A Calculus for Classical Pseudo-Differential Operators with Non-Smooth Symbols

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Abstract

A calculus for classical pseudo-differential operators having coefficients in L^2-Sobolev spaces is presented. The standard elements of pseudo-differential calculi such as compositions, adjoints, invariance under coordinate changes and continuity between Sobolev spaces are implemented. One main feature leading to elliptic regularity and non-linear microlocal analysis is that parametrices to elliptic operators can be constructed within the calculus.

The constructions presented below set down the several structural aspects of a pseudo-differential calculus for operators with non-smooth coefficients. These constructions also serve as a forerunner for a calculus for non-classical operators with more involved estimates intended to publication in a further paper.

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Contents

1 Introduction

The objective of this paper is to establish a pseudo-differential calculus for classical operators having symbols with coefficients in certain L^2-Sobolev spaces.

Since their invention in the 60's by J. J. Kohn, L. Nirenberg, and L. Hörmander, pseudo-differential operators have become an important tool in the study of solutions to linear partial differential equations. They were used in the proofs of existence and regularity results, deriving energy estimates, constructions of parametrices, and many other wide-ranging applications. Over the last two decades, non-linear partial differential equations have received more and more attention. In order to make pseudo-differential techniques also available in the investigation of non-linear partial differential equations, one is necessarily forced to develop certain elements of pseudo-differential theory for operators having symbols with restricted regularity.

A great deal of applications in partial differential equations leads to classical pseudo-differential operators. Thus classical operators have always been a particular part of the program. Although once the general pseudo-differential calculus is established one has it especially for classical operators, it turns out that as a rule proving results for non-classical operators is much more involved than for classical ones. For the non-regular calculus this distinction becomes significant. Therefore, in this paper we develop the pseudo-differential calculus for classical operators first. It permits us to work out in detail the structural elements of the calculus (which are the same as for the non-classical calculus) so making the main ideas transparent.

However, in the theory of partial differential equations on several occasions one is forced to leave the range of applicability of classical operator calculus and to take advantage of pseudo-differential technique in its full strength. Therefore it is desirable to generalize the constructions given below to obtain a calculus for non-classical operators in which one is mainly interested in. This is subject of a further paper (see [25]).

Several authors contributed to pseudo-differential operators with non-regular symbols under quite different aspects. In order to mention only some topics, the mapping properties of pseudo-differential operators with non-smooth symbols, the mapping properties of the adjoint operators and commutator estimates were studied, e.g., in [4], [5], [15], [16], [20] and many other places. Also further elements of pseudo-differential calculus like compositions and asymptotic expansions were treated, e.g., in [2], [3], [7], [12].

An important rôle was played by Bony's paradifferential calculus, see [3], [17]. Here it had been for the first time that a pseudo-differential calculus for operators with non-smooth symbols was realized. Moreover, additionally to the complete calculus a parametrix construction for the elliptic operators was given in full generality. The remainder terms were characterized by their mapping properties.

In [1], [2], M. Beals and M. Reed proposed a calculus for pseudo-differential operators with non-smooth symbols having coefficients in certain L^2-Sobolev spaces. Their exposition is distinguished by the very simple estimates used. In this paper we fall back on some of the ideas exploited there and develop them further. In the result we obtain a calculus which on the one hand is sufficiently general for applications in non-linear partial differential equations and in which on the other hand the simplicity of the approach of M. Beals and M. Reed is preserved. One main feature is that parametrices to elliptic operators can be constructed within the calculus. Especially this point is important in non-linear microlocal analysis and, in [1], [2], it had not been completely realized.

Before we describe the content of the paper in more detail we want to discuss some of the ideas involved in the constructions. The coefficients of the operators must be at least continuous. Discontinuities in the coefficients lead with necessity to symbolic levels additionally to the principal one which then, e.g., enter in ellipticity conditions to provide only one argument. Continuity of coefficients, however, immediately causes the next observation, namely that asymptotic expansions that typically appear in pseudo-differential theory have to break off after finitely many steps. For instance, in case of smooth coefficients the symbol $c(x,\eta)$ of the composition $b(x,D)\,a(x,D)$ allows the asymptotic expansion $c(x,\eta) \sim \sum_{|\alpha|\geq 0}(1/\alpha!)\,\partial_\eta^\alpha b(x,\eta)\,D_x^\alpha a(x,\eta)$, and various differentiations with respect to x arise.

The main parts of operators in the calculus of M. Beals and M. Reed were of the form

$$\sum_{j=1}^{M} a_j(x)\,p_j(x,D) \tag{1.1}$$

for some $M < \infty$, $a_j \in H^s(\mathbb{R}^n)$ for s sufficiently large, and $p_j \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$. Then it is obvious that a general parametrix construction is possible only after an appropriate completion of expressions of the kind (1.1) is chosen. This immediately leads to tensor products. Our approach is based on the work with completed tensor products on the symbolic level.

In [25], the constructions rely on the weak symbol topology, τ, on symbol classes $S^m(\mathbb{R}^n; E)$ introduced in [24], e.g., for E being a Fréchet space. Recall that for the space $S^m_\tau(\mathbb{R}^n;E)$ we have $S^m_\tau(\mathbb{R}^n;E) = S^m_\tau(\mathbb{R}^n)\tilde{\otimes}_\epsilon E$. Moreover, a mapping from $S^m_\tau(\mathbb{R}^n;E)$, where E stands for certain coefficient space, into some space of bounded operators between Sobolev spaces realizing concrete pseudo-differential operators is continuous in general only when the latter space carries the strong operator topology. In this paper the considerations are based on the symbol classes $S^m_{cl}(\mathbb{R}^n;E)$. Here we have $S^m_{cl}(\mathbb{R}^n;E) = S^m_{cl}(\mathbb{R}^n)\tilde{\otimes}_\pi E$ due to the fact that $S^m_{cl}(\mathbb{R}^n)$ is a nuclear space. For that reason, arguments simplify considerably. So we are now allowed to work with the natural Fréchet topology on the symbol classes, hence deducing boundedness of the operators under consideration in a direct manner. The simplifications give us the possibility to introduce the several elements of the calculus without disturbing about the more difficult topological considerations used in [25] in estimating the remainders.

In the paper we make use of standard notation in the theory of pseudo-differential operators, the reader is referred to classical textbooks dealing with this subject, e.g., [6], [11], [21]. In order to be definite, let $x \in \mathbb{R}^n$ be the space variable, $\xi,\,\eta \in \mathbb{R}^n$ be frequency variables. Introduce the Fourier transform

$$Fu(\xi) = \hat{u}(\xi) = \int e^{-ix\xi}u(x)\,dx$$

such that the inverse Fourier transform becomes $F^{-1}_{\xi\to x}\hat{u}(\xi) = (2\pi)^{-n}\int e^{ix\xi}\hat{u}(\xi)\,d\xi$. For brevity, in what follows, we forget about the factor $(2\pi)^{-n}$ and work in Fourier space with a renormalized Lebesgue measure, i.e., $d\xi$ is the usual Lebesgue measure times $(2\pi)^{-n}$, and

$$F^{-1}_{\xi\to x}\hat{u}(\xi) = \int e^{ix\xi}\hat{u}(\xi)\,d\xi.$$

Then, with symbols $p(\xi, x, \eta)$, we associate operators $p(D, x, D)$,

$$(p(D,x,D)u)\hat{}(\xi) = \int \hat{p}(\xi, \xi - \eta, \eta)\hat{u}(\eta)\, d\eta, \tag{1.2}$$

where $\hat{p}(\xi, \zeta, \eta) = F_{x\to\zeta}\{p(\xi, x, \eta)\}$. We shall see later on that this definition makes perfectly good sense for our symbol classes under consideration. Intuitively, the operator $p(D, x, D)$ is applied to u from the right to the left acting first by differentiation, then by multiplication, and finally by differentiation again. That holds precisely if $p(\xi, x, \eta)$ has product form, i.e., $p(\xi, x, \eta) = p_1(\xi)\, a(x)\, p_0(\eta)$, where $p_0(\eta)$, $p_1(\eta)$ are certain symbols with constant coefficients, and $a(x)$ is multiplication by some coefficient. For symbols $p(x, \eta)$, $q(\xi, x)$, at least formally, (1.2) becomes the standard operator convention in pseudo-differential theory, i.e.,

$$p(x, D)u(x) = \int e^{-ix\xi} p(x, \xi)\hat{u}(\xi)\, d\xi$$

and $q(D, x)u(x) = \int e^{-i(x-y)\xi} q(\xi, y)u(y)\, dy\, d\xi$.

The plan of the paper is as follows: In Section 2 we discuss classical vector-valued symbols used in the sequel to establish the calculus for classical operators. Then Section 3 is devoted to the introduction of the several operator classes and the derivation of their basic properties. Compositions, adjoints, commutators, and mapping properties between Sobolev spaces are treated. Special emphasize is put on the accurate description of the mapping properties of the remainders. Also in this section M. Beals and M. Reed's estimate is reproduced and the basic technique in estimating the remainders is established. Finally, in Section 4, some further topics are grouped together like the parametrix construction and elliptic regularity, invariances under coordinate changes and operators on manifolds, and the equivalence of ellipticity and the Fredholm property on compact manifolds. In the notes at the end we comment on some additional material.

2 Classical Symbols

In this section we discuss several aspects of classical pseudo-differential symbols. Our standpoint is to treat symbols having their coefficients in certain function spaces as vector-valued ones. The classes $S^m_{cl}(\mathbb{R}^n; E)$ are defined if E is a Fréchet space.

First we are concerned with abstract vector-valued classical symbols. The main result, $S^m_{cl}(\mathbb{R}^n; E) = S^m_{cl}(\mathbb{R}^n)\hat{\otimes}_\pi E$, is stated in Proposition 2.4. Then we deal with symbols depending on two covariables which are separately classical in both covariables. In a final subsection, we specify the results previously obtained to symbols having their coefficients in Sobolev spaces. In addition, auxiliary symbol classes arising during subsequent calculations, e.g., symbols depending on three covariables, symbols one coefficient of which belongs to $C^\infty_b(\mathbb{R}^n)$ and so on, are briefly introduced.

2.1 Abstract Vector-Valued Symbols

In treating classical symbols it is convenient to replace the elliptic symbol $\langle\xi\rangle^r \in S^r(\mathbb{R}^n)$ usally used, where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, by a classical symbol which shall again be denoted by $\langle\xi\rangle^r$, i.e., $\langle\xi\rangle^r \in S^r_{cl}(\mathbb{R}^n)$. We choose $\xi \mapsto \langle\xi\rangle$ to be a smoothed norm function, i.e., $\langle\xi\rangle$ is positive on $\mathbb{R}^n$ satisfying $\langle\xi\rangle = |\xi|$ for all $\xi \in \mathbb{R}^n$, $|\xi| \geq C$, and some constant $C > 0$. The symbol estimates are not effected by this substitute. When dealing with classical symbols we also need 0-excision functions ψ, i.e., functions $\psi \in C^\infty(\mathbb{R}^n)$ satisfying $\psi(\xi) = 0$ for $|\xi| \leq C_1$, $\psi(\xi) = 1$ for $|\xi| \geq C_2$, and some constants C_1, C_2, where $C_2 > C_1 > 0$.

Let E be a Fréchet space with fundamental semi-norm system $\{\|\ \|_l\}_{l\in\mathbb{N}}$. Recall that a fundamental semi-norm system for $S^m(\mathbb{R}^n; E)$ is given by

$$S^m(\mathbb{R}^n; E) \ni a \mapsto \sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{-m+|\alpha|} \|\partial^\alpha_\xi a(\xi)\|_l \tag{2.1}$$

for all $\alpha \in \mathbb{N}^n$, $l \in \mathbb{N}$. $S^m(\mathbb{R}^n; E)$ equipped with the resulting locally convex topology is a Fréchet space. The space $S^{-\infty}(\mathbb{R}^n; E) = \bigcap_{m\in\mathbb{R}} S^m(\mathbb{R}^n; E)$ of E-valued symbols of order $-\infty$ is the Schwartz space $\mathcal{S}(\mathbb{R}^n; E)$, and we have $S^{-\infty}(\mathbb{R}^n; E) = S^{-\infty}(\mathbb{R}^n)\tilde{\otimes}_\pi E$.

If E is a Fréchet space, then it is possible to form asymptotic sums in $S^m(\mathbb{R}^n; E)$.

2.1 Proposition. *Let E be a Fréchet space. Let $\{m_j\}_{j\in\mathbb{N}} \subset \mathbb{R}$ be a sequence satisfying $m_j \to -\infty$ as $j \to \infty$. Suppose further that we are given symbols $a_j \in S^{m_j}(\mathbb{R}^n; E)$, $j = 0, 1, 2, \ldots$ Then there is a symbol $a \in S^m(\mathbb{R}^n; E)$, $m = \max_{j\in\mathbb{N}} m_j$, such that for every $r \in \mathbb{R}$ there exists $M \in \mathbb{N}$, $M \geq 1$, such that*

$$a - \sum_{j=0}^{M-1} a_j \in S^r(\mathbb{R}^n; E). \tag{2.2}$$

a is uniquely determined modulo $S^{-\infty}(\mathbb{R}^n; E)$.

Proof: We make the ansatz

$$a(\xi) = \sum_{j=0}^{\infty} \psi(c_j\xi)\, a_j(\xi) \tag{2.3}$$

with ψ being a 0-excision function. Then it is routine procedure to check that the reals $c_j > 0$ converging to 0 sufficiently fast can be chosen in such a way that, for every $M \in \mathbb{N}$, the sum $\sum_{j=M}^{\infty} \psi(c_j\xi)a_j$ converges absolutely in $S^{m'_M}(\mathbb{R}^n; E)$, where $m'_M = \max_{j\geq M} m_j$. Then, for every $M \in \mathbb{N}$, $M \geq 1$, we have that the symbol

$$a - \sum_{j=0}^{M-1} a_j = - \sum_{j=0}^{M-1} \big(1 - \psi(c_j\xi)\big)a_j + \sum_{j=M}^{\infty} \psi(c_j\xi)\, a_j$$

belongs to $S^r(\mathbb{R}^n; E)$ provided that $r \geq m'_M$, which shows that (2.2) is valid. The uniqueness statement is obvious. □

In the case that (2.2) holds we also write

$$a \sim \sum_{j=0}^{\infty} a_j.$$

Now we are in a position to introduce classical E-valued symbols.

2.2 Definition. *Let E be a Fréchet space, $m \in \mathbb{R}$. Then the space $S^m_{cl}(\mathbb{R}^n; E)$ of classical symbols of order m consists of all functions $a \in S^m(\mathbb{R}^n; E)$ admitting asymptotic expansions into homogeneous components, i.e., there are homogeneous functions $a_{(m-j)}$ in $S^{(m-j)}(\mathbb{R}^n \setminus 0; E)$, $j = 0, 1, 2, \ldots$, such that*

$$a(\xi) \sim \sum_{j=0}^{\infty} \psi(\xi)\, a_{(m-j)}(\xi) \tag{2.4}$$

holds for an arbitrary 0-excision function ψ.

Notice that the homogeneous components of a symbol $a \in S^m_{cl}(\mathbb{R}^n; E)$ are uniquely determined. The space $S^{(m)}(\mathbb{R}^n \setminus 0; E)$ is defined as the space of all functions $a \in C^\infty(\mathbb{R}^n \setminus \{0\}; E)$ which are homogeneous of order m, i.e., which satisfy $a(\lambda\xi) = \lambda^m a(\xi)$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, $\lambda > 0$.

Next a suitable Fréchet topology for $S^m_{cl}(\mathbb{R}^n; E)$ is introduced. The spaces $S^{(m)}(\mathbb{R}^n \setminus 0; E)$ are topologized by identifying them with $C^\infty(S^{n-1}; E)$, S^{n-1} being the unit sphere in $\mathbb{R}^n$. In particular, $S^{(m)}(\mathbb{R}^n \setminus 0)$ is a nuclear Fréchet space. Now, once a 0-excision function ψ is fixed, we have natural continuous injections we have natural continuous injections

$$S^{(m)}(\mathbb{R}^n; E) \oplus S^{m-1}(\mathbb{R}^n; E) \hookrightarrow S^m(\mathbb{R}^n; E), \quad (a_{(m)}, a_{m-1}) \mapsto \psi(\xi)a_{(m)} + a_{m-1} \tag{2.5}$$

leading, for all $M \in \mathbb{N}$, to continuous mappings

$$\bigoplus_{j=0}^{M} S^{(m-j)}(\mathbb{R}^n; E) \oplus S^{m-M-1}(\mathbb{R}^n; E) \to \bigoplus_{j=0}^{M-1} S^{(m-j)}(\mathbb{R}^n; E) \oplus S^{m-M}(\mathbb{R}^n; E)$$

with the mapping in the first M components being the identity. The space $S^m_{cl}(\mathbb{R}^n; E)$ is algebraically a projective limit,

$$S^m_{cl}(\mathbb{R}^n; E) = \text{proj-lim} \bigoplus_{j=0}^{M-1} S^{(m-j)}(\mathbb{R}^n; E) \oplus S^{m-M}(\mathbb{R}^n; E) \tag{2.6}$$

with the limit is extended over $M \to \infty$, and $S^m_{cl}(\mathbb{R}^n; E)$ is equipped with the projective limit topology. This definition is independent of the choice made on ψ.

From the representation (2.6) we draw some conclusions. For example, for all $M \in \mathbb{N}$, we have that

$$S^m_{cl}(\mathbb{R}^n; E) = \bigoplus_{j=0}^{M-1} S^{(m-j)}(\mathbb{R}^n; E) \oplus S^{m-M}_{cl}(\mathbb{R}^n; E) \tag{2.7}$$

holds in a topological sense. In particular, $S^{m-M}_{cl}(\mathbb{R}^n; E)$ is a complemented subspace in $S^m_{cl}(\mathbb{R}^n; E)$. Similarly, it is seen that $S^{-\infty}(\mathbb{R}^n; E)$ carries the topology induced by $S^m_{cl}(\mathbb{R}^n; E)$. However, $S^{-\infty}(\mathbb{R}^n; E)$ is closed but not complemented in $S^m_{cl}(\mathbb{R}^n; E)$.

Next we will recognize that $S^m_{cl}(\mathbb{R}^n)$ is a nuclear space.

2.3 Proposition. *$S^m_{cl}(\mathbb{R}^n)$ is a nuclear Fréchet space.*

Proof: Setting $E = S^m_{cl}(\mathbb{R}^n)$, we have to show that for every continuous semi-norm p on E there is a continuous semi-norm q on E, $q \geq p$, such that the canonical mapping $\hat{E}_q \to \hat{E}_p$ is nuclear. Thereby, $\hat{E}_p$ denotes the local Banach space to a given semi-norm p, i.e., the completion of the space $E/\ker p$ normed in a canonical way.

We start with the representation

$$S^m_{cl}(\mathbb{R}^n) = \operatorname{proj\,lim} \bigoplus_{j=0}^{M-1} S^{(m-j)}(\mathbb{R}^n) \oplus S^{m-M}(\mathbb{R}^n).$$

Let p be a continuous semi-norm on E. We may suppose that, for some M, p is a continuous semi-norm living on $\bigoplus_{j=0}^{M-1} S^{(m-j)}(\mathbb{R}^n) \oplus S^{m-M}(\mathbb{R}^n)$. Since in the direct sum the first M summands are nuclear, we may further suppose that p is a continuous semi-norm living on $S^{m-M}(\mathbb{R}^n)$. Assume that, for some $r \geq 1$, only estimates of derivatives up to order $r-1$ are involved in the semi-norm p. Then we can choose q to be a continuous semi-norm on $\bigoplus_{j=0}^{r-1} S^{(m-M-j)}(\mathbb{R}^n) \oplus S^{m-M-r}(\mathbb{R}^n)$ living on the first r summands such that q pulled back to $S^m_{cl}(\mathbb{R}^n)$ estimates p from above. Again using the nuclearity of $\bigoplus_{j=0}^{r-1} S^{(m-M-j)}(\mathbb{R}^n)$ we conclude that the mapping $\hat{E}_q \to \hat{E}_p$ is nuclear. □

The following proposition is basic in proving continuity between Sobolev spaces of operators arising in the classical calculus. It is interesting that its proof can be based on the weak symbol topology that has been introduced in Part I. Another proof independent of the weak symbol topology shall be given in the notes at the end of the paper.

