IRRATIONALITY OF THE MODULI SPACES OF POLARIZED ABELIAN SURFACES

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MPI / 94-26

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ABSTRACT. The moduli space of abelian surfaces with non-principal polarization of type $(1,t)$ is a three dimensional quasi-projective variety. In the paper we construct holomorphic sections of the canonical line bundle on a smooth compact model of it using the theory of Siegel and Jacobi modular forms. It is proved that the moduli spaces of abelian surfaces with polarization of type $(1,t)$, where t is a natural number more than or equal to 13 and $t \neq 14, 15, 16, 20, 24, 30, 36$, are not unirational.

§1. MODULI SPACES OF ABELIAN SURFACES.

Let $S = \mathbb{C}^2/L$ be an abelian surface, where L is a free $\mathbb{Z}$-module of rank 4, and let $\mathcal{L}$ be a ample line bundle on it (i.e., a line bundle for which the sections of some power embed the surface in a projective space). The integral alternating bilinear form W representing the first Chern class of the line bundle $\mathcal{L}$ may be reduced to the following normal form

$$c_1(\mathcal{L}) = W_T = \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix},$$

where $t_1, t_2 \in \mathbb{N}$ and t_1 is a divisor of t_2. The pair $(S, \mathcal{L})$ is called a polarized abelian surface. The pair (t_1, t_2) uniquely determined by $\mathcal{L}$ is called the type of polarization. The period matrix Ω_S of S, which is the matrix whose rows are the basis vectors of L, may be written as

$$\Omega_S = \begin{pmatrix} Z_S \\ T \end{pmatrix},$$

where Z_S belongs to the Siegel upper half space of degree two

$$\mathbb{H}_2 = \{Z = {}^tZ \in M_2(\mathbb{C}),\ \operatorname{Im}(Z) > 0\}.$$

The point Z_S is defined up to the action of the group of linear isomorphisms of L preserving the alternating form W_T. Let us define the integral symplectic group of this form

$$Sp(W_T, \mathbb{Z}) = \{g \in M_4(\mathbb{Z}) :\ gW_T{}^tg = W_T\}.$$

Supported by Max-Planck-Institut für Mathematik in Bonn.

This group is conjugate to the following subgroup of the usual rational symplectic group $Sp_4(\mathbb{Q})$ (the case of $T = \mathbf{1}_2$)

$$\Gamma^{(m)}[T] = I_T^{-1} Sp(W_T, \mathbb{Z}) I_T \subset Sp_4(\mathbb{Q}), \quad \text{where } I_T = \begin{pmatrix} 1_2 & 0 \\ 0 & T \end{pmatrix}.$$

(The index m means "moduli".) For example, if $T = \operatorname{diag}(1, t)$ (i.e., $t_1 = 1,\ t_2 = t$) this group consists of the following elements

$$\Gamma^{(m)}[t] = \left\{ \begin{pmatrix} * & * & * & t* \\ t* & * & t* & t* \\ * & * & * & t* \\ * & t^{-1}* & * & * \end{pmatrix} \in Sp_4(\mathbb{Q}) \right\}, \tag{1.1}$$

where all $*$ denote integral numbers. The real symplectic group acts on the Siegel upper-half space as the group of fractional-linear transformations

$$M < Z >= (AZ + B)(CZ + D)^{-1}, \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{R}).$$

The quotient space

$$\mathcal{A}_T = \Gamma^{(m)}[T] \setminus \mathbb{H}_2$$

is the coarse moduli space of abelian surfaces with polarization of type (t_1, t_2) (see [I], [HKW1]). Without loss of generality we shall consider only the case of the polarizations of type $(1, t)$ and we put instead of T the index t in all our notation like W_t, I_t, $\Gamma^{(m)}[t]$, $\mathcal{A}_t$.

It is known, that $\mathcal{A}_1$ (the moduli space of abelian surfaces with principal polarization), $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{A}_5$, $\mathcal{A}_7$, $\mathcal{A}_9$ are rational or unirational (see [I], [O'G], [BL]). For $t = 5$ a finite covering of $\mathcal{A}_5$ is connected with the famous Horrocks-Mumford vector bundle (see [HKW1]). In this paper we prove the following main theorem, which is quite opposite to these examples.

Theorem 1. *Let $\widetilde{\mathcal{A}}_t$ be a non-singular model of a compactification of the moduli space $\mathcal{A}_t$ of abelian surfaces with polarization of type $(1, t)$. The variety $\widetilde{\mathcal{A}}_t$ is not unirational if $t \geq 13$ and $t \neq 14$, 15, 16, 18, 20, 24, 30, 36.*

To prove this result we shall construct sections of the canonical line bundle of the variety $\widetilde{\mathcal{A}}_t$ using the theory of Siegel modular forms. It will give us an estimation from below of the geometrical genus of the variety.

Theorem 2. *Let $p_g(t) = \dim_{\mathbb{C}} H^{3,0}(\widetilde{\mathcal{A}}_t)$ be the geometrical genus of a smooth compactification $\widetilde{\mathcal{A}}_t$ of the moduli space of abelian surfaces with polarization of type $(1, t)$. The following inequality is valid*

$$p_g(t) \geq \sum_{j=1}^{t-1} \{2j + 2\}_{12} - \lfloor \frac{j^2}{4t} \rfloor,$$

where

$$\{m\}_{12} = \begin{cases} \lfloor \frac{m}{12} \rfloor & \text{if } m \not\equiv 2 \mod 12, \\ \lfloor \frac{m}{12} \rfloor - 1 & \text{if } m \equiv 2 \mod 12, \end{cases}$$

and $\lfloor x \rfloor$ *is the integral part of* x.

For instance $p_g(13) \geq 1$, $p_g(29) \geq 2$, $p_g(37) \geq 4$, $p_g(53) \geq 5$. Theorem 1 is a corollary of Theorem 2. The genus $p_g(t)$ could be zero only for the special values of t mentioned in Theorem 1. The estimation of Theorem 2 gives us even more.

Corollary 1. *The geometrical genus of $\mathcal{A}_t$ tends to infinity as $t \to \infty$. More exactly, $\frac{p_g(t)}{t} > C$ as $t \to \infty$, where the constant C does not depend on t.*

For any smooth compact variety X one may define the Kodaira dimension of X through the transcendence degree of its canonical ring

$$\kappa(X) = \text{tr. deg} \bigoplus_{n=0}^{\infty} H^0(X, nK_X) \ - 1.$$

The Kodaira dimension is a birational invariant of X. If the geometrical genus of X is strictly greater than one, then the Kodaira dimension of X is positive. The estimation of Theorem 2 gives us the following

Corollary 2. *The Kodaira dimension of $\widetilde{\mathcal{A}}_t$ is positive if $t \geq 29$ and $t \neq 30$, 32, 35, 36, 40, 42, 48, 60.*

It is known that the Kodaira dimension $\kappa(X)$ is not more than the dimension of the variety X. If $\kappa(X) = \dim X$, then X is said to be of general type.

