ON THE COEFFICIENTS OF ARTIN-WEIL L-FUNCTIONS

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Resumè

Let $L(s,\rho) = \sum_{n=1}^{\infty} a_n(\rho)n^{-s}$ be an L-function associated to a representation ρ of the Weil group of a number field. We give an asymptotic formula for the sum $\sum_{n<x} a_n(\rho)$, as $x \longrightarrow \infty$, with an error term of the same shape as in the Primidealsatz. The main difficulty is due to the fact that $|a_n(\rho)|$ is comparably large, while the known zero free region for $L(s,\rho)$ is as narrow as the one for Hecke's L-functions "mit Größencharakteren". It has been overcomed by reducing the problem to an asymptotic estimate for the sum $\sum_{n<x} |a_n(\rho)|^2$ of the coefficients of the scalar product of $L(s,\rho)$ and $L(s,\rho^*)$, where ρ^* denotes the representation contragradient to ρ. The consequences of Artin's Conjecture and the Extended Riemann Hypothesis are discussed.

Let k be a number field of degree $d = [k : \mathbb{Q}]$ over $\mathbb{Q}$ and let

$$\rho : W(k) \longrightarrow GL(V)$$

be a finite dimensional complex representation of the Weil group $W(k)$ in a vector space V. One defines the L-function associated to ρ by

$$L(s,\chi) = \prod_{\mathfrak{p}} \det(1 - \rho(\sigma_{\mathfrak{p}})|_{V_{\mathfrak{p}}} |\mathfrak{p}|^{-s})^{-1} \quad , \tag{1}$$

where the product is extended over all the prime ideals $\mathfrak{p}$ of k, $|\mathfrak{p}| := N_{k/\mathbb{Q}}\mathfrak{p}$, $\sigma_{\mathfrak{p}}$ is the Frobenius class at $\mathfrak{p}$ and

$$V_{\mathfrak{p}} = \{x | \quad \rho(\tau)x = x \quad , \tau \in I_{\mathfrak{p}} \quad , x \in V\}$$

is the subspace of $I_{\mathfrak{p}}$ - invariant vectors, $I_{\mathfrak{p}}$ is the inertia subgroup of $W(k)$ at $\mathfrak{p}$, $\chi = \operatorname{tr} \rho$. The Euler product (1) converges absolutely for $\operatorname{Re} s > 1$ and can be developed in a Dirichlet series

$$L(s,\chi) = \sum_{\mathfrak{n}} \alpha_{\mathfrak{n}}(\chi) \, |\mathfrak{n}|^{-s} \quad , \quad \operatorname{Re} s > 1 \quad ,$$

where $\mathfrak{n}$ varies over integral ideals of k. The main object of this paper is an asymptotic estimate, as $x \to \infty$, for the sum

$$A(x,\chi) = \sum_{|\mathfrak{n}|<x} \alpha_{\mathfrak{n}}(\chi) \quad .$$

It is convenient to consider at the same time the sums

$$A(x,\vec{\chi}) = \sum_{|\mathfrak{n}|<x} \prod_{i=1}^{r} a_{\mathfrak{n}}(\chi_i) \quad ,$$

where $\vec{\chi} = (\chi_1,\ldots,\chi_r)$, $\chi_i = \mathrm{tr}\rho_i$, ρ_i is a representation of $W(k_i)$ for an extension $k_i \supseteq k$ of finite degree, and $L(s,\chi_i)$ is developed in a Dirichlet series

$$L(s,\chi_i) = \sum_{\mathfrak{n}} a_{\mathfrak{n}}(\chi_i) \; |\mathfrak{n}|^{-s}$$

over k, $1 \leq i \leq r$.

<u>Theorem 1.</u> There are an effectively computable constant $c(\vec{\chi}) > 0$ and a polynomial $P(\vec{\chi},t)$ such that

$$|A(x,\vec{\chi}) - xP(\vec{\chi},\log x)| < \exp(-c(\vec{\chi})\sqrt{\log x}\;).$$

We write

$$A(x,\vec{\chi}) = xP(\vec{\chi}, \log x) + R(x,\vec{\chi}) \tag{2}$$

and show that if the representation $p = \theta_1 \otimes \ldots \otimes \theta_r$, where θ_i denotes the representation of $W(k)$ induced by ρ_i , satisfies Artin's conjecture, then one can find $\gamma > 0$ such that

$$R(x,\vec{\chi}) = O(x^{1-\gamma}) \quad . \tag{AH.3}$$

Moreover, under the Grand Riemann Hypothesis (that is, assuming that any relevant L-function "mit Größencharakteren" has no zeroes in the half-plane $\mathrm{Re}\ s > \frac{1}{2}$)

$$R(x,\vec{\chi}) = O_\varepsilon(x^{\frac{1}{2}+\varepsilon}) \qquad \text{(RH.4)}$$

for any $\varepsilon > 0$. Finally (§ 4) we discuss the prime number theorem for non-Abelian characters and make a few bibliographical remarks.

