Noncommutative tori and the Riemann–Hilbert correspondence

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Abstract

We study the interplay between noncommutative tori and noncommutative elliptic curves through a category of equivariant differential modules on $\mathbb{C}^*$. We functorially relate this category to the category of holomorphic vector bundles on noncommutative tori as introduced by Polishchuk and Schwarz and study the induced map between the corresponding K-theories. In addition, there is a forgetful functor to the category of noncommutative elliptic curves of Soibelman and Vologodsky, as well as the forgetful functor to the category of vector bundles on $\mathbb{C}^*$ with regular singular connections.

The category that we consider has the nice property of being a Tannakian category, hence it is equivalent to the category of representations of an affine group scheme. Via an equivariant version of the Riemann–Hilbert correspondence we determine this group scheme to be (the algebraic hull of) $\mathbb{Z}^2$.

Introduction

Noncommutative geometry in its various forms has come to the forefront of mathematical research lately and noncommutative tori constitute perhaps the most extensively studied class of examples of noncommutative differentiable manifolds. They were introduced by Connes during the early eighties [5] and were systematically studied by Connes [5], Rieffel [30, 31] and others. Recently Polishchuk and Schwarz have provided a new perspective on them which is quite amenable to techniques in algebraic geometry [27, 26]. At the same time Soibelman and Vologodsky have introduced noncommutative elliptic curves as certain equivariant categories of coherent sheaves [33]. The guiding principle behind both constructions is replacing a mathematical object by its category of appropriately defined representations, *viz.,* vector bundles with connections in the former case, denoted by $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$, and coherent sheaves in the latter, denoted by $\mathcal{B}_q$, where $q = e^{2\pi i\theta}$ and θ is an irrational number.

In this article we try to connect the above two constructions by introducing an intermediate category $\mathcal{B}^\tau_q$. Besides the existence of a forgetful functor from $\mathcal{B}^\tau_q$ to $\mathcal{B}_q$ (as the notation might suggest), we construct a faithful and exact functor from $\mathcal{B}^\tau_q$ to $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$. It turns out to be well-adapted to the Tannakian formalism. In fact, our main result is that it is a Tannakian category and via an equivariant version of the Riemann–Hilbert correspondence we show that it is equivalent to the category of finite dimensional representations of (the algebraic hull of) $\mathbb{Z}^2$ (see Theorem 19). This allows us to describe the K-theory of $\mathcal{B}_q^\tau$ as the free abelian group generated by two copies of $\mathbb{C}^*$ (see Corollary 21). This paper is organized as follows.

In the first section we briefly review the main results of [27], including the basic definitions and examples. We also discuss the rudiments of noncommutative tori, which are relevant for our purposes as it is known that there are several ways of looking at them. We also show that there is a certain *modularity* property satisfied by the categories $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ (see Proposition 2).

In the second section we first provide a motivation for the definition of the categories $\mathcal{B}_q^\tau$ and then construct a faithful and exact functor from $\mathcal{B}_q^\tau$ into $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$. We also give a description of the image of our functor and discuss the induced map on the K-theories of the corresponding categories. There is a canonical forgetful functor from $\mathcal{B}_q^\tau$ to $\mathcal{B}_q$.

In the third section we start by briefly recalling some preliminaries of Tannakian categories. We explain the structure of a Tannakian category on the category $\mathcal{B}_q^\tau$ and prove an equivariant version of the Riemann–Hilbert correspondence on $\mathbb{C}^*$. Via this correspondence, we find that $\mathcal{B}_q^\tau$ is equivalent to the category of finite dimensional representations of $\mathbb{Z}^2$. As a consequence we are able to compute the K-theory of $\mathcal{B}_q^\tau$.

We conclude with a motivation for a possible notion of the fundamental group of noncommutative tori (see Remark 22) and with a discussion on the degeneration of the complex structure on noncommutative tori (see Remark 23).

Convention. In this article, unless otherwise stated, θ is always assumed to be irrational and τ in the lower half plane as in [27]. The ground field is also assumed to be $\mathbb{C}$.

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1 Preliminaries

Let E be an elliptic curve and $\mathrm{Coh}(E)$ denote the category of coherent sheaves on it. We start by recalling some basic facts about $\mathrm{Coh}(E)$ and the t-structures on its derived category. We also briefly review the rudiments of noncommutative tori and holomorphic bundles on them.

1.1 Coherent sheaves on elliptic curves

For $\mathcal{F} \in \mathrm{Coh}(E)$ we write $\mathrm{rk}(\mathcal{F})$ for the generic rank and $\chi(\mathcal{F})$ for the Euler characteristic of $\mathcal{F}$. Since E has genus 1, by the Riemann-Roch Theorem the degree of $\mathcal{F}$ is the same as its Euler characteristic $\chi(\mathcal{F}) := \dim_{\mathbb{C}}\mathrm{Hom}(\mathcal{O}_E, \mathcal{F}) - \dim_{\mathbb{C}}\mathrm{Ext}^1(\mathcal{O}_E, \mathcal{F})$. Thus the slope of a coherent sheaf $\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}$ is the rational number $\frac{\chi(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}$ or infinity, when its rank is zero.

One extends the notion of slope to the objects in the derived category by defining rank (resp. Euler characteristic) of a complex as the alternating sum of the ranks (resp. Euler characteristics) of the individual terms of the corresponding cohomology complex. A coherent sheaf $\mathcal{F}$ is called *semistable* (resp. *stable*) if for any non-trivial exact sequence $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ one has $\mu(\mathcal{F}') \leqslant \mu(\mathcal{F})$ (resp. $\mu(\mathcal{F}') < \mu(\mathcal{F})$) or equivalently $\mu(\mathcal{F}) \leqslant \mu(\mathcal{F}'')$ (resp. $\mu(\mathcal{F}) < \mu(\mathcal{F}'')$).