The Kodaira dimension is known for some moduli spaces of abelian surfaces for which the moduli group is a subgroup of the Siegel modular group $Sp_4(\mathbb{Z})$. The group $\Gamma^{(m)}[t]$ is conjugate to a subgroup of $Sp_4(\mathbb{Z})$ only in the case of perfect squares. If $t = d^2$, then

$$\Gamma^{(m)}[d^2] \cong \left\{ \begin{pmatrix} * & d* & * & d* \\ d* & * & d* & * \\ * & d* & * & d* \\ d* & * & d* & * \end{pmatrix} \in Sp_4(\mathbb{Z}) \right\}$$

(compare with (1.1)). Thus the moduli space $\mathcal{A}_{d^2}$ is a finite covering of the rational variety $\mathcal{A}_1$ of abelian surfaces with principal polarization. Using this fact K. O'Grady proved, that $\mathcal{A}_{p^2}$ is of general type for any prime $p \geq 17$ (see [O'G]).

The geometrical type of the moduli space $\mathcal{A}_p^{lev}$ of abelian surfaces with polarization of type $(1,p)$ (p is a prime) and with an additional level structure is investigated by K. Hulek and G. K. Sankaran in [HS]. $\mathcal{A}_p^{lev}$ is isomorphic to $\Gamma_p^{lev} \setminus \mathbb{H}_2$, where

$$\Gamma_p^{lev} = \left\{ g \in Sp_4(\mathbb{Z});\ \ g - \mathbf{1}_4 = \begin{pmatrix} * & * & * & p* \\ p* & p* & p* & p^2* \\ * & * & * & p* \\ * & * & * & p* \end{pmatrix} \in Sp_4(\mathbb{Z}) \right\}.$$

The moduli space $\mathcal{A}_p^{lev}$ is a ramified covering of the moduli space $\mathcal{A}_p$ of degree $\frac{p(p^2-1)}{2}$ and a ramified covering of the rational moduli space $\mathcal{A}_1$ of degree $\frac{p(p^4-1)}{2}$. In [HS] they proved that for $p \geq 41$ the variety $\mathcal{A}_p^{lev}$ is of general type, using the classification of singularities of a toroidal compactification of $\Gamma_p \setminus \mathbb{H}_2$ given in [HKW2].

In this paper we shall construct examples of cusp forms of weight 3 with respect to the paramodular group $\Gamma^{(m)}[t]$, which give us canonical differential forms on $\widetilde{\mathcal{A}}_t$. Our method does not depend on the resolution of singularities of a compactification of the moduli space. Moreover, using our result it is possible to improve the mentioned results of [O'G] and [HS]. We shall come back to these questions in a future paper.

The statement of Theorem 1 for a square-free t was proved in [G1] using a reduction to a very particular case of moduli spaces of polarized $K3$ surfaces. In this paper we follow a more direct method.

The author is indebted to D. Zagier and G. K. Sankaran for helpful conversations. I am grateful to the Max-Planck-Institut für Mathematik in Bonn for hospitality and excellent working conditions.

§2 Siegel modular froms and Jacobi lifting

We remind the definition of modular forms with respect to $Sp_4(\mathbb{Z})$.

Definition. *A holomorphic function $F(Z)$ on the Siegel upper half-space $\mathbb{H}_2$ is called a Siegel modular form of weight k with respect to $Sp_4(\mathbb{Z})$ if the following condition is satisfied*

$$F|_k\, g(Z) := J(g,Z)^{-k} F(g < Z >) = F(Z), \qquad J(g,Z) = \det(CZ + D),$$

for any $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{Z})$.

Analogously one can define the space $\mathfrak{M}_k(\Gamma)$ of all modular forms of weight k with respect to an arithmetic subgroup Γ of $Sp_4(\mathbb{Q})$. In this chapter we construct modular forms with respect to the group

$$\Gamma[t] := \left\{ \begin{pmatrix} * & t* & * & * \\ * & * & * & t^{-1}* \\ * & t* & * & * \\ t* & t* & t* & * \end{pmatrix} \in Sp_2(\mathbb{Q}) \right\} = {}^t\Gamma^{(m)}[t],$$

where all $*$ denote integral numbers and ^{t} means the transposition.

Let us take the decomposition of the matrix $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2$, where $\tau, \omega \in \mathbb{H}_1$ lie in the usual upper half-plane. The Fourier-Jacobi expansion of F is its Fourier expansion with respect to the variable ω

$$F(\tau, z, \omega) = f_0(\tau) + \sum_{m\geq 1} f_m(\tau, z)\exp(2\pi i\, m\omega),$$

where $\tau \in \mathbb{H}_1$ and $z \in \mathbb{C}$. The function $f_m(\tau, z)$ being SL_2-modular form in τ for a fixed z and a Jacobi function in z is an example of Jacobi modular forms of index m. We shall construct a lifting from the space of Jacobi modular forms of index t to the space of modular forms with respect to the group $\Gamma[t]$ on the Siegel upper-half space $\mathbb{H}_2$.

To define this lifting we need the notions of Jacobi forms and corresponding Hecke rings.

The Satake compactification of the quotient space $Sp_4(\mathbb{Z})\setminus\mathbb{H}_2$ has two boundary components: the curve $SL_2(\mathbb{Z})\setminus\mathbb{H}_1$ and the point ∞. The function $f_0(\tau)$ is equal to the restriction of the modular form $F(Z)$ to the boundary curve. The Fourier-Jacobi expansion, describing the behaviour of the form F near the boundary curve, corresponds to the Fourier expansion with respect to the maximal parabolic subgroup defining the boundary curve (see [BB]). From this point of view the function $\tilde{f}(Z) = f_m(\tau, z)\exp(2\pi i\,\omega)$ is nothing else but a modular form with respect to the parabolic subgroup

$$\Gamma_\infty = \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in Sp_4(\mathbb{Z}) \right\}$$

consisting of the elements which preserve an isotropic line:

$$(\tilde{f}|_k\gamma)(Z) = \tilde{f}(Z) \quad \text{for } \gamma \in \Gamma_\infty.$$

It is easy to see that

$$\Gamma_\infty/\{\pm 1_2\} \cong SL_2(\mathbb{Z}) \ltimes H(\mathbb{Z}),$$

where $H(\mathbb{Z})$ is the integral Heisenberg group, i.e., the central extension of the abelian group $\mathbb{Z}\times\mathbb{Z}$

$$0 \to \mathbb{Z} \to H(\mathbb{Z}) \to \mathbb{Z}\times\mathbb{Z} \to 0.$$

We have the following realizations of these two groups as subgroups of Γ_∞:

$$SL_2(\mathbb{Z}) \cong \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in Sp_4(\mathbb{Z}) \right\},$$

$$H(\mathbb{Z}) \cong \left\{ \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & r \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \in Sp_4(\mathbb{Z}) \right\}. \tag{2.1}$$

The group $\Gamma_\infty/\{\pm 1_2\}$ is called the Jacobi group in [EZ].