To fix our notation let us recall that the Weil group $W(k)$ is defined as a projective limit over all the finite Galois extensions $E \supseteq k$ of the groups $W(E|k)$. Any $W(E|k)$ fits in an exact sequence

$$1 \longrightarrow C_E \longrightarrow W(E|k) \longrightarrow G(E|k) \longrightarrow 1 \quad ,$$

where C_E is the idèle-class group of E and $G(E|k)$ is the Galois group of E over k . Since $C_E \cong \mathbb{R}_+ \times C^\circ_E$, where $\mathbb{R}_+$ denotes the multiplicative group of positive real numbers and C°_E is compact, we have

$$W(E|k) \cong \mathbb{R}_+ \times W^\circ(E|k)$$

with compact $W^\circ(E|k)$. Moreover,

$$W(E|k) \cong W(k)/W(E)^c$$

where $W(E)^c$ denotes the closure of the commutant of $W(E)$. Any continuous finite dimensional representation ρ of $W(k)$ contains $W(E)^c$ in its kernel for some $E \supseteq k$, therefore it can be regarded as a representation of $W(E|k)$. If, moreover, $\mathbb{R} \subseteq \mathrm{Ker}\ \rho$, so that actually ρ is a representation of a compact group $W^\circ(E|k)$, we say that it is normalised. Let Rep (k) denote the set of continuous finite dimensional normalised complex representations of $W(k)$ and let $X(k)$ be the set of characters of such representations. We denote by $Gr(k)$ the subset of one-dimensional representations in $Rep(k)$; it follows that $Gr(k)$ coincides with the set of normalised Größencharaktere, that is, continuous homomorphisms of C_k in the unit circle trivial on $\mathbb{R}_+$.

Lemma 1. Let $\chi \in X(k)$. One can find finite extensions $E_i \supseteq k$, $1 \leq i \leq \ell$, and Größencharacters $\Psi_i \in Gr(E_i)$ such that

$$L(s,\chi) = \prod_{i=1}^{\mu} L(s,\Psi_i)^{e_i} \quad , \tag{5}$$

where $e_i = \begin{cases} 1 & , i \leq \mu_1 \\ -1 & , i > \mu_1 \end{cases}$, $0 < \mu_1 \leq \mu$.

Proof. It is classical, [20], p. 33-34.

Let S_1 and S_2 denote the sets of real and complex places of k, respectively; let $\chi \in Gr(k)$ and let $\chi_{\mathfrak{p}}$ be the $\mathfrak{p}$-component of χ, $\mathfrak{p} \in S_1 \cup S_2$. We write

$$\chi_{\mathfrak{p}}(x) = |x|^{it_{\mathfrak{p}}} \left(\frac{x}{|x|}\right)^{\alpha_{\mathfrak{p}}} \quad , \quad x \in k_{\mathfrak{p}} \, ,$$

where $k_{\mathfrak{p}}$ is the completion of k at $\mathfrak{p}$, $\alpha_{\mathfrak{p}} \in \mathbb{Z}$, $t_{\mathfrak{p}} \in \mathbb{R}$, and, moreover, $\alpha_{\mathfrak{p}} \in \{0,1\}$ for $\mathfrak{p} \in S_1$. Let

$$s_{\mathfrak{p}}(\chi) = \begin{cases} it_{\mathfrak{p}} + \alpha_{\mathfrak{p}} , & \mathfrak{p} \in S_1 \\ 2it_{\mathfrak{p}} + \frac{|\alpha_{\mathfrak{p}}|}{2} , & \mathfrak{p} \in S_2 \end{cases} ,$$

and let f and D denote the conductor of χ and the discriminant of k , respectively. We define

$$\alpha_{\circ}(\chi) = |D| \cdot N_{k/Q} f \quad ,$$

and

$$\alpha(\chi) = \prod_{\mathfrak{p} \in S_1} (|s_{\mathfrak{p}}(\chi)| + 1) \prod_{\mathfrak{p} \in S_2} (|s_{\mathfrak{p}}(\chi)|^2 + 1) .$$

Let

$$L_{\infty}(s,\chi) = \prod_{\mathfrak{p} \in S_1 \cup S_2} G_{\mathfrak{p}}(s + s_{\mathfrak{p}}(\chi)) ,$$

where $G_{\mathfrak{p}}(s) = \begin{cases} \pi^{-S/2} \, \Gamma(s/2) , & \mathfrak{p} \in S_1 \\ (2\pi)^{1-s} \, \Gamma(s) , & \mathfrak{p} \in S_2 \end{cases}$ .For $\chi \in X$ satisfying (5)

let

$$L_{\infty}(s,\chi) = \prod_{i=1}^{\mu} L_{\infty}(s,\Psi_i)^{e_i} \quad , \quad \alpha(\chi) = \prod_{i=1}^{\mu} \alpha(\Psi_i) \quad ,$$

$$\alpha_{\circ}(\chi) = \prod_{i=1}^{\mu} \alpha_{\circ}(\Psi_i)^{e_i}$$

and let $\Lambda(s,\chi) = L(s,\chi) \, L_{\infty}(s,\chi)$. Then

$$\Lambda(s,\chi) = W(\chi)\alpha_\circ(\chi)^{\frac{1}{2}-s}\ \Lambda(1-s,\bar{\chi})\ ,\quad |W(\chi)| = 1\ . \tag{6}$$

To deduce (6) (cf. [20]) one uses the functional equations for the Abelian L-functions (see, e.g., [21], p. 133) and the identity (5). Let

$$\nu(\chi) = \text{card}\ \{i|\Psi_i = 1\ ,\ i \leq \mu_1\} - \text{card}\ \{i|\Psi_i = 1\ ,\ i > \mu_1\}\ .$$

The function

$$s \longmapsto L(s,\chi)$$

has at $s = 1$ a pole of order $\nu(\chi)$ whenever $\nu(\chi) > 0$. One can show that $\nu(\chi) \geq 0$ for $\chi \in X(k)$.