It is well-known that every coherent sheaf on E splits as the direct sum of its torsion and torsion free parts. Since E is smooth, projective and of dimension 1, every torsion free coherent sheaf is locally free. The following result from [16] gives us a good understanding of the indecomposable objects of $\mathrm{Coh}(E)$, which can be shown to be semistable.
Let X be a projective curve. Then for any $\mathcal{F} \in \mathrm{Coh}(E)$ there exists a unique filtration:

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} = \mathcal{F} \tag{1}$$

such that,

- $\mathcal{A}_i := \mathcal{F}_{i+1}/\mathcal{F}_i$ for $0 \leqslant i \leqslant n$ are semistable and
- $\mu(\mathcal{A}_0) > \mu(\mathcal{A}_1) > \cdots > \mu(\mathcal{A}_n)$.

The filtration above is called the *Harder–Narasimhan filtration* of $\mathcal{F}$ and the graded quotients $\mathcal{A}_i$ of the Harder–Narasimhan Filtration are called the *semistable factors* of $\mathcal{F}$.

Remark 1. Over an elliptic curve, every coherent sheaf is isomorphic to the direct sum of its semistable factors, which is a consequence of the special Calabi–Yau property. A $\mathbb{C}$-linear abelian category of dimension 1 is said to have the Calabi–Yau property if the Ext^1 sets are isomorphic to the dual Hom sets as vector spaces.

1.2 Torsion pairs and t-structures

We now recall the definition of a torsion pair in an abelian category and its associated t-structure as in [15]. The notations employed here are local and should not be confused with their appearances in different forms elsewhere. Let $(\mathcal{T}, \mathcal{F})$ be a pair of full subcategories of an abelian category $\mathcal{A}$. We say that $(\mathcal{T}, \mathcal{F})$ is a *torsion pair* in $\mathcal{A}$ if the following conditions are satisfied:

1. $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
2. For all $X \in \mathcal{A}$ there exists $t(X) \in \mathcal{T}$ and a short exact sequence in $\mathcal{A}$

   $$0 \longrightarrow t(X) \longrightarrow X \longrightarrow X/t(X) \longrightarrow 0$$

   such that $X/t(X) \in \mathcal{F}$.

Due to condition 1 the exact sequence in condition 2 is unique up to isomorphism. Let $\mathcal{C}$ be a triangulated category. Following [2] a *t-structure* $(\mathcal{C}^{\leqslant 0}, \mathcal{C}^{\geqslant 0})$ on $\mathcal{C}$ is a pair of full subcategories of $\mathcal{C}$ such that the following conditions are satisfied:

Define $\mathcal{C}^{\leqslant n} := \mathcal{C}^{\leqslant 0}[-n]$ and $\mathcal{C}^{\geqslant n} := \mathcal{C}^{\geqslant 0}[-n]$ for all $n \in \mathbb{N}$.

1. $\mathrm{Hom}(X,Y)=0$ for all $X\in\mathcal{C}^{\leqslant 0}$ and $Y\in\mathcal{C}^{\geqslant 1}$.
2. $\mathcal{C}^{\leqslant 0}\subset\mathcal{C}^{\leqslant 1}$ and $\mathcal{C}^{\geqslant 1}\subset\mathcal{C}^{\geqslant 0}$.
3. For all $X\in\mathcal{C}$ there exist $X'\in\mathcal{C}^{\leqslant 0}$ and $X''\in\mathcal{C}^{\geqslant 1}$ such that

$$X'\longrightarrow X\longrightarrow X''\longrightarrow X'[1]$$

is a distinguished triangle in $\mathcal{C}$.

Given a t-structure $(\mathcal{C}^{\leqslant 0},\mathcal{C}^{\geqslant 0})$ on $\mathcal{C}$ we denote by $\mathcal{H}$ the full subcategory $\mathcal{C}^{\leqslant 0}\cap\mathcal{C}^{\geqslant 0}$ of $\mathcal{C}$ and call it the *heart* of the t-structure. Let $(\mathcal{T},\mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$. Then

$$\begin{array}{rcl}\mathcal{D}^{\leqslant 0} & = & \{X^\bullet\in D^b(\mathcal{A})\,|\,H^i(X^\bullet)=0, i>0, H^0(X^\bullet)\in\mathcal{T}\}\\ \mathcal{D}^{\geqslant 0} & = & \{X^\bullet\in D^b(\mathcal{A})\,|\,H^i(X^\bullet)=0, i<-1, H^{-1}(X^\bullet)\in\mathcal{F}\}\end{array}$$

defines a t-structure on $D^b(\mathcal{A})$.

Example 1. *The most obvious t-structure on $D^b(\mathcal{A})$ is the standard one, which is*

$$\begin{array}{rcl}\mathcal{D}^{\leqslant 0} & = & \{X^\bullet\in D^b(\mathcal{A})\,|\,H^i(X^\bullet)=0, i>0\}\\ \mathcal{D}^{\geqslant 0} & = & \{X^\bullet\in D^b(\mathcal{A})\,|\,H^i(X^\bullet)=0, i<0\}\end{array}$$

The heart is clearly equivalent to $\mathcal{A}$.