Definition. *A holomorphic function*

$$\phi(\tau, z):\ \mathbb{H}_1 \times \mathbb{C} \to \mathbb{C}$$

is called a Jacobi form of index $m \in \mathbb{N}$ and weight k if the function $\widetilde{\phi}(Z) = \phi(\tau, z)exp(2\pi i\, m\omega)$ on the Siegel upper half-space $\mathbb{H}_2$ is a modular form of weight k with respect to the integral parabolic subgroup Γ_∞:

1. $\widetilde{\phi}|_k M = \widetilde{\phi}$ *for any* $M \in \Gamma_\infty$*;*

2. The function has the Fourier expansion

$$\phi(\tau,\, z) = \sum_{\substack{n,l \in \mathbb{Z},\ n \geq 0 \\ 4nm \geq l^2}} f(n,l) \exp\left(2\pi i \left(n\tau + lz\right)\right), \tag{2.2}$$

i.e., $\widetilde{\phi}(Z)$ *is holomorphic at "∞".*

We call the function ϕ a Jacobi cusp form if we have the strict inequality $N = \begin{pmatrix} n & l/2 \\ l/2 & m \end{pmatrix} > 0$ in the last summation. We shall denote the space of all Jacobi forms or all Jacobi cusp forms of index m and weight k by $\mathfrak{M}^J_{k,m}$ or $\mathfrak{S}^J_{k,m}$.

For generators of the group Γ_∞ a Jacobi form ϕ of weight k and index m satisfies the following functional equations

$$\phi(\tau,\, z) = (c\tau + d)^{-k} \exp\left(-\frac{2\pi i c m z^2}{c\tau + d}\right)\phi\left(\frac{a\tau + b}{c\tau + d},\, \frac{z}{c\tau + d}\right),$$

$$\phi(\tau, z) = \exp\left(2\pi i\, m(\lambda^2 \tau + 2\lambda z)\right)\phi(\tau, z + \lambda\tau + \mu)$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and any $\mu, \lambda \in \mathbb{Z}$. This shows us that the definition is equivalent to the definition given in [EZ].

The construction of the lifting will be described in terms of a Hecke ring of the parabolic subgroup Γ_∞. Note here, that Γ_∞ is not reductive. We shall consider this ring as a non-commutative extension of the Hecke ring of $Sp_4(\mathbb{Z})$. First of all let us recall the definition of an abstract Hecke ring.

Definition. A pair $(\Gamma,\, G)$, where Γ is a subgroup of a semigroup G, is called a Hecke pair if any double coset $\Gamma g \Gamma$ $(g \in G)$ is the union of a finite number of left and right cosets relative to Γ. The Hecke ring $\mathcal{H}(\Gamma,\, G)$ of the pair $(\Gamma,\, G)$ is the Γ-invariant subspace of the $\mathbb{Q}$-vector space consisting of all formal finite linear combinations $X = \sum_i a_i \Gamma g_i$ $(a_i \in \mathbb{Q},\ g_i \in G)$, where a representation of the group Γ on this space is defined by the right multiplication $X \to X \cdot \gamma = \sum_i a_i \Gamma(g_i \gamma)$. For any two elements of this space $X = \sum_i a_i \Gamma h_i$ and $Y = \sum_j b_j \Gamma g_j$ their product is defined by $X \cdot Y = \sum_{i,j} a_i b_j \Gamma(h_i g_j)$. The product is independent of the choice of representatives g_i, h_j and $\mathcal{H}(\Gamma,\, G)$ is an associative ring.

It is evident that the elements $\Gamma g \Gamma = \sum_i \Gamma g_i$ $(g \in G)$ form a basis of the vector space $\mathcal{H}(\Gamma,\, G)$.

Let us define two Hecke rings

$$\mathcal{H}(\Gamma) = \mathcal{H}_{\mathbb{Q}}(Sp_4(\mathbb{Z}), GSp_4(\mathbb{Q})) \qquad \text{and} \qquad \mathcal{H}(\Gamma_\infty) = \mathcal{H}_{\mathbb{Q}}(\Gamma_\infty, G\Gamma_\infty(\mathbb{Q})),$$

where

$$GSp_4(\mathbb{Q}) = \{g \in M_4(\mathbb{Q}) :\ {}^t g W_1 g = \mu(g) W_1, \quad \mu(g) \in \mathbb{Q}^+\}$$

is the group of rational symplectic similitudes and $G\Gamma_\infty(\mathbb{Q})$ its parabolic subgroup of type Γ_∞. If $X \in \mathcal{H}(\Gamma)$, then according to the elementary divisors theorem one can represent X in the form $X = \sum_i a_i \Gamma g_i$, where $g_i \in G\Gamma_\infty(\mathbb{Q})$ and $a_i \in \mathbb{Q}$. It easy to see that the map

$$Im : X = \sum_i a_i \Gamma g_i \to \sum_i a_i \Gamma_\infty g_i \tag{2.3}$$

is a homomorphic embedding of the Hecke ring $\mathcal{H}(\Gamma)$ into $\mathcal{H}(\Gamma_\infty)$ (see [G3], [G4] for more general constructions) and we shall identify the ring $\mathcal{H}(\Gamma)$ with its image in $\mathcal{H}(\Gamma_\infty)$. Thus the Hecke ring of the parabolic group might be viewed as its non-commutative extension.

The ring $\mathcal{H}(\Gamma_\infty)$ contains also a subring isomorphic to the Hecke ring $\mathcal{H}(SL_2(\mathbb{Z}))$:

$$\mathcal{H}(\Gamma) \xrightarrow{Im} \mathcal{H}(\Gamma_\infty) \xleftarrow{j_-} \mathcal{H}(SL_2(\mathbb{Z})). \tag{2.4}$$

It is enough to define the embedding j_- for the generators

$$T(p) = SL_2(\mathbb{Z})\mathrm{diag}(1,p)SL_2(\mathbb{Z}) \quad \text{and} \quad T(p,p) = SL_2(\mathbb{Z})\mathrm{diag}(p,p)SL_2(\mathbb{Z})$$

of the ring $\mathcal{H}(SL_2(\mathbb{Z}), M_2^+(\mathbb{Z}))$, where $M_2^+(\mathbb{Z})$ is the semigroup of integral matrices with positive determinant. By definition we have

$$\begin{aligned} T_-(p) &:= j_-(T(p)) = \Gamma_\infty \mathrm{diag}(1,p,p,1)\Gamma_\infty, \\ T_-(p,p) &:= j_-(T(p,p)) = \Gamma_\infty \mathrm{diag}(p,p^2,p,1)\Gamma_\infty. \end{aligned}$$

The statement that the mapping j_- is a homomorphic embedding is clear, because there is a one-to-one correspondence between the left cosets in the decomposition of the double cosets $T(p)$, $T(p,p)$ and $T_-(p)$, $\Lambda_-(p)$ (see the proof of Lemma 2.1 below).

Our point of view of the ring $\mathcal{H}(\Gamma_\infty)$ is as an extension of the given Hecke ring of Sp_4 connected with some arithmetical properties of local L-functions of the symplectic group. For instance, the local L-function of Sp_4 splits in factors over the ring $\mathcal{H}(\Gamma_\infty)$, which correspond to the local L-function of SL_2(see [G2], [G3]).