Artin's conjecture (AH). The function

$$F : s \longmapsto L(s,\chi) - \frac{\omega(\chi,s)}{(s-1)^{\nu(\chi)}} \tag{AH.7}$$

it holomorphic in $\mathbb{C}$ for some polynomial $\omega(\chi,s)$ of degree $\nu(\chi)-1$ (assumed to be equal to zero when $\nu(\chi)= 0$) .

Let k_i be a finite extension of k , $\rho_i \in \text{Rep}(k_i)$, $1 \leq i \leq r$, $\chi_i = \text{tr}\,\rho_i$ and $\vec{\chi} =(\chi_1,\ldots,\chi_r)$. Consider a Dirichlet series

$$L(s,\vec{\chi}) = \sum_{\mathfrak{n}} |\mathfrak{n}|^{-s} \prod_{i=1}^{r} a_{\mathfrak{n}}(\chi_i)\ ,\ \text{Re}\ s > 1 \tag{8}$$

over k . Let $\theta_i = \text{Ind}\ (W(k_i), W(k), \rho_i)$ and $\rho = \theta_1 \otimes \ldots \otimes \theta_r$,

$\chi = \operatorname{tr} \rho$. Write

$$L(s,\vec{\chi}) = L(s,\chi) \cdot \ell(s,\vec{\chi}), \quad \operatorname{Re} s > 1 \ . \tag{9}$$

<u>Proposition 1.</u> The function $s \to \ell(s,\vec{\chi})$ has a holomorphic continuation to the half-plane $\operatorname{Re} s > \frac{1}{2}$ and, moreover,

$$\ell(s,\vec{\chi}) = O_\varepsilon(1) \quad \text{when} \quad \operatorname{Re} s > \frac{1}{2} + \varepsilon \tag{10}$$

for any $\varepsilon > 0$.

We deduce this Proposition from the following lemma on convolutions. Consider r Dirichlet series

$$L_i(s) = \sum_{\mathfrak{n}} a_i(\mathfrak{n}) |\mathfrak{n}|^{-s} \quad , \quad \operatorname{Re} s > 1 \ , \quad 1 \le i \le r \ .$$

Suppose that

$$L_i(s) = \prod_{\mathfrak{p}} Q_{\mathfrak{p}}^i \left(|\mathfrak{p}|^{-s} \right)^{-1} , \quad \operatorname{Re} s > 1 \ ,$$

where

$$Q_{\mathfrak{p}}^i(t) = \prod_{j=1}^{n_i} (1 - \alpha_{ji}(\mathfrak{p})\, t) \in \mathbb{C}[t] \ .$$

Let, moreover, $|\alpha_{ji}(\mathfrak{p})| = 1$ for $\mathfrak{p} \notin S$ and let S be a finite set of primes; let

$$B(\mathfrak{p}) = \{\alpha_{j_1 1}(\mathfrak{p}) \ldots \alpha_{j_r r}(\mathfrak{p}) \mid 1 \le j_1 \le n_1, \ldots , 1 \le j_r \le n_r\} \quad .$$

Define two Dirichlet series

$$L_\circ(s) = \prod_{\mathfrak{p}} Q_{\mathfrak{p}}(|\mathfrak{p}|^{-s})^{-1} \quad ,$$

where $Q_{\mathfrak{p}}(t) = \prod_{\alpha \in B(\mathfrak{p})} (1 - \alpha t)$, and

$$L(s) = \sum_{\mathfrak{n}} |\mathfrak{n}|^{-s} \left(\prod_{i=1}^{r} a_i(\mathfrak{n}) \right) .$$

Lemma 2. The function

$$s \longrightarrow L(s) \cdot L_\circ(s)^{-1} = \ell(s)$$

can be holomorphically continued to the half-plane $\mathrm{Re}\ s > \frac{1}{2}$ and, moreover,

$$\ell(s) = O_\varepsilon(1) \quad \text{when} \quad \mathrm{Re}\ s > \frac{1}{2} + \varepsilon \tag{11}$$

for any $\varepsilon > 0$.

Proof. Let $\mathfrak{p} \notin S$. One can show by a formal computation (cf.[14], lemma 5) that

$$\sum_{m=0}^{\infty} t^m \prod_{i=1} a_i(\mathfrak{p}^m) = \frac{R_{\mathfrak{p}}(t)}{Q_{\mathfrak{p}}(t)} \quad , \quad R_{\mathfrak{p}}(t) \equiv 1 \pmod{t^2} \tag{12}$$

in the ring $\mathbb{C}\left[[t]\right]$ of formal power series, where $R_{\mathfrak{p}}$ is a polynomial of degree smaller than $\prod_{i=1}^{r} n_i$. Since the set S is finite, it follows from (12) that the product

$$\ell(s) = \prod_{\mathfrak{p} \notin S} R_{\mathfrak{p}}(|\mathfrak{p}|^{-s})^{-1} \prod_{\mathfrak{p} \in S} \left(Q_{\mathfrak{p}}(|\mathfrak{p}|^{-s}) \sum_{m=0}^{\infty} |\mathfrak{p}|^{-sm} \prod_{i=1}^{r} a_i(\mathfrak{p}^m) \right)$$

converges absolutely when $\operatorname{Re} s > \frac{1}{2}$. This proves the lemma.

2. We prove Theorem 1 in this paragraph. The following lemmas give the analytic information needed to deduce the estimate for $A(x,\vec{\chi})$.

Lemma 3. Let $\chi \in Gr(k)$ and $\varepsilon > 0$. There exists $C(\varepsilon) > 0$ such that

$$| L(u + it,\chi)| < C(\varepsilon)(2 + |t|)^{\frac{d}{2}(1+\varepsilon-u)} \alpha_{\circ}(\chi)\, \alpha(\chi)^{\frac{1+\varepsilon-u}{2}} \tag{13}$$

for $|t| > 1, t \in \mathbb{R}$ and any u in the interval $-\varepsilon \leq u \leq 1 + \varepsilon$.