Example 2. *As some interesting examples, we provide the t-structures considered by Polishchuk in [26]. Let $\mathcal{A}=\mathrm{Coh}(X_\tau)$, where $X_\tau=\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and set $D^b(X_\tau)=D^b(\mathcal{A})$. First we recall the definition of a torsion pair $(\mathrm{Coh}_{>\theta},\mathrm{Coh}_{\leqslant\theta})$ in $\mathrm{Coh}(X_\tau)$, where $\theta\in(\mathbb{R}\setminus\mathbb{Q})\cap[0,1)$.*

$$\begin{array}{rcl}\mathrm{Coh}_{>\theta} & = & \{F\in\mathrm{Coh}(X_\tau)\mid \textit{all semistable factors of } F \textit{ have slope} >\theta\}\\ \mathrm{Coh}_{\leqslant\theta} & = & \{F\in\mathrm{Coh}(X_\tau)\mid \textit{all semistable factors of } F \textit{ have slope} \leqslant\theta\}\end{array}$$

Note that the torsion sheaves, having slope $=\infty$, belong to $\mathrm{Coh}_{>\theta}$. The associated t-structure is given by

$$\begin{array}{rcl}D^{\theta,\leqslant 0} & = & \{K^\bullet\in D^b(X_\tau)\mid H^{>0}(K^\bullet)=0, H^0(K^\bullet)\in\mathrm{Coh}_{>\theta}\}\\ D^{\theta,\geqslant 0} & = & \{K^\bullet\in D^b(X_\tau)\mid H^{<-1}(K^\bullet)=0, H^{-1}(K^\bullet)\in\mathrm{Coh}_{\leqslant\theta}\}\end{array}$$

whose heart is denoted by $\mathcal{C}^{\theta,\tau}$. It is shown in [26] that $\mathcal{C}^{\theta,\tau}$ has cohomological dimension 1 and that it is derived equivalent to $\mathrm{Coh}(X_\tau)$. The latter assertion follows from the fact that the torsion pairs under consideration are cotilting, which is sufficient due to Proposition 5.4.3 of [3]. At $\theta=\infty$ one puts the standard t-structure on $D^b(X_\tau)$, whose heart is just $\mathrm{Coh}(X_\tau)$.

1.3 Holomorphic bundles on noncommutative tori

The noncommutative torus is a particular case of a transformation group C^*-algebra, with $\mathbb{Z}$ acting continuously on the C^*-algebra $C(\mathbb{S}^1)$ of continuous functions on the circle. Pimsner and Voiculescu [23] and separately Rieffel [30] studied their K-theory, while Connes analysed their differential structure [5]. We will work with the smooth noncommutative torus, which is a dense Fréchet subalgebra of this transformation group C^*-algebra.

Let θ be an irrational real number. The algebra of smooth functions $\mathcal{A}_\theta$ on the noncommutative torus $\mathbb{T}_\theta$ consists of elements of the form $\sum_{(n_1,n_2)\in\mathbb{Z}^2} a_{n_1,n_2} U_1^{n_1} U_2^{n_2}$ with $(n_1, n_2) \longrightarrow a_{n_1,n_2}$ rapidly decreasing and U_1, U_2 are unitaries satisfying the commutation relation

$$U_2 U_1 = exp(2\pi i\theta) U_1 U_2 \tag{2}$$

A less *ad hoc* definition of $\mathcal{A}_\theta$ is given as a *smooth crossed product*. This is the smooth analogue of the aforementioned transformation group C^*-algebra. Let $C^\infty(\mathbb{S}^1)$ be the Fréchet $*$-algebra of smooth functions on the circle with the family of seminorms given by

$$\|f\|_\alpha = \sup_{s\in\mathbb{S}^1} |\partial_s^\alpha f(s)| \,.$$

We equip this algebra with a smooth action α of $\mathbb{Z}$ by automorphisms given by $\alpha_n(f)(s) = f(s-n\theta)$. We consider the vector space $\mathcal{S}(\mathbb{Z}, C^\infty(\mathbb{S}^1))$ of sequences on $\mathbb{Z}$ of rapid decay that take values in $C^\infty(\mathbb{S}^1)$. In other words, $\mathcal{S}(\mathbb{Z}, C^\infty(\mathbb{S}^1))$ consists of $C^\infty(\mathbb{S}^1)$-valued sequences $\{f_n\}_{n\in\mathbb{Z}}$ such that

$$\|f\|_{\alpha,\beta} = \sup_n (1 + |n|^\beta) \|f_n\|_\alpha,$$

is finite for all α and β. We introduce the following convolution product and involution on $\mathcal{S}(\mathbb{Z}, C^\infty(\mathbb{S}^1))$,

$$\begin{aligned} (f * g)_n &= \sum_{m\in\mathbb{Z}} f_m \alpha_m (g_{n-m}) \,, \\ (f^*)_n &= \alpha_n \left(f_{-n}^*\right) . \end{aligned} \tag{3}$$

The Fréchet $*$-algebra $(\mathcal{S}(\mathbb{Z}, C^\infty(\mathbb{S}^1)), *, ^*)$ is denoted by $C^\infty(\mathbb{S}^1) \rtimes_\theta \mathbb{Z}$ and is called the smooth crossed product of $C^\infty(\mathbb{S}^1)$ by $\mathbb{Z}$.

It is well-known that the Fourier transform maps an element in $\mathcal{S}(\mathbb{Z})$ isomorphically to an element in $C^\infty(\mathbb{S}^1)$. Under this identification, we have the isomorphism $\mathcal{S}(\mathbb{Z}, C^\infty(\mathbb{S}^1)) \simeq C^\infty(\mathbb{S}^1 \times \mathbb{S}^1)$ as vector spaces. The above convolution product on the generating unitaries U_1 and U_2 of $C^\infty(\mathbb{S}^1 \times \mathbb{S}^1)$ translates to the defining relation of Eqn. (2) of $\mathcal{A}_\theta$. Hence, $\mathcal{A}_\theta \simeq C^\infty(\mathbb{S}^1) \rtimes_\theta \mathbb{Z}$.