2.4 Proposition. *Let E be a Fréchet space, $m \in \mathbb{R}$. Then*

$$S^m_{cl}(\mathbb{R}^n; E) = S^m_{cl}(\mathbb{R}^n)\tilde{\otimes}_\pi E. \tag{2.8}$$

Proof: The assertion (2.8) is implied by the following calculation:

$$S^m_{cl}(\mathbb{R}^n; E) \quad = \quad \text{proj-lim} \bigoplus_{j=0}^{M-1} S^{(m-j)}(\mathbb{R}^n; E) \oplus S^{m-M}_{\tau}(\mathbb{R}^n; E)$$

$$\begin{aligned} &= \text{proj-lim}\left(\bigoplus_{j=0}^{M-1} S^{(m-j)}(\mathbb{R}^n)\hat{\otimes}_\epsilon E\right) \oplus \left(S_\tau^{m-M}(\mathbb{R}^n)\hat{\otimes}_\epsilon E\right) \\ &= \text{proj-lim}\left(\bigoplus_{j=0}^{M-1} S^{(m-j)}(\mathbb{R}^n) \oplus S_\tau^{m-M}(\mathbb{R}^n)\right)\hat{\otimes}_\epsilon E \\ &= S_{cl}^m(\mathbb{R}^n)\hat{\otimes}_\pi E. \end{aligned}$$

Hereby in the first line we have used the continuity of the embedding $S_\tau^m(\mathbb{R}^n; E) \hookrightarrow S^{m'}(\mathbb{R}^n; E)$ for $m' > m$, in the second and the forth line the nuclearity of the spaces $S^{(m-j)}(\mathbb{R}^n)$ and $S_{cl}^m(\mathbb{R}^n)$, respectively, and in the third line the fact that the injective tensor product is well-behaved under forming projective limits. □

2.2 Multiple Classical Symbols

The constructions in the classical calculus prompt to symbols $p(\xi, x, \eta)$ which are separately classical in both covariables $(\xi, \eta) \in \mathbb{R}^{2n}$. Here we introduce the corresponding notions. The spaces $S^{m,m'}(\mathbb{R}^n \times \mathbb{R}^n; E)$ have been considered in [24]. Recall that a fundamental semi-norm system is given by

$$S^{m,m'}(\mathbb{R}^n \times \mathbb{R}^n; E) \ni a \mapsto \sup_{\xi,\eta \in \mathbb{R}^n} \langle\xi\rangle^{-m'+|\alpha|} \langle\eta\rangle^{-m+|\beta|} \|\partial_\xi^\alpha \partial_\eta^\beta a(\xi, \eta)\|_l \tag{2.9}$$

for E being a Fréchet space with fundamental semi-norm system $\{\| \ \|_l\}_{l\in\mathbb{N}}$.

As in Proposition 2.1 one shows that it is possible to form double indexed asymptotic sums in $S^{m,m'}(\mathbb{R}^n \times \mathbb{R}^n; E)$.

2.5 Proposition. *Let E be a Fréchet space. Let $\{m_j\}_{j\in\mathbb{N}}$, $\{m'_k\}_{k\in\mathbb{N}}$ be sequences of reals satisfying $m_j \to -\infty$ as $j \to \infty$, $m'_k \to -\infty$ as $k \to \infty$. Suppose further that we are given symbols $a_{jk} \in S^{m_j,m'_k}(\mathbb{R}^n \times \mathbb{R}^n; E)$ for $j, k = 0, 1, 2, \ldots$*

Then there is a symbol $a \in S^{m,m'}(\mathbb{R}^n \times \mathbb{R}^n; E)$, where $m = \max_{j\in\mathbb{N}} m_j$, $m' = \max_{k\in\mathbb{N}} m'_k$, such that for all r, $r' \in \mathbb{R}$ there exist M, $M' \in \mathbb{N}$, $M \geq 1$, $M' \geq 1$, such that

$$a - \sum_{j,k=0}^{M-1,M'-1} a_{jk} \in S^{r,r'}(\mathbb{R}^n \times \mathbb{R}^n; E). \tag{2.10}$$

a is uniquely determined modulo $S^{-\infty,-\infty}(\mathbb{R}^n \times \mathbb{R}^n; E)$.

In the case that (2.10) holds we write

$$a \sim \sum_{j,k=0}^{\infty} a_{jk}.$$

Notice that in (2.10) asymptotic summation can take place first in one index and then in the other leading to the same result. That means, e.g., if we put

$$a_j \sim \sum_{k=0}^{\infty} a_{jk}, \tag{2.11}$$

where this asymptotic sum exists in $S^{m_j,m'}(\mathbb{R}^n \times \mathbb{R}^n; E)$ with the result unique modulo $S^{m_j,-\infty}(\mathbb{R}^n \times \mathbb{R}^n; E)$, then we can asymptotically sum up the a_j's in $S^{m,m'}(\mathbb{R}^n \times \mathbb{R}^n; E)$ with the result unique modulo $S^{-\infty,m'}(\mathbb{R}^n \times \mathbb{R}^n; E)$, and we find

$$a \sim \sum_{j=0}^{\infty} a_j. \tag{2.12}$$

The definition of the classes $S^{m,m'}_{cl}(\mathbb{R}^n \times \mathbb{R}^n; E)$ is the following one:

2.6 Definition. *Let E be a Fréchet space, m, $m' \in \mathbb{N}$.*

Then the space $S^{m,m'}_{cl}(\mathbb{R}^n \times \mathbb{R}^n; E)$ consists of all functions $a \in S^{m,m'}(\mathbb{R}^n \times \mathbb{R}^n; E)$ for which, for $j, k = 0, 1, 2, \ldots$, there are symbols $a_{(m-j),(m'-k)} \in S^{(m-j),(m'-k)}((\mathbb{R}^n \setminus 0) \times (\mathbb{R}^n \setminus 0); E)$ homogeneous of multi-order $(m-j, m'-k)$ such that

$$a(\xi, \eta) \sim \sum_{j,k=0}^{\infty} \psi(\xi, \eta)\, a_{(m-j),(m'-k)}(\xi, \eta) \tag{2.13}$$

holds for an arbitrary 0-excision function ψ.

In (2.13), the homogeneous components $a_{(m-j),(m'-k)}(\xi, \eta)$ are uniquely determined. The space $S^{(m),(m')}((\mathbb{R}^n \setminus 0) \times (\mathbb{R}^n \setminus 0); E)$ is defined as the space of all functions $a \in C^\infty((\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\}); E)$ satisfying $a(\lambda\xi, \mu\eta) = \lambda^{m'}\mu^m a(\xi, \eta)$ for all ξ, $\eta \in \mathbb{R}^n \setminus \{0\}$, $\lambda > 0$, $\mu > 0$.

In order to topologize $S^{m,m'}_{cl}(\mathbb{R}^n \times \mathbb{R}^n; E)$ we provide the spaces $S^{(m),(m')}((\mathbb{R}^n \setminus 0) \times (\mathbb{R}^n \setminus 0); E)$ with Fréchet topologies by identifying them with $C^\infty(S^{n-1} \times S^{n-1}; E)$. Moreover, for m, $m' \in \mathbb{R}$, M, $M' \in \mathbb{N}$, we have natural continuous injections

$$\begin{aligned} &\bigoplus_{j,k=0}^{M-1,M'-1} S^{(m-j),(m'-k)}((\mathbb{R}^n \setminus 0) \times (\mathbb{R}^n \setminus 0); E) \oplus S^{m-M,m'-M'}(\mathbb{R}^n \times \mathbb{R}^n; E) \\ &\hookrightarrow S^{m,m'}(\mathbb{R}^n \times \mathbb{R}^n; E), \quad (a_{(m),(m')}, \ldots, a_{(m-M+1),(m'-M'+1)}, a_{m-M,m-M'}) \mapsto \\ &\qquad \sum_{j,k=0}^{M-1,M'-1} \psi(\xi, \eta) a_{(m-j),(m'-k)} + a_{m-M,m'-M'}. \end{aligned}$$

Then $S^{m,m'}_{cl}(\mathbb{R}^n \times \mathbb{R}^n; E)$ becomes equipped with the projective limit topology

$$\begin{aligned} &S^{m,m'}_{cl}(\mathbb{R}^n \times \mathbb{R}^n; E) \\ &= \operatorname{proj\,lim} \bigoplus_{j,k=0}^{M-1,M'-1} S^{(m-j),(m'-k)}((\mathbb{R}^n \setminus 0) \times (\mathbb{R}^n \setminus 0); E) \\ &\qquad \oplus S^{m-M,m'-M'}(\mathbb{R}^n \times \mathbb{R}^n; E). \end{aligned} \tag{2.14}$$

2.7 Proposition. *Let E be a Fréchet space, m, $m' \in \mathbb{N}$.*

Then we have

$$S^{m,m'}_{cl}(\mathbb{R}^n \times \mathbb{R}^n; E) = S^m_{cl}(\mathbb{R}^n; S^{m'}_{cl}(\mathbb{R}^n; E)). \tag{2.15}$$

Proof: The proof of (2.15) is a straightforward, but lengthy exercise in employing the representations of $S_{cl}^{m}(\mathbb{R}^n; S_{cl}^{m'}(\mathbb{R}^n; E))$, $S_{cl}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^n; E)$ as projective limits according to (2.6) and (2.14), respectively. □

Especially, from Propositions 2.4 and 2.7 we get

$$S_{cl}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^n; E) = S_{cl}^{m}(\mathbb{R}^n)\tilde{\otimes}_\pi S_{cl}^{m'}(\mathbb{R}^n)\tilde{\otimes}_\pi E. \tag{2.16}$$

Notice also for further reference that the linearization of the continuous bilinear mapping

$$S_{cl}^{m}(\mathbb{R}^n) \times S_{cl}^{m'}(\mathbb{R}^n; E) \to S_{cl}^{m+m'}(\mathbb{R}^n; E), \quad (a, a') \mapsto a\, a'$$

extends by continuity to a continuous surjective mapping

$$S_{cl}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^n; E) \to S_{cl}^{m+m'}(\mathbb{R}^n; E), \quad a(\xi,\eta) \mapsto a(\xi,\eta)_{|\eta=\xi}. \tag{2.17}$$

2.3 Coefficients in Sobolev Spaces

We particularize the results on abstract vector-valued symbol classes to symbol classes used subsequently. We will be mainly concerned with symbols having their coefficients in L^2-Sobolev spaces, $H^s(\mathbb{R}^n)$, where $s > \frac{n}{2}$. This means, in particular, that the coefficients are at least bounded and continuous.

In the case that the coefficient space is $\mathcal{F}(\mathbb{R}^n)$, e.g., $\mathcal{F} = H^s$, H^s_{loc}, C_b^∞, we shall denote $S^m(\mathbb{R}^n; \mathcal{F}(\mathbb{R}^n)) = \mathcal{F}S^m(\mathbb{R}^n \times \mathbb{R}^n)$, $S_{cl}^m(\mathbb{R}^n; \mathcal{F}(\mathbb{R}^n)) = \mathcal{F}S_{cl}^m(\mathbb{R}^n \times \mathbb{R}^n)$, $S^{m,m'}(\mathbb{R}^{2n}; \mathcal{F}(\mathbb{R}^n)) = \mathcal{F}S^{m,m'}(\mathbb{R}^n \times \mathbb{R}^{2n})$ etc., where the first set of coordinates refers to the space variable x and the second set of coordinates to the frequency variables ξ, η. As mentioned in the introduction, for symbols $p \in \mathcal{F}S_{cl}^m(\mathbb{R}^n \times \mathbb{R}^n)$ we adopt to different operator conventions, $p(x, D)$ and $p(D, x)$. In the first case the symbol is denoted by $p(x, \eta)$, in the second case by $p(\xi, x)$.

To describe later on the behaviour of operators under compositions, we introduce further symbol classes. For s, $s' \in \mathbb{R}$, the space $H^{s,s'}(\mathbb{R}^n \times \mathbb{R}^n)$ consists of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^{2n})$ satisfying

$$\langle\xi\rangle^{s'} \langle\eta\rangle^{s} \hat{u}(\xi,\eta) \in L^2(\mathbb{R}^n \times \mathbb{R}^n). \tag{2.18}$$

$H^{s,s'}(\mathbb{R}^n \times \mathbb{R}^n)$ is the Hilbert space tensor product $H^s(\mathbb{R}^n)\hat{\otimes}_H H^{s'}(\mathbb{R}^n)$. Note that for $u \in H^{s,s'}(\mathbb{R}^n \times \mathbb{R}^n)$ we have

$$v(x) = u(x,y)_{|y=x} \in H^{s'}(\mathbb{R}^n),$$

if $|s'| \le s$, $s > \frac{n}{2}$. This is seen by writing $\hat{v}(\xi) = \int \hat{u}(\xi-\eta,\eta)\, d\eta$.

Let m, m', $m'' \in \mathbb{R}$, s, $s' \in \mathbb{R}$, $s > \frac{n}{2}$, $s' > \frac{n}{2}$. Then the symbol class $H^{s,s'}S^{m,m',m''}(\mathbb{R}^{2n} \times \mathbb{R}^{3n})$ consists of all functions $p(\xi, x, \zeta, y, \eta) \in C^\infty(\mathbb{R}^{5n})$ satisfying

$$\sup_{(\xi,\zeta,\eta)\in\mathbb{R}^{3n}} \langle\xi\rangle^{-m''+|\alpha|} \langle\zeta\rangle^{-m'+|\beta|} \langle\eta\rangle^{-m+|\gamma|} \left\|\partial_\xi^\alpha \partial_\zeta^\beta \partial_\eta^\gamma u(\xi,\cdot,\zeta,\cdot,\eta)\right\|_{H^{s,s'}} < \infty. \tag{2.19}$$

This definition is in complete analogy to those in (2.1), (2.9).

For symbols in (2.19) we choose the following operator convention:

$$(p(D,x,D,x,D)u)^{\wedge}(\xi) = \int \hat{p}(\xi,\xi-\zeta,\zeta,\zeta-\eta,\eta)\hat{u}(\eta)\,d\zeta\,d\eta. \qquad (2.20)$$

The first $\hat{}$ under the integral sign refers to the partial Fourier transform of $p(\xi,x,\zeta,y,\eta)$ with respect to x, y.

The classes $H^{s,s'}S^{m,m',m''}(\mathbb{R}^{2n}\times\mathbb{R}^{3n})$ have been introduced for the reason that for $p_0 \in H^{s_0}S^{m_0,m_0'}(\mathbb{R}^n\times\mathbb{R}^{2n})$, $p_1 \in H^{s_1}S^{m_1,m_1'}(\mathbb{R}^n\times\mathbb{R}^{2n})$ we have

$$p(D,x,D,x,D) = p_1(D,x,D)p_0(D,x,D)$$

with symbol $p(\xi,x,\zeta,y,\eta) = p_1(\xi,x,\zeta)p_0(\zeta,y,\eta) \in H^{s_0,s_1}S^{m_0,m_0'+m_1,m_1'}(\mathbb{R}^{2n}\times\mathbb{R}^{3n})$.

The definition of the symbol classes $H^{s,s'}S_{cl}^{m,m',m''}(\mathbb{R}^{2n}\times\mathbb{R}^{3n})$ is obvious by the foregoing considerations. Only note that

$$H^{s,s'}S_{cl}^{m,m',m''}(\mathbb{R}^{2n}\times\mathbb{R}^{3n}) = H^{s,s'}(\mathbb{R}^{2n})\tilde{\otimes}_\pi S_{cl}^{m,m',m''}(\mathbb{R}^{3n})$$

with $S_{cl}^{m,m',m''}(\mathbb{R}^{3n})$ being the class of symbols with constant coefficients which are classical separately in all covariables $(\xi,\zeta,\eta)\in\mathbb{R}^{3n}$.

In the parametrix construction as well as in the discussion of invariance under coordinate changes we shall encounter symbol classes $H^sC_b^\infty S_{cl}^{m,m',m''}(\mathbb{R}^{2n}\times\mathbb{R}^{3n})$, $C_b^\infty H^{s'}S_{cl}^{m,m',m''}(\mathbb{R}^{2n}\times\mathbb{R}^{3n})$ etc. These classes consist of symbols depending on five coordinates, $p(\xi,x,\zeta,y,\eta)$. As explained in [24], no additional complications arise if one of the coefficients belong to the space $C_b^\infty(\mathbb{R}^n)$ such that we do not further comment on these classes. Note only that $H^sC_b^\infty$ stands for the space $C_b^\infty(\mathbb{R}^n)\tilde{\otimes}_\epsilon H^s(\mathbb{R}^n) = C_b^\infty(\mathbb{R}^n)\hat{\otimes}_2 H^s(\mathbb{R}^n)$, where subscript 2 indicates the tensor product associated with the Banach ideal of 2-factorable operators, in agreement with the choice made for $H^{s,s'}$ (for a discussion of tensor products see [8]). Here, as usual, we adopt the convention that operators are applied from the left to the right such that the first symbol in $H^sC_b^\infty$ refers to the space from which the coefficient depending on y is, while the second symbol means the space from which the coefficient depending on x is.

3 Elements of the Calculus

In this section we introduce the several operator classes. Thereby, we first only regard operators globally defined on $\mathbb{R}^n$. A distinction is made between the classes further on called the standard operator classes and operator classes arising from these standard classes through conjugation with powers of a fixed elliptic operator. Mapping properties between Sobolev spaces are treated as well as compositions, adjoints, and commutators. We take care in the precise description of the mapping properties of the remainders.

In Subsections 3.1–3.3 we introduce the operator classes and list some of their basic properties. Proofs are postponed to Subsection 3.4, where the basic technique for showing continuity between Sobolev spaces is developed. The remainders in Subsection 3.1 are not of the most general form, but their mapping properties are modelled on those of the other components appearing in an asymptotic expansion. We prefer instead, in Subsection 3.2, to motivate the introduction of the general remainder classes by a first example. Afterwards, the remainder classes so obtained turn out to be an integral part of the calculus. Subsection 3.5 is devoted to compositions, adjoints, and commutators. For commutators, one of the operators involved has C^∞-coefficients. A thorough discussion of the general case will be given in [25].

A motivation for the special appearance of the operators discussed below has been given in the introduction.

3.1 The Standard Operator Classes

The following lemma is needed in order to assign to formal expressions appearing in Definition 3.2 operators acting between Sobolev spaces.

3.1 Lemma. *Let $s, m \in \mathbb{R}$, $s > \frac{n}{2}$. Then, for $p(x,\eta) \in H^sS^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$, $p(x,D)$ induces a continuous operator*

$$p(x,D) : H^{t+m}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{3.1}$$

for all $t \in \mathbb{R}$, $|t| \leq s$.

The standard operator classes, $\mathcal{A}^{(m),d}_{s,cl}(\mathbb{R}^n)$, incorporate three parameters s, m, d: s stands for the smoothness of coefficients, m is the order of operators, and d denotes the length of asymptotic expansions. We require $s > \frac{n}{2} + d$, since coefficients should be continuous and bounded. Imposing a restriction on the real variable t in the form $t = -s$ to $s - 2d$, as in the next definition, we mean that t varies in the closed interval $[-s, s-2d]$.

3.2 Definition. *Let $s, m \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$.*

Then $\mathcal{A}^{(m),d}_{s,cl}(\mathbb{R}^n)$ denotes the class of all operators P which can be written in the form

$$P = \sum_{j=0}^{d-1} p_j(x,D) + P_d, \tag{3.2}$$

where $p_j(x,\eta) \in H^{s-j}S^{m-j}_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$ *for* $j = 0, 1, \ldots, d-1$ *and*

$$P_d \in \bigcap_{t=-s}^{s-2d} \mathcal{L}(H^{t+m}(\mathbb{R}^n), H^{t+d}(\mathbb{R}^n)). \tag{3.3}$$

We add some remarks. The right-hand side in (3.2) provides finite asymptotic expansions of operators in $\mathcal{A}^{(m),d}_{s,cl}(\mathbb{R}^n)$ into operators belonging to $\mathcal{A}^{(m-j),d-j}_{s-j,cl}(\mathbb{R}^n)$ for $j = 0, 1, \ldots, d$, by forgetting the first $j-1$ summands. The remainder class for $\mathcal{A}^{(m),d}_{s}(\mathbb{R}^n)$, where the latter shall be introduced in [25], is characterized by property (3.3). Therefore, $\mathcal{A}^{(m-d),0}_{s-d,cl}(\mathbb{R}^n) = \mathcal{A}^{(m-d),0}_{s-d}(\mathbb{R}^n)$.

Denoting the operator in (3.2) by $\sum_{j<d} P_j + P_d$, where $P_j = p_j(x, D)$ for $j = 0, 1, \ldots, d-1$, we see that

$$P_j \in \bigcap_{t=-s}^{s-2j} \mathcal{L}(H^{t+m}(\mathbb{R}^n), H^{t+j}(\mathbb{R}^n)) \tag{3.4}$$

according to Lemma 3.1. The assumptions on P_d in (3.3) are derived from that fact.