We have the following representation of the ring $\mathcal{H}(\Gamma_\infty)$ on the space of functions, which are invariant with respect to $|_k$-action of the parabolic subgroup Γ_∞:

$$F \to F|_k X = \sum_i \mu(g_i)^{2k-3} a_i \, J(g_i, Z)^{-k} F(g_i < Z >), \tag{2.5}$$

for any $X = \sum_i a_i \Gamma_\infty g_i \in \mathcal{H}(\Gamma_\infty)$. We keep the same normalizing factor as for the Hecke operators for $Sp_4(\mathbb{Z})$.

Lemma 2.1. *Let $\phi(\tau, z) \in \mathfrak{M}_{k,t}^J$ be a Jacobi modular form of weight k and index t. Let us denote by $T_-(m)$ the j_--image in $\mathcal{H}(\Gamma_\infty)$ of the standard Hecke element*

$$T^{SL}(m) = \sum_{\substack{ef=m \\ e|f}} SL_2(\mathbb{Z})\mathrm{diag}(e,f)SL_2(\mathbb{Z}).$$

Then

$$\phi|_k\, T_-(m) := \big(\widetilde{\phi}|_k\, T_-(m)\big)(Z)\, exp(-2\pi i\, mt\omega) \in \mathfrak{M}^J_{k,mt}. \tag{2.6}$$

Proof. The holomorphic function on $\mathbb{H}_2$, $\big(\widetilde{\phi}|_k\, T_-(m)\big)(Z)$ is invariant with respect to the action of Γ_∞. Let us take the standard decomposition of the Hecke element $T(m)$ into the sum of the left cosets

$$T^{SL}(m) = \sum_{\substack{ad=m\\ b \bmod d}} SL_2(\mathbf{Z})\begin{pmatrix} a & b\\ 0 & d\end{pmatrix},$$

thus

$$T_-(m) = \sum_{\substack{ad=m\\ b \bmod d}} \Gamma_\infty \begin{pmatrix} a & 0 & b & 0\\ 0 & m & 0 & 0\\ 0 & 0 & d & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

By the definition

$$\begin{pmatrix} a & 0 & b & 0\\ 0 & m & 0 & 0\\ 0 & 0 & d & 0\\ 0 & 0 & 0 & 1\end{pmatrix} < \begin{pmatrix} \tau & z\\ z & \omega\end{pmatrix} >= \begin{pmatrix} \frac{a\tau+b}{d} & az\\ az & m\omega\end{pmatrix},$$

therefore the function

$$\big(\widetilde{\phi}|_k\, T_-(m)\big)(Z) = m^{2k-3} \sum_{\substack{ad=m\\ b \bmod d}} d^{-k}\phi(\frac{a\tau+b}{d}, az)\exp\,(2\pi i\, mt\omega)$$

(compare with the definition (2), §4 in [EZ]) corresponds to a Jacobi form of index mt.

There are two other types of commutative rings of Hecke operators acting on the space of Jacobi forms. They keep the index of Jacobi invariant or divide it by m (see [EZ], [G3], [G4]).

In the next theorem we construct an injective map from the space of Jacobi forms of index $t \geq 1$ and weight k (i.e., from the space of modular forms on the parabolic subgroup Γ_∞) into the space of modular forms with respect to the paramodular group $\Gamma[\,t\,]$ of level t.

Theorem 3. *Let $\phi(\tau, z)$ be a Jacobi cusp form of weight k and index $t \geq 1$ with the Fourier expansion*

$$\phi(\tau,\, z) = \sum_{\substack{n,l\in\mathbf{Z},\ n>0\\ 4nt>l^2}} f(n,l)\exp\,(2\pi i\,(n\tau + lz)).$$

Then the following function

$$G_\phi(\tau, z, \omega) = \sum_{m=1}^{\infty} m^{2-k}\,\big(\phi\,|_k\, T_-(m)\big)(\tau, z)\exp\,(2\pi i\, tm\omega)$$

is a cusp form of weight k with respect to the paramodular group $\Gamma[t]$.

Firstly we make some remarks about this theorem.

If index $t=1$, the map $\phi \to G_\phi$ coincides with the well-known Maass or Saito-Kurokawa lifting (see [EZ]). The theorem shows that the Maass lifting is only the first member in the infinite series of liftings connected with Jacobi forms. Thus for any Siegel modular form

$$F(\tau, z, \omega) = \sum_{m \geq 1} f_m(\tau, z) \exp(2\pi i m\omega)$$

we can construct an infinite series of lifted functions F_{f_m}, that define a "section" of the following infinite product

$$F \to (G_{f_m}) \in \prod_{m \in \mathbb{N}} \mathfrak{M}_k(\Gamma[m]).$$

We may rewrite at least formally the definition of the form G_ϕ using multiplicative notation. Let $\widetilde{\phi}(Z) = \phi(\tau, z)\exp(2\pi i\, t\omega)$. Then formally

$$\begin{aligned} G_\phi(Z) = \widetilde{\phi}\,|_k \sum_{m=1}^{\infty} m^{2-k} T_-(m) = \widetilde{\phi}\,|_k \prod_p (\sum_{\delta=1}^{\infty} p^{(2-k)\delta} T_-(p^\delta)) = \\ \widetilde{\phi}\,|_k \prod_p (1 - T_-(p)p^{2-k} + T_-(p,p)p^{3-2k})^{-1}, \end{aligned} \tag{2.7}$$

where the p-factor in the infinite product is the j_--image of the Hecke polynomial $Q_p^{SL}(X) = 1 - T(p)X + pT(p,p)X^2$ for $SL_2(\mathbb{Z})$. We cannot define the global operator L-function $j_-(L^{SL_2}(k-2)) = \prod_p j_-(Q_p(p^{2-k}))^{-1}$, but (2.7) shows us that the form G_ϕ has a multiplicative structure. From this point of view, the function $G_\phi(Z)$ is a generalization of the classical even theta-function! To make this remark clear let us define the theta-series in the same terms. Let $\mathcal{H}^{(0)}(SL_2) = \mathcal{H}(SL_2(\mathbb{Z}), SL_2(\mathbb{Q}))$ be the "even" Hecke of SL_2 and $\mathcal{H}(\Gamma_0) = \mathcal{H}(\Gamma_0(\mathbb{Z}), \Gamma_0(\mathbb{Q}))$ be the Hecke ring of its parabolic subgroup $\Gamma_0 = \left\{ \begin{pmatrix} \pm 1 & x \\ 0 & \pm 1 \end{pmatrix} \right\}$. As in the case of $Sp_4(\mathbb{Z})$ (see (2.3)) we can define an embedding of the first Hecke ring into the second $\mathcal{H}(SL_2) \to \mathcal{H}(\Gamma_0)$. We may continue the comparison with (2.4) and define an embedding of the multiplicative semigroup $\mathbb{N}^{-1}$ or, more general, the formal group ring $\mathbb{Q}[\mathbb{N}^{-1}]$ (we have to distinguish generators and coefficients!) into the Hecke ring $\mathcal{H}(\Gamma_0)$. By definition

$$n^{-1} \xrightarrow{j_-} [n^{-1}] := \Gamma_0 \begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix} \Gamma_0 = \Gamma_0 \begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix} \in \mathcal{H}(\Gamma_0).$$

The ring $\mathbb{Q}[\mathbb{N}^{-1}]$ is the Hecke ring $\mathcal{H}(\{1\}, \mathbb{N}^{-1})$ of the trivial group, consisting only of the identity. Thus we have the full analogy with the situation described in (2.4).