The two basic derivations δ_1 and δ_2 acting on $\mathcal{A}_\theta$ are as follows,

$$\delta_j \left(\sum_{(n_1,n_2)\in\mathbb{Z}^2} a_{n_1,n_2} U_1^{n_1} U_2^{n_2} \right) = 2\pi i \sum_{(n_1,n_2)\in\mathbb{Z}^2} n_j a_{n_1,n_2} U_1^{n_1} U_2^{n_2}; \qquad (j = 1, 2).$$

Equivalently, one can define δ_1 and δ_2 by $\delta_j(U_i) = 2\pi i \delta_{ij} U_i$ which is then extended to the whole of $\mathcal{A}_\theta$ by applying the Leibniz rule.

The derivations δ_1 and δ_2 are the infinitesimal generators of the action of a commutative torus $\mathbb{T}^2$ on $\mathcal{A}_\theta$ by automorphisms. Inside the complexified Lie algebra generated by δ_1 and δ_2, we are interested in the vector parametrized by two complex numbers ω_1 and ω_2. We denote

$$\delta_\omega = \omega_1\delta_1 + \omega_2\delta_2.$$

If $\omega = (\tau, 1)$ we also set $\delta_\tau = \delta_\omega$, which is the so-called *complex structure* on $\mathcal{A}_\theta$ already present in [9].

1.3.1 The category of holomorphic bundles on $\mathbb{T}_\theta$

The Serre-Swan Theorem [34] establishes an equivalence between the category of vector bundles over a topological space M and finitely generated projective modules (henceforth, for brevity, referred to as finite projective modules) over $C(M)$ (for the smooth analogue of the Serre-Swan Theorem we refer the readers to Chapter 2 of [14]). In this spirit, it makes sense to *define* vector bundles over the noncommutative torus $\mathbb{T}_\theta$ as finite projective right $\mathcal{A}_\theta$-modules.

In [5], Connes has constructed finite projective modules over $\mathcal{A}_\theta$ that are labelled by a tuple $(c, d) \in \mathbb{Z}^2$. Later, in [30] Rieffel has shown that this set, in fact, exhausts the complete set of finite projective modules over $\mathcal{A}_\theta$ (up to isomorphism).

We generalise the category considered by Polishchuk and Schwarz slightly by defining the objects of the category $\mathsf{Vect}(\mathbb{T}^\omega_\theta)$ to be finite projective right $\mathcal{A}_\theta$-modules carrying a *holomorphic structure* which is a lifting of δ_ω. More precisely, a holomorphic structure on a finite projective $\mathcal{A}_\theta$-module E is given by a $\mathbb{C}$-linear connection $\nabla : E \longrightarrow E$ satisfying the Leibniz rule,

$$\nabla(ea) = \nabla(e)a + e\delta_\omega(a); \qquad (\forall e \in E, a \in \mathcal{A}_\theta). \tag{4}$$

A morphism $h : E \longrightarrow E'$ is said to be *holomorphic* if it commutes with the connection, *i.e.*, $\nabla_{E'}(he) = h\nabla_E(e)$. These are the morphisms of the category.

One defines the cohomology groups H^0 (resp. H^1) of $\mathcal{A}_\theta$ with respect to a holomorphic bundle E, equipped with a connection ∇, as the kernel (resp. cokernel) of ∇.

If $\omega = (\tau, 1)$, then $\mathsf{Vect}(\mathbb{T}^\omega_\theta)$ reduces to the category of holomorphic bundles $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$ as introduced in [27].

Proposition 2. *(a) If g is an element in $SL(2,\mathbb{Z})$, then* $\mathsf{Vect}(\mathbb{T}^{g\omega}_\theta) \simeq \mathsf{Vect}(\mathbb{T}^\omega_\theta)$.

(b) If $\omega_2 \neq 0$ and $\tau = \frac{\omega_1}{\omega_2}$, then $\mathsf{Vect}(\mathbb{T}^\omega_\theta) \simeq \mathsf{Vect}(\mathbb{T}^\tau_\theta)$.

Proof. (a) Given a $g \in SL(2,\mathbb{Z})$, we construct a $*$-automorphism σ of $\mathcal{A}_\theta$ such that $\sigma^{-1}\delta_\omega\sigma = \delta_{g\omega}$. Evidently, it is enough to do this for the generators of $SL(2,\mathbb{Z})$, *i.e.*, $g_1 = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $g_2 = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$. For g_1, $\delta_{g_1\omega} = (\omega_1 + \omega_2)\delta_1 + \delta_2$. We define $\sigma_1 : \mathcal{A}_\theta \longrightarrow \mathcal{A}_\theta$ as $\sigma_1(U_1) = U_1U_2$, $\sigma_1(U_2) = U_2$. One may easily check that $\sigma_1(U_1)$ and $\sigma_1(U_2)$ satisfy the commutation relation of $\mathcal{A}_\theta$ as in Eqn. (2) and also that

$$
\begin{aligned}
\sigma_1^{-1}\delta_\omega\sigma_1(U_1) &= \sigma_1^{-1}\delta_\omega(U_1U_2)\\
&= \sigma_1(\omega_1\delta_1+\omega_2\delta_2)(U_1U_2)\\
&= \sigma_1(2\pi i\omega_1U_1U_2+2\pi i\omega_2U_1U_2)\\
&= 2\pi i(\omega_1+\omega_2)\sigma_1^1(U_1U_2)\\
&= 2\pi i(\omega_1+\omega_2)U_1\\
&= ((\omega_1+\omega_2)\delta_1+\delta_2)U_1\\
&= \delta_{g_1\omega}(U_1).
\end{aligned}
$$