From Lemma 3.1 we obtain:

3.3 Proposition. *Let* $s, m \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$. *Then*

$$P \in \bigcap_{t=-s}^{s-d} \mathcal{L}(H^{t+m}(\mathbb{R}^n), H^{t}(\mathbb{R}^n)) \tag{3.5}$$

for $P \in \mathcal{A}^{(m),d}_{s,cl}(\mathbb{R}^n)$.

We further introduce operator classes $\mathcal{B}^{(m),d}_{s,cl}(\mathbb{R}^n)$ consisting of the formal adjoints to operators in $\mathcal{A}^{(m),d}_{s,cl}(\mathbb{R}^n)$. More precisely, we define:

3.4 Definition. *Let* $s, m \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$.

Then $\mathcal{B}^{(m),d}_{s,cl}(\mathbb{R}^n)$ *denotes the class of all operators* Q *which can be written in the form*

$$Q = \sum_{j=0}^{d-1} q_j(D, x) + Q_d, \tag{3.6}$$

where $q_j(\xi, x) \in H^{s-j}S^{m-j}_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$ *for* $j = 0, 1, \ldots, d-1$ *and*

$$Q_d \in \bigcap_{t=-s+2d}^{s} \mathcal{L}(H^{t-d}(\mathbb{R}^n), H^{t-m}(\mathbb{R}^n)). \tag{3.7}$$

For $\mathcal{B}^{(m),d}_{s,cl}(\mathbb{R}^n)$, similar remarks apply as for $\mathcal{A}^{(m),d}_{s,cl}(\mathbb{R}^n)$. In particular, the analogue of Lemma 3.1 is valid, i.e., for $q(\xi, x) \in H^sS^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$, $q(D, x)$ induces a bounded operator

$$q(D, x) : H^t(\mathbb{R}^n) \to H^{t-m}(\mathbb{R}^n) \tag{3.8}$$

for all $t \in \mathbb{R}$, $|t| \leq s$. For the mapping properties of the operator Q in (3.6) we find

$$Q \in \bigcap_{t=-s+d}^{s} \mathcal{L}(H^t(\mathbb{R}^n),\, H^{t-m}(\mathbb{R}^n)).$$

3.2 One Example

Before we proceed we discuss an example. It is about a second-order partial differential operator in divergence form. A motivation for introducing pseudo-differential operators is that one is interested in the nature of parametrices to partial differential operators of the described kind in case these operators are elliptic.

3.5 Example. Consider the linear partial differential operator

$$A = -\sum_{j,k=1}^{n} \partial_j a_{jk}(x)\partial_k + \sum_{k=1}^{n} a_k(x)\partial_k - \sum_{j=1}^{n} \partial_j a'_j(x) + a(x)\,, \tag{3.9}$$

where

$$a_{jk}(x),\ a_j(x),\ a'_j(x),\ a(x) \in H^s(\mathbb{R}^n)$$

for certain $s \in \mathbb{R}$,

$$s > \frac{n}{2} + 2.$$

Then

$$A \in \mathcal{A}^{(2),1}_{s,cl}(\mathbb{R}^n). \tag{3.10}$$

Next we rise the question what happens if one considers a given operator in $\mathcal{A}^{(m),d}_{s,cl}(\mathbb{R}^n)$ as an operator with asymptotic expansion shorten by 1. Obviously, we have

$$\mathcal{A}^{(m),d}_{s,cl}(\mathbb{R}^n) \not\subseteq \mathcal{A}^{(m),d-1}_{s,cl}(\mathbb{R}^n) \tag{3.11}$$

for any integer $d \geq 1$. The next example shows that (3.11) is due to a "lack" of smoothness of coefficients in the remainder term.

3.6 Example. Consider again the linear partial differential operator from (3.9). Under the above assumptions we also have

$$A \in \mathcal{A}^{(2),2}_{s,cl}(\mathbb{R}^n). \tag{3.12}$$

The minimal hypotheses on the coefficients under which (3.10), (3.12) are true under the restriction that coefficients should belong to L^2-Sobolev spaces are shown in the table:

	$\mathcal{A}^{(2),2}_{s,cl}(\mathbb{R}^n)$	$\mathcal{A}^{(2),1}_{s,cl}(\mathbb{R}^n)$
$a_{jk}(x)$	$H^s(\mathbb{R}^n)$	$H^s(\mathbb{R}^n)$
$a_j(x)$	$H^{s-1}(\mathbb{R}^n)$	$H^{s-1}(\mathbb{R}^n)$
$a'_j(x)$	$H^{s-1}(\mathbb{R}^n)$	$H^s(\mathbb{R}^n)$
$a(x)$	$H^{s-2}(\mathbb{R}^n)$	$H^{s-1}(\mathbb{R}^n)$

The operator classes $\mathcal{A}_{s,cl}^{(m),d,d'}(\mathbb{R}^n)$ are defined as follows:

3.7 Definition. *Let $s, m \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $|d - d'| \leq 2s - 2d$.*
Then $\mathcal{A}_{s,cl}^{(m),d,d'}(\mathbb{R}^n)$ denotes the class of all operators P which can be written in the form

$$P = \sum_{j=0}^{d-1} p_j(x, D) + P_d,$$

where $p_j(x,\eta) \in H^{s-j}S_{cl}^{m-j}(\mathbb{R}^n \times \mathbb{R}^n)$ for $j = 0, 1, \ldots, d-1$ and

$$P_d \in \bigcap_{t=-s+d-\min(d,d')}^{s-d-\max(d,d')} \mathcal{L}(H^{t+m}(\mathbb{R}^n),\ H^{t+d}(\mathbb{R}^n)). \tag{3.13}$$

Property (3.13) means that the scale on which the operators act is shortened by $|d - d'|$, from above if $d' > d$, from below if $d' < d$. For $d' = d$ we get $\mathcal{A}_{s,cl}^{(m),d,d}(\mathbb{R}^n) = \mathcal{A}_{s,cl}^{(m),d}(\mathbb{R}^n)$. The mapping properties in (3.5) are changed into

$$P \in \bigcap_{t=-s+d-\min(d,d')}^{s-\max(d,d')} \mathcal{L}(H^{t+m}(\mathbb{R}^n),\ H^{t}(\mathbb{R}^n)). \tag{3.14}$$

Note that $|d' - d| \leq 2s - 2d$ is required because otherwise the class $\mathcal{A}_{s,cl}^{(m),d,d'}(\mathbb{R}^n)$ would be empty.

With the enlarged operator classes thus defined, the inclusion

$$\mathcal{A}_{s,cl}^{(m),d}(\mathbb{R}^n) \subseteq \mathcal{A}_{s,cl}^{(m),d-1,d}(\mathbb{R}^n), \tag{3.15}$$

substitutes (3.11). Later on we shall see that the classes $\mathcal{A}_{s,cl}^{(m),d,d'}(\mathbb{R}^n)$ also appear, e.g., in the composition of operators and consequently are an integral part of the calculus.

We have again operator classes $\mathcal{B}_{s,cl}^{(m),d,d'}(\mathbb{R}^n)$ defined to consist of the formal adjoints to operators in $\mathcal{A}_{s,cl}^{(m),d,d'}(\mathbb{R}^n)$. These classes are given as in Definition 3.4 with property (3.7) replaced by

$$Q_d \in \bigcap_{t=-s+d+\max(d,d')}^{s-d+\min(d,d')} \mathcal{L}(H^{t-d}(\mathbb{R}^n),\ H^{t-m}(\mathbb{R}^n)). \tag{3.16}$$

3.3 The Full Operator Classes

The operator classes $\mathcal{A}_{s,cl}^{(m),d,d'}(\mathbb{R}^n)$ are preserved under compositions. However, in verifying that fact we encounter additional operator classes. Furthermore, as we shall see, the parametrix to an elliptic operator in $\mathcal{A}_{s,cl}^{(m),d}(X)$, e.g., for X being a closed compact manifold, belongs to the operator class $\mathcal{A}_{s,cl}^{(0),(-m),d}(X)$. Thus we are going to complete the operator classes introduced so far with respect to these operations.

We start with an analogue to Lemma 3.1 and (3.8):

3.8 Lemma. *Let $s, m, m' \in \mathbb{R}$, $s > \frac{n}{2}$. Then, for $p(\xi, x, \eta) \in H^s S_{cl}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^n)$, $p(D, x, D)$ induces a continuous operator*

$$p(D, x, D) : H^{t+m}(\mathbb{R}^n) \to H^{t-m'}(\mathbb{R}^n) \tag{3.17}$$

for all $t \in \mathbb{R}$, $|t| \leq s$.

3.9 Definition. *Let $s, m, m', d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $|d - d'| \leq 2s - 2d$. Then $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ denotes the class of all operators P of the form*

$$P = \sum_{j=0}^{d-1} p_j(D, x, D) + P_d\,, \tag{3.18}$$

where $p_j(\xi, x, \eta) \in H^{s-j} S_{cl}^{m-j,m'}(\mathbb{R}^n \times \mathbb{R}^{2n})$ for $j = 0, 1, \ldots, d-1$ and

$$P_d \in \bigcap_{t=-s+d-\min\{d,d'\}}^{s-d-\max\{d,d'\}} \mathcal{L}(H^{t+m}(\mathbb{R}^n),\, H^{t-m'+d}(\mathbb{R}^n)). \tag{3.19}$$

Similar remarks apply as for the classes $\mathcal{A}_{s,cl}^{(m),d}(\mathbb{R}^n)$. For example, (3.18) gives a finite asymptotic expansion into operators belonging to $\mathcal{A}_{s-j,cl}^{(m-j),(m'),d-j,d'-j}(\mathbb{R}^n)$ for $j = 0, 1, \ldots, d$. For the mapping properties we find

$$P \in \bigcap_{t=-s+d-\min\{d,d'\}}^{s-\max\{d,d'\}} \mathcal{L}(H^{t+m}(\mathbb{R}^n),\, H^{t-m'}(\mathbb{R}^n)) \tag{3.20}$$

for $P \in \mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$. The total order is $m + m'$.
The classes for $m' = 0$ are the same as before:

3.10 Proposition. *Let $s, m, d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $|d - d'| \leq 2s - 2d$. Then*

$$\mathcal{A}_{s,cl}^{(m),d,d'}(\mathbb{R}^n) = \mathcal{A}_{s,cl}^{(m),(0),d,d'}(\mathbb{R}^n). \tag{3.21}$$

Proof: The inclusion $\mathcal{A}_{s,cl}^{(m),d,d'}(\mathbb{R}^n) \subseteq \mathcal{A}_{s,cl}^{(m),(0),d,d'}(\mathbb{R}^n)$ is obvious. To obtain the other direction it suffices to deal with $d' = d$. But then the assertion follows from the proof of Proposition 3.15 given below, in the special case $m' = 0$, $r = 0$. □

$\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ arises from $\mathcal{A}_{s,cl}^{(m+m'),d,d'}(\mathbb{R}^n)$ through conjugation with $\langle D\rangle^{m'}$:

3.11 Proposition. *Let s, m, $d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $|d - d'| \leq 2s - 2d$. Then*

$$\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n) = \langle D\rangle^{m'} \mathcal{A}_{s,cl}^{(m+m'),d,d'}(\mathbb{R}^n) \langle D\rangle^{-m'}. \tag{3.22}$$

Proof: The relation $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n) = \langle D\rangle^{m'} \mathcal{A}_{s,cl}^{(m+m'),(0),d,d'}(\mathbb{R}^n) \langle D\rangle^{-m'}$ follows from the definition. Proposition 3.10 then yields the conclusion. □

In an analogous manner the classes $\mathcal{B}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ are defined. An operator Q belongs to $\mathcal{B}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ if it has the form

$$Q = \sum_{j=0}^{d-1} q_j(D, x, D) + Q_d, \tag{3.23}$$

where $q_j \in H^{s-j}S_{cl}^{m,m'-j}(\mathbb{R}^n \times \mathbb{R}^{2n})$ for $j = 0, 1, \dots, d-1$ and

$$Q_d \in \bigcap_{t=-s+d+\max\{d,d'\}}^{s-d+\min\{d,d'\}} \mathcal{L}(H^{t+m-d}(\mathbb{R}^n), H^{t-m'}(\mathbb{R}^n)). \tag{3.24}$$

We have $\mathcal{B}_{s,cl}^{(m),d,d'}(\mathbb{R}^n) = \mathcal{B}_{s,cl}^{(0),(m),d,d'}(\mathbb{R}^n)$. The adjoint to an operator in $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ belongs to $\mathcal{B}_{s,cl}^{(m'),(m),d,d'}(\mathbb{R}^n)$.

We conclude this subsection by topologizing the operator classes $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ anticipating explanations given in Subsection 4.1 on symbols. $\mathcal{A}_{s-d}^{(m-d),(m'),0,d'-d}(\mathbb{R}^n)$ is a Banach space by interpolation. An operator $P \in \mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ has uniquely determined homogeneous symbols $p_j \in H^{s-j}S^{(m+m'-j)}(T^*\mathbb{R}^n \setminus 0)$ for $j = 0 \dots, d-1$ (see Definition 4.2) leading to a representation of $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ as a direct sum:

$$\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n) = \bigoplus_{j=0}^{d-1} H^{s-j}S^{(m+m'-j)}(T^*\mathbb{R}^n \setminus 0) \oplus \mathcal{A}_{s-d}^{(m-d),(m'),0,d'-d}(\mathbb{R}^n). \tag{3.25}$$

Notice a certain indetermination contained in the composition (3.25) consisting in another possible choice of the 0-excision function ψ in fixing the splitting in (4.3). It is, however, plain that the topology of the locally convex direct sum is independent of that choice. Then $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ becomes equipped with the resulting Fréchet topology.

3.4 The Basic Technique

The results leading to the different conclusions concerning the operator calculus will be derived by applying Taylor's formula to produce the asymptotic expansions and then estimating the remainder terms showing that the remainders obey the right mapping properties. Thereby, in the latter step we encounter expressions of the following kind: For given functions $G(\xi,\eta)$, $g(\xi,\eta)$ we consider

$$(Th)(\xi) = \int G(\xi,\eta) g(\xi-\eta,\eta) h(\eta)\, d\eta \tag{3.26}$$

for $h \in L^2(\mathbb{R}^n)$. We want to find conditions under which T realizes a bounded operator on $L^2(\mathbb{R}^n)$.

In [2], M. Beals and M. Reed made the following simple observation. For the sake of completeness we indicate the proof:

3.12 Lemma. *Let $G(\xi,\eta)$, $g(\xi,\eta)$ be measurable functions on $\mathbb{R}^n \times \mathbb{R}^n$. Suppose that*

$$\sup_\xi \int |G(\xi,\eta)|^2\, d\eta = C_G^2 < \infty, \quad \sup_\eta \int |g(\xi,\eta)|^2\, d\xi = C_g^2 < \infty$$

or

$$\sup_\eta \int |G(\xi,\eta)|^2\, d\xi = C_G^2 < \infty, \quad \sup_\xi \int |g(\xi-\eta,\eta)|^2\, d\eta = C_g^2 < \infty.$$

Then (3.26) *defines a bounded operator on $L^2(\mathbb{R}^n)$ satisfying*

$$\|Th\|_{L^2} \leq C_G C_g \|h\|_{L^2}. \tag{3.27}$$

Proof: We only treat the case when the first of the assumptions is fulfilled. The other proof is similar. For $v \in L^2(\mathbb{R}^n)$, one estimates

$$\begin{aligned}
\left|\int Th(\xi)v(\xi)\, d\xi\right| &= \left|\int G(\xi,\eta)g(\xi-\eta,\eta)h(\eta)v(\xi)\, d\eta\, d\xi\right| \\
&\leq \left\{\int |G(\xi,\eta)|^2 |v(\xi)|^2\, d\eta\, d\xi\right\}^{1/2} \left\{\int |g(\xi-\eta,\eta)|^2 |h(\eta)|^2\, d\eta\, d\xi\right\}^{1/2} \\
&\leq C_G \|v\|_{L^2} C_g \|h\|_{L^2},
\end{aligned}$$

which implies that (3.27) holds. □

To be able to apply Lemmas 3.12 we need the following statement in which certain Sobolev exponents are regarded:

3.13 Lemma. *Let s, t, $r \in \mathbb{R}$. Suppose that, for some $\delta > 0$, $\min\{s, t, s+t-\frac{n}{2}-\delta\} \geq 0$ and*

$$r \leq \min\{s, t, s+t-\frac{n}{2}-\delta\}$$

hold. Then

$$\sup_{\xi\in\mathbb{R}^n} \int \frac{\langle\xi\rangle^{2r}}{\langle\xi-\eta\rangle^{2s}\langle\eta\rangle^{2t}}\, d\eta < \infty. \tag{3.28}$$

Proof: The proof follows by splitting the integral in (3.28) into four integrals over the regions $\{(\xi,\eta)\in\mathbb{R}^{2n};\ |\xi-\eta|\le 1,\ |\eta|\le 1\}$, $\{(\xi,\eta)\in\mathbb{R}^{2n};\ |\xi-\eta|>1,\ |\eta|\le 1\}$, $\{(\xi,\eta)\in\mathbb{R}^{2n};\ |\xi-\eta|\le 1,\ |\eta|>1\}$, and $\{(\xi,\eta)\in\mathbb{R}^{2n};\ |\xi-\eta|>1,\ |\eta|>1\}$, respectively. □

Now we come to the announced proofs of Lemma 3.1 and Lemma 3.8 so establishing the basic technique used later on. Of course, it would suffice only to establish Lemma 3.8, but we use the proof of Lemma 3.1 to familiarize the reader with the technique used in [2].

Proof of Lemma 3.1: We present two different proofs. The first one makes direct use of the estimate of M. Beals and M. Reed and works only in the case when $t\ge 0$, whereas the second one additionally uses considerations involving the projective tensor product.

(a) Assume that $t\ge 0$. Under this assumption we are going to show that the L^2-norm of $\langle\xi\rangle^{t+m}\hat{u}(\xi)$ yields an upper bound for the L^2-norm of $\langle\xi\rangle^t(p(x,D)u)\hat{}(\xi)$.

To do so write

$$\langle\xi\rangle^t(p(x,D)u)\hat{}(\xi)=\int\frac{\langle\xi\rangle^t}{\langle\xi-\eta\rangle^s\langle\eta\rangle^t}\langle\xi-\eta\rangle^s\hat{p}(\xi-\eta,\eta)\langle\eta\rangle^{-m}\langle\eta\rangle^{t+m}\hat{u}(\eta)\,d\eta. \tag{3.29}$$

and apply Lemma 3.12 with

$$G(\xi,\eta)=\frac{\langle\xi\rangle^t}{\langle\xi-\eta\rangle^s\langle\eta\rangle^t},\quad g(\xi,\eta)=\langle\xi\rangle^s\hat{p}(\xi,\eta)\langle\eta\rangle^{-m},$$

where the first of its assumptions is fulfilled.

(b) Exemplary we treat the case $t<0$.