We can interpret $\mathbb{Z}$-periodic functions of the complex variable τ as automorphic functions with respect to the parabolic subgroup $\Gamma_0 \subset SL_2(\mathbb{Z})$ (compare with the definition of the Jacobi forms). If we take the representation of the Hecke ring $\mathcal{H}(\Gamma_0)$ on the space of $\mathbf{Z}$-periodic functions (i.e., automorphic with respect to Γ_0) we obtain, for instance, that $\exp(2\pi i\,\tau)|[n^{-1}] = \exp(2\pi i\,n^2\tau)$. As a consequence, we can represent the classical theta-function as a sum over a semigroup of the Hecke operators $\{[n^{-1}],\ n \in \mathbb{N}\}$ instead of as a sum over the lattice $\mathbb{Z}$. Namely,

$$\theta(\tau) = \sum_{n\in\mathbf{Z}} \exp(2\pi i\,n^2\tau) = 1 + 2 \sum_{[n^{-1}]\in\mathcal{H}(\{1\},\mathbb{N}^{-1})} \exp(2\pi i\,\tau)|[n^{-1}],$$

or using the same formal notation as above we have

$$\theta(\tau) = 1 + 2\exp(2\pi i\,\tau)|\prod_p (1-[p^{-1}])^{-1} = 1 + 2\exp(2\pi i\,\tau)|\,j_-(\zeta(1)).$$

We see that the lifting (2.7) is a generalization of the last formal identity, where we have taken the formal operator Hecke L-function $j_-(L^{SL_2}(k-2))$ instead of the operator Rieman zeta-function $j_-(\zeta(1))$. (Note once more that these function do not exist.) Using this analogy between the theta-series and the Jacobi lifting we constructed a holomorphic analytic continuation of Spin-L-function of the Siegel modular form for $Sp(4)$ taking the Rankin-Selberg convolution of a Siegel modular form with a lifted form (see [G2]).

We note that there exists a variant of the Jacobi lifting for Eisenstein series. In this case we have to restrict ourselves with weights $k \geq 4$.

Proof of Theorem 3. The convergence of the series defining G_ϕ follows from the estimation of Jacobi cusp forms of weight k and index t on $\mathbb{H}_1 \times \mathbb{C}$:

$$|\phi(\tau,z)| < Cv^{-\frac{k}{2}}\exp(2\pi t y^2/v),$$

where $v = \operatorname{Im}\tau > 0$, $y = \operatorname{Im} z$ and the constant C does not depend on τ and z. To prove the last inequality we take the function

$$\phi^*(\tau,z) = v^{\frac{k}{2}}\exp(-2ty^2/v)|\phi(\tau,z)|.$$

The function ϕ^* is Γ_∞-invariant and is bounded on any compact subset in $\mathbb{H}_1 \times \mathbb{C}$. We may take the following realization of the fundamental domain of Γ_∞ on $\mathbb{H}_1 \times \mathbb{C}$

$$\mathcal{D} = \{(\tau, \alpha\tau + \beta):\ -1 \leq \alpha, \beta \leq 1,\ \tau \in SL_2(\mathbb{Z}) \setminus \mathbb{H}_1\}.$$

The function ϕ^* is bounded on the set $\{\tau \in \mathcal{D}, \operatorname{Im}\tau > C\}$ since $\phi^*(\tau,z) \to 0$ as $v \to \infty$ for any cusp form.

The function $G_\phi(Z)$ is given by its Fourier-Jacobi expansion. The Jacobi forms in the summation have indexes divided by t (see Lemma 2.1). Thus G_ϕ is invariant with respect to the parabolic subgroup $\Gamma_\infty[t] = \Gamma_\infty(\mathbb{Q}) \cap \Gamma[t]$, which is generated by the integral parabolic subgroup Γ_∞ and the element

$$\nabla(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which belongs to the center of the rational Heisenberg group (see (2.1)) Let us calculate the Fourier expansion of G_ϕ at infinity. In accordance with the proof of Lemma 2.1

$$\phi|_k T_-(m) = m^{2k-3} \sum_{ad=m} d^{-k} \sum_{\substack{n,\,l \\ 4tn>l^2}} \sum_{b \bmod d} f(n,l) \exp\left(2\pi i\left(n\frac{a\tau+b}{d} + laz\right)\right).$$

Taking the sum over $b \bmod d$ and over all m we get

$$\begin{aligned} G_\phi(Z) &= \sum_{m\geq 1} \sum_{ad=m} a^{k-1} \sum_{\substack{n,\,l \\ 4tdn_1>l^2}} f(dn_1,l) \exp\left(2\pi i\left(n_1 a\tau + alz + adt\omega\right)\right) \\ &= \sum_{\substack{n,\,l \\ 4tmn>l^2}} \sum_{a|(n,l,m)} a^{k-1} f\left(\frac{nm}{a^2}, \frac{l}{a}\right) \exp\left(2\pi i\left(n\tau + lz + mt\omega\right)\right). \end{aligned}$$

This expansion shows us that $G_\phi(\tau,z,\omega)$ is invariant under the change of the variables $\{\tau \to t\omega,\ z \to z,\ \omega \to t^{-1}\tau\}$. The element

$$V_t = \begin{pmatrix} {}^tU_t & 0 \\ 0 & U_t \end{pmatrix}, \quad \text{where } U_t = \begin{pmatrix} 0 & \sqrt{t}^{-1} \\ \sqrt{t} & 0 \end{pmatrix},$$

realizes this transformation. Hence

$$G_\phi|_k V_t = (-1)^k G_\phi. \tag{2.8}$$

Moreover we have $G_\phi|_k J_t = G_\phi$, where

$$J_t = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t^{-1} \\ -1 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 \end{pmatrix}$$

is the element from the group $\Gamma[t]$, since

$$V_t I V_t I = J_t, \qquad \text{where } I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_\infty.$$

We finish the proof that G_ϕ is a modular form with the next lemma.