Similarly, for U_2 one may check that the actions of δ_ω and $\delta_{g_1\omega}$ agree. For g_2, $\delta_{g_2\omega} = -\omega_2\delta_1 + \omega_1\delta_2$ and we define $\sigma_2(U_1) = U_2^{-1}$, $\sigma_2(U_2) = U_1$. Once again one can easily check that the new generators satisfy Eqn. (2) and that the actions of δ_ω and $\delta_{g_2\omega}$ agree on U_1 and U_2. Explicitly, the functor sends $(\mathcal{A}_\theta, \delta_\omega)$ to $(\mathcal{A}_\theta, \delta_{g_i\omega})$, $i = 1, 2$, and twists the module structure by σ_i, $i = 1, 2$, *i.e.,* $e \cdot a := e\sigma_i(a)$, $i = 1, 2$ and $e \in E$. One verifies that ∇ on E is compatible with $\delta_{g_i\omega}$, $i = 1, 2$, with respect to the twisted module structure. Indeed,

$$
\begin{aligned}
\nabla(e\cdot a) &= \nabla(e\sigma_i(a))\\
&= \nabla(e)\sigma_i(a) + e\delta_\omega(\sigma_i(a))\\
&= \nabla(e)a + e\sigma_i(\delta_{g_i\omega}(a))\\
&= \nabla(e)\cdot a + e\cdot\delta_{g_i\omega}(a)
\end{aligned}
$$

where $e \in E$, $a \in \mathcal{A}_\theta$ and $i = 1, 2$.

(b) In our notation, $\delta_\tau = \frac{\delta_\omega}{\omega_2}$. Sending each ∇ to $\nabla' := \frac{\nabla}{\omega_2}$ makes ∇' automatically compatible with δ_τ. More precisely, the functor sends $(\mathcal{A}_\theta, \delta_\omega)$ to $(\mathcal{A}_\theta, \delta_\tau)$ and (E, ∇) to (E, ∇'). □

1.3.2 The derived category

This discussion is included just for the sake of completeness. The readers may easily ignore it and skip to Remark 3.

The derived category of holomorphic bundles is *defined* as the cohomology category of a DG category (or a differential graded category), which is a category with the Hom sets carrying a structure of a differential graded complex of $\mathbb{C}$-vector spaces (see [17] for more details). The corresponding cohomology category is obtained by replacing the Hom complexes by their cohomologies. The DG category in consideration, denoted by $\mathcal{C}(\theta, \tau)$, consists of objects of $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$, labelled by an integer indicating its translation degree. The Hom's in $\mathcal{C}(\theta, \tau)$ are given by a differential complex conjured up from the connection on the Hom's in $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$. Note that the Hom's in $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ also carry a module structure over some noncommutative torus (not necessarily $\mathbb{T}_\theta$).

Polishchuk and Schwarz construct a functor from the DG category $\mathcal{C}(\theta, \tau)$ to $D^b(X_\tau)$ and show that the induced functor on the cohomology category is fully faithful and that the image of $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ lies in the heart of the t-structure corresponding to θ (*cf.* Example 2). Then Polishchuk shows that this functor actually induces an equivalence between $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ and the heart [25], whose derived category is again equivalent to $D^b(X_\tau)$ [26]. This implies that $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ is abelian and its derived category is equivalent to $D^b(X_\tau)$ via the Polishchuk–Schwarz functor, denoted by $\mathcal{S}_\tau : H^0\mathcal{C}(\theta, \tau) \longrightarrow D^b(X_\tau)$.

Remark 3. The functor $\mathcal{S}_\tau$ actually induces an equivalence between $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$ and $\mathcal{C}^{-\theta^{-1},\tau}$. Observe that $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\theta = -\theta^{-1}$ says that $A_{-\theta^{-1}}$ is Morita equivalent to $\mathcal{A}_\theta$.
Summarising, one has

$$\mathsf{Vect}(\mathbb{T}^\tau_\theta) \cong \mathsf{Vect}(\mathbb{T}^\tau_{-\theta^{-1}}) \cong \mathcal{C}^{\theta,\tau} \text{ and } D^b(\mathcal{C}^{\theta,\tau}) \cong D^b(X_\tau). \tag{5}$$

2 Equivariant coherent sheaves and $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$

At this point, we would like to adopt the philosophy of Manin as explained in [19]. Heuristically, one considers the quotient $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}+\theta\mathbb{Z})$, with $\theta \in \mathbb{R}\setminus\mathbb{Q}$ and $\tau \in \mathbb{H}^-$. On one side, this space can be regarded as $\mathbb{T}_\theta = \mathbb{C}/(\mathbb{Z}+\theta\mathbb{Z})$ modulo an *infinitesimal action* of $\tau\mathbb{Z}$ described by the derivation δ_τ on $\mathbb{T}_\theta$. More precisely, one considers the category $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$ defined above; it consists of finite projective modules over $C^\infty(\mathbb{T}_\theta)$ equipped with a connection ∇ covering δ_τ. Taking the quotient in the other order, one obtains $X_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with an analogue of the infinitesimal action of $\theta\mathbb{Z}$ on it which amounts to putting a t-structure on $D^b(X_\tau)$ that depends on θ (see Example 2) and taking its heart $\mathcal{C}^{\theta,\tau}$. The equivalence between the two categories $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$ and $\mathcal{C}^{\theta,\tau}$ is the agreement of the two quotient spaces by looking at the category of representations of the two objects.