Writing $p(x,\eta)=a(x)p_0(\eta)$ with $a\in H^s(\mathbb{R}^n)$, $p_0\in S^m_{cl}(\mathbb{R}^n)$, Lemma 3.12 applies to (3.29) with

$$G(\xi,\eta)=\frac{\langle\eta\rangle^{-t}}{\langle\xi-\eta\rangle^s\langle\xi\rangle^{-t}}\langle\eta\rangle^{-m}p_0(\eta),\quad g(\xi,\eta)=\langle\xi\rangle^s\hat{a}(\xi),$$

with now the second assumption fulfilled, showing that the bilinear mapping

$$H^s(\mathbb{R}^n)\times S^m_{cl}(\mathbb{R}^n)\to\mathcal{L}(H^{t+m}(\mathbb{R}^n),H^t(\mathbb{R}^n)),\quad (a,p_0)\mapsto a(x)\,p_0(D)$$

is continuous. Equivalently, the linearization of the last mapping,

$$H^s(\mathbb{R}^n)\otimes_\pi S^m_{cl}(\mathbb{R}^n)\to\mathcal{L}(H^{t+m}(\mathbb{R}^n),H^t(\mathbb{R}^n)),\quad a\otimes p_0\mapsto a(x)\,p_0(D)$$

is continuous and extends by continuity, in view of Proposition 2.4, to a continuous mapping

$$H^sS^m_{cl}(\mathbb{R}^n\times\mathbb{R}^n)\to\mathcal{L}(H^{t+m}(\mathbb{R}^n),H^t(\mathbb{R}^n)).$$

Thereby, the symbol $p(x,\xi)$ is mapped to the operator $p(x,D)$, since this is true on $H^s(\mathbb{R}^n)\otimes S^m_{cl}(\mathbb{R}^n)$ and the mapping $H^sS^m_{cl}(\mathbb{R}^n\times\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$, where $p\mapsto p(x,D)u$, is continuous for each $u\in\mathcal{S}(\mathbb{R}^n)$. □

Notice that the first proof also does not work for symbols $p(\xi,x,\eta)$ even in the case that $t\ge 0$. But the second proof does, and we obtain:

Proof of Lemma 3.8: We apply again Lemma 3.12. But this time we only provide the functions $G(\xi,\eta)$, $g(\xi,\eta)$ showing continuity of the trilinear mapping

$$\begin{aligned} S^{m'}_{cl}(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \times S^m_{cl}(\mathbb{R}^n) &\to \mathcal{L}(H^{t+m}(\mathbb{R}^n), H^{t-m'}(\mathbb{R}^n)), \\ (p_1, a, p_0) &\mapsto p_1(D)\, a(x)\, p_0(D). \end{aligned}$$

In case $t \geq 0$

$$G(\xi,\eta) = \frac{\langle\xi\rangle^t}{\langle\xi-\eta\rangle^s\langle\eta\rangle^t}\,\langle\xi\rangle^{-m'}p_1(\xi)\,\langle\eta\rangle^{-m}p_0(\eta), \quad g(\xi,\eta) = \langle\xi\rangle^s\hat{a}(\xi),$$

in case $t < 0$

$$G(\xi,\eta) = \frac{\langle\eta\rangle^{-t}}{\langle\xi-\eta\rangle^s\langle\xi\rangle^{-t}}\,\langle\xi\rangle^{-m'}p_1(\xi)\,\langle\eta\rangle^{-m}p_0(\eta), \quad g(\xi,\eta) = \langle\xi\rangle^s\hat{a}(\xi),$$

where in any case $h(\eta) = \langle\eta\rangle^{t+m}\hat{u}(\eta)$. □

Finally, we provide a natural companion to Lemma 3.12 which we use in the proof of Proposition 3.20. For that we consider the formal expression

$$(Th)(\xi) = \int G(\xi,\zeta,\eta)g(\xi-\zeta,\zeta-\eta,\eta)h(\eta)\,d\zeta\,d\eta, \tag{3.30}$$

for given functions $G(\xi,\zeta,\eta)$, $g(\xi,\zeta,\eta)$.

3.14 Lemma. *Let $G(\xi,\zeta,\eta)$, $g(\xi,\zeta,\eta)$ be measurable functions on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. Suppose that*

$$\sup_\xi \int |G(\xi,\zeta,\eta)|^2\,d\eta\,d\zeta = C_G^2 < \infty, \quad \sup_\eta \int |g(\xi,\zeta,\eta)|^2\,d\xi\,d\zeta = C_g^2 < \infty$$

or

$$\sup_\eta \int |G(\xi,\zeta,\eta)|^2\,d\zeta\,d\xi = C_G^2 < \infty, \quad \sup_\xi \int |g(\xi-\zeta,\zeta-\eta,\eta)|^2\,d\zeta\,d\eta = C_g^2 < \infty.$$

Then (3.30) *defines a bounded operator on $L^2(\mathbb{R}^n)$ satisfying*

$$\|Th\|_{L^2} \leq C_G C_g \|h\|_{L^2}. \tag{3.31}$$

A possible generalization of Lemma 3.13 is as follows. Let s_0, s_1, t, $r \in \mathbb{R}$ satisfying $\min\{s_0, s_1, t, s_0+s_1-\frac{n}{2}-\delta, s_0+t-\frac{n}{2}-\delta, s_1+t-\frac{n}{2}-\delta, s_0+s_1+t-n-2\delta\} \geq 0$ and

$$\begin{aligned} r \leq \min\{s_0, s_1, t, s_0+s_1-\frac{n}{2}-\delta, s_0+t-\frac{n}{2}-\delta, \\ s_1+t-\frac{n}{2}-\delta, s_0+s_1+t-n-2\delta\} \end{aligned} \tag{3.32}$$

for some $\delta > 0$. Then

$$\sup_{\xi\in\mathbb{R}^n}\int \frac{\langle\xi\rangle^{2r}}{\langle\xi-\zeta\rangle^{2s_1}\langle\zeta-\eta\rangle^{2s_0}\langle\eta\rangle^{2t}}\,d\zeta\,d\eta < \infty. \tag{3.33}$$

A proof follows by noting that for $k = \min\{s_0, t, s_0+t-\frac{n}{2}-\delta\}$ the quantity $\min\{s_1, k, s_1+k-\frac{n}{2}-\delta\}$ is equal to the right-hand side of (3.32). Therefore,

$$\begin{aligned}&\sup_{\xi\in\mathbb{R}^n}\int \frac{\langle\xi\rangle^{2r}}{\langle\xi-\zeta\rangle^{2s_1}\langle\zeta-\eta\rangle^{2s_0}\langle\eta\rangle^{2t}}\,d\zeta\,d\eta\\ &\qquad\le \sup_{\xi\in\mathbb{R}^n}\int \frac{\langle\xi\rangle^{2r}}{\langle\xi-\zeta\rangle^{2s_1}\langle\zeta\rangle^{2k}}\,d\zeta\ \sup_{\zeta\in\mathbb{R}^n}\int \frac{\langle\zeta\rangle^{2k}}{\langle\zeta-\eta\rangle^{2s_0}\langle\eta\rangle^{2t}}\,d\eta < \infty.\end{aligned}$$

3.5 Change of Representation, Compositions, and Adjoints

After introducing the several operator classes in previous subsections we come now to further elements of the classical operator calculus like compositions, adjoints, and commutators. First we clarify what happens when one representation is changed for another.

The particulars in the subsequent proof are carried out in detail making use of the techniques developed in Subsection 3.4. Later in similar proofs we shall confine ourselves to certain steps, e.g., we indicate the changes to the proof of Proposition 3.15.

3.15 Proposition. *Let $s, m, m', r \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2}+d$, $|m'-r| \le 2s-2d$. Then we have*

$$\mathcal{A}_{s,cl}^{(m),(m'),d}(\mathbb{R}^n) \subseteq \mathcal{A}_{s,cl}^{(m+m'-r),(r),d,d+m'-r}(\mathbb{R}^n). \tag{3.34}$$

In particular, we have $\mathcal{A}_{s,cl}^{(m),(m'),d,d-m'+r}(\mathbb{R}^n) = \mathcal{A}_{s,cl}^{(m+m'-r),(r),d,d+m'-r}(\mathbb{R}^n)$.

To prove Proposition 3.15 we need two lemmas. The proof of the first one is straightforward.

3.16 Lemma. *Let $r \in \mathbb{R}$, $r \ge 0$. Then there exist symbols $\chi_0, \chi_1 \in S^{0,0}(\mathbb{R}^n\times\mathbb{R}^n)$ such that*

$$\langle\xi\rangle^r = \chi_0(\xi,\eta)\,\langle\eta\rangle^r + \chi_1(\xi,\eta)\,\langle\xi-\eta\rangle^r \tag{3.35}$$

holds. In case $\langle\xi\rangle^r$, $\langle\eta\rangle^r$ are classical symbols (as we always assume) $\chi_0(\xi,\eta)$, $\chi_1(\xi,\eta)$ could be chosen to belong to $S_{cl}^{0,0}(\mathbb{R}^n\times\mathbb{R}^n)$.

The main step in the proof of Proposition 3.15 consists in establishing the next lemma.

3.17 Lemma. *Let $s, m, m', r \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2}+d$, $-2s+m'+d \le r \le 2s+m'-2d$. Let further $p(\xi,x,\eta) \in H^sS_{cl}^{m,m'}(\mathbb{R}^n\times\mathbb{R}^{2n})$. Then we have*

$$p(D,x,D) \in \begin{cases} \mathcal{A}_{s,cl}^{(m+m'-r),(r),d,m'-r}(\mathbb{R}^n) & \text{if } d \le m'-r \le 2s-d,\\ \mathcal{A}_{s,cl}^{(m+m'-r),(r),d}(\mathbb{R}^n) & \text{if } 0 \le m'-r \le d,\\ \mathcal{A}_{s,cl}^{(m+m'-r),(r),d,d+m'-r}(\mathbb{R}^n) & \text{if } -2s+2d \le m'-r \le 0.\end{cases} \tag{3.36}$$

Proof: First we show that, for $|d - m' + r| \leq 2s - 2d$, we have

$$p(D,x,D) \in \mathcal{A}_{s,cl}^{(m+m'-r),(r),d,m'-r}(\mathbb{R}^n). \tag{3.37}$$

We set $p(\xi,x,\eta) = \langle\xi\rangle^r q(\xi,x,\eta)$ with $q(\xi,x,\eta) \in H^s S_{cl}^{m,m'-r}(\mathbb{R}^n \times \mathbb{R}^{2n})$. Then

$$\begin{aligned}(p(D,x,D)u)\hat{}(\xi) &= \int \langle\xi\rangle^r \hat{q}(\xi,\xi-\eta,\eta)\hat{u}(\eta)\,d\eta \\ &= \int \sum_{|\alpha|<d} \frac{\langle\xi\rangle^r}{\alpha!}(\xi-\eta)^\alpha(\partial_\xi^\alpha q)\hat{}(\eta,\xi-\eta,\eta)\hat{u}(\eta)\,d\eta \\ &\quad + \int d \sum_{|\alpha|=d} \frac{\langle\xi\rangle^r}{\alpha!}\left\{\int_0^1 (1-t)^{d-1}(D_x^\alpha\partial_\xi^\alpha q)\hat{}(\eta+t(\xi-\eta),\xi-\eta,\eta)\,dt\right\}\hat{u}(\eta)\,d\eta.\end{aligned} \tag{3.38}$$

Thus

$$p(D,x,D) = \sum_{|\alpha|<d} \frac{1}{\alpha!}\langle D\rangle^r \big(D_x^\alpha\partial_\xi^\alpha q(\xi,x,\eta)\big)_{|\xi=\eta}(x,D) + R,$$

where for the symbols, for $|\alpha| < d$, we have

$$\frac{\langle\xi\rangle^r}{\alpha!}\,(D_x^\alpha\partial_\xi^\alpha q)(\xi,x,\eta)_{|\xi=\eta} \in H^{s-|\alpha|}S_{cl}^{m+m'-r-|\alpha|,r}(\mathbb{R}^n \times \mathbb{R}^{2n}).$$

Hence it remains to prove that the remainder R given by the third line in (3.38) is a bounded operator from $H^{t+m+m'-r}(\mathbb{R}^n)$ to $H^{t-r+d}(\mathbb{R}^n)$ for each $t \in [-s+d-\min\{d, m'-r\}, s-d-\max\{d,m'-r\}]$.

According to our general procedure we prove it when the symbol $p(\xi,x,\eta) \in H^s S_{cl}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^{2n})$ has product form, i.e., $q(\xi,x,\eta) = q_1(\xi)\,a(x)\,q_0(\eta)$ with $a \in H^s(\mathbb{R}^n)$, $q_0 \in S_{cl}^m(\mathbb{R}^n)$, $q_1 \in S_{cl}^{m'-r}(\mathbb{R}^n)$. In that case the defining formula for the remainder R becomes

$$\begin{aligned}&(Ru)\hat{}(\xi) \\ &= \int d \sum_{|\alpha|=d} \frac{\langle\xi\rangle^r}{\alpha!}\left\{\int_0^1 (1-t)^{d-1}(\partial_\xi^\alpha q_1)(\eta+t(\xi-\eta))\,dt\right\}(D_x^\alpha a)\hat{}(\xi-\eta)q_0(\eta)\hat{u}(\eta)\,d\eta.\end{aligned} \tag{3.39}$$

Now the expression $\int_0^1 (1-t)^{d-1}(\partial_\xi^\alpha q_1)(\eta+t(\xi-\eta))\,dt$ can be rewritten as $k_0(\xi,\eta)\,\langle\xi\rangle^{m'-r-d} + k_1(\xi,\eta)\,\langle\eta\rangle^{m'-r-d}$, where k_0, $k_1 \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. This is seen by writing

$$k_0(\xi,\eta) = \frac{k(\xi,\eta)\,\chi(\{(\xi,\eta);\ \langle\xi\rangle < \langle\eta\rangle\})}{\langle\xi\rangle^{m'-r-d}}, \quad k_1(\xi,\eta) = \frac{k(\xi,\eta)\,\chi(\{(\xi,\eta);\ \langle\xi\rangle \geq \langle\eta\rangle\})}{\langle\eta\rangle^{m'-r-d}}$$

in case $m'-r-d < 0$, and

$$k_0(\xi,\eta) = \frac{k(\xi,\eta)\,\chi(\{(\xi,\eta);\ \langle\xi\rangle \geq \langle\eta\rangle\})}{\langle\xi\rangle^{m'-r-d}}, \quad k_1(\xi,\eta) = \frac{k(\xi,\eta)\,\chi(\{(\xi,\eta);\ \langle\xi\rangle < \langle\eta\rangle\})}{\langle\eta\rangle^{m'-r-d}}$$

in case $m'-r-d \geq 0$, respectively, where we have set $k(\xi,\eta) = \int_0^1 (1-t)^{d-1}(\partial_\xi^\alpha q_1)(\eta + t(\xi-\eta))\,dt$, and $\chi(M)(\xi,\eta)$ is the characteristic function of a set $M \subseteq \mathbb{R}^{2n}$.

In this situation Lemma 3.12 applies. For the first summand, the first assumption in Lemma 3.12 is fulfilled in case $t \in [-m'+r, s-d-m'+r]$ if we set

$$h(\eta) = \langle\eta\rangle^{t+m+m'-r}\hat{u}(\eta), \quad g(\xi,\eta) = \langle\xi\rangle^{s-d}(D_x^\alpha a)\hat{\ }(\xi),$$

$$G(\xi,\eta) = k_0(\xi,\eta)\,\frac{\langle\xi\rangle^{t+m'-r}}{\langle\xi-\eta\rangle^{s-d}\langle\eta\rangle^{t+m'-r}}\,\langle\eta\rangle^{-m}q_0(\eta),$$

whereas the second assumption is fulfilled in case $t \in [-s+d-m'+r, -m'+r]$ if we set

$$h(\eta) = \langle\eta\rangle^{t+m+m'-r}\hat{u}(\eta), \quad g(\xi,\eta) = \langle\xi\rangle^{s-d}(D_x^\alpha a)\hat{\ }(\xi),$$

$$G(\xi,\eta) = k_0(\xi,\eta)\,\frac{\langle\eta\rangle^{-t-m'+r}}{\langle\xi-\eta\rangle^{s-d}\langle\xi\rangle^{-t-m'+r}}\,\langle\eta\rangle^{-m}q_0(\eta).$$

This shows that this part of the remainder R defines a bounded operator acting from $H^{t+m+m'-r}(\mathbb{R}^n)$ to $H^{t-r+d}(\mathbb{R}^n)$ for any $t \in [-s+d-m'+r, s-d-m'+r]$.

In a similar fashion we argue for the second summand. This time the first assumption in Lemma 3.12 is fulfilled in case $t \in [-d, s-2d]$ if we set

$$h(\eta) = \langle\eta\rangle^{t+m+m'-r}\hat{u}(\eta), \quad g(\xi,\eta) = \langle\xi\rangle^{s-d}(D_x^\alpha a)\hat{\ }(\xi),$$

$$G(\xi,\eta) = k_1(\xi,\eta)\,\frac{\langle\xi\rangle^{t+d}}{\langle\xi-\eta\rangle^{s-d}\langle\eta\rangle^{t+d}}\,\langle\eta\rangle^{-m}q_0(\eta),$$

whereas the second assumption is fulfilled in case $t \in [-s, -d]$ if we set

$$h(\eta) = \langle\eta\rangle^{t+m+m'-r}\hat{u}(\eta), \quad g(\xi,\eta) = \langle\xi\rangle^{s-d}(D_x^\alpha a)\hat{\ }(\xi),$$

$$G(\xi,\eta) = k_1(\xi,\eta)\,\frac{\langle\eta\rangle^{-t-d}}{\langle\xi-\eta\rangle^{s-d}\langle\xi\rangle^{-t-d}}\,\langle\eta\rangle^{-m}q_0(\eta).$$

Thus the second part of R defines a bounded operator acting from $H^{t+m+m'-r}(\mathbb{R}^n)$ to $H^{t-r+d}(\mathbb{R}^n)$ for any $t \in [-s, s-2d]$.

All in all we have obtained that

$$R \in \bigcap_{-s+d-\min\{d,m'-r\}}^{s-d-\max\{d,m'-r\}} \mathcal{L}(H^{t+m+m'-r}(\mathbb{R}^n), H^{t-r}(\mathbb{R}^n))$$

as required, in the special case that the symbol $p(\xi,x,\eta)$ has product form.

Since Lemma 3.12 also provides us with corresponding estimates, we have actually shown that the multilinear mapping

$$\begin{array}{rcl} S_{cl}^{m'}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)\times S_{cl}^m(\mathbb{R}^n) & \to & \mathcal{A}_{s-d}^{(m+m'-r-d),(r),0,m'-r-d}(\mathbb{R}^n), \\ (\langle\xi\rangle^r q_1, a, q_0) & \mapsto & R \end{array} \tag{3.40}$$

is continuous, with R given by (3.39). Hence, according to the properties of the projective tensor product, the linearization of (3.40) is continuous as mapping

$$S_{cl}^{m'}(\mathbb{R}^n)\otimes_\pi H^s(\mathbb{R}^n)\otimes_\pi S_{cl}^m(\mathbb{R}^n) \to \mathcal{A}_{s-d}^{(m+m'-r-d),(r),0,m'-r-d}(\mathbb{R}^n)$$

and may be extended by continuity to a continuous mapping

$$H^sS_{cl}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^{2n}) \to \mathcal{A}_{s-d}^{(m+m'-r-d),(r),0,m'-r-d}(\mathbb{R}^n). \tag{3.41}$$

Thereby, it is seen that the symbol $p(\xi, x, \eta) \in H^sS_{cl}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^{2n})$ is mapped to the operator R given by the third line in (3.38), since the mapping $H^sS_{cl}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^{2n}) \to \mathcal{S}'(\mathbb{R}^n)$, $p \mapsto \int d \sum_{|\alpha|=d} \langle\xi\rangle^r/\alpha! \{\int_0^1 (1-t)^{d-1}(D_x^\alpha \partial_\xi^\alpha q)\hat{}(\eta + t(\xi-\eta), \xi-\eta, \eta)\, dt\}\hat{u}(\eta)\, d\eta$, $p = \langle\xi\rangle^r q$, is continuous for each $u \in \mathcal{S}(\mathbb{R}^n)$. Thus (3.37) follows.

To conclude the proof, for a symbol $p(\xi, x, \eta) \in H^sS_{cl}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^{2n})$ we write

$$p(\xi, x, \eta) = \langle\eta\rangle^d q(\xi, x, \eta) = \chi_0(\xi, \eta)\langle\xi\rangle^d q(\xi, x, \eta) + \chi_1(\xi, \eta)\langle\xi - \eta\rangle^d q(\xi, x, \eta)$$

with $q \in H^sS_{cl}^{m-d,m'}(\mathbb{R}^n \times \mathbb{R}^{2n})$, and $\chi_0, \chi_1 \in S_{cl}^{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$ according to Lemma 3.16. Now $\langle\xi\rangle^d q(\xi, x, \eta) \in H^sS_{cl}^{m-d,m'+d}(\mathbb{R}^n \times \mathbb{R}^{2n})$, and the operator with symbol $\langle\xi - \eta\rangle^d q(\xi, x, \eta)$ can be written as an operator with symbol $h(\xi, x, \eta) \in H^{s-d}S_{cl}^{m-d,m'}(\mathbb{R}^n \times \mathbb{R}^{2n})$ if we put $h(\xi, x, \eta) = F_{\zeta\to x}^{-1}\{\langle\zeta\rangle^d \hat{q}(\xi, \zeta, \eta)\}$. Therefore, by (3.37) the operator $p(D, x, D)$ has been represented as the sum of an operator in $\mathcal{A}_{s,cl}^{(m+m'-r),(r),d,d+m'-r}(\mathbb{R}^n)$ and an operator in $\mathcal{A}_{s-d}^{(m+m'-r-d),(r),0,m'-r}(\mathbb{R}^n)$ if $|m' - r| \le 2s - 2d$. That is, in this case we have seen that

$$p(D, x, D) \in \mathcal{A}_{s,cl}^{(m+m'-r),(r),d,d+m'-r}(\mathbb{R}^n). \tag{3.42}$$

(3.36) follows from a discussion of the several cases resulting from (3.37), (3.42). □

Notice that the choices of $g(\xi, \eta)$, $G(\xi, \eta)$ made in the proof of Lemma 3.17 can be recorded, e.g., for the first case in the form

$$g(\xi - \eta, \eta) = \langle\xi - \eta\rangle^{s-d}(D_x^\alpha a)\hat{}(\xi - \eta),$$

$$G(\xi, \eta) = k_0(\xi, \eta)\, \frac{\langle\xi\rangle^{t+m'-r}}{\langle\xi - \eta\rangle^{s-d}\langle\eta\rangle^{t+m'-r}}\, \langle\eta\rangle^{-m} q_0(\eta)$$

leaving it open whether $t + m' - r \ge 0$ holds or not and whether in the assumptions of Lemma 3.12 integration of the square of the modulus of $g(\xi - \eta, \eta)$ takes place with respect to ξ and η, respectively.