Proof. It follows from Theorem 5 in [17].

From now on we assume that $\chi = \operatorname{tr} \rho$, $\rho \in \operatorname{Rep}(k)$ and χ satisfies (5) . Let

$$n(\chi) = \sum_{i=1}^{\mu} [E_i : \mathbb{Q}] \quad . \tag{14}$$

Lemma 4. The function $F(u + it)$ defined by (7) is holomorphic and has no zeroes in the region

$$u \geq 1 - \varphi(t)^{-1} \quad , \quad t \in \mathbb{R} , \tag{15}$$

where

$$\varphi(t) = \begin{cases} a \quad \log\, (\alpha_{\circ}(\chi)\, \alpha(\chi)\, (|t| + 2)) & \text{when} \quad t \geq |t_{\circ}| \\ a \quad \log\, (\alpha_{\circ}(\chi)\, \alpha(\chi)\, (|t_{\circ}| + 2)) & \text{when} \quad t \leq |t_{\circ}| \end{cases}$$

for some a , $t_\circ$ depending only on the fields E_i , $1 \le i \le \mu$ (the constant a may depend on the conductors of those Ψ_i for which $\Psi_i^2 = 1$).

Proof. In view of (5) , this assertion follows from known zero-free region for Abelian L-functions $L(s,\Psi_i)$, $1 \le i \le \mu$ (see, e.g., [5], p. 512, lemma 2, or [12], p. 15, lemma 2).

Lemma 5. In the region (15) the function (7) can be estimated as follows:

$$F(u + it) = O_\delta(\alpha_\circ(\chi)\ (2 + |t|)^\delta\ \alpha(\chi)^\delta\ ) \tag{16}$$

for any $\delta > 0$, where the O-constants depend only on the fields E_i (and δ), and $t_\circ$ is chosen to be small enough (depending on δ).

Proof. By lemma 4, the function $g(s) = \log F(s)$ is holomorphic in (15). Moreover, $\mathrm{Re}\ g(s) = \log |F(s)|$, so that, by (13) and (5),

$$\mathrm{Re}\ g(s) \le \log \left[C(\varepsilon)(2+|t|)^{\frac{n}{2}(1+\varepsilon-u)} (\alpha_\circ(\chi)\alpha(\chi))^{\frac{\mu(1+\varepsilon-u)}{2}} \right] \tag{17}$$

$$+ \log\left(1 + \left|\frac{\omega(\chi,s)}{(s-1)^{\nu(\chi)}}\right|\right) ,$$

where $s = u + it$,$-\varepsilon \le u \le 1 + \varepsilon$, $t \in \mathbb{R}$, $\varepsilon > 0$. Choose $\delta > 0$. By (17), one can choose $t_\circ > 0$ in such a way that

$$\mathrm{Re}\ g(s) \le \log\left[C_1(\delta)\ (2 + |t|)^\delta\ \alpha_\circ(\chi)\alpha(\chi)^\delta\right] \tag{18}$$

for $s = u + it$ satisfying (15). We cite now the following lemma from the theory of functions (see, e.g., [16] , p. 383, Satz 4.2) : let $R > 0$ and assume that $g(s)$ is holomorphic in the circle $\{s \mid |s - s_0| \le R\}$, let $\operatorname{Re} g(s) \le M$ in this circle, then

$$|g(s) - g(s_0)| \le 2(M - \operatorname{Re} g(s_0)) \frac{r}{R-r} \tag{19}$$

whenever $|s - s_0| \le r < R$. We choose $s_0 = 1 + \frac{1}{2\varphi(t)} + it$, $R = \varphi(t)^{-1}$, $r = \frac{3}{4\varphi(t)}$. Combining (18) and (19) one deduces the statement of the lemma.

We need two general results from the theory of Dirichlet series. Let $f(s)$ be a function meromorphic in the region

$$\mathcal{B} = \{u + it \mid u \ge 1 - \frac{b}{\log B(|t| + 2)}, \quad t \in \mathbb{R}\} \tag{20}$$

and satisfying the following conditions:

a) in the half-plane $\operatorname{Re} s > 1$ it is given by an absolutely convergent Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad ;$$

b) there exists an integer $\nu \ge 0$ and a polynomial $\omega(s)$ such that $\omega(s) = 0$ when $\nu = 0$, $\omega(s)$ is of degree $\nu - 1$ when $\nu \ge 1$, the function

$$F(s) = f(s) - \frac{\omega(s)}{(s-1)^{\nu}}$$

is holomorphic in $\mathcal{B}$;

c) $F(u+it) = O(B(|t|+2)^{\gamma})$ in $\mathcal{B}$ for some γ such that $0<\gamma<1$.

Let $A(x) = \sum_{n<x} a_n$ for $x>0$, $x \notin \mathbb{Z}$.

Lemma 6. There is a polynomial P , of degree $\nu - 1$ when $\nu \geq 1$ and equal to zero when $\nu = 0$, such that

$$A(x) = x \cdot P(\log x) + O(x \exp(-c_1 \frac{\log x}{\sqrt{\log x} + \log B})) + O(\sum_{x \leq n < \beta \cdot x} |a_n|), \tag{21}$$

where $c_1 > 0$ and

$$\beta = 1 + \exp\left(-c_1 \frac{\log x}{\sqrt{\log x} + \log B}\right) .$$

Lemma 7. If $a_n \geq 0$ for every n , then

$$A(x) = x \cdot P(\log x) + O\left(x \exp\left(-c_1 \frac{\log x}{\sqrt{\log x} + \log B}\right)\right) , \tag{22}$$

in notation of lemma 6 .