Lemma 2.2. *The group $\Gamma_\infty[t]$ and the element J_t generate $\Gamma[t]$.*

Proof. It is more natural to prove this lemma in terms of the integral paramodular group. We have to show that

$$J = I_t^{-1} J_t I_t = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}, \quad I_t^{-1}\Gamma_\infty[t] I_t = \{g = \begin{pmatrix} 1 & 0 & b & xt \\ -y & 1 & x & z \\ 0 & 0 & 1 & yt \\ 0 & 0 & 0 & 1 \end{pmatrix}\}, \tag{2.9}$$

where $x, y, z \in \mathbb{Z}$, generate the group

$$\Gamma^{(i)}[t] := I_t^{-1}\Gamma[t]I_t = \{g \in M_4(\mathbb{Z}) : \ {}^t g W_t g = W_t\}.$$

(The index ${}^{(i)}$ means "integral".) Note that we consider here the transposition of the paramodular group $Sp(W_t, \mathbb{Z})$ from §1. For the elements of $I_t^{-1}\Gamma_\infty[t]I_t$ we have

$$JgJ^{-1} = \begin{pmatrix} 1 & yt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -xt & 1 & 0 \\ -x & -z & -y & 1 \end{pmatrix}. \tag{2.10}$$

The group $SL_2(\mathbb{Z})$ is generated by the elements $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, thus $< \Gamma_\infty[t], J >$ contains two copies of $SL_2(\mathbb{Z})$. The first copy is a subgroup of $I_t^{-1}\Gamma_\infty[t]I_t$ defined in (2.1) and the second is the group of elements of type

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}. \tag{2.11}$$

The skew-symmetric form W_t defines the following scalar product on $\mathbb{Z}^4$:

$$< X, Y >_t = \det \begin{pmatrix} x_1 & x_3 \\ y_1 & y_3 \end{pmatrix} + t\det \begin{pmatrix} x_2 & x_4 \\ y_2 & y_4 \end{pmatrix},$$

where $X = {}^t(x_1, x_2, x_3, x_4)$.

For any $X \in \mathbb{Z}^4$ we denote by $\operatorname{div}(X)$ the natural number generates the ideal $\{< X, Y >_t, \ Y \in \mathbb{Z}^4\}$. The integral paramodular group leaves the linear form $< \cdot, \cdot >_t$ invariant, therefore

$$\operatorname{div}(gX) = \operatorname{div}(X) \qquad \text{for} \quad g \in \Gamma^{(i)}[t].$$

This is a divisor of the level of the group t.

Any primitive vector (the greatest common divisor of its entries has to be equal to one) could be reduced by the multiplication by an element of type (2.9) to a vector X with $g.c.d.\,(x_2, \ x_4) = 1$. Using the elements of SL_2-types we may reduce the vector X to the form $X = {}^t(x, 1, 0, 0)$. Using an element (2.10) we reduce x mod t. Consequently, we have proved the following

Lemma 2.3. *For a primitive integral vector $X \in \mathbb{Z}^4$ the orbit $\Gamma^{(i)}[t] \cdot X$ contains an element of the form ${}^t(d, 1, 0, 0)$, where $d = \operatorname{div}(X)$ is a divisor of t.*

Let us take $\gamma \in \Gamma^{(i)}[t]$ and denote by X_j the j-th column of the matrix γ and by X_{ij} its i-th element. X_2 is a primitive vector of the lattice $\mathbb{Z}$. According to the definition of the group $\Gamma^{(i)}[t]$ we have $< X_2, X_1 >_t = 0$, $< X_2, X_3 >_t = 0$, $< X_2, X_4 >_t = t$. Reducing X_2 to the form ${}^t(d, 1, 0, 0)$ and taking into account the last three equalities one can see, that $d = t$ (if not, then the third row of γ is not primitive). Multiplying by an element of type (2.11) we can reduce X_2 to the form ${}^t(0,1,0,0)$. An element $\gamma \in \Gamma^{(i)}[t]$ with the such second column belongs to the parabolic subgroup $I_t^{-1}\Gamma_\infty[t]I_t$. The lemma is proved.

Remark. The proof shows us that any $\gamma \in \Gamma^{(i)}[t]$ has the form

$$\gamma = \begin{pmatrix} * & t* & * & t* \\ * & * & * & * \\ * & t* & * & t* \\ * & * & * & * \end{pmatrix} \in M_4(\mathbb{Z}).$$

This explain the particular form of the elements from the group $\Gamma[t]$, which is conjugate to $\Gamma^{(i)}[t]$.

To prove that the lifting of a Jacobi cusp form gives us a cusp form with respect to the group $\Gamma[t]$ we need to describe the Satake compactification of the quotient space $\Gamma[t]\backslash\mathbb{H}_2$ (see, for example, [HKW1], where the case of prime t was considered, or [BB], [N] for the general theory of the Satake compactification).

The Siegel upper-half plane $\mathbb{H}_2$ is isomorphic to a boundary domain

$$\mathbb{D}_2 = \{W = {}^tW \in M_2(\mathbb{C});\ \mathbf{1}_2 - W\overline{W} > 0\}.$$

This isomorphism is given by the Cayley transformation

$$Z \to (Z - i\mathbf{1}_2)(Z + i\mathbf{1}_2)^{-1},$$

which one may use to define the corresponding action of $Sp_4(\mathbb{R})$ on $\mathbb{D}_2$. A maximal connected complex analytic set in the boundary $\partial\mathbb{D}_2 = \overline{\mathbb{D}}_2 \setminus \mathbb{D}_2$ is called a proper boundary component of $\mathbb{D}_2$. If $g \in \mathrm{Sp}_4(\mathbb{R})$, then either $g(F) = F$ or the intersection of $g(F)$ with F is empty.

There is a one-to-one correspondence between the sets of boundary components and the sets of isotropic subspaces in the four dimensional vector space $\mathbb{R}^4$ equipped with the standard symplectic form $J = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}$. For $W \in \overline{\mathbb{D}}_2$ we denote by $\mathrm{Is}\,(W)$ the real linear space dual to $\mathrm{Ker}\,\Psi_W$, where

$$\Psi_W : \mathbb{R}^4 \to \mathbb{C}^2, \quad X \to X \cdot \begin{pmatrix} i(\mathbf{1}_2 + W) \\ \mathbf{1}_2 - W \end{pmatrix}.$$

One can prove, that $\mathrm{Is}\,(W)$ is isotropic with respect to the skew-symmetric form J and does not depend on the point W, but only on the boundary component B containing W. Moreover, $\mathrm{Is}\,(g(B)) = g(\mathrm{Is}\,(B))$, where g acts on $\mathbb{R}^4$, considered as a set of column vectors. In the case of the group Sp_4 we have two types of the proper boundary components: points and components of dimension one.

A boundary component B' is said to be adjacent to another boundary component B if $B' \subset \overline{B}$. It is equivalent to say, that $\mathrm{Is}(B') \supset \mathrm{Is}(B)$.

For example, we have the following standard boundary components $B_0 = \{\mathbf{1}_2\}$ with $\mathrm{Is}\,(B_0) = {}^t(1,0,0,0)\mathbb{R} + {}^t(0,1,0,0)\mathbb{R}$ and $B_1 = \{\begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix};\ |w| < 1\}$ with $\mathrm{Is}\,(B_1) = {}^t(0,1,0,0)\mathbb{R}$. It is clear, that B_0 is adjacent to B_1.

A boundary component B is called rational if the isotropic space $\mathrm{Is}(B)$ or, equivalently, its stabilizing subgroup

$$\mathcal{P}(B) = \{g \in \mathrm{Sp}_4(\mathbb{R}) : \; g(B) = B\}$$

is defined over $\mathbb{Q}$. If B is an arbitrary rational boundary component of $\mathbb{D}_2$, then there are an index $i = 0$ or 1 and an element $g \in \mathrm{Sp}_4(\mathbb{Z})$ such that $g(B_i) = B$.