Proof of Proposition 3.15: Let $P \in \mathcal{A}_{s,cl}^{(m),(m'),d}(\mathbb{R}^n)$. According to Definition 3.9 write $P = \sum_{j<d} P_j + P_d$ with $P_j = p_j(D, x, D)$, $p_j \in H^{s-j}S_{cl}^{m-j,m'}(\mathbb{R}^n \times \mathbb{R}^{2n})$ for $j = 0, 1, \ldots, d-1$. By the foregoing lemma (see especially (3.42)) we have

$$P_j \in \mathcal{A}_{s-j,cl}^{(m+m'-j-r),(r),d-j,d+m'-j-r}(\mathbb{R}^n)$$

for $j = 0, \ldots, d-1$, so it remains deal with P_d.

We show that $\mathcal{A}_{s-d}^{(m-d),(m'),0,-m'+r}(\mathbb{R}^n) = \mathcal{A}_{s-d}^{(m+m'-r-d),(r),0,m'-r}(\mathbb{R}^n)$ holds, i.e.,

$$\bigcap_{t=-s+d-\min\{d,d-m'+r\}}^{s-d-\max\{d,d-m'+r\}} \mathcal{L}(H^{t+m}(\mathbb{R}^n), H^{t+d}(\mathbb{R}^n))$$

$$= \bigcap_{t=-s+d-\min\{d,d+m'-r\}}^{s-d-\max\{d,d+m'-r\}} \mathcal{L}(H^{t+m+m'-r}(\mathbb{R}^n), H^{t-r+d}(\mathbb{R}^n)).$$

To do so we have to verify $s-d-\max\{d,d-m'+r\} = s-d+m'-r-\max\{d,d+m'-r\}$, $-s+d-\min\{d,d-m'+r\} = -s+d+m'-r-\min\{d,d+m'-r\}$. The first relation is implied by $\max\{d,d-m'+r\} = 2d-m'+r-\min\{d,d-m'+r\} = 2d-m'+r+\max\{-d,-d+m'-r\} = -m'+r+\max\{d,d+m'-r\}$, the second relation follows in an analogous manner.

Therefore,

$$P_d \in \mathcal{A}_{s-d}^{(m-d),(m'),0}(\mathbb{R}^n) \subseteq \mathcal{A}_{s-d}^{(m-d),(m'),0,-m'+r}(\mathbb{R}^n) = \mathcal{A}_{s-d}^{(m+m'-r-d),(r),0,m'-r}(\mathbb{R}^n),$$

which proves the first part of the proposition.

The second part follows from

$$\begin{aligned} \mathcal{A}_{s,cl}^{(m),(m'),d,d-m'+r}(\mathbb{R}^n) &= \mathcal{A}_{s,cl}^{(m),(m'),d}(\mathbb{R}^n) + \mathcal{A}_{s-d}^{(m-d),(m'),0,-m'+r}(\mathbb{R}^n) \\ &\subseteq \mathcal{A}_{s,cl}^{(m+m'-r),(r),d,d+m'-r}(\mathbb{R}^n) \end{aligned}$$

by what which has been already proved, and

$$\mathcal{A}_{s,cl}^{(m+m'-r),(r),d,d+m'-r}(\mathbb{R}^n) \subseteq \mathcal{A}_{s,cl}^{(m),(m'),d,d-m'+r}(\mathbb{R}^n)$$

by the same argument. □

In our next result it is asserted that $\mathcal{B}$-classes constitute merely another representation for $\mathcal{A}$-classes.

3.18 Proposition. *Let $s, m, m', d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2}+d$, $|d+m'-r| \le 2s-2d$. Then we have*

$$\mathcal{A}_{s,cl}^{(m),(m'),d}(\mathbb{R}^n) \subseteq \mathcal{B}_{s,cl}^{(m+m'-r),(r),d,r-m'}(\mathbb{R}^n). \tag{3.43}$$

In particular, we have $\mathcal{A}_{s,cl}^{(m),(m'),d,r-m'}(\mathbb{R}^n) = \mathcal{B}_{s,cl}^{(m+m'-r),(r),d,r-m'}(\mathbb{R}^n)$.

Proof: For $P \in \mathcal{A}_{s,cl}^{(m),(m'),d}(\mathbb{R}^n)$ we have to show that $P \in \mathcal{B}_{s,cl}^{(m+m'-r),(r),d,r-m'}(\mathbb{R}^n)$. This is obviously true if $P \in \mathcal{A}_{s-d}^{(m-d),(m'),0}(\mathbb{R}^n)$.

Hence we may assume $P = p(D,x,D)$ for some $p(\xi,x,\eta) \in H^sS_{cl}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^{2n})$. But then we obtain

$$\begin{aligned} (Pu)\hat{}(\xi) &= \int \hat{q}(\xi,\xi-\eta,\eta)\langle\eta\rangle^{m+m'-r}\hat{u}(\eta)\,d\eta \\ &= \int \sum_{|\alpha|<d} \frac{(-1)^{|\alpha|}}{\alpha!}(\xi-\eta)^\alpha(\partial_\eta^\alpha q)\hat{}(\xi,\xi-\eta,\xi)\langle\eta\rangle^{m+m'-r}\hat{u}(\eta)\,d\eta \;+ \end{aligned}$$

$$\int d\sum_{|\alpha|=d}\frac{(-1)^d}{\alpha!}\left\{\int_0^1 (1-t)^{d-1}(D_x^\alpha\partial_\eta^\alpha q)\hat{}(\xi,\xi-\eta,\xi-t(\xi-\eta))\,dt\right\}\langle\eta\rangle^{m+m'-r}\hat{u}(\eta)\,d\eta,$$

where we have set $p(\xi,x,\eta) = q(\xi,x,\eta)\langle\eta\rangle^{m+m'-r}$, $q(\xi,x,\eta) \in H^sS_{cl}^{-m'+r,m'}(\mathbb{R}^n \times \mathbb{R}^{2n})$. Thus

$$P = \sum_{|\alpha|<d}\frac{(-1)^{|\alpha|}}{\alpha!}\left(D_x^\alpha\partial_\eta^\alpha q(\xi,x,\eta)\right)_{|\eta=\xi}(D,x)\,\langle D\rangle^{m+m'-r} + R,$$

where for the symbols, for $|\alpha| < d$, we have

$$\frac{(-1)^{|\alpha|}}{\alpha!}\,(D_x^\alpha \partial_\eta^\alpha q)(\xi,x,\eta)_{|\eta=\xi}\,\langle\eta\rangle^{m+m'-r} \in H^{s-|\alpha|}S_{cl}^{m+m'-r,r-|\alpha|}(\mathbb{R}^n\times\mathbb{R}^{2n}).$$

To treat the remainder we write $q(\xi,x,\eta) = q_1(\xi)\,a(x)\,q_0(\eta)$ with $a \in H^s(\mathbb{R}^n)$, $q_0 \in S_{cl}^{-m'+r}(\mathbb{R}^n)$, $q_1 \in S_{cl}^{m'}(\mathbb{R}^n)$. The defining formula for the remainder term becomes

$$\begin{aligned}(Ru)\hat{}(\xi) \;=\; \int d\sum_{|\alpha|=d}\frac{(-1)^d}{\alpha!}&\left\{\int_0^1 (1-t)^{d-1}(\partial_\eta^\alpha q_0)(\xi - t(\xi-\eta))\,dt\right\}\\ &\times q_1(\xi)(D_x^\alpha a)\hat{}(\xi-\eta)\langle\eta\rangle^{m+m'-r}\hat{u}(\eta)\,d\eta.\end{aligned}$$

The expression $\int_0^1(1-t)^{d-1}(\partial_\eta^\alpha q_0)(\xi - t(\xi-\eta))\,dt$ can be rewritten in the form $k_0(\xi,\eta)$ $\langle\xi\rangle^{-m'+r-d} + k_1(\xi,\eta)\,\langle\eta\rangle^{-m'+r-d}$ with certain k_0, $k_1 \in L^\infty(\mathbb{R}^n\times\mathbb{R}^n)$. Now Lemma 3.12 applies with

$$h(\eta) = \langle\eta\rangle^{t+m+m'-r-d}\hat{u}(\eta), \quad g(\xi,\xi-\eta) = \langle\xi-\eta\rangle^{s-d}(D_x^\alpha a)\hat{}(\xi-\eta),$$

$$G(\xi,\eta) = k_0(\xi,\eta)\frac{\langle\xi\rangle^{t-d}}{\langle\xi-\eta\rangle^{s-d}\langle\eta\rangle^{t-d}}\langle\xi\rangle^{-m'}q_1(\xi),$$

and

$$G(\xi,\eta) = k_1(\xi,\eta)\frac{\langle\xi\rangle^{t+m'-r}}{\langle\xi-\eta\rangle^{s-d}\langle\eta\rangle^{t+m'-r}}\langle\xi\rangle^{-m'}q_1(\xi),$$

respectively, yielding that

$$R \in \bigcap_{-s+d+\max\{d,r-m'\}}^{s-d+\min\{d,r-m'\}} \mathcal{L}(H^{t+m+m'-r-d}(\mathbb{R}^n), H^{t-r}(\mathbb{R}^n)),$$

i.e., we get $P \in \mathcal{B}_{s,cl}^{(m+m'-r),(r),d,r-m'}(\mathbb{R}^n)$.

By symmetry we further have $\mathcal{B}_{s,cl}^{(m+m'-r),(r),d}(\mathbb{R}^n) \subseteq \mathcal{A}_{s,cl}^{(m),(m'),d,r-m'}(\mathbb{R}^n)$, and it is easily checked that $\mathcal{A}_{s-d}^{(m-d),(m'),0,r-m'-d}(\mathbb{R}^n) = \mathcal{B}_{s-d}^{(m+m'-r),(r-d),0,r-m'-d}(\mathbb{R}^n)$. The second assertion follows. □

The second part of Proposition 3.18 can equivalently be formulated as

$$\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n) = \mathcal{B}_{s,cl}^{(m-d'),(m'+d'),d,d'}(\mathbb{R}^n), \tag{3.44}$$

which is seen by setting $r = m' + d'$. Recall that the classes $\mathcal{B}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ have been designed to incorporate the formal adjoints to operators in $\mathcal{A}_{s,cl}^{(m'),(m),d,d'}(\mathbb{R}^n)$. Thus as a further corollary to Proposition 3.18 we obtain that $P \in \mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ implies $P^* \in \mathcal{A}_{s,cl}^{(m'+d'),(m-d'),d,d'}(\mathbb{R}^n)$. We call a representation in $\mathcal{B}$-classes an adjoint representation.

Relation (3.44) can be used to find statements for $\mathcal{B}$-classes from the analogous ones for $\mathcal{A}$-classes. For example, an analogue to Lemma 3.17 is that for s, m, m', $r \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $-2s + m' + 2d \le r \le 2s + m' - d$ we have

$$p(D,x,D) \in \begin{cases} \mathcal{B}_{s,cl}^{(m+m'-r),(r),d,d+m'-r}(\mathbb{R}^n) & \text{if } d \le d+m'-r \le 2s-d,\\ \mathcal{B}_{s,cl}^{(m+m'-r),(r),d}(\mathbb{R}^n) & \text{if } 0 \le d+m'-r \le d,\\ \mathcal{B}_{s,cl}^{(m+m'-r),(r),d,2d+m'-r}(\mathbb{R}^n) & \text{if } -2s+2d \le d+m'-r \le 0 \end{cases}$$

provided that $p(\xi,x,\eta)\in H^sS_{cl}^{m,m'}(\mathbb{R}^n\times\mathbb{R}^{2n})$.

Next the behaviour under compositions is established.

3.19 Proposition. *Let s, m_0, m_0', m_1, $m_1'\in\mathbb{R}$, $d\in\mathbb{N}$, $s>\frac{n}{2}+d$, $|m_0'+m_1|\le 2s-2d$. Then we have*

$$\mathcal{A}_{s,cl}^{(m_0),(m_0'),d}(\mathbb{R}^n)\cdot\mathcal{A}_{s,cl}^{(m_1),(m_1'),d}(\mathbb{R}^n)\subseteq\mathcal{A}_{s,cl}^{(m_0+m_0'+m_1),(m_1'),d,d+(m_0'+m_1)}(\mathbb{R}^n).\tag{3.45}$$

In (3.45), the composition on the left-hand side is understood in the opposite direction, i.e., we mean that $P\in\mathcal{A}_{s,cl}^{(m_0),(m_0'),d}(\mathbb{R}^n)$, $Q\in\mathcal{A}_{s,cl}^{(m_1),(m_1'),d}(\mathbb{R}^n)$ implies $Q\,P\in\mathcal{A}_{s,cl}^{(m_0+m_0'+m_1),(m_1'),d,d+(m_0'+m_1)}(\mathbb{R}^n)$.

To prove Proposition 3.19 we first show the following lemma:

3.20 Lemma. *Let s, m, m', $m''\in\mathbb{R}$, $d\in\mathbb{N}$, $s>\frac{n}{2}+d$, $-2s+2d\le m'\le 2s-d$. Then, for $p\in H^{s,s}S_{cl}^{m,m',m''}(\mathbb{R}^{2n}\times\mathbb{R}^{3n})$, we have*

$$p(D,x,D,x,D)\in\begin{cases}\mathcal{A}_{s,cl}^{(m+m'),(m''),d,m'}(\mathbb{R}^n) & \text{if } d\le m'\le 2s-d,\\ \mathcal{A}_{s,cl}^{(m+m'),(m''),d}(\mathbb{R}^n) & \text{if } 0\le m'\le d,\\ \mathcal{A}_{s,cl}^{(m+m'),(m''),d,d+m'}(\mathbb{R}^n) & \text{if } -2s+2d\le m'\le 0.\end{cases}\tag{3.46}$$

Proof: We first show (3.46) under the assumption that we have already proved it when $m'=0$. By (2.16), (2.17), we may assume that $p(\xi,x,\zeta,y,\eta)$ is given in product form, i.e., $p(\xi,x,\zeta,y,\eta)=p_1(\xi,x,\zeta)\,p_0(\zeta,y,\eta)$, where $p_0\in H^sS_{cl}^{m,m'}(\mathbb{R}^n\times\mathbb{R}^{2n})$, $p_1\in H^sS_{cl}^{0,m''}(\mathbb{R}^n\times\mathbb{R}^{2n})$.

By Lemma 3.17 we have $p_0(D,x,D)\in\mathcal{A}_{s,cl}^{(m+m'),(0),d,l}(\mathbb{R}^n)$ with $l=m'$ if $d\le m'\le 2s-d$, $l=d$ if $0\le m'\le d$, and $l=d+m'$ if $-2s+2d\le m'\le 0$. Thus we write $p_0(D,x,D)=\sum_{j=0}^{d-1}p_{0j}(D,x,D)+P_{0d}$, where $p_{0j}\in H^{s-j}S_{cl}^{m+m'-j,0}(\mathbb{R}^n\times\mathbb{R}^{2n})$, $P_{0d}\in\mathcal{A}_{s-d}^{(m+m'-d),(0),0,l-d}(\mathbb{R}^n)$. We find $p_1(D,x,D)\,p_{0j}(D,x,D)\in\mathcal{A}_{s-j,cl}^{(m+m'-j),(m''),d-j}(\mathbb{R}^n)$ for $j=0,1,\ldots,d-1$, whereas $p_1(D,x,D)\,P_{0d}\in\mathcal{A}_{s-d}^{(m+m'-d),(m''),0,l-d}(\mathbb{R}^n)$ by direct calculation.

It remains to prove the validity of (3.46) in case $m'=0$. We first prove it when $m'=d$. Thus let $p\in H^{s,s}S_{cl}^{m,d,m''}(\mathbb{R}^{2n}\times\mathbb{R}^{3n})$, where we assume that $p(\xi,x,\zeta,y,\eta)=p_2(\xi)\,a_1(x)\,p_1(\zeta)\,a_0(y)\,p_0(\eta)$, a_0, $a_1\in H^s(\mathbb{R}^n)$, $p_0\in S_{cl}^m(\mathbb{R}^n)$, $p_1\in S_{cl}^d(\mathbb{R}^n)$, $p_2\in S_{cl}^{m''}(\mathbb{R}^n)$. We write

$$\begin{aligned}(p(D,x,D,x,D)u)\hat{}(\xi)&=\int\hat{p}(\xi,\xi-\zeta,\zeta,\zeta-\eta,\eta)\hat{u}(\eta)\,d\zeta\,d\eta\\ &=\int\sum_{|\alpha|<d}\frac{1}{\alpha!}(\zeta-\eta)^\alpha(\partial_\zeta^\alpha p)\hat{}(\xi,\xi-\zeta,\eta,\zeta-\eta,\eta)\hat{u}(\eta)\,d\zeta\,d\eta\\ &+\int d\sum_{|\alpha|=d}\frac{1}{\alpha!}\left\{\int(1-t)^{d-1}(D_y^\alpha\partial_\zeta^\alpha p)\hat{}(\xi,\xi-\zeta,\eta+t(\zeta-\eta),\zeta-\eta,\eta)\,dt\right\}\hat{u}(\eta)\,d\zeta\,d\eta\end{aligned}\tag{3.47}$$

and obtain

$$p(D,x,D,x,D)=\sum_{|\alpha|<d}\frac{1}{\alpha!}\left(D_y^\alpha\partial_\zeta^\alpha p)(\xi,x,\zeta,y,\eta)\right)_{|y=x,\,\zeta=\eta}(D,x,D)+R,$$

where the remainder R is given by

$$(Ru)\hat{}(\xi)=\int d\sum_{|\alpha|=d}\frac{1}{\alpha!}\left\{\int(1-t)^{d-1}(\partial_\zeta^\alpha p_1)(\eta+t(\zeta-\eta))\,dt\right\}$$
$$\times\,\hat{a}_1(\xi-\zeta)\,(D_y^\alpha a_0)\hat{}(\zeta-\eta)\,p_2(\xi)\,p_0(\eta)\,\hat{u}(\eta)\,d\zeta\,d\eta.$$

For the symbols, for $|\alpha|<d$, we have

$$\frac{1}{\alpha!}\,(D_y^\alpha\partial_\zeta^\alpha p)(\xi,x,\zeta,y,\eta)_{|y=x\,\zeta=\eta}\in H^{s-|\alpha|}S_{cl}^{m+d-|\alpha|,m''}(\mathbb{R}^n\times\mathbb{R}^{2n}),$$

whereas to estimate the remainder we apply Lemma 3.14 with $t\in[-s,s-2d]$ and

$$h(\eta)=\langle\eta\rangle^{t+m+d}\hat{u}(\eta),$$
$$g(\xi-\zeta,\zeta-\eta,\eta)=\langle\xi-\zeta\rangle^s\hat{a}_1(\xi-\zeta)\langle\zeta-\eta\rangle^{s-d}(D_x^\alpha a_0)\hat{}(\zeta-\eta),$$
$$G(\xi,\zeta,\eta)=k(\zeta,\eta)\,\frac{\langle\xi\rangle^{t+d}}{\langle\xi-\zeta\rangle^s\langle\zeta-\eta\rangle^{s-d}\langle\eta\rangle^{t+d}}\,\langle\xi\rangle^{-m''}p_2(\xi)\,\langle\eta\rangle^{-m}p_0(\eta),$$

with $k(\zeta,\eta)=\int(1-t)^{d-1}(\partial_\zeta^\alpha p_1)(\eta+t(\zeta-\eta))\,dt$ bounded, getting $R\in\mathcal{A}_{s-d}^{(m),(m''),0}(\mathbb{R}^n)$. This shows (3.46) for $m'=d$. Now let $p\in H^{s,s}S_{cl}^{m,0,m''}(\mathbb{R}^{2n}\times\mathbb{R}^{3n})$. Then (3.46) for $m'=0$ follows by writing $p(\xi,x,\zeta,y,\eta)=\langle\eta\rangle^d q(\xi,x,\zeta,y,\eta)=\chi_0(\zeta,\eta)\,\langle\zeta\rangle^d q(\xi,x,\zeta,y,\eta)+\chi_1(\zeta,\eta)\,\langle\zeta-\eta\rangle^d q(\xi,x,\zeta,y,\eta)$ with $q\in H^{s,s}S_{cl}^{m-d,0,m''}(\mathbb{R}^{2n}\times\mathbb{R}^{3n})$, χ_0, $\chi_1\in S_{cl}^{0,0}(\mathbb{R}^{2n})$. □

Note that in the proof of Lemma 3.20 we have obtained as a byproduct

$$p(D,x,D,x,D)\in\mathcal{A}_{s,cl}^{(m+m'),(m''),d,d+m'}(\mathbb{R}^n)\tag{3.48}$$

for $|m'|\le 2s-2d$.