These assertions are of quite classical origin; we are following the lines of [8] , § 242, Satz 62. (*

Proof of (21) and (22).

Let $A_1(x) = \sum_{n<x} a_n \log \frac{x}{n}$. Since

(* The author is grateful to Professor H. Delange for an alternative proof of lemma 7.

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s^2}\,ds = \begin{cases} 0 & , \text{ when } y<1 \\ \log y , & \text{when } y>1 \end{cases}$$

for any $c>1$, $y>0$ and in view of a) , one obtains

$$A_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s^2}\,f(s)\,ds \quad . \tag{23}$$

Conditions a)-c) allows to move the contour of integration in (23) to the line

$$u = 1 - \frac{b}{\log B\,(|t|+2)} \quad .$$

This gives

$$A_1(x) = x\cdot P_1(\log x) + O(x \exp(-c_2 \frac{\log x}{\sqrt{\log x + \log B}})) \tag{24}$$

for some $c_2>0$, where P_1 is a polynomial of degree $\nu - 1$ (equal to zero when $\nu = 0$). Let $\beta>1$ and write

$$A_1(x\cdot\beta) = \sum_{n<x\beta} a_n \log\frac{x\beta}{n} = A(x)\cdot\log\beta + A_1(x) + \sum_{x\leq n<x\beta} a_n \log\frac{x\beta}{n} \quad . \tag{25}$$

Combining (24) and (25) one obtains the assertion of lemma 6.

Let now $a_n \geq 0$ for every n . Then, by (25),

$$A(x) \leq \frac{A_1(x\cdot\beta) - A_1(x)}{\log\beta} \tag{26}$$

Analogously,

$$A_1(x\beta^{-1}) = \sum_{n < x\beta^{-1}} a_n \log \frac{x}{\beta n} = \sum_{n < x\beta^{-1}} a_n \log \frac{x}{n} + \log \frac{1}{\beta} \sum_{n < x\beta^{-1}} a_n ,$$

and, since

$$\log \beta \sum_{x\beta^{-1} \le n<x} a_n \ge \sum_{x\beta^{-1} \le n<x} a_n \log \frac{x}{n} ,$$

we see that

$$A(x) \geqq \frac{A_1(x) - A_1(x\beta^{-1})}{\log \beta} \tag{27}$$

The inequalities (26) and (27) when combined with (24) imply (22). It follows from lemma 5 and Proposition 1 that the function $L(s,\vec{\chi})$ defined by (8) satisfies conditions a)-c) of lemma 6 with $B = \alpha_\circ(\chi)\alpha(\chi)$, therefore

$$A(x,\vec{\chi}) = x \cdot P(\vec{\chi}, \log x) + O\left(x \exp(-c_1 \frac{\log x}{\sqrt{\log x} + \log(\alpha_\circ(\chi)\alpha(\chi))})\right)$$
$$+ O\left(\sum_{x \le n < \beta x} |a_n(\vec{\chi})|\right) ,$$

where $a_n(\vec{\chi}) = \sum_{|\pi|=n} \prod_{i=1}^{r} a_\pi(\chi_i)$, $c_1 > 0$, and $P(\vec{\chi},t)$ is a polynomial of degree $\nu(\chi) - 1$, $\chi = \operatorname{tr} \rho, \rho = \bigoplus_{i=1}^{r} \theta_i$. Let us consider now a sequence of of characters $\vec{\chi}' = (\chi_1, \bar{\chi}_1, \ldots, \chi_r, \bar{\chi}_r)$, where $\bar{\chi}_i$ is the character of the representation contragradient to χ_i , so that

$$a_n(\overrightarrow{\chi'}) = \sum_{|\mathfrak{n}|=n} \prod_{i=1}^{r} |a_{\mathfrak{n}}(\chi_i)|^2 \geq 0 \quad .$$

Therefore, by lemma 7 ,

$$A(x,\overrightarrow{\chi'}) = x\cdot P(\overrightarrow{\chi'}, \log x) + O\left(x \exp(-c_2 \frac{\log x}{\sqrt{\log x} + \log(\alpha_\circ(\chi)\alpha(\chi))})\right) \qquad (28)$$

with some $c_2 > 0$. On the other hand,

$$\sum_{x\leq n<\beta x} |a_n(\vec{\chi})| \leq \sum_{x\leq|\mathfrak{n}|<\beta x} \prod_{i=1}^{r} |a_{\mathfrak{n}}(\chi_i)| \quad .$$

Therefore, by Schwartz's inequality,

$$\sum_{x\leq n<\beta x} |a_n(\vec{\chi})| \leq \left(\sum_{x\leq n<\beta x} a_n(\overrightarrow{\chi'})\right)^{\frac{1}{2}} \left(\sum_{x\leq|\mathfrak{n}|<\beta x} 1\right)^{\frac{1}{2}} \quad .$$

