2}{3} r \widehat{g}(t).$$

Here $r = r(t)$ a scalar function, which is the *average scalar curvature* of the metric $\widehat{g}$. The metric $\widehat{g}$ has constant volume, this metric is called the *normalized solution* of the Ricci flow.

What is it good for? Suppose for a moment that $\widehat{g}$ is a fixed point of the NRF, then $\widehat{g}'(t) = 0$ and hence the Ricci tensor is a scalar multiple of the metric tensor, i.e. the metric $\widehat{g}$ is *Einsten.* In dimension 3, Einsten metrics are metrics of constant (sectional) curvature, hence M is geometric!

Ricci flow as an analogue of the heat flow. Consider the metric tensor $g_{ij}(x)$ in a normal coordinate $(x = (x^i))$ near zero, then

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ipjq} x^p x^q + O(|x|^3)$$

where R_{ipjq} is the Riemann curvature tensor (and I use the Einstein summation notation). Hence for the usual (Euclidean) laplacian Δ we have:

$$\Delta g_{ij}(0) = -\frac{1}{3} Ric_{ij}.$$

Thus, up to the higher order terms, the Ricci flow equation is the heat equation

$$g'_{ij}(t) \approx -6\Delta g_{ij}$$

on the space of symmetric 3×3 matrices.

The good news:

1. Short-time existence theorem (Hamilton, 1982, [8]; de Turck, 1983, [5]): There exists $T > 0$ such that given the initial condition $g(0)$ the RF equation has a (unique) solution for $t \in [0, T)$.

2. Positive curvature solutions (Hamilton, 1982, [8]): If $g(0)$ has positive curvature then the RF equation has a solution for $t \in [0, \infty)$, the solution has positive curvature and the normalized solution converges to a constant curvature metric. In particular, M is geometric.

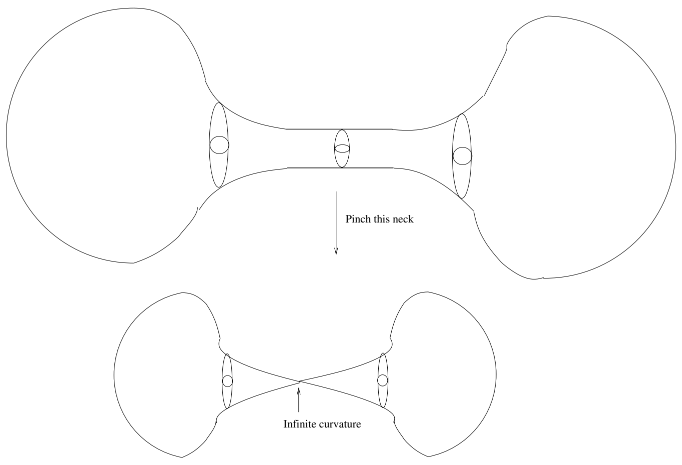

Figure 1: *Neck pinching.*

3. Geometric decomposition at infinity (Hamilton, 1999, [10]): Suppose that RF equation has a solution for all $t > 0$ and the curvature tensor Rm of the normalized solution $\widehat{g}$ has operator norm bounded by some time-independent constant: $|Rm(t)| \le Const$ for all t. Then GC holds for M. Moreover, as $t \to \infty$, $(M, \widehat{g}(t))$ splits as $M_{thick} \cup M_{thin}$ along incompressible tori, where $(M_{thick}, \widehat{g}(t))$ converges to a disjoint union of finite-volume complete hyperbolic manifolds and M_{thin} *collapses* and is homeomorphic to a graph-manifold.

The bad news: If M contains essential spheres then NRF blows up in a finite amount of time.

Example: Take two copies of the round sphere S^3 and connect them by a thin neck. The neck will get pinched (under RF) in a finite time. Figure 1.

Figure 2: *Blow up times.*

Hamilton-Perelman Approach to GC via Ricci flow:

Part 1. Show that the forming singularities of the normalized solution at the blow-up times T_i (say, near the first finite blow-up time T_1) are *standard*: "neck" or "cap". See Figure 3.

To identify the location of neck and caps look "around" points of $(M, g(t))$, $t = T_i - \epsilon$, where the norm $|Rm|$ is maximal.

Remark: Hamilton was unable to justify this step of the program, in particular, he was unable to rule out a singularity of the form $S^1\times$ 2-dimensional "sigar soliton" (steady-state) solution of the Ricci flow. Step 1 was done by Perelman in his 1-st paper, [14].