Now we are prepared to give the proof of Proposition 3.19.

P r o o f o f P r o p o s i t i o n 3.19: From (3.48) it follows that, for $p\in H^sS_{cl}^{m_0,m_0'}(\mathbb{R}^n\times\mathbb{R}^{2n})$, $q\in H^sS_{cl}^{m_1,m_1'}(\mathbb{R}^n\times\mathbb{R}^{2n})$, we have

$$q(D,x,D)\,p(D,x,D)\in\mathcal{A}_{s,cl}^{(m_0+m_0'+m_1),(m_1'),d,d+m_0'+m_1}(\mathbb{R}^n)\tag{3.49}$$

for the symbol of the operator $q(D,x,D)\,p(D,x,D)$ is $q(\xi,x,\zeta)\,p(\zeta,y,\eta)$.

Now (3.49) implies that

$$\mathcal{B}_{s,cl}^{(m_0),(m_0'),d}(\mathbb{R}^n)\cdot\mathcal{A}_{s,cl}^{(m_1),(m_1'),d}(\mathbb{R}^n)\subseteq\mathcal{A}_{s,cl}^{(m_0+m_0'+m_1),(m_1'),d,m_0'+m_1}(\mathbb{R}^n).$$

This is seen by choosing $P=\sum_{j<d}P_j+P_d\in\mathcal{B}_{s,cl}^{(m_0),(m_0'),d}(\mathbb{R}^n)$, $Q=\sum_{k<d}Q_k+Q_d\in\mathcal{A}_{s,cl}^{(m_1),(m_1'),d}(\mathbb{R}^n)$, where $P_j=p_j(D,x,D)$, $p_j\in H^{s-j}S_{cl}^{m_0,m_0'-j}(\mathbb{R}^n\times\mathbb{R}^{2n})$ for $j=0,1,\dots,$

$d-1$, $Q_k = q_k(D,x,D)$, $q_k \in H^{s-k}S_{cl}^{m_1-k,m_1'}(\mathbb{R}^n \times \mathbb{R}^{2n})$ for $k = 0,1,\ldots,d-1$. Then, for $j+k = l < d$, we obtain

$$Q_k\, P_j \in \mathcal{A}_{s-l,\,cl}^{(m_0+m_0'+m_1-l),\,(m_1'),\,d-l,\,d+m_0'+m_1-l}(\mathbb{R}^n),$$

whereas, for $j+k \geq d$, it is checked that

$$Q_k\, P_j \in \mathcal{A}_{s-d}^{(m_0+m_0'+m_1-d),\,(m_1'),\,0,\,d+m_0'+m_1-d}(\mathbb{R}^n).$$

Finally, Proposition 3.18 yields

$$\begin{aligned} \mathcal{A}_{s,\,cl}^{(m_0),\,(m_0'),\,d}(\mathbb{R}^n) \cdot \mathcal{A}_{s,\,cl}^{(m_1),\,(m_1'),\,d}(\mathbb{R}^n) &= \mathcal{B}_{s,\,cl}^{(m_0-d),\,(m_0'+d),\,d}(\mathbb{R}^n) \cdot \mathcal{A}_{s,\,cl}^{(m_1),\,(m_1'),\,d}(\mathbb{R}^n) \\ &\subseteq \mathcal{A}_{s,\,cl}^{(m_0+m_0'+m_1),\,(m_1'),\,d,\,d+m_0'+m_1}(\mathbb{R}^n). \end{aligned}$$

The proof is furnished. □

Note the equality

$$\mathcal{A}_{s,\,cl}^{(m_0+m_1+m_0'),\,(m_1'),\,d,\,d+(m_1+m_0')}(\mathbb{R}^n) = \mathcal{A}_{s,\,cl}^{(m_0),\,(m_1+m_0'+m_1'),\,d,\,d-(m_1+m_0')}(\mathbb{R}^n). \tag{3.50}$$

which is valid by Proposition 3.15. Further note that the composition result for the $\mathcal{B}$-classes is

$$\mathcal{B}_{s,\,cl}^{(m_0),\,(m_0'),\,d}(\mathbb{R}^n) \cdot \mathcal{B}_{s,\,cl}^{(m_1),\,(m_1'),\,d}(\mathbb{R}^n) \subseteq \mathcal{B}_{s,\,cl}^{(m_0+m_0'+m_1),\,(m_1'),\,d,\,d+(m_0'+m_1)}(\mathbb{R}^n). \tag{3.51}$$

Finally in this subsection we come to a discussion about commutators. Although it is possible to describe commutators in the calculus in general the formulas arising for remainders are complicated. Therefore, we confine ourselves to the special case when one of the operators involved has smooth coefficients and postpone the general case to [25]. The special case is sufficient, e.g., in treating semi-linear partial differential equations.

The results on commutators could be achieved by considerations similar to those above, but we prefer to take advantage of the elements of the calculus developed so far. In doing so, we have to anticipate two points of our later discussion: The first one is that to construct globally parametrices to elliptic operators within the calculus we adjoin the pseudo-differential calculus of classical operators having their coefficients in $C_b^\infty(\mathbb{R}^n)$ to our calculus, i.e., we work in the operator classes

$$L_{cl}^{m+m'}(\mathbb{R}^n) + \mathcal{A}_{s,\,cl}^{(m),\,(m'),\,d}(\mathbb{R}^n). \tag{3.52}$$

Here $L_{cl}^{m+m'}(\mathbb{R}^n)$ is the space of classical pseudo-differential operators of order $m+m'$ defined on $\mathbb{R}^n$ with uniform symbol estimates in the space variables. Then it is easy to see that our non-smooth calculus obeys the ideal property in the calculus given by (3.52), e.g., we have

$$L_{cl}^{m_0}(\mathbb{R}^n) \cdot \mathcal{A}_{s,\,cl}^{(m),\,(m'),\,d}(\mathbb{R}^n) \subseteq \mathcal{A}_{s,\,cl}^{(m+m_0),\,(m'),\,d}(\mathbb{R}^n).$$

The second point to anticipate is that operators have a symbolic structure. That point will be discussed in detail in Subsection 4.1.

3.21 Proposition. *Let* $s, m, m', m_0 \in \mathbb{R}$, $d \in \mathbb{R}$, $s > \frac{n}{2}$, $d \geq 1$, $|m_0| \leq 2s - 2d$. *Then, for* $A \in \mathcal{A}_{s,cl}^{(m),(m'),d}(\mathbb{R}^n)$, $P \in L_{cl}^{m_0}(\mathbb{R}^n)$, *we have*

$$[P, A] = P\,A - A\,P \in \mathcal{A}_{s-1,cl}^{(m+m_0-1),(m'),d-1,d+m_0-1}(\mathbb{R}^n). \tag{3.53}$$

Proof: We have $A\,P \in \mathcal{A}_{s,cl}^{(m+m_0),(m'),d}(\mathbb{R}^n)$ and $P\,A \in \mathcal{A}_{s,cl}^{(m),(m'+m_0),d}(\mathbb{R}^n)$. By Proposition 3.15, $\mathcal{A}_{s,cl}^{(m),(m'+m_0),d}(\mathbb{R}^n) \subseteq \mathcal{A}_{s,cl}^{(m+m_0),(m'),d,d+m_0}(\mathbb{R}^n)$. Furthermore, the principal symbol of the operator $P\,A - A\,P \in \mathcal{A}_{s,cl}^{(m+m_0),(m'),d,d+m_0}(\mathbb{R}^n)$ vanishes. Hence, $[P, A] \in \mathcal{A}_{s-1,cl}^{(m+m_0-1),(m'),d-1,d+m_0-1}(\mathbb{R}^n)$. □

4 Further Elements

In this section, several elements of the classical operator calculus are further developed. The selection made owns to the author's decision. In Subsection 4.1, principal symbols and subordinated homogeneous components for complete symbols are introduced. The results obtained there are used in the parametrix construction for elliptic operators which is performed in Subsection 4.2. Elliptic regularity and the Fredholm property for operators on closed compact manifolds are dealt with in Subsection 4.4. Before, in Subsection 4.3, we discuss coordinate invariance and operators on manifolds.

4.1 Symbolic structure

Next we become acquainted with the symbolic structure of operators in $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$. For those operators we have principal symbols as well as complete symbols, the latter circumstance is due to the fact that we are working on $\mathbb{R}^n$.

4.1 Lemma. *Let s, m, m', $d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $d \geq 1$, $|d - d'| \leq 2s - 2d$.*

Let $p(\xi, x, \eta) \in H^sS_{cl}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^{2n})$. Then $p(D, x, D)$ belongs to $\mathcal{A}_{s-1,cl}^{(m-1),(m'),d-1,d'-1}(\mathbb{R}^n)$ if and only if $p(\xi, x, \eta) \in H^sS_{cl}^{m-1,m'}(\mathbb{R}^n \times \mathbb{R}^{2n})$.

Proof: First assume $p(D, x, D) \in \mathcal{A}_{s-1,cl}^{(m-1),(m'),d-1,d'-1}(\mathbb{R}^n)$. Writing $p(D, x, D) = \langle D\rangle^{m'}(q_0(x, D) + Q')$, where $q_0 \in H^sS_{cl}^{m+m'}(\mathbb{R}^n \times \mathbb{R}^n)$, $Q' \in \mathcal{A}_{s-1,cl}^{(m+m'-1),d-1,d'-1}(\mathbb{R}^n)$, we reduce to the case that $m' = 0$ and $p \in H^sS_{cl}^{m}(\mathbb{R}^n \times \mathbb{R}^n)$.

By assumption, there exists some $t \in [-s + d - \min\{d, d'\}, s - \max\{d, d'\} - 1]$ such that $p(x, D) \in \mathcal{L}(H^{t+m}(\mathbb{R}^n), H^{t+1}(\mathbb{R}^n))$. Now suppose $p_0(x, \xi) \not\equiv 0$, where p_0 denotes the principal part of p. Choose some $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ such that $p_0(x_0, \xi_0) \neq 0$. Further choose $u \in H^{t+m}(\mathbb{R}^n)$, $u \notin H_{ml}^{t+m+1}(x_0, \xi_0)$, where the latter means u is not in $H^{t+m+1}(\mathbb{R}^n)$ microlocally at (x_0, ξ_0). Then, for $\phi \in C_0^\infty(\mathbb{R}^n)$, $\chi \in S_{cl}^0(\mathbb{R}^n)$ with ϕ supported in a small neighbourhood of x_0, $\phi(x_0) \neq 0$, and χ supported in a small conic neighbourhood of ξ_0, $|\chi(\xi)| \geq c$ for $\xi \in \mathbb{R}^n$ in some smaller conic neighbourhood of ξ_0, $|\xi| \geq C$, we find a symbol $q \in S_{cl}^{-m}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $q(x, \xi)\, p(x, \xi) = \phi(x)\chi(\xi)$. By Proposition 3.19 we have $q(x, D)\, p(x, D) = r_0(x, D) + R'$, where $r_0(x, \xi) = q(x, \xi)\, p(x, \xi)$ and $R' \in \mathcal{A}_{s-1,cl}^{(-1),d-1}(\mathbb{R}^n)$. This implies $\phi(x)\chi(D)u = q(x, D)\, p(x, D)u - R'u \in H^{t+m+1}(\mathbb{R}^n)$, which contradicts $u \notin H_{ml}^{t+m+1}(x_0, \xi_0)$. Hence, $p_0(x, \xi) \equiv 0$, i.e., $p \in H^sS_{cl}^{m-1}(\mathbb{R}^n \times \mathbb{R}^n)$.

The reverse direction is straightforward. □

4.2 Definition. *Let s, m, m', $d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $d \geq 1$, $|d - d'| \leq 2s - 2d$.*

Let further $P \in \mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$. Then $(p_0, p_1, \ldots, p_{d-1})$, where $H^{s-j}S^{(m+m'-j)}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$ for $j = 0, 1, \ldots, d-1$, is called the complete symbol of P if

$$P - \langle D\rangle^{m'} \sum_{j=0}^{d-1} (\psi(\eta)\, p_j(x, \eta))(x, D)\, \langle D\rangle^{-m'} \in \mathcal{A}_{s-d}^{(m-d),(m'),0,d'-d}(\mathbb{R}^n) \tag{4.1}$$

holds, with ψ being an arbitrary 0-excision function. $p_0 \in H^s S^{(m+m')}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$ *is called the principal symbol of P.*

Each operator $P \in \mathcal{A}^{(m),(m'),d,d'}_{s,cl}(\mathbb{R}^n)$ has a complete symbol found by arranging the homogeneous components of the symbols appearing in Definition 3.2 of the operator $\langle D\rangle^{-m'} P \langle D\rangle^{m'} \in \mathcal{A}^{(m+m'),d,d'}_{s,cl}(\mathbb{R}^n)$. As a consequence of Lemma 4.1 we obtain uniqueness of the components of the complete symbol.

For $Q \in \mathcal{B}^{(m),(m'),d,d'}_{s,cl}(\mathbb{R}^n)$, define the complete symbol as d-tuple $(q_0, q_1, \dots, q_{d-1})$, where $q_j \in H^{s-j} S^{(m+m'-j)}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$ for $j = 0, \dots, d-1$, such that

$$Q - \langle D\rangle^{-m} \sum_{j=0}^{d-1} \big(\psi(\xi)\, q_j(\xi, x)\big)(D, x) \langle D\rangle^{m} \in \mathcal{B}^{(m),(m'-d),0,d'-d}_{s-d}(\mathbb{R}^n) \tag{4.2}$$

holds. $q_0 \in H^s S^{(m+m')}(\mathbb{R}^n \times (\mathbb{R}^n \times 0))$ is called the principal symbol of Q.

Principal symbols behave in the same manner as they do in the case of pseudo-differential operators with C_b^∞-coefficients. We list some of their properties in Proposition 4.3. Proofs follow by examining the asymptotic expansions provided in the proofs given in Subsection 3.5.

4.3 Proposition. *The principal symbol of an operator is independent of the representation chosen for this operator in one of the classes* $\mathcal{A}^{(m),(m'),d,d'}_{s,cl}(\mathbb{R}^n)$, *i.e., it does not change altering the representation in accordance with* Proposition 3.15. *It even does not change passing to the adjoint representation according to* Proposition 3.18.

Under compositions of operators principal symbols are multiplied. The principal symbol of the adjoint to an operator is the complex conjugate of the principal symbol of that operator. The principal symbol of a commutator is the Poisson bracket of the principal symbols of the operators involved times $1/i$.

Proof: We must be careful in compositions, since the proof of Lemma 3.20 has been rather implicit. But checking the single steps, the proof can also be accomplished in that case. □

The subordinated components of the complete symbol depend on the chosen representation. They also depend on the elliptic operator, $\langle D\rangle$, the powers of which enter into the reduction to the standard operator classes. We shall provide, in [25], some formulas demonstrating how subordinated components alter when the representation is changed. The corresponding formulas for the classical situation are produced by collecting homogeneous components in the right way.

In the sequel let the symbol $\mathfrak{m}$ be an abbreviation for the data set (m, m', d, d'). Let further the set $(m-j, m', d-j, d'-j)$ be abbreviated by $\mathfrak{m} - j$.

As a conclusion to Lemma 4.1 we obtain the short exact principal symbol sequence:

4.4 Proposition. *Let* s, m, m', $d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $d \geq 1$, $|d - d'| \leq 2s - 2d$. *Then we have a short exact split sequence*

$$0 \longrightarrow \mathcal{A}^{\mathfrak{m}-1}_{s-1,cl}(\mathbb{R}^n) \longrightarrow \mathcal{A}^{\mathfrak{m}}_{s,cl}(\mathbb{R}^n) \stackrel{\sigma}{\longrightarrow} H^s S^{(m+m')}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)) \longrightarrow 0 \tag{4.3}$$

with the natural injection and σ being the principal symbol mapping.

Using subordinated symbols, for $j = 0, 1, \ldots, d-1$, we obtain short exact split sequences:

$$0 \longrightarrow \mathcal{A}^{m-j-1}_{s-j-1,\,cl}(\mathbb{R}^n) \longrightarrow \mathcal{A}^{m}_{s,\,cl}(\mathbb{R}^n) \longrightarrow \bigoplus_{k=0}^{j} H^s S^{(m+m'-k)}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)) \longrightarrow 0. \quad (4.4)$$

Notice that for $j = d-1$ the sequence (4.4) is a reformulation of (3.25) thus completing the discussion around the topologization of $\mathcal{A}^{(m),(m'),d,d'}_{s,\,cl}(\mathbb{R}^n)$.

4.2 The Parametrix Construction

Now we come to the parametrix construction for elliptic operators within the calculus. To start with record that $a(x) \to 0$ holds as $|x| \to \infty$ for $a \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2}$. For that reason it is impossible to obtain uniform ellipticity estimates for operators in $\mathcal{A}^{(m),(m'),d}_{s,\,cl}(\mathbb{R}^n)$. Moreover, the identity $\mathrm{Id} = \mathrm{op}(1)$ does not belong to the calculus.

To get round this problem, we adjoin the classical pseudo-differential operator calculus, as mentioned in Subsection 3.5. Thus we work in the operator classes

$$L^{m+m'}_{cl}(\mathbb{R}^n) + \mathcal{A}^{(m),(m'),d,d'}_{s,\,cl}(\mathbb{R}^n). \quad (4.5)$$

Here $L^{m+m'}_{cl}(\mathbb{R}^n)$ is the space of classical pseudo-differential operators of order $m + m'$ which are defined on $\mathbb{R}^n$ with uniform symbol estimates in the space variables, i.e., the coefficients are taken from $C^\infty_b(\mathbb{R}^n)$. Recall that the non-smooth calculus has the ideal property in the larger calculus given by (4.5), e.g., we have

$$L^{m_0}_{cl}(\mathbb{R}^n) \cdot \mathcal{A}^{(m),(m'),d,d'}_{s,\,cl}(\mathbb{R}^n) \subseteq \mathcal{A}^{(m+m_0),(m'),d,d'}_{s,\,cl}(\mathbb{R}^n).$$

Notice that on C^∞-manifolds, X, to be considered in the Subsection 4.3, which corresponds to coefficients from local Sobolev spaces, $H^s_{loc}(X)$, we have

$$L^{m+m'}_{cl}(X) \subseteq \mathcal{A}^{(m),(m'),d,d'}_{s,\,cl}(X). \quad (4.6)$$

4.5 Definition. *Let s, m, m', $d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $d \geq 1$, $|d - d'| \leq 2s - 2d$.*

Then an operator $A \in L^{m+m'}_{cl}(\mathbb{R}^n) + \mathcal{A}^{(m),(m'),d,d'}_{s,\,cl}(\mathbb{R}^n)$ is called elliptic if it has an elliptic principal symbol $a_0(x,\xi) \in C^\infty_b S^{(m+m')}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)) + H^s S^{(m+m')}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$, i.e., $a_0(x,\xi)$ satisfies the estimate

$$|a_0(x,\xi)| \geq \delta\, |\xi|^{m+m'} \quad (4.7)$$

for all $(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$, and certain constant $\delta > 0$.

4.6 Definition. *Let s, m, m', $d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $d \geq 1$, $|d - d'| \leq 2s - 2d$.*

Let an operator $A \in L_{cl}^{m+m'}(\mathbb{R}^n) + \mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ *be given. Then* $P \in L_{cl}^{-m-m'}(\mathbb{R}^n) + \mathcal{A}_{s,cl}^{(-m'),(-m),d,d'}(\mathbb{R}^n)$ *is called a parametrix to* A *if it satisfies*

$$PA - \mathrm{Id} \in \mathcal{A}_{s-d}^{(m-d),(-m),0,d'-d}(\mathbb{R}^n), \tag{4.8}$$

$$AP - \mathrm{Id} \in \mathcal{A}_{s-d}^{(-m'-d),(m'),0,d'-d}(\mathbb{R}^n). \tag{4.9}$$

Uniqueness modulo $\mathcal{A}_{s-d}^{(-m'-d),(-m),0,d'-d}(\mathbb{R}^n)$ is immediate from (4.8), (4.9) if a parametrix exists.