By (28),

$$\sum_{x\leq n<\beta x} a_n(\overrightarrow{\chi'}) = A(x\ ,\overrightarrow{\chi'}) - A(x,\overrightarrow{\chi'}) = O(x \exp(- \frac{c_1 \log x}{\sqrt{\log} x + \log(\alpha_\circ(\chi)\alpha(\chi))}))$$

$$+ O\left((\beta - 1)x\cdot(\log x)^{\nu'}\right) \quad ,$$

where ν' denotes the degree of the polynomial $P(\overrightarrow{\chi'},t)$. Since, by a theorem of Landau's ,

$$\sum_{|\mathfrak{n}|<x} 1 = \omega_k x + O(x^{1-\frac{2}{d+1}}) \ ,$$

where ω_k denotes the residue of the zeta-function of k , we have

$$\sum_{x \le |\pi| < \beta x} 1 = O((\beta - 1)x) \ .$$

Combining these estimates one obtains

$$A(x,\vec{\chi}) = P(\vec{\chi},\log x) + O\left(x \exp(-c \frac{\log x}{\sqrt{\log x} + \log(\alpha_\circ(\chi)\alpha(\chi))})\right) \ , \qquad (29)$$

where $c > 0$ depends only on the fields E_i and possibly on the conductors of those Ψ_i in (5) for which $\Psi_i^2 = 1$; as before, χ denotes the character of $\rho = \bigotimes_{i=1}^{r} \theta_i$.

Estimate (29) implies the inequality of Theorem 1.

3. Artin's conjecture or Riemann's Hypothesis (RH) allow both to improve the errow term in (29) and to simplify the reasoning. Let us assume first that

$$L(s,\Psi_i) \neq 0 \quad \text{for} \quad \text{Re } s > \frac{1}{2} \quad , \quad 1 \le i \le r \quad , \qquad \text{(RH.30)}$$

for each of Größencharaktere Ψ_i occuring in (5). A classical argument [10] (or [19], p. 282-283) shows that (30) implies the Lindelöf's hypothesis:

$$|L(u + it,\Psi_i)|^{e_i} = O_\varepsilon((\alpha_\circ(\chi)\alpha(\chi)|t|)^\varepsilon) \quad \text{when} \quad u \ge \frac{1}{2} + \varepsilon \qquad \text{(RH.31)}'$$

for any of the functions

$$s \longmapsto L(s,\Psi_i) \ , \ 1 \le i \le \mu \ ,$$

in (5). In particular, for any $\varepsilon > 0$ and $\sigma_\circ > \frac{1}{2}$ there exists $C(\varepsilon,\sigma_\circ)$ such that

$$|L(u + it,\chi)| < C(\varepsilon,\sigma_\circ)\ (\alpha_\circ(\chi)\alpha(\chi)|t|)^{\varepsilon} \tag{RH 31}$$

when $u \geq \sigma_\circ$, $t \in \mathbb{R}$, $|t| \geq 1$. On the other hand, since the series (8) converges absolutely for $\operatorname{Re} s > 1$, we have

$$a_n(\vec{\chi}) = 0_\varepsilon(n^{\varepsilon}) \quad ,$$

so that (see, e.g., [16], p. 376, equation (3.3))

$$A(x,\vec{\chi}) = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \frac{x^s}{s} L(s,\vec{\chi})\, ds + 0_\varepsilon\left(\frac{x^{1+\varepsilon}\log x}{T}\right) \ , \quad \varepsilon > 0 \ . \tag{32}$$

By Proposition 1 and assumption (30), the function $L(s,\vec{\chi})$ has no singularities in the half-plane $\operatorname{Re} s > \frac{1}{2}$, save for a pole of order $\nu(\chi)$ at $s = 1$ (when $\nu(\chi) > 0$). Moreover, by (31) and (10), one can find $C_1(\varepsilon,\sigma_\circ)$ such that

$$|\ L\ (u + it,\vec{\chi})| < C_1(\varepsilon,\sigma_\circ)(\alpha_\circ(\chi)\alpha(\chi)\ |t|)^{\varepsilon} \tag{RH 33}$$

whenever $u \geq \sigma_\circ > \frac{1}{2}$, $|t| \geq 1$, $t \in \mathbb{R}$, $\varepsilon > 0$. Applying Cauchy's theorem on residues to the contour of integration consisting of the lines

$$\{u \pm ix^{\frac{1}{2}}\ |\ \tfrac{1}{2} + \varepsilon \leq u \leq 1 + \varepsilon\},\ \{1+\varepsilon + it|\ -x^{\frac{1}{2}} \leq t \leq x^{\frac{1}{2}}\} \quad ,$$

$$\{\tfrac{1}{2} + \varepsilon + it\ |\ -x^{\frac{1}{2}} \leq t \leq x^{\frac{1}{2}}\}$$

one deduces from (32) and (33) equation (2) and an estimate

$$R(x,\vec{\chi}) = O_{\varepsilon}(x^{\frac{1}{2}}(\alpha_0(\chi)\alpha(\chi)x)^{\varepsilon}) \ , \ \varepsilon > 0, \qquad \text{(RH 34)}$$

with the O-constant depending on the fields E_i in (5) (and $\varepsilon > 0$). We have just proved the following assertion.

Theorem 2. If each of the L-functions $L(s,\Psi_i)$, $1 \leq i \leq \mu$, occuring in (5) satisfies (RH 30), then equation (2) holds with the estimate (RH 34) for the error term.

If ρ satisfies Artin's conjecture, one can use the functional equation to prove an analogue of (13) and then, moving the contour of integration in (32) to the critical strip, to deduce (AH.3). Since the Artin's conjecture has been proved to hold for many representations, e.g., for Größencharacters, for monomial representations, for tensor products of monomial representations, and for certain two dimensional representations with $\mu_1 < \mu$, it seems desirable to go into details. We start with the following lemma of which lemma 3 is a special case.