Let us now describe another way of obtaining a quotient, based on the observation that there is an honest action of $\theta\mathbb{Z}$ on X_τ and hence on $\mathrm{Coh}(X_\tau)$. Indeed, the point $\theta\, mod\,(\mathbb{Z}+\tau\mathbb{Z})$ on X_τ lies on the real axis of the fundamental domain of the torus and its action is restricted to the circle obtained by folding this axis. In fact, the action given by translations of θ on X_τ transforms to the action of multiplication by powers of $q = e^{2\pi i\theta}$ under the Jacobi uniformisation, *i.e.*, $z \longmapsto qz$ on $\mathbb{C}^*/\tilde{q}^{\mathbb{Z}}$, $\tilde{q} = e^{2\pi i\tau}$. Once again we are confronted with a double quotient problem, where the actions commute. Namely, it is the improper action of the group $q^{\mathbb{Z}}$ on X_τ, which is itself obtained by the free and proper action of the group $\tilde{q}^{\mathbb{Z}}$ on $\mathbb{C}^*$ (both actions are by multiplication). Soibelman and Vologodsky have described the *quotient space* $\mathbb{C}^*/q^{\mathbb{Z}}$ in terms of their noncommutative elliptic curves $\mathcal{B}_q$ in [33] (strictly speaking, our definition differs from theirs by an exchange $q \leftrightarrow q^{-1}$). In the formal case when $|q| < 1$ analogues of such objects have been investigated in [1]. The category $\mathcal{B}_q$ is nothing but the category of $q^{\mathbb{Z}}$-equivariant (analytic) coherent sheaves on $\mathbb{C}^*$ (or equivalently, the category of modules over the crossed product algebra $\mathcal{O}(\mathbb{C}^*) \rtimes_q \mathbb{Z}$, which are finitely presentable over $\mathcal{O}(\mathbb{C}^*)$). It follows from Lemma 3.2 of [33] that for any $M \in \mathcal{B}_q$ the underlying $\mathcal{O}(\mathbb{C}^*)$-module is free. However, there are interesting actions of $\theta\mathbb{Z}$ or $q^{\mathbb{Z}}$ on the free modules with respect to which they are equivariant. Let us denote by α the induced action by automorphisms of $\theta\mathbb{Z}$ on $\mathcal{O}(\mathbb{C}^*)$:

$$\alpha(f)(z) = f(q^{-1}z); \qquad (z \in \mathbb{C}^*, q = e^{2\pi i\theta}).$$

Here, we have understood the notation $\alpha := \alpha(1)$ for the generator of $\mathbb{Z}$, so that $\alpha(n) = \alpha^n$. What is lacking in this picture is an nfinitesimal action in terms of δ_τ and compatible connections, which accounts for the remaining $\tau\mathbb{Z}$ quotient operation. To this end, we define a derivation on $\mathcal{O}(\mathbb{C}^*)$ by $\delta = \tau z\frac{d}{dz}$. It is this infinitesimal action by δ that will turn out to be the appropriate replacement for the infinitesimal action of the group $\tau\mathbb{Z}$.

2.1 The category $\mathcal{B}^\tau_q$

Our goal in this section is to define a category alluded to before, which is somehow 'in between' the categories $\mathsf{Vect}(\mathbb{T}^\tau_\theta)$ introduced by Polishchuk and Schwarz and $\mathcal{B}_q$ by Soibelman and Vologodsky.

More precisely, we would like to construct a category $\mathcal{B}_q^\tau$ that is functorially related to both of these categories. At the same time, we would like to stay as close as possible to the setting of the Riemann–Hilbert correspondence. The discussion above motivates us to define the following category as a description of the quotient of $\mathcal{B}_q$ by the infinitesimal action of $\tau\mathbb{Z}$.

Definition 4. *The category $\mathcal{B}_q^\tau$ consists of triples (M, σ, ∇), where*

- *M is a finitely presentable $\mathcal{O}(\mathbb{C}^*)$-module, i.e., there is an exact sequence,*

$$\mathcal{O}(\mathbb{C}^*)^m \longrightarrow \mathcal{O}(\mathbb{C}^*)^n \longrightarrow M \longrightarrow 0.$$

- *σ is a representation of $\theta\mathbb{Z}$ on M covering the action α of $\theta\mathbb{Z}$ on $\mathcal{O}(\mathbb{C}^*)$, i.e.,*

$$\sigma(m \cdot f) = \sigma(m) \cdot \alpha(f); \qquad (m \in M, f \in \mathcal{O}(\mathbb{C}^*)).$$

- *∇ is a $\theta\mathbb{Z}$-equivariant connection on M covering the derivation $\delta = \tau z \frac{d}{dz}$ on $\mathcal{O}(\mathbb{C}^*)$, i.e., it satisfies,*

$$\begin{aligned} \nabla(m \cdot f) &= \nabla(m) \cdot f + m \cdot \delta(f), \\ \nabla(\sigma(m)) &= \sigma(\nabla(m)), \end{aligned}$$

  for all $m \in M, f \in \mathcal{O}(\mathbb{C}^*)$.

In addition, we impose that the connection ∇ is a regular singular connection *on M, that is, there exists a module basis $\{e_1, \ldots, e_n\}$ of M for which the holomorphic functions $z^{-1}A_{ij}$ $(i, j = 1, \ldots, n)$ defined by $A_{ij}e_j = \nabla(e_i)$ have simple poles at 0. We call $A = (A_{ij})$ the* matrix of the connection *with respect to that module basis.*

The morphisms in this category are equivariant $\mathcal{O}(\mathbb{C}^*)$-module maps that are compatible with the connections. We will also write $M = (M, \sigma, \nabla)$ when no confusion can arise. For two objects M and N we denote by $\mathrm{Hom}^{\theta\mathbb{Z},\delta}_{\mathcal{O}(\mathbb{C}^*)}(M, N)$ the $\mathbb{C}$-linear vector space of morphisms between them.

The uniqueness of the matrix $A = A_{ij}$ (after the choice of a module basis $\{e_i\}$ for M) is due to the fact that the modules M in $\mathcal{B}_q^\tau$ turn out to be free as $\mathcal{O}(\mathbb{C}^*)$-modules. This was observed in [33, Lemma 2] and used the fact that the sheaf $M \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^*}$ must be torsion free due to $\theta\mathbb{Z}$-equivariance. Hence it is locally free on $\mathbb{C}^*$ and thus a trivial vector bundle. The $\mathcal{O}(\mathbb{C}^*)$-module of its global sections is then clearly free. This freeness as $\mathcal{O}(\mathbb{C}^*)$-modules can be translated into freeness as $\theta\mathbb{Z}$-equivariant $\mathcal{O}(\mathbb{C}^*)$-modules as follows. Suppose that $M \simeq V \otimes_{\mathbb{C}} \mathcal{O}(\mathbb{C}^*)$ as $\mathcal{O}(\mathbb{C}^*)$-modules with V a complex vector space. Via this identification there is an induced action of $\theta\mathbb{Z}$ on $V \otimes_{\mathbb{C}} \mathcal{O}(\mathbb{C}^*)$ making this an isomorphism of $\theta\mathbb{Z}$-equivariant $\mathcal{O}(\mathbb{C}^*)$-modules.