Preliminary to Proposition 4.8 we establish a lemma providing the formal Neumann series argument required in the parametrix construction:

4.7 Lemma. *Let* $s, m \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $d \geq 1$.
Further assume that $C \in L_{cl}^{-1}(\mathbb{R}^n) + \mathcal{A}_{s-1,cl}^{(m-1),(-m),d-1}(\mathbb{R}^n)$. *Then*

$$(\mathrm{Id} - C)(\mathrm{Id} + C + C^2 + \ldots + C^{d-1}) = \mathrm{Id} \tag{4.10}$$

is satisfied modulo $\mathcal{A}_{s-d}^{(m-d),(-m),0}(\mathbb{R}^n)$.

Proof: In view of Proposition 3.45, we obtain by induction

$$C^j \in L_{cl}^{-j}(\mathbb{R}^n) + \mathcal{A}_{s-1,cl}^{(m-j),(-m),d-1,d-j}(\mathbb{R}^n) \tag{4.11}$$

for $j = 0, 1, \ldots, d$. Then the assertion follows from

$$\mathcal{A}_{s-1,cl}^{(m-j),(-m),d-1,d-j}(\mathbb{R}^n) \subseteq \mathcal{A}_{s-j,cl}^{(m-j),(-m),d-j}(\mathbb{R}^n)$$

and

$$(\mathrm{Id} - C)(\mathrm{Id} + C + C^2 + \ldots + C^{d-1}) = \mathrm{Id} - C^d. \qquad \square$$

Notice that in the proof of Lemma 4.7 we get another interpretation for d' additionally appearing in Definition 3.9. Now it yields that the expansion given by (4.11) is asymptotic.

4.8 Proposition. *Let* $s, m, m', d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $d \geq 1$, $|d - d'| \leq 2s - 2d$.
Then, for an operator $A \in L_{cl}^{m+m'}(\mathbb{R}^n) + \mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$, *the following conditions are equivalent:*

(a) *A is elliptic in the sense of* Definition 4.5.

(b) *There exists a parametrix* $P \in L_{cl}^{-m-m'}(\mathbb{R}^n) + \mathcal{A}_{s,cl}^{(-m'),(-m),d,d'}(\mathbb{R}^n)$ *to A.*

Proof: It suffices to deal with $A \in L_{cl}^{m+m'}(\mathbb{R}^n) + \mathcal{A}_{s,cl}^{(m),(m'),d}(\mathbb{R}^n)$, since

$$\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n) = \mathcal{A}_{s,cl}^{(m),(m'),d}(\mathbb{R}^n) + \mathcal{A}_{s-d}^{(m-d),(m'),0,d'-d}(\mathbb{R}^n).$$

Suppose that A is elliptic. Let $a_0(x,\xi) \in C_b^\infty S^{(m+m')}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)) + H^s S^{(m+m')}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$ be the principal symbol of A. It is clear that, under condition (4.7), $p_0(x,\xi) = a_0(x,\xi)^{-1}$ belongs to $C_b^\infty S^{(-m-m')}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)) + H^s S^{(-m-m')}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$. Thus using the short exact sequence (4.3) (and its analogue for classical pseudo-differential operators with coefficients in $C_b^\infty(\mathbb{R}^n)$) we find an operator $P_1 \in L_{cl}^{-m-m'}(\mathbb{R}^n) + \mathcal{A}_{s,cl}^{(-m'),(-m),d}(\mathbb{R}^n)$ having $p_0(x,\xi)$ as its principal symbol. Then by the composition rules for principal symbols and the short exact sequence (4.3) again we conclude that

$$C = \mathrm{Id} - P_1 A \in L_{cl}^{-1}(\mathbb{R}^n) + \mathcal{A}_{s-1,cl}^{(m-1),(-m),d-1}(\mathbb{R}^n).$$

Therefore, by Lemma 4.7, $P = (\mathrm{Id} + C + C^2 + \ldots + C^{d-1})\, P_1$ is a left parametrix to A. In a similar manner, a right parametrix to A can be constructed. The argument is completed by noting that a left parametrix and a right parametrix if they exist are equal modulo $\mathcal{A}_{s-d}^{(-m'-d),(-m),0}(\mathbb{R}^n)$.

The reverse direction follows from the composition rules for principal symbols. □

We conclude this subsection by remarking that the parametrix construction in Proposition 4.8 can also be achieved either only locally, or on a closed compact manifold as discussed in Subsection 4.4, or right form the beginning in larger classes of operators taking their coefficients in H^s_{loc}-spaces. But the element of adjoining the smooth calculus to the non-smooth one is familiar in the analysis of non-linear partial differential equations; thus we have decided to do the constructions in the indicated way.

4.3 Coordinate Changes and Operators on Manifolds

In this subsection we discuss the invariance of operator classes $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ under coordinate changes and introduce the classes $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(X)$, X being a C^∞-manifold.

Our first goal is to explain the action of the pseudo-differential operator $p(D,x,D)$ for $p \in H^s_{loc} S_{cl}^{m,m'}(X \times \mathbb{R}^{2n})$, $s > \frac{n}{2}$, and X being an open set in $\mathbb{R}^n$. The expression $p(D,x,D)$ will in general not be defined. This corresponds to compositions in the C^∞-theory in which one of two operators has to be properly supported. Here we have to make a similar assumption. We discuss only a rather special case, which, however, suffices for the applications we have in mind.

Notice that, for $\phi \in C_0^\infty(\mathbb{R}^n)$, we have $\phi(x)\, p(\xi,x,\eta) \in H^s S_{cl}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^{2n})$.

4.9 Lemma. *Let s, m, $m' \in \mathbb{R}$, $s > \frac{n}{2}$. Let further $p(\xi,x,\eta) \in H^s_{loc} S_{cl}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^n)$.*

(a) *Suppose that the Fourier transform $F^{-1}_{\xi\to\rho}\{p(\xi,x,\eta)\}$ is compactly support in ρ uniformly in (x,η). Then $p(D,x,D)$ induces a continuous operator*

$$p(D,x,D) : H^{t+m}(\mathbb{R}^n) \to H^{t-m'}_{loc}(\mathbb{R}^n) \tag{4.12}$$

for all $t \in \mathbb{R}$, $|t| \le s$

(b) *Suppose that the Fourier transform $F^{-1}_{\eta\to\rho'}\{p(\xi,x,\eta)\}$ is compactly support in ρ' uniformly in (ξ,x). Then $p(D,x,D)$ induces a continuous operator*

$$p(D,x,D) : H^{t+m}_{comp}(\mathbb{R}^n) \to H^{t-m'}(\mathbb{R}^n) \tag{4.13}$$

for all $t \in \mathbb{R}$, $|t| \leq s$.

Proof: (a) Suppose that $\mathrm{supp}(F^{-1}_{\xi\to\rho}\{p(\xi,x,\eta)\}) \subseteq K \times \mathbb{R}^n \times \mathbb{R}^n$ for some compact set $K \subset \mathbb{R}^n$. Then, for ϕ, $\phi' \in C_0^\infty(\mathbb{R}^n)$ satisfying $(\mathrm{supp}\phi - \mathrm{supp}\phi') \cap K = \emptyset$, we have

$$\phi(x)\,\big(\phi'(y)\,p(\xi,y,\eta)\big)(D,x,D) = 0.$$

Let $u \in H^{t+m}(\mathbb{R}^n)$ for $t \in \mathbb{R}$, $|t| \leq s$. In order to define $p(D,x,D)u$, we define $p(D,x,D)u$ on K' for any compact set $K' \subset \mathbb{R}^n$. To this end, choose functions ϕ, $\phi' \in C_0^\infty(\mathbb{R}^n)$ such that $\phi = 1$ on K', $\phi' = 1$ in a neighbourhood of $K - \mathrm{supp}\phi$ and set

$$p(D,x,D)u = \phi(x)\big(\phi'(y)\,p(\xi,y,\eta)\big)(D,x,D)u$$

on K'. It is seen that this definition is independent of the functions ϕ, ϕ' with the above properties, consistent on intersections of compact sets K' and agrees with the usual definition in the case that $p \in H^s S^{m,m'}_{cl}(\mathbb{R}^n \times \mathbb{R}^{2n})$.

(b) Now suppose that $\mathrm{supp}(F^{-1}_{\eta\to\rho'}\{p(\xi,x,\eta)\}) \subseteq \mathbb{R}^n \times \mathbb{R}^n \times K$ for some compact set $K \subset \mathbb{R}^n$. Then, for ϕ, $\phi' \in C_0^\infty(\mathbb{R}^n)$ satisfying $(\mathrm{supp}\phi - \mathrm{supp}\phi') \cap K = \emptyset$, we have

$$\big(p(\xi,x,\eta)\,\phi(x)\big)(D,x,D)\,\phi'(x) = 0.$$

Let $u \in H^{t+m}_{comp}(\mathbb{R}^n)$ for $t \in \mathbb{R}$, $|t| \leq s$. We choose functions ϕ, $\phi' \in C_0^\infty(\mathbb{R}^n)$ such that $\phi' = 1$ on $\mathrm{supp}u$, $\phi = 1$ in a neighbourhood of $K + \mathrm{supp}\phi'$ and set

$$p(D,x,D)u = \big(p(\xi,x,\eta)\,\phi(x)\big)(D,x,D)\phi'(x)u.$$

It is seen that this definition is independent of the functions ϕ, ϕ' with the above properties and agrees with the usual definition in the case that $p \in H^s S^{m,m'}_{cl}(\mathbb{R}^n \times \mathbb{R}^{2n})$. □

Now let $p(\xi,x,\eta) \in H^s_{loc} S^{m,m'}_{cl}(X \times \mathbb{R}^n)$ with X being an open set in $\mathbb{R}^n$. As a corollary to Lemma 4.9 we obtain that we can give a meaning to the expression $p(D,x,D)$ as a continuous operator

$$p(D,x,D) : H^{t+m}_{comp}(X) \to H^{t-m'}_{loc}(X) \tag{4.14}$$

for all $t \in \mathbb{R}$, $|t| \leq s$, if one of the assumptions of Lemma 4.9 is satisfied.

Notice that for symbols $p(\xi,x,\eta) \in H^s S^{m,m'}_{cl}(\mathbb{R}^n \times \mathbb{R}^{2n})$ we can always assume that one or both assumptions of Lemma 4.9 are satisfied. To see this, choose a 0-excision function ψ and consider, e.g.,

$$F_{\rho\to\xi}\big\{(1-\psi(\rho))F^{-1}_{\xi\to\rho}\{p(\xi,x,\eta)\}\big\} \in H^s S^{m,m'}_{cl}(\mathbb{R}^n \times \mathbb{R}^{2n}). \tag{4.15}$$

This symbol fulfills assumption (a), while the symbol $F_{\rho\to\xi}\big\{\psi(\rho)\,F^{-1}_{\xi\to\rho}\{p(\xi,x,\eta)\}\big\}$ leads to an operator which belongs to $\mathcal{L}(H^{-s+m}(\mathbb{R}^n), H^t_{loc}(\mathbb{R}^n))$ for any $t \in \mathbb{R}$.

We need a technical lemma:

4.10 Lemma. *Let* $s, m, m', d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $|d - d'| \leq 2s - 2d$. *Let further* $P \in \mathcal{A}^{(m),(m'),d,d'}_{s,cl}(\mathbb{R}^n)$ *and suppose that, for the kernel* K *of* P, *we have that* $K(x,y) = 0$

for $x,\, y \in \mathbb{R}^n$, $|x-y| \le \delta$, *and some* $\delta > 0$. *Then* $P \in \mathcal{A}_{s-d}^{(m-d),(m'),0,d'-d}(\mathbb{R}^n)$.

Proof: We may reduce to the case $m' = 0$ and $P \in \mathcal{A}_{s,cl}^{(m),d,d'}(\mathbb{R}^n)$. Then it is sufficient to prove that $P \in \mathcal{A}_{s-1,cl}^{(m-1),d-1,d'-1}(\mathbb{R}^n)$ if $d \ge 1$.

Write $P = p_0(x,D) + P'$ with $p_0 \in H^s S_{cl}^m(\mathbb{R}^n \times \mathbb{R}^n)$, $P' \in \mathcal{A}_{s-1,cl}^{(m-1),d-1,d'-1}(\mathbb{R}^n)$. Let K_0 be the kernel of $p_0(x,D)$ and K' be the kernel of P'. Then $K_0(x,y) = L_0(x, x-y)$, where $L_0(x,\rho) = F^{-1}_{\eta\to\rho}\{p_0(x,\eta)\}$. In particular, $\psi(\rho)\, L_0(x,\rho) \in H^s(\mathbb{R}^n)\tilde{\otimes}_\pi \mathcal{S}(\mathbb{R}^n)$ for an arbitrary 0-excision function ψ. Choosing $\psi \in C^\infty(\mathbb{R}^n)$ in such a way that $\psi(\rho) = 1$ for $|\rho| \ge \delta$ we arrive at a decomposition $(1-\psi(x-y))K_0(x,y) + \psi(x-y)K_0(x,y) + K'(x,y)$ for the kernel K of P, where $(1-\psi(x-y))K_0(x,y)$ and $\psi(x-y)K_0(x,y) + K'(x,y)$ give rise to operators in $\mathcal{A}_{s-1,cl}^{(m-1),d-1,d'-1}(\mathbb{R}^n)$, since $(1-\psi(x-y))K_0(x,y)$ is supported in $|x-y| \le \delta$. Hence, we have $P \in \mathcal{A}_{s-1,cl}^{(m-1),d-1,d'-1}(\mathbb{R}^n)$. □

The transition to the operator classes mentioned in the end of the previous subsection is accomplished by the next lemma:

4.11 Lemma. *Let* $s,\, m,\, m',\, d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2}$, $|d-d'| \le 2s-2d$.
Let further, for some open subset $X \subset \mathbb{R}^n$, $P : C_0^\infty(X) \to \mathcal{D}'(X)$ *be a linear continuous operator such that for any* $\phi,\, \phi' \in C_0^\infty(X)$ *the operator* $\mathcal{S}(\mathbb{R}^n) \ni u \mapsto \phi' P(\phi u) \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$. *Then there are symbols* $p_j(\xi,x,\eta) \in H_{loc}^{s-j} S_{cl}^{m-j,m'}(X \times \mathbb{R}^{2n})$ *for* $j = 0,1,\ldots,d-1$ *satisfying one of the conditions of* Lemma 4.9 *and an operator*

$$P_d \in \bigcap_{t=-s+d-\min\{d,d'\}}^{s-d-\max\{d,d'\}} \mathcal{L}(H_{comp}^{t+m}(X), H_{loc}^{t-m'+d}(X))$$

such that

$$P = \sum_{j=0}^{d-1} p_j(D,x,D) + P_d. \tag{4.16}$$

In the notation of the operators $\mathcal{S}(\mathbb{R}^n) \ni u \mapsto \phi' P(\phi u) \in \mathcal{S}'(\mathbb{R}^n)$ in Lemma 4.11 we have omitted the extensions to $\mathbb{R}^n$ and the restrictions to X, respectively.

Proof of Lemma 4.11: Choose a partition $\{\phi_k\}_{k\in\mathbb{N}}$ of unity on X and choose functions $\psi_k \in C_0^\infty(X)$ such that $\phi_k\psi_k = \phi_k$ for all $k \in \mathbb{N}$ and $\{\mathrm{supp}\psi_k\}_{k\in\mathbb{N}}$ is a locally finite cover of X. Then, for $u \in C_0^\infty(X)$, we have

$$Pu = \sum_{k=0}^{\infty} P(\phi_k u) = \sum_{k=0}^{\infty} \psi_k P(\phi_k u) + \sum_{k=0}^{\infty}(1-\psi_k)P(\phi_k u). \tag{4.17}$$

By Lemma 4.10, we have $\phi_l\,(1-\psi_k)P\phi_k \in \mathcal{A}_{s-d}^{(m-d),(m'),0,d'-d}(\mathbb{R}^n)$ for all $k,\, l \in \mathbb{N}$. Thus, $(1-\psi_k)P\phi_k \in \bigcap_{t=-s+d-\min\{d,d'\}}^{s-d-\max\{d,d'\}} \mathcal{L}(H^{t+m}(X), H_{loc}^{t-m'+d}(X))$ for all $k \in \mathbb{N}$. Moreover, for $u \in C_0^\infty(X)$, the sum $\sum_{j=0}^{\infty}(1-\psi_k)P(\phi_k u)$ is finite. Consequently,

$$\sum_{k=0}^{\infty}(1-\psi_k)P\phi_k \in \bigcap_{t=-s+d-\min\{d,d'\}}^{s-d-\max\{d,d'\}} \mathcal{L}(H_{comp}^{t+m}(X), H_{loc}^{t-m'+d}(X)).$$

By assumption, the operators $\psi_k P\phi_k$ belong to $\mathcal{A}^{(m),(m'),d,d'}_{s,cl}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$. Write $\psi_k P\phi_k = \sum_{j=0}^{d-1} p_{jk}(D,x,D)+P_{jk}$, where $p_{jk} \in H^s S^{m-j,m'}_{cl}(\mathbb{R}^n \times \mathbb{R}^{2n})$ for $j = 0, 1, \dots, d-1$, $k \in \mathbb{N}$, and $P_{dk} \in \mathcal{A}^{(m-d),(m'),0,d'-d}_{s-d}(\mathbb{R}^n)$. We can arrange that the $p_{jk}(\xi,x,\eta)$'s satisfy, e.g., the first assumption of Lemma 4.9 and, for fixed j, their supports in x form a locally finite cover of X uniformly in (ξ,η). For the kernels of the P_{dk}'s, we may assume that their supports, both in x and y, form locally finite covers of X. Then, on X,

$$p_j(\xi,x,\eta) = \sum_{k=0}^{\infty} p_{jk}(\xi,x,\eta) \in H^s_{loc} S^{m-j,m'}_{cl}(X \times \mathbb{R}^{2n})$$

satisfies the first assumption of Lemma 4.9. Thus, for $j = 0, 1, \dots, d-1$, these symbols give rise to operators $p_j(D,x,D) : H^{t+m}_{comp}(X) \to H^{t-m'}_{loc}(X)$ for every $t \in \mathbb{R}$, $|t| \le s$. Eventually, the sum $\sum_{k=0}^{\infty} P_{dk}$ exist, and

$$\sum_{k=0}^{\infty} P_{dk} \in \bigcap_{t=-s+d-\min\{d,d'\}}^{s-d-\max\{d,d'\}} \mathcal{L}(H^{t+m}_{comp}(X), H^{t-m'+d}_{loc}(X)).$$

The proof is finished. □

Next we are concerned with the invariance of $\mathcal{A}^{(m),(m'),d,d'}_{s,cl}(\mathbb{R}^n)$ under global changes of coordinates of $\mathbb{R}^n$. Let $\kappa : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism of $\mathbb{R}^n$. In the sequel, we shall always assume that there are constants c_1, $c_2 > 0$ and $c_\alpha > 0$ for $\alpha \in \mathbb{N}^n$ such that

$$c_1 \le |\det \kappa'(x)| \le c_2 \tag{4.18}$$

and

$$|\partial_x^\alpha \kappa(x)| \le c_\alpha \tag{4.19}$$

hold for all $x \in \mathbb{R}^n$. Recall that, for functions $u \in \mathcal{S}(\mathbb{R}^n)$ and distributions $v \in \mathcal{S}'(\mathbb{R}^n)$, the pull-back $\kappa^* u$ is defined by $\kappa^* u(x) = u(\kappa(x))$ and the push-forward $\kappa_* v$ by

$$\langle \kappa_* v, \phi \rangle = \langle v, \kappa^* \phi \, |\det \kappa'| \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^n).$$

By (4.18), (4.19), we have $\kappa^* : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ and $\kappa_* : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. Furthermore, $\kappa_* = (\kappa^{-1})^*$ holds on functions, since distributions are transformed as 1-densities.

For $P : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, we define the operator $P_\kappa : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by

$$P_\kappa(\phi) = \kappa_* P(\kappa^* \phi). \tag{4.20}$$

4.12 Proposition. *Let s, m, m', $d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $|d - d'| \le 2s - 2d$.*

Let further $\kappa : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism satisfying (4.18), (4.19). *Then, for $P \in \mathcal{A}^{(m),(m'),d,d'}_{s,cl}(\mathbb{R}^n)$, we have*

$$P_\kappa \in \mathcal{A}^{(m),(m'),d,d'}_{s,cl}(\mathbb{R}^n). \tag{4.21}$$

Proof: It suffices to show that the conclusion holds for the operator classes $\mathcal{A}^{(m),d,d'}_{s,cl}(\mathbb{R}^n)$, since

$$\mathcal{A}^{(m),(m'),d,d'}_{s,cl}(\mathbb{R}^n) = \langle D\rangle^{m'} \mathcal{A}^{(m+m'),d,d'}_{s,cl}(\mathbb{R}^n) \langle D\rangle^{-m'},$$

and the behaviour of classical pseudo-differential operators with coefficients in $C_b^\infty(\mathbb{R}^n)$ under coordinate changes is known.