Lemma 8. If $L(s,\chi)$ satisfies (AH.7), then

$$L(u + it,\chi) = O_\eta\left(|t|^{(1+\eta-u)\frac{n(\chi)}{2}} (\alpha_0(\chi)\alpha(\chi))^{\frac{1+\eta-u}{2}}\right) \qquad \text{(AH.35)}$$

whenever $t \in \mathbb{R}$, $|t| > 1$, $0 < \eta < \frac{1}{2}$, $-\eta \leq u \leq 1 + \eta$.

Proof. Let $0 < \eta \leq \frac{1}{2}$ and $f(t) = \log \left| \frac{L_\infty(1+\eta-it,\bar{\chi})}{L_\infty(-\eta+it,\chi)} \right|$, $t \in \mathbb{R}$,

where χ is a Größencharakter in a field E of degree $m = [E:Q]$.
By definition,

$$f(t) = \sum_{\mathfrak{p} \in S_1(E) \cup S_2(E)} \log \left| \frac{G_{\mathfrak{p}}(1+\eta - it + \overline{s_{\mathfrak{p}}})}{G_{\mathfrak{p}}(-\eta + it + s_{\mathfrak{p}})} \right| ,$$

so that

$$|f(t)| \le \sum_{\mathfrak{p} \in S_1(E)} \left| \log \left| \frac{\Gamma(\frac{1}{2}(1+\eta+\alpha_{\mathfrak{p}} - i(t+t_{\mathfrak{p}}))}{\Gamma(\frac{1}{2}(-\eta+\alpha_{\mathfrak{p}} + i(t+t_{\mathfrak{p}}))} \right| \right| \log \pi +$$

$$+ \log(2\pi) \sum_{\mathfrak{p} \in S_2(E)} \left| \log \left| \frac{\Gamma(1+\eta+\frac{1}{2}|\alpha_{\mathfrak{p}}| - i(t+t_{\mathfrak{p}}))}{\Gamma(-\eta+\frac{1}{2}|\alpha_{\mathfrak{p}}| - i(t+t_{\mathfrak{p}}))} \right| \right| .$$

Taking into account that $\alpha_{\mathfrak{p}} \in \{0,1\}$ for $\mathfrak{p} \in S_1(E)$ and applying the formula $\Gamma(s+1) = s\,\Gamma(s)$ to reduce successively the real part of the argument of the Γ-functions in the second sum to $1+\eta+\{\frac{|\alpha_{\mathfrak{p}}|}{2}\}$ in the numerator and to $-\eta+\{\frac{|\alpha_{\mathfrak{p}}|}{2}\}$ in the denominator, where $\{\alpha\}$ denotes the fractional part of $\alpha \in \mathbb{R}$ (so that $0 \le \{\alpha\} < 1$), one obtains by means of Stirling's formula for Γ-function

$$|f(t)| \le \frac{1+2\eta}{2} \sum_{\mathfrak{p} \in S_1} \log |t+t_{\mathfrak{p}}| + \sum_{\mathfrak{p} \in S_2} \Bigg[(1+2\eta) \log|t+t_{\mathfrak{p}}| +$$

$$\sum_{1 \le k < \frac{|\alpha_{\mathfrak{p}}|}{2}} \log \left| \frac{k+\eta - i(t+t_{\mathfrak{p}})}{k-(\eta+1) + i(t+t_{\mathfrak{p}})} \right| \Bigg] + O(1)$$

with an O-constant depending on $m = [E:Q]$. As an elementary consequence of this estimate, it follows that

$$|f(t)| \le \left(\frac{1}{2} + \eta\right) (\log \alpha(\chi) + m \log (1 + |t|)) + O(1). \qquad (36)$$

It follows from (36) that

$$\frac{L_\infty(1+\eta-it,\overline{\chi})}{L_\infty(-\eta+it,\chi)} = O_\eta\left((\alpha(\chi)(1+|t|))^{n(\frac{1}{2}+\eta)}\right), \tag{37}$$

where, by definition, $n = \sum_{i=1}^{\mu} [E_i:\mathbb{Q}]$. By (6) and (37),

$$L(-\eta+it,\chi) = O_\eta\left(\alpha_0(\chi)\ (\alpha(\chi)\ (1+|t|))^{n(\frac{1}{2}+\eta)}\right), \tag{38}$$

since $L(1+\eta-it,\overline{\chi}) = O_\eta(1)$ for any $\eta>0$. We cite the following result from the theory of functions (cf.[3], p. 92):

Let $F(s)$ be a function regular in the strip $\alpha \le \mathrm{Re}\, s \le \beta$ and $F(u+it) \le \exp(\exp(\gamma|t|))$ for $\alpha \le u \le \beta$, $t \in \mathbb{R}$ with $\gamma < \frac{\pi}{\beta-\alpha}$. If $F(\alpha+it) \le U(1+|t|)^a$, $F(\beta+it) \le V$ for $t \in \mathbb{R}$ and $U>1$, $V>1$, $a>0$, then, for $\alpha \le u \le \beta$, $t \in \mathbb{R}$, we have

$$F(u+it) = O\left(U^{\frac{\beta-u}{\beta-\alpha}}\ V^{\frac{u-\alpha}{\beta-\alpha}}\ (1+|t|)^{\frac{a(\beta-u)}{\beta-\alpha}}\right) \tag{39}$$

Properly adjusting the constants one deduces (AH.35) from (38), (39) and (AH.7).