We call the union

$$\mathbb{D}_2^{(r)} = \coprod_B B \coprod \mathbb{D}_2,$$

where B runs over the proper rational boundary components of $\mathbb{D}_2$, the rational closure of $\mathbb{D}_2$.

There is a realization of $\mathbb{D}_2$ as an unbounded Siegel domain of third kind $\mathbb{D}(B)$ corresponding to a boundary component B. Let us consider $\mathbb{D}_2$ as a subset of its compact dual $\mathbb{D}_2^c$, which can be realized as the space of all two-dimensional isotropic vector subspaces of $\mathbb{C}^4$. For any boundary component B there exists the "partial Cayley transformation" with respect to B which transforms B into the variety at infinity $\mathbb{D}_2^c \setminus \mathrm{Sym}(2)$, where $\mathrm{Sym}(2)$ is isomorphic to the space of the complex (2×2)-symmetric matrices, and $\mathbb{D}_2$ onto the unbounded domain in $\mathrm{Sym}(2)$ (see [BB]). For example, for the boundary component B_1

$$\mathbb{D}(B_1) = \{Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2, \quad \tau \in \mathbb{H}_1 \approx B_1\}.$$

Let us take a cylindrical topology on $\mathbb{D}_2^{(r)}$. In this topology we have the following notion of convergence:

$$\mathbb{H}_2 \ni Z_n = \begin{pmatrix} \tau_n & z_n \\ z_n & \omega_n \end{pmatrix} \to \tau \in \mathbb{H}_1 \approx B_1 \quad \text{as } n \to \infty$$

iff $\tau_n \to \tau$ and $\mathrm{Im}\,\omega_n - (\mathrm{Im}\,\tau_n)^{-1}(\mathrm{Im}\,z_n)^2 \to \infty$. It is known that the group $\Gamma[t]$ acts on $\mathbb{D}_2^{(r)}$ properly discontinuously in the cylindrical topology, the quotient $\Gamma[t] \setminus \mathbb{D}_2^{(r)}$ with its quotient topology is compact and $\Gamma[t] \setminus \mathbb{D}_2$ is an open dense subset of it. $\Gamma[t] \setminus \mathbb{D}_2^{(r)}$ is called the Satake-Baily-Borel compactification.

If F is a modular form of weight k with respect to $\Gamma[t]$, then it extends to a modular form on a 1-dimensional boundary component B. We denote this form as $\Phi_B(F)$. This is the modular form of weight k with respect to the group $\Gamma(B) \subset \mathrm{SL}_2(\mathbb{Q})$ which is the image of $\Gamma[t] \cap \mathcal{P}(B)$ in $\mathcal{P}(B)/\mathcal{S}(B)$, where $\mathcal{S}(B)$ is the subgroup of the elements in $\mathcal{P}(B)$ acting trivially on B. We call the operator Φ_B the Siegel operator corresponding to the boundary component B. For example,

$$\Phi_{B_1}(F) = f_0(\tau) = \lim_{v\to\infty} F(\begin{pmatrix} \tau & 0 \\ 0 & iv \end{pmatrix}).$$

$f_0(\tau)$ is the modular form with respect to $\Gamma(B_1) = \mathrm{SL}_2(\mathbb{Z})$ and is equal to the zeroth coefficient of the Fourier-Jacobi expansion of F with respect to ω (i.e., the expansion along B_1). The following formal property follows directly from the definition

$$\Phi_{gB}(F) \circ g = \Phi_B(F|_k g)$$

for any $g \in \mathrm{Sp}_4(\mathbb{Q})$, where $F|_k g$ is the modular from of weight k with respect to $g^{-1}\Gamma[t]g$.

Definition. *Let F be a modular form with respect to the paramodular group $\Gamma[t]$. If $\Phi_B(F) = 0$ for any 1-dimensional boundary component B, then F is called a cusp form.*

We have seen that to understand the structure of the Satake compactification we need to define the action of the paramodular group $\Gamma[t]$ on the set of rational isotropic subspaces. In Lemma 2.3 we have described the $\Gamma[t]$-orbits on the integral lattice. A vector $l = {}^t(v_1, v_2, v_3, tv_4) \in L_t$ is called primitive, if v_1, v_2, v_3, v_4 are coprime. We may reformulate the statement of Lemma 2.3 in the following form

Lemma 2.3'. *For any primitive vector $l \in L_t$ the orbit $\Gamma[t]l$ contains one and only one element of the forms ${}^t(1,0,0,0)$, ${}^t(0,1,0,0)$ or ${}^t(d,1,0,0)$, where d is a non-trivial ($\neq 1, t$) divisor of t.*

Corollary 2.4. *The number of the 1-dimensional components of the Satake compactification of the quotient $\Gamma[t] \setminus \mathbb{H}_2$ is equal to the number of divisors of t.*

Corollary 2.5. *For any 1-dimensional boundary component B there exist a "diagonal" element*

$$M = \begin{pmatrix} D & 0 \\ 0 & {}^tD^{-1} \end{pmatrix} \in Sp_4(\mathbb{Q})$$

and $\gamma \in \Gamma[t]$ such that $M\gamma(B) = B_1$.

Let us finish the proof of Theorem 3.

Proof that G_ϕ is a cusp form. Let us take a 1-dimensional boundary component B. In accordance with Corollary 2.5 there exists a diagonal element M and $\gamma \in \Gamma[t]$ such that $M\gamma(B) = B_1$. In accordance with the definition of the Siegel operator

$$\Phi_B(G_\phi) \circ (M\gamma)^{-1} = \Phi_{B_1}(G_\phi|_k\gamma^{-1}M^{-1}) = \Phi_{B_1}(G_\phi|_kM^{-1}).$$

The form G_ϕ is defined by its Fourier-Jacobi expansion, corresponding to the standard boundary component B_1. The restriction $\Phi_{B_1}(G_\phi) \equiv 0$ and the Fourier expansion of G_ϕ at B_0 (this is the 0-cusp of the 1-dimensional boundary component B_1)

$$G_\phi(Z) = \sum_{N>0} a(N) \exp\left(2\pi i \operatorname{tr}(NZ)\right)$$

has no coefficients with degenerate N, because ϕ is a cusp form. The action of any "diagonal" element M keeps the type of Fourier expansion, i.e., the function $G_\phi|_kM$ has no Fourier coefficients with degenerate indices. Thus $\Phi_{B_1}(G_\phi|_kM) \equiv 0$ and $G_\phi(Z)$ is a cusp form. Theorem 3 is proved.

In the proof of the theorem we have seen that the lifting has an additional invariant property (see (2.8)). The group

$$\Gamma^*[t] = \Gamma[t] \cup \Gamma[t]V_t \subset Sp_4(\mathbb{R})$$

is a normal extension of the group $\Gamma[t]$ of index 2. In the case of $t = d^2$ this group is conjugate to a subgroup of $Sp_4(\mathbb{Z})$.