The mapping $\kappa^* : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ extends by continuity to an isomorphism $\kappa^* : H^t(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$ for every $t \in \mathbb{R}$. Thus $A \in \mathcal{A}^{(m-d),0,d'-d}_{s-d}(\mathbb{R}^n)$ implies $A_\kappa \in \mathcal{A}^{(m-d),0,d'-d}_{s-d}(\mathbb{R}^n)$. Therefore, we may assume that $P = p(x, D)$ holds for some symbol $p(x,\eta) \in H^s S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$. Then, for the symbol p_κ of the operator $P_\kappa = p_\kappa(x, D)$, we claim that

$$p_\kappa(\kappa(x),\eta) = e^{-i\kappa(x)\eta} p(x, D) e^{i\kappa(x)\eta}. \tag{4.22}$$

This is immediate for symbols in product form, i.e., $p(x,\eta) = a(x)p_0(\eta)$ with $a \in H^s(\mathbb{R}^n)$, $p_0 \in S^m_{cl}(\mathbb{R}^n)$, from the corresponding result in the C^∞-situation (see, e.g., [11, Theorem 18.1.17]). In that manner we obtain a continuous bilinear mapping

$$H^s(\mathbb{R}^n) \times S^m_{cl}(\mathbb{R}^n) \to H^s S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n), \quad (a, p_0) \mapsto a(\kappa^{-1}(x))\,(p_0)_\kappa(\eta)$$

showing that $p_\kappa(x,\eta)$ defined by (4.22) indeed belongs to $H^s S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$ for $p(x,\eta) \in H^s S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$ by invoking the usual tensor product argument. That $p_\kappa(x,\eta)$ is actually the symbol of P_κ now follows from the fact that it is true on $S^m_{cl}(\mathbb{R}^n) \otimes H^s(\mathbb{R}^n)$ and that the mapping $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, $u \to p_\kappa(x, D)u$, is continuous for each $u \in \mathcal{S}(\mathbb{R}^n)$. □

In the proof of Proposition 4.12 we have shown that the principal symbols of P and P_κ are interrelated by

$$(p_\kappa)_0(\kappa(x),\eta) = p_0(x, {}^t\kappa'(x)\eta), \tag{4.23}$$

i.e., as usual, the principal symbol behaves like a function defined on $T^*(\mathbb{R}^n) \setminus \{0\}$.

In the situation considered in (4.22) we get as in the case of coefficients in $C_b^\infty(\mathbb{R}^n)$ that

$$p_\kappa(\kappa(x),\eta) \sim \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} (\partial_\eta^\alpha p)(x, {}^t\kappa'(x)\eta)\, D_y^\alpha (e^{i\rho_x(y)\eta})_{|y=x} \tag{4.24}$$

in $H^s S^m(\mathbb{R}^n \times \mathbb{R}^n)$ holds, where $\rho_x(y) = \kappa(y) - \kappa(x) - \kappa'(x)(y - x)$. Recall that the $D_y^\alpha(e^{i\rho_x(y)\eta})_{|y=x}$'s are polynomials in η of degree less than or equal to $|\alpha|/2$ with coefficients in $C_b^\infty(\mathbb{R}^n)$. One obtains, from (4.24), formulas for the components of lower order of the complete symbol of P_κ by replacing p in (4.24) by the complete symbol of P and ordering terms with respect to homogeneity.

We go now over to the operator classes $\mathcal{A}^{(m),(m'),d,d'}_{s,cl}(X)$. From now on, let X denote a C^∞-manifold.

4.13 Definition. *Let s, m, m', $d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $|d - d'| \leq 2s - 2d$.*

Then $\mathcal{A}^{(m),(m'),d,d'}_{s,cl}(X)$ denotes the class of all operators $P : C_0^\infty(X) \to \mathcal{D}'(X)$ such that for every chart (Y,κ) and arbitrary ϕ, $\phi' \in C_0^\infty(Y)$

$$(\phi' P \phi)_\kappa \in \mathcal{A}^{(m),(m'),d,d'}_{s,cl}(\mathbb{R}^n). \tag{4.25}$$

holds.

Note that, by Proposition 4.12, it is enough to ask (4.25) only for a collection of charts (Y_k, κ_k) such that the $Y_k \times Y_k$'s cover $X \times X$. Further note that, for $X = \mathbb{R}^n$, the operator classes $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(X)$ introduced in Definition 4.13 are different from the classes $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ considered earlier in that respect that the behaviour of coefficients is not now restricted as $|x| \to \infty$.

Below we summarize some of the properties of the operator classes $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(X)$ which are immediate from our foregoing considerations.

4.14 Proposition. *Let* $s, m, m', r, m_0, m_0', m_1, m_1', d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2}+d$, $|d-d'| \leq 2s-2d$, $|m'-r| \leq 2s-2d$, $|m_0'+m_1| \leq 2s-2d$.

Then the following properties are valid:

(a) *We have*

$$\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(X) \subseteq \bigcap_{t=-s+d-\min\{d,d'\}}^{s-\max\{d,d'\}} \mathcal{L}(H_{comp}^{t+m}(X), H_{loc}^{t-m'}(X)), \tag{4.26}$$

and the remainder classes are completely characterized by that property, i.e.,

$$\mathcal{A}_{s-d}^{(m-d),(m'),0,d'-d}(X) = \bigcap_{t=-s+d-\min\{d,d'\}}^{s-d-\max\{d,d'\}} \mathcal{L}(H_{comp}^{t+m}(X), H_{loc}^{t-m'+d}(X)). \tag{4.27}$$

In (4.26), *for properly supported operators, one can replace either* $H_{comp}^{t+m}(X)$ *by* $H_{loc}^{t+m}(X)$ *or* $H_{loc}^{t-m'}(X)$ *by* $H_{comp}^{t-m'}(X)$.

(b) *We have*

$$\mathcal{A}_{s,cl}^{(m),(m'),d}(X) \subseteq \mathcal{A}_{s,cl}^{(m+m'-r),(r),d,d+m'-r}(X). \tag{4.28}$$

(c) *The adjoint to an operator in* $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(X)$ *belongs to* $\mathcal{A}_{s,cl}^{(m'+d'),(m-d'),d,d'}(X)$.

(d) *For the composition of* $P \in \mathcal{A}_{s,cl}^{(m_0),(m_0'),d}(X)$, $Q \in \mathcal{A}_{s,cl}^{(m_1),(m_1'),d}(X)$, *where one of these operators is properly supported, we have*

$$Q\,P \in \mathcal{A}_{s,cl}^{(m_0+m_0'+m_1),(m_1'),d,d+(m_0'+m_1)}(X). \tag{4.29}$$

In order to define the homogeneous principal symbol for operators in $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(X)$ we introduce the space $H_{loc}^s S^{(m+m')}(T^*X \setminus 0)$ as the space of all functions $p(x,\xi)$ on $T^*X \setminus 0$ which are homogeneous of order $m+m'$ in the fibres and which, in any chart on X, belong to $H_{loc}^s S^{(m+m')}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$. It is plain that an operator $P \in \mathcal{A}_{s,cl}^{(m),(m'),d,d'}(X)$ has a uniquely defined principal symbol $\sigma_P(x,\xi) \in H_{loc}^s S^{(m)}(T^*X \setminus 0)$. As in (4.3), we obtain a short exact split sequence, i.e.,

$$0 \longrightarrow \mathcal{A}_{s-1,cl}^{m-1}(X) \longrightarrow \mathcal{A}_{s,cl}^{m}(X) \xrightarrow{\sigma} H^s S^{(m+m')}(T^*X \setminus 0) \longrightarrow 0, \tag{4.30}$$

where $\mathfrak{m} = (m, m', d, d')$. An operator $A \in \mathcal{A}_{s,cl}^{(m),(m'),d,d'}(X)$ is called elliptic if its principal symbol $\sigma_A(x,\xi)$ never vanishes on $T^*X \setminus 0$. In that case, using a partition of unity, a parametrix $P \in \mathcal{A}_{s,cl}^{(-m'),(-m),d,d'}(X)$ to P can be constructed, i.e., an operator P satisfying

$$PA - \mathrm{Id} \in \mathcal{A}_{s-d}^{(m-d),(-m),0,d'-d}(X), \quad AP - \mathrm{Id} \in \mathcal{A}_{s-d}^{(-m'-d),(m'),0,d'-d}(X). \tag{4.31}$$

Conversely, from the existence of a parametrix we conclude on the ellipticity of A. Again, natural Fréchet topologies on the operator classes $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(X)$ are introduced also by using a partition of unity.

4.4 Elliptic Regularity, Parametrices, and the Fredholm Property

In this subsection we quote results when X is a closed compact manifold. Then the spaces $H^t_{loc}(X)$, $H^t_{comp}(X)$ become replaced by $H^t(X)$. Note that, for $t' > t$, the embedding $H^{t'}(X) \hookrightarrow H^t(X)$ is compact. The spaces $H^s_{loc}S^{(m)}(T^*X \setminus 0)$ are now denoted by $H^sS^{(m)}(T^*X \setminus 0)$.

In case X is compact, ellipticity is equivalent to the Fredholm property:

4.15 Proposition. *Let s, m, m', $d' \in \mathbb{R}$, $d \in \mathbb{N}$, $s > \frac{n}{2} + d$, $d \geq 1$, $|d - d'| \leq 2s - 2d$.*

Let X be a closed compact C^∞-manifold. Then, for an operator $A \in \mathcal{A}_{s,cl}^{(m),(m'),d,d'}(X)$, the following conditions are equivalent:

(a) *A is elliptic.*

(b) *The operator $A : H^{t+m}(X) \to H^{t-m'}(X)$ is Fredholm for some (and then for all) $t \in [-s + d - \min\{d, d'\}, s - \max\{d, d'\}]$.*

In that case, there exists a parametrix $P \in \mathcal{A}_{s,cl}^{(-m'),(-m),d,d'}(X)$ to A. Moreover, elliptic regularity holds, i.e., $u \in H^{-s+m+d-\min\{d,d'\}}(X)$, $Au \in H^{t-m'}(X)$ for some $t \in [-s + d - \min\{d, d'\}, s - \max\{d, d'\}]$ implies that $u \in H^{t+m}(X)$.

Proof: Suppose that $A : H^{t+m}(X) \to H^{t-m'}(X)$ is a Fredholm operator for certain $t \in [-s + d - \min\{d, d'\}, s - \max\{d, d'\}]$. By order reduction and other manipulations we may assume that $m = m' = 0$, $d = d'$, $t = 0$, and $A \in \mathcal{A}_{s,cl}^{(0),d}(X)$.

We use a device from [18] to recover the principal symbol $\sigma_A(x,\xi)$ of A: Given $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$, there exists a family of unitary operators R_λ, $\lambda > 0$, on $L^2(\mathbb{R}^n)$ such that, for $u \in L^2(\mathbb{R}^n)$, $R_\lambda u \to 0$ weakly in $L^2(\mathbb{R}^n)$ as $\lambda \to \infty$ and

$$R_\lambda^{-1}(A + K)R_\lambda u \to \sigma_A(x_0, \xi_0)u \text{ in } L^2(\mathbb{R}^n) \text{ as } \lambda \to \infty \tag{4.32}$$

for $A \in \mathcal{A}_{s,cl}^{(0),d}(\mathbb{R}^n)$ and any compact operator K on $L^2(\mathbb{R}^n)$. A family of operators R_λ, $\lambda > 0$, obeying these properties is given by

$$R_\lambda u(x) = \lambda^{n/4} e^{i\lambda x\xi_0} u(\lambda^{1/2}(x - x_0)). \tag{4.33}$$

For details, see [18, Section 2.3.4].

Using a partition of unity, we find, for fixed $(x_0, \xi_0) \in T^*X \setminus 0$, a family of isomorphisms R_λ, $\lambda > 0$, on $L^2(X)$ such that R_λ, R_λ^{-1} are uniformly bounded in norm for $\lambda > 0$ independently of $(x_0, \xi_0) \in T^*X \setminus 0$, further, for $u \in L^2(X)$, $R_\lambda u \to 0$ weakly in $L^2(X)$ as $\lambda \to \infty$ and

$$R_\lambda^{-1}(A+K)R_\lambda u \to \sigma_A(x_0, \xi_0)u \text{ in } L^2(X) \text{ as } \lambda \to \infty$$

for $A \in \mathcal{A}_{s,cl}^{(0),d}(X)$ and any compact operator K on $L^2(X)$. Then, if $A \in \mathcal{A}_{s,cl}^{(0),d}$ is a Fredholm operator on $L^2(X)$ and $P \in \mathcal{L}(L^2(X))$ is a Fredholm parametrix to A, i.e., $P\,A - \mathrm{Id}$, $A\,P - \mathrm{Id}$ are compact operators, we get, for $u \in L^2(X)$ and $K = P\,A - \mathrm{Id}$,

$$\|u\| \le C\,\|R_\lambda u\| = C\,\|(PA-K)\,R_\lambda u\| \le C\,\|R_\lambda^{-1} A R_\lambda u\| + C\,\|K R_\lambda u\|,$$

where $\|\ \|$ is the norm on $L^2(X)$ and $C > 0$ is some generic constant. Now, if λ tends to ∞, we find, for each $u \in L^2(X)$, $u \neq 0$,

$$0 < \|u\| \le C\,|\sigma_A(x_0, \xi_0)|\ \|u\|, \tag{4.34}$$

with $C > 0$ being independent of $(x_0, \xi_0) \in T^*X \setminus 0$, yielding the ellipticity of A.

Conversely, if $A \in \mathcal{A}_{s,cl}^{(m),(m'),d,d'}(X)$ is elliptic, then a parametrix exists. The existence of a parametrix implies elliptic regularity and the Fredholm property. □

5 Notes and Remarks

We conclude with further notes on topics discussed in previous sections. Our exposition in this paper is based on the classical stock of pseudo-differential calculus. For references concerning the theory when the operators have smooth coefficients, the reader is referred to standard textbooks, e.g., [6], [11], [21]. The theory of classical pseudo-differential operators goes back to the classical paper [14]. A little later, in [10], the invariance of the calculus under coordinate changes was proved and the general calculus was invented.

We add references which are close to the subject treated in the body of the paper. Further references may be found in papers cited in the bibliography.

Section 2. The main attribute avoiding the complications in the treatment of non-classical operators in [25] is the nuclearity of $S^m_{cl}(\mathbb{R}^n)$. This property was recognized by F. Mantlik (Dortmund). He observed that, under radial compactification of $\mathbb{R}^n$ and appropriate renormalization, $S^m_{cl}(\mathbb{R}^n)$ is transformed into the space $C^\infty(\mathbb{B}^n)$. Here $\mathbb{B}^n$ is the closed unit ball in $\mathbb{R}^n$, and expansions of symbols in $S^m_{cl}(\mathbb{R}^n)$ into homogeneous components correspond to Taylor expansions near the boundary $\partial\mathbb{B}^n$.

The topologization of the symbol spaces $S^m_{cl}(\mathbb{R}^n)$ is taken from [19] and represents in a closed form the construction given there (cf. (2.6)). The proof announced for Proposition 2.4 that does not rely on the result $S^m_\tau(\mathbb{R}^n; E) = S^m_\tau(\mathbb{R}^n)\tilde{\otimes}_\epsilon E$ is as follows: We have that $S^m_{cl}(\mathbb{R}^n) \otimes E$ is algebraically a subspace of $S^m_{cl}(\mathbb{R}^n; E)$. The induced topology is that of the injective tensor product. $S^m_{cl}(\mathbb{R}^n; E)$ becomes a subspace of $\mathcal{L}((S^m_{cl}(\mathbb{R}^n))', E)$ via the mapping $a \mapsto (\Phi \mapsto \langle\Phi, a\rangle)$. Here $(S^m_{cl}(\mathbb{R}^n))'$ is the strong dual to $S^m_{cl}(\mathbb{R}^n)$. Note that functionals $\Phi \in (S^m_{cl}(\mathbb{R}^n))'$ can be applied to symbols $a \in S^m_{cl}(\mathbb{R}^n; E)$ yielding elements in E. It is seen that $S^m_{cl}(\mathbb{R}^n; E)$ carries the topology induced from $\mathcal{L}((S^m_{cl}(\mathbb{R}^n))', E)$ when the latter is equipped with the topology of uniform convergence on all bounded subsets of $(S^m_{cl}(\mathbb{R}^n))'$. We further have $\mathcal{L}((S^m_{cl}(\mathbb{R}^n))', E) = S^m_{cl}(\mathbb{R}^n)\tilde{\otimes}_\epsilon E$ (see, e.g., [13]), since $S^m_{cl}(\mathbb{R}^n)$ is a nuclear Fréchet space. Therefore,

$$S^m_{cl}(\mathbb{R}^n) \otimes_\pi E \subseteq S^m_{cl}(\mathbb{R}^n; E) \subseteq \mathcal{L}((S^m_{cl}(\mathbb{R}^n))', E) = S^m_{cl}(\mathbb{R}^n)\tilde{\otimes}_\pi E,$$

and $S^m_{cl}(\mathbb{R}^n; E) = S^m_{cl}(\mathbb{R}^n)\tilde{\otimes}_\pi E$.

The symbol classes $S^{m,m'}_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$ were introduced by T. Hirschmann in connection with a pseudo-differential calculus on $\mathbb{R}^n$ for operators having symbols the coefficients of which satisfy certain exit conditions at infinity. From (2.11), (2.12) and the same with the rôles of j, k interchanged it is seen that the definition given in [9] agrees with that one used above.

Section 3. The structural aspects of the pseudo-differential calculus which have been considered follow the general framework of a pseudo-differential theory. In particular, we had to take care in two respects: only finite asymptotic expansions are allowed in the calculus, and the components in these asymptotic expansions are generally of the form $p(D, x, D)$. The latter requirement causes that so-called spectral conditions otherwise to impose on the symbols are avoided, as it has been already mentioned in the introduction.

The symbols considered in this paper are infinitely differentiable in the covariables. Sometimes one is interested in symbols satisfying weaker differentiability conditions. Here we did not go into this question, and no attempts were made to obtain optimal results. In

[25], after the non-classical operator calculus will have been established, we will come back to that topic and will make several comments on it.

It is natural to provide the classes $\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n)$ with suitable Fréchet topologies. For example, using these Fréchet topologies one confirms oneself that the inclusions stated in Propositions 3.10, 3.15, 3.18 and 3.19 hold in a topological sense. The operator classes $\mathcal{A}_s^{(m),(m'),d,d'}(\mathbb{R}^n)$ to be introduced in [25] will turn out to be Banach spaces mainly due to the fact that only finitely many derivatives with respect to the covariables are needed in estimates. In any case we have

$$\mathcal{A}_{s,cl}^{(m),(m'),d,d'}(\mathbb{R}^n) \hookrightarrow \mathcal{A}_s^{(m),(m'),d,d'}(\mathbb{R}^n),$$

and the embedding is continuous.

Section 4. We compare the pseudo-differential calculus for operators with non-smooth coefficients developed in this paper to other possible alternatives. In our calculus, the operators have coefficients in L^2-Sobolev spaces $H^s(\mathbb{R}^n)$, while, roughly speaking, in Bony's paradifferential calculus the coefficients are taken from Hölder-Zygmund spaces. One advantage in our approach manifests in the simpler estimates used not built on Littlewood-Paley decompositions. Moreover, e.g., in view of applications to quasi-linear hyperbolic equations it is desirable to demand coefficients in L^2-Sobolev spaces, since solutions are looked for in energy spaces. There are, however, applications in nonlinear partial differential equations which oblique one to leave the range of applicability of L^2-theory. In such instances it would be better to have a calculus at one's disposal in which the coefficients are permitted in more general function spaces, e.g., in Besov- or Bessel-potential spaces. In a future paper, we shall lead into such pseudo-differential calculi along the lines stressed in this paper. In the global parametrix construction in Subsection 4.2 we had to adjoin the operator classes $L_{cl}^m(\mathbb{R}^n)$ to our calculus to obtain uniform ellipticity estimates, since functions in $H^s(\mathbb{R}^n)$ vanish at infinity. This setting-up and also some trouble in formulating the commutator results would be prevented when working from the beginning with coefficients, e.g., in Hölder-Zygmund spaces.

A further subject not touched upon in this paper concerns microlocal analysis, but it has been already used in embryo in the proof of Lemma 4.1. Questions related to microlocal analysis and similar questions are also intended to publication in a future paper. Here we only note that the expected effect that proofs simplify considerably in dealing with classical instead of non-classical operators again occurs.

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