Write

$$A(x,\chi) = x\cdot P(\chi,\log x) + R(x,\chi). \tag{40}$$

<u>Theorem 3.</u> If $L(s,\chi)$ satis fies (AH.7), then

$$R(x,\chi) = O_\varepsilon(\alpha(\chi)\alpha_\circ(\chi)x^{1-\frac{2}{n+2}+\varepsilon}) \quad , \tag{AH.41}$$

if, moreover, $\chi = \prod_{i=1}^{r} \operatorname{tr}\theta_i$ is the character associated to $\vec{\chi}$ as in (9), then

$$R(x,\vec{\chi}) = O_\varepsilon(\alpha(\chi)\alpha_\circ(\chi)x^{1-\frac{2}{n+4}+\varepsilon}) \; , \tag{AH.42}$$

for any $\varepsilon > 0$ and with $n = n(\chi)$.

Proof. The estimate (AH.41) follows from (AH.35) and (32) by the theorem of Cauchy's on residues applied to the contour of integration consisting of the lines

$$\ell_1 = \{1+\varepsilon+ti \mid -T \leq t \leq T\} \;,\; \{it \mid -T \leq t \leq T\},\; \{u \pm iT \mid 0 \leq 1 \leq 1+\varepsilon\}.$$

To deduce (AH.42) one uses (AH.35), (9) and (10) and applies the theorem on residues to the contour consisting of ℓ_1 and the lines

$$\{\tfrac{1}{2}+\varepsilon+it \mid -T \leq t \leq T\},\; \{u \pm iT \mid \tfrac{1}{2}+\varepsilon \leq u \leq 1+\varepsilon\} \quad .$$

Remark. Throughout this paragraph the constants implied by the O-symbols may depend on the fields E_i, $1 \leq i \leq \mu$.

Corollary. If $\vec{\chi} = (\chi_1,\dots,\chi_r)$, $\chi_i \in Gr(k_i)$, then (42) holds unconditionally.

Proof. By a theorem of Mackey's , [11], $\rho = \theta_1 \otimes \dots \otimes \theta_r$ being a product of monomial representations is equivalent to a direct sum of monomial representations, therefore ρ satisfies (5) with $\mu_1 = \mu$. In particular, (AH.7) holds for $\chi = \mathrm{tr}\,\rho$ (cf. [14] and [15], Appendix).

4. Write $\chi(\mathfrak{p}^m) = \mathrm{tr}\left(\rho(\sigma_{\mathfrak{p}})|_{V_{\mathfrak{p}}}\right)$. As an easy consequence of (5), one deduces the following assertion.

Proposition 2. Let χ satisfy (5). Then

$$\sum_{|\mathfrak{p}|<x} \chi(\mathfrak{p}) = \sum_{i=1}^{\mu} e_i \sum_{|\mathfrak{q}_i|<x} \Psi_i(\mathfrak{q}_i) + O(x^{\frac{1}{2}}) , \tag{43}$$

where $\mathfrak{q}_i$ varies over the prime divisors of E_i, $1 \leq i \leq \mu$.

Proof. Taking the logarithm of both sides in (5) one obtains for $\mathrm{Re}\, s > 1$:

$$\sum_{m\geq 1} \sum_{\mathfrak{p}} \frac{\chi(\mathfrak{p}^m)}{m} |\mathfrak{p}|^{-ms} = \sum_{i=1}^{\mu} e_i \sum_{m\geq 1, \mathfrak{q}_i} \frac{\Psi_i(\mathfrak{q}_i^m)}{m} |\mathfrak{q}_i|^{-ms} \tag{44}$$

In particular, equating the coefficients in (44) one obtains

$$\sum_{|\mathfrak{p}|=p} \chi(\mathfrak{p}) = \sum_{i=1}^{\mu} e_i \sum_{|\mathfrak{q}_i|=p} \Psi_i(\mathfrak{q}_i) \tag{45}$$

for any prime p in $\mathbb{Q}$; estimate (43) is an obvious consequence of (45). Write

$$\sum_{|\mathfrak{q}_i|<x} \Psi_i(\mathfrak{q}_i) = g(\Psi_i) \int_2^x \frac{du}{\log u} + r(x,\Psi_i), \quad g(\Psi_i) = \begin{cases} 0, & \Psi_i \neq 1 \\ 1, & \Psi_i = 1 \end{cases}$$

Since $\sum_{i=1}^{\mu} e_i g(\Psi_i) = \nu(\chi)$, it follows from (43) that

$$\sum_{|\mathfrak{p}|<x} \chi(\mathfrak{p}) = \nu(\chi) \int_2^x \frac{d\,u}{\log u} + r(x,\chi) \quad , \tag{46}$$

where

$$|r(x,\chi)| \leq \sum_{i=1}^{\mu} |r_i(x,\Psi_i)| + O(x^{\frac{1}{2}}) \; . \tag{47}$$

From the known estimates for $r(x,\Psi_i)$, $\Psi_i \in Gr(E_i)$ (cf. [5], p.513, lemma 4, or [12], p. 16, lemma 7) one deduces , in view of (47), that

$$r(x,\chi) = O\left(x \exp(-\gamma \frac{\log x}{\sqrt{\log x} + \log \alpha(\chi)})\right) \quad \text{with} \quad \gamma > 0 \tag{48}$$

Moreover, recent results [4],[13] allow to improve on the estimate (48). The reader can consult [18] (and references therein) for interesting applications of the non-Abelian prime number theorem.

By the methods developed in [6],[7] (cf. also [14]) one can continue $L(s,\vec{\chi})$ defined by (8) to the half-plane $\operatorname{Re} s > 0$ (but, in general, not beyond the line $\operatorname{Re} s = 0$). We should like to refer to [1] ,[2], [22] for some results related to the ones discussed here and to [9] for comprehensive information concerning Artin's conjecture.

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