Corollary 2.6. *Let us assume that weight k of a Jacobi form ϕ is even. Then the lifting G_ϕ, constructed in Theorem 3, is a modular form of weight k with respect to the group $\Gamma^*[t]$.*

Note that the group $\Gamma^*[p]$ is the maximal real normal extension of the paramodular group $\Gamma[p]$ for a prime p.

§3 The quotients of symmetric domains

Basic to the geometric applications of the theory of automorphic forms is the fact that automorphic forms of special weights correspond to sections of canonical or pluricanonical line bundles on algebraic varieties.

Let $H^0(\mathcal{A}_t,\, \Omega_3(\mathcal{A}_t))$ be the space of holomorphic 3-forms on the moduli space $\mathcal{A}_t = \Gamma^{(m)}[t] \setminus \mathbb{H}_2$. For any $\omega \in H^0(\mathcal{A}_t,\, \Omega_3(\mathcal{A}_t))$ we may write it in the form

$$\omega = F(Z)dZ, \qquad dZ = d\tau \wedge dz \wedge d\omega,$$

where $F \in \mathfrak{M}_3(\Gamma^{(m)}[t])$ is a modular form of weight 3 with respect to $\Gamma^{(m)}[t]$.

The complex variety $\mathcal{A}_t$ is not compact and has a lot of singularities. We have the following nice criterion, due to E. Freitag, about continuation of canonical differential forms on a singular variety to its non-singular model. This criterion is quite general, but we formulate it only for the case of $\mathcal{A}_t$.

Criterion. *(Freitag) An element $\omega \in H^0(\mathcal{A}_t,\, \Omega_3(\mathcal{A}_t))$ can be extended to a canonical differential form on a non-singular model $\widetilde{\mathcal{A}}_t$ of a compactification of $\mathcal{A}_t$ if and only if the differential form ω is square integrable.*

Proof. See [F], Hilfsatz 3.2.1.

It is well known that $\omega_F = F(Z)dZ$ is square-integrable for the cusp form F. Thus we have the following identity for the geometrical genus of the variety $\widetilde{\mathcal{A}}_t$:

$$p_g(\widetilde{\mathcal{A}}_t) = h^{3,0}(\widetilde{\mathcal{A}}_t) = \dim_{\mathbb{C}} \mathfrak{M}_3^{(cusp)}(\Gamma^{(m)}[t]).$$

Lemma 3.1. *For any k and t the following identity is valid:*

$$\dim_{\mathbb{C}} \mathfrak{M}_3^{(cusp)}(\Gamma^{(m)}[t]) = \dim_{\mathbb{C}} \mathfrak{M}_3^{(cusp)}(\Gamma[t]).$$

Proof. Let us define

$$C_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix}.$$

It is easy to see, that $\Gamma^{(m)}[t] = C_t^{-1}\Gamma[t]C_t$. Thus $F|_k C_t$ is a cusp form with respect to $\Gamma^{(m)}[t]$, iff F is a cusp form with respect to $\Gamma^{(m)}[t]$.

Now we may prove Theorem 2. The lifting of Theorem 3 is injective, because the first Fourier-Jacobi coefficient of the lifted form G_ϕ does not vanish identically.

Consequently, we have the following estimation for the geometrical genus of the moduli variety

$$p_g(\widetilde{\mathcal{A}}_t) \geq \dim_{\mathbb{C}} \mathfrak{S}^J_{3,t}.$$

In accordance with results of [EZ], [SZ], the dimension of the space of the Jacobi cusp forms is given by the following formula

$$\dim_{\mathbb{C}} \mathfrak{S}^J_{k,t} = \begin{cases} \sum_{j=1}^{t} \{k+2j\}_{12} - \lfloor \frac{j^2}{4t} \rfloor, & k \text{ is even,} \\ \sum_{j=1}^{t-1} \{k+2j-1\}_{12} - \lfloor \frac{j^2}{4t} \rfloor, & k \text{ is odd,} \end{cases}$$

with

$$\{m\}_{12} = \begin{cases} \lfloor \frac{m}{12} \rfloor & \text{if } m \not\equiv 2 \mod 12, \\ \lfloor \frac{m}{12} \rfloor - 1 & \text{if } m \equiv 2 \mod 12. \end{cases}$$

This gives us the estimation of Theorem 2. It is evident that $\dim_{\mathbb{C}} \mathfrak{S}^J_{k,t} = O(t)$ as $t \to \infty$.

The same formulae show us that there exist cusp forms of weight 2 with respect to the paramodular group. Using this fact we may prove the following

Theorem 4. *Let Γ be a subgroup of the paramodular group $\Gamma^{(m)}[t]$ which has no element of the finite order. If the dimension of the space of Jacobi cusp forms of weight 2 and index t is positive, then the quotient space $\mathcal{M} = \Gamma \setminus \mathbb{H}_2$ is of general type.*

Proof. Let $\widetilde{\mathcal{M}}$ be a toroidal compactification of $\mathcal{M}$. Since the group Γ is torsion free, then according to Tai's criterion (see [SC]) a pluricanonical differential form

$$\omega_F = F(Z)(dZ)^{\otimes n} \in H^0(\mathcal{M}, \Omega_3(\mathcal{M})^{\otimes n}),$$

where $F(Z)$ is a modular form of weight $3n$, could be extended to the compactification $\widetilde{\mathcal{M}}$ if $F(Z)$ vanishes on the boundary with order n. Let $G \in \mathfrak{M}_2^{(cusp)}(\Gamma^{(m)}[t])$ be a cusp form $G(Z)$ of weight 2 which is the lifting of a Jacobi cusp form of weight 2 and index t. For any modular form $F \in \mathfrak{M}_{3n}(\Gamma)$ of weight $3n$ we have that the form $(G(Z))^{3n}F(Z) \in \mathfrak{M}_{9n}(\Gamma)$ vanishes on the boundary with order $3n$. Thus

$$\dim_{\mathbb{C}} H^0(\widetilde{\mathcal{M}}, \Omega_3(\widetilde{\mathcal{M}})^{\otimes n}) \geq \dim_{\mathbb{C}} \mathfrak{M}_{3n}(\Gamma).$$

The last dimension could be easily estimated by Mumford's extension of the Hirzebruch's proportionality principal (see [M] and [T]). At the end we have that

$$\dim_{\mathbb{C}} H^0(\widetilde{\mathcal{M}}, \Omega_3(\widetilde{\mathcal{M}})^{\otimes n}) \geq Cn^3,$$

where the constant C does not depend on n. Therefore the variety $\widetilde{\mathcal{M}}$ is of general type.

We may formulate a natural conjecture

Conjecture. *The moduli space $\mathcal{A}_t$ of abelian surfaces with a polarization of type $(1,t)$ is of general type if the dimension of the space of the Jacobi cusp forms of weight 2 and index t is positive.*

The first such value of t is 37. The maximal t for which $\dim_{\mathbb{C}} \mathfrak{S}^J_{2,t} = 0$ is $t = 180$.

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