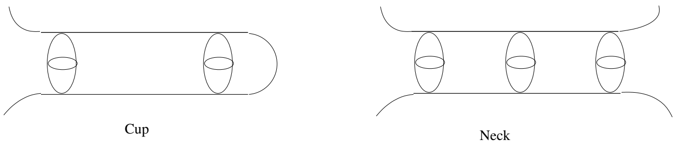

Figure 3: *Standard singularities.*

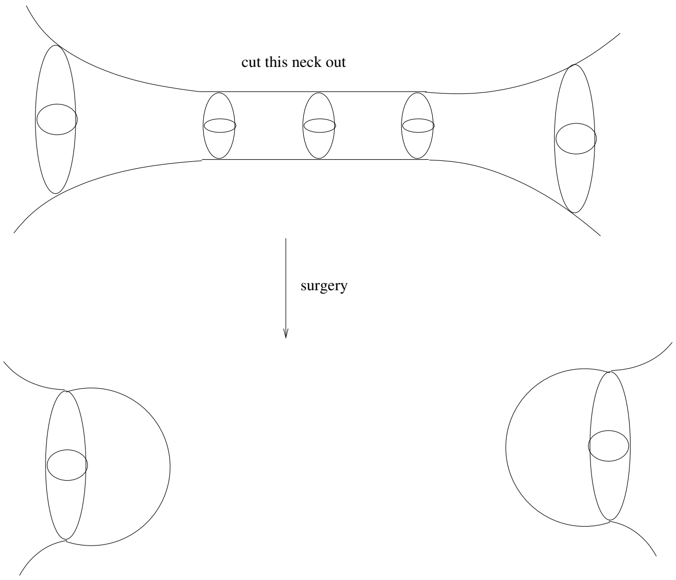

Figure 4: *Surgery.*

Part 2. Do *surgery*: Cut out necks and cups from M near T_i and replace them with (carefully chosen) spherical cups of bounded curvature. See Figure 4.

In the resulting manifold through away components of positive curvature; call the result M^{sur}. Continue the flow on M^{sur}.

Part 3. Show that there are only finitely many (Hamilton)/ locally finitely many (Perelman)[3] blow-up times T_i.

Remark. It follows from Kneser's theorem that after finitely many surgeries the topology of M^{sur} (which is possibly "extinct"=empty) does not change.

Part 4. Show that as $t \gg 0$, the manifold M^{sur} splits (along incompressible tori) into $M_{thick} \cup M_{thin}$ where the metrics on the components of the thick part converge to finite volume complete hyperbolic metrics; the thin part collapses as $t \to \infty$.

Part 5. Show that M_{thin} is a graph-manifold.

Part 1 is covered by the 1-st Perelman's paper [14]; parts 2, 3 and 4 are covered by the second paper [14]; part 5 should appear in the third paper by Perelman. Part 5 is partly covered by the preprint of Shioya and Yamaguchi [17].

See also notes by Kleiner and Lott [12] where they fill in some of the details in 1-st Perelman's paper [14].

[3]In the original version of [14] Perelman was saying that there are only finitely many blow-up times. Later, he modified the claim to local finiteness.

3 Beyond the GC.

Assume that GC holds.

Corollary 1. *Suppose that M is aspherical ($\pi_i(M) = 0, i \geq 2$). Then:*
1. The universal cover of M is diffeomorphic to $\mathbb{R}^3$.
2. Homotopy-equivalence $M \approx N$ implies diffeomorphism $M = N$. Thus M is determined by its fundamental group.

Remark. *(1) fails (even in the topological setting) for manifolds of dimension ≥ 4, as was shown by Mike Davis.*
(2) fails for manifolds of dimension ≥ 4. Topological version of this is known in higher dimensions as Borel Conjecture. It was verified by Farrell and Jones in a number of cases.

Corollary 2. *Suppose that M is compact. Then:*
1. All three algorithmic problems for $\pi_1(M)$ (i.e. the word, conjugacy and isomorphism problem in the class of 3-manifold groups) are solvable.
2. The homeomorphism problem (PL setting) is solvable for 3-manifolds.
3. If $\pi_1(M)$ is amenable then it is "elementary amenable" (and, moreover, is virtually solvable).

Note that 1, 2 and 3 fail for 4-manifolds.

Corollary 3. *One can "order" hyperbolic 3-manifolds by their volume: This is analogous to ordering surfaces by their genus.*

The problem with this "order" is that given v there could be several (but only finitely many) hyperbolic manifolds with the volume v. Also, the set of hyperbolic volumes in $\mathbb{R}_+$ is not discrete, although the appearence of accumulation points is relatively well-understood (hyperbolic Dehn surgery).

Several remaining open problems:

1. Relate topology of a hyperbolizable 3-manifold M with the geometric properties of the hyperbolic metric on M. For instance: Kashaev's conjecture; how to predict Margulis tubes; how to predict arithmeticity of $\pi_1(M)$, etc.

2. Virtual problems, e.g. the virtual $b_1(M) > 0$ problem, i.e. if M is hyperbolic then it admits a finite cover with positive first Betti number. This problem is still open even in the case of arithmetic π_1, although much was proven in the works of Millson, Li, Clozel, Lubotzky and others.

3. $PD(3)$ groups: Show that 3-dimensional Poincare duality groups G are 3-manifold groups. Much of the *Coarse Approach* (see [11, Chapter 20]) to the GC makes sense in this setting, but since we have no smooth structure to speak of, analytic methods are not available. Note however that even "Haken" case (i.e. when G splits as an amalgam) is open, even assuming that G is, say, amenable!

4. Algorithmic aspects: Once GC is known to hold for M, there is a rigorous but extremely inefficient algorithm for constructing geometric structures on the simple pieces of M. The most difficult case here is when M is hyperbolic. On the other hand, there are very efficient algorithms (Weeks; Casson) for constructing hyperbolic structure on M. However there is no theoretical justification for their work (can a combinatorial version of Ricci flow help here?). *Find a (justified and efficient) algorithm for geometrizing 3-manifolds.* The same applies to the recognition problem for 3-manifolds: The fastest (probabilistic) algorithm to tell two 3-manifolds apart is to check that they are hyperbolic, compute their volumes and show that the volumes are different. Can this be converted to a rigorous procedure? (One needs of course more invariants in addition to the volume.)

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