Let $\mathcal{B}^\tau$ denote the category of pairs (V, ∇) with V a vector bundle on $\mathbb{C}^*$ and ∇ a regular singular connection on V associated to $\delta = \tau z \frac{d}{dz}$. By the above remarks on the modules M in $\mathcal{B}_q^\tau$, there is a functor from $\mathcal{B}_q^\tau$ to $\mathcal{B}^\tau$ which forgets the action of $\theta\mathbb{Z}$. Thanks to Deligne [10] (see also, for instance, Theorem 1.1 and the paragraph after Remark 1.2 of [18]), we know that the category $\mathcal{B}^\tau$ is equivalent to the category of finite dimensional representations of the fundamental group $\pi_1(\mathbb{C}^*, z') \simeq \mathbb{Z}$ with a base point z'. This result motivates the regular singularity condition we have imposed on the connections in Definition 4.

In Section 3 we will enhance this Riemann–Hilbert correspondence to an equivariant version and show that a similar statement holds for $\mathcal{B}_q^\tau$. Let us first proceed to examine some of the properties of $\mathcal{B}_q^\tau$ and its relation with the other two categories, *viz.*, $\mathcal{B}_q$ and $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$.

Proposition 5. *The category $\mathcal{B}_q^\tau$ is an abelian category.*

Proof. It is proven in Proposition 3.3 of [33] that the category $\mathcal{B}_q$ is abelian. One observes readily that there is a faithful functor (forgetting the connection) from $\mathcal{B}_q^\tau$ to $\mathcal{B}_q$. Suppose that $f : M \to N$ is a morphism in $\mathcal{B}_q^\tau$. Since it is also a morphism in $\mathcal{B}_q$, both $\ker f$ and $\mathrm{coker} f$ are equivariant $\mathcal{O}(\mathbb{C}^*)$-modules. Moreover, the map f intertwines the connections on M and N and hence induces compatible connections on $\ker f$ and $\mathrm{coker} f$ making them objects in $\mathcal{B}_q^\tau$. □

We now view $\mathcal{A}_\theta$ as a module over $\mathcal{O}(\mathbb{C}^*)$ via the homomorphism

$$\begin{aligned} \psi : \mathcal{O}(\mathbb{C}^*) &\to \mathcal{A}_\theta \\ \sum_{n\in\mathbb{Z}} f_n z^n &\mapsto \sum_{n\in\mathbb{Z}} f_n U_1^n. \end{aligned}$$

This is well-defined since a sequence f_n of exponential decay is certainly a Schwartz sequence.

Remark 6. The map is essentially restricting a holomorphic function on $\mathbb{C}^*$ to the unit circle. In fact, it is injective since, if a holomorphic function vanishes on the unit circle, it must vanish on the whole of $\mathbb{C}^*$. Note that $\mathcal{A}_\theta$ is not finitely generated over $\mathcal{O}(\mathbb{C}^*)$ and hence not an element of $\mathcal{B}_q$ or $\mathcal{B}_q^\tau$.

Proposition 7. *The following association defines a right exact functor, denoted ψ_*, from $\mathcal{B}_q^\tau$ to $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$. For an object (M, σ, ∇) in $\mathcal{B}_q^\tau$ we define an object $(\tilde{M}, \tilde{\nabla})$ in $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ by*

$$\begin{aligned} \tilde{M} &= M \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{A}_\theta \\ \tilde{\nabla} &= 2\pi i\ \nabla \otimes 1 + 1 \otimes (\tau\delta_1 + \delta_2) \\ &= 2\pi i\ \nabla \otimes 1 + 1 \otimes \delta_\tau. \end{aligned}$$

Proof. Observe that $\psi(2\pi i\delta f) = \tau\delta_1(\psi(f))$ as follows from the definitions of δ and δ_1. Moreover, the image of $\mathcal{O}(\mathbb{C}^*)$ under the map ψ lies in the kernel of the derivation δ_2 on the noncommutative torus (since $\delta_2(U_1)$ is vanishing). Hence one can add δ_2 to $\tau\delta_1$ making $\tilde{\nabla}$ a connection on $\tilde{M}$ covering δ_τ. □

Note that by a simple adjustification one can actually define a right exact functor from $\mathcal{B}_q^{\omega_1}$ to $\mathsf{Vect}(\mathbb{T}_\theta^\omega)$. We also claim that, in fact,

Proposition 8. *The module $\mathcal{A}_\theta$ over $\mathcal{O}(\mathbb{C}^*)$ via the map ψ is flat.*

Proof. The algebra $\mathcal{O}(\mathbb{C}^*)$ is a commutative integral domain, since holomorphic functions cannot have disjoint support. Further, from Corollary 3.2 of [24] one concludes that the global Ext dimension of $\mathcal{O}(\mathbb{C}^*)$ is 1. Hence it is a *Prüfer domain, i.e.,* a domain in which all finitely generated non-zero ideals are invertible. Indeed, Theorem 6.1 of [13] says that a (fractional) ideal in a domain is invertible if and only if it is projective and, since $\mathcal{O}(\mathbb{C}^*)$ has Ext dimension 1, given any finitely generated ideal I, applying $\mathrm{Hom}(-, M)$ to the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ for an arbitrary M, one finds that $\mathrm{Ext}^1(I, M) = 0$, *i.e.,* I is projective. It is known that a module over a Prüfer domain is flat if and only if it is torsion free (see, *e.g.,* Theorem 1.4 *ibid.*). So we only need to check torsion freeness. We identify $\mathcal{A}_\theta$ as a module over $\mathcal{O}(\mathbb{C}^*)$ with $\mathcal{S}(\mathbb{Z}, C^\infty(\mathbb{S}^1))$ and represent each element as a sequence $\{g_n\}_{n\in\mathbb{Z}}$, $g_n \in C^\infty(\mathbb{S}^1)$, refer to the discussion before (3). The image of the map ψ clearly lies in $C^\infty(\mathbb{S}^1)$, which is identified with the functions supported at the identity element of $\mathbb{Z}$. In other words, for all $f \in \mathcal{O}(\mathbb{C}^*)$, $\psi(f)$ is of the form $\{f_n\}$, where $f_n = 0$ unless $n = 0$. Now consider any $g = \{g_n\} \in \mathcal{A}_\theta$ and suppose that some non-zero $f \in \mathrm{Ann}(\{g_n\})$, *i.e.,* $g * \psi(f) = \{g_n\alpha_n(f_0)\} = 0$. This implies that $g_n(z)f_0(q^{-n}z) = 0$ for all n, $|z| = 1$. Being the restriction of a holomorphic function on $\mathbb{C}^*$, $f_0(q^{-n}z)$ has a discrete zero set on the unit circle. A smooth function on $\mathbb{S}^1$ cannot have a discrete set of points as support and hence each $g_n(z)$ must be identically zero. Thus, whenever an element in $\mathcal{A}_\theta$ has a non-zero element in its annihilator ideal, the element is itself zero. Hence $\mathcal{A}_\theta$ is torsion free from which the result follows. □

Corollary 9. *The base change functor ψ_* induced by the homomorphism ψ is exact and faithful.*

Proof. From the previous Proposition we conclude that the functor sends an exact sequence of $\mathcal{O}(\mathbb{C}^*)$-modules to an exact sequence of $\mathcal{A}_\theta$-modules. However, the fact that ψ_* is a functor from $\mathcal{B}_q^\tau$ to $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ says that the induced morphisms respect the induced connections. For the faithfulness, identify each object $M \in \mathcal{B}_q^\tau$ with $V \otimes \mathcal{O}(\mathbb{C}^*)$ with V a vector space; similarly write $M = V \otimes \mathcal{O}(\mathbb{C}^*)$. A morphism in $\mathcal{B}_q^\tau$ from M to M' is then given by an element in $\mathrm{Hom}_\mathbb{C}(V, V') \otimes_\mathbb{C} \mathcal{O}(\mathbb{C}^*)$, whereas a morphism in $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ between $\tilde{M}$ and $\tilde{M}'$ is given by an element in $\mathrm{Hom}_\mathbb{C}(V, V') \otimes_\mathbb{C} \mathcal{A}_\theta$. The functor ψ_* acts on these element by $1 \otimes \psi$ and since ψ is injective, it follows that ψ_* is injective on morphisms. □

Remark 10. However, the functor is not full. It is certainly not essentially surjective as the underlying $\mathcal{A}_\theta$-modules of the objects in the image are all free, whilst $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ has modules which are not free.

The main Theorem of [25] says that the category generated by succesive extensions of all *standard* holomorphic bundles, as defined in [27], over $\mathbb{T}_\theta^\tau$ ($\mathbb{T}_\theta$ equipped with the derivation δ_τ) is already all of $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$. Let us recall the definition of a standard holomorphic bundle (or as standard module) in the special case when the underlying module is just $\mathcal{A}_\theta$. Given any fixed $z' \in \mathbb{C}$, the connection $\nabla_{z'}$ is defined by

$$\nabla_{z'}(a) = \delta_\tau(a) + 2\pi i z' \cdot a, \qquad (a \in \mathcal{A}_\theta).$$

The tuple $E_1^{z'} := (\mathcal{A}_\theta, \nabla_{z'})$ is by definition a standard module. Let us denote the full subcategory of $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ generated by successive extensions of standard modules of the form $E_1^{z'}$, $z' \in \mathbb{C}$ by $\mathrm{FrVect}(\mathbb{T}_\theta^\tau)$. Since the extension of two free modules is again free, it is clear that the underlying $\mathcal{A}_\theta$-module of all objects of $\mathrm{FrVect}(\mathbb{T}_\theta^\tau)$ is free.

Lemma 11. *With respect to a suitable basis each object of* $\mathrm{FrVect}(\mathbb{T}_\theta^\tau)$ *is of the form* $(\mathcal{A}_\theta^n, \delta + A)$, *where A is an $n \times n$ upper triangular matrix in $M_n(\mathcal{A}_\theta)$ with diagonal entries in $\mathbb{C}$.*

Proof. It is known that given any finitely generated projective module M over $\mathcal{A}_\theta$ and a fixed connection ∇ compatible with δ_τ, all other compatible connections are of the form $\nabla + \phi$, $\phi \in \mathrm{End}_{\mathcal{A}_\theta}(M)$. This follows easily from the Leibniz rule (4). Since M is of the form $\mathcal{A}_\theta^n$, ϕ is determined by a matrix $A \in M_n(\mathcal{A}_\theta)$. Let

$$0 \longrightarrow (\mathcal{A}_\theta, \nabla_{z'}) \xrightarrow{\iota} (\mathcal{A}_\theta^2, \delta + A) \xrightarrow{\pi} (\mathcal{A}_\theta, \nabla_{z''}) \longrightarrow 0 \tag{6}$$

be a holomorphic extension in $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$. Write $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with entries $a, b, c, d \in \mathcal{A}_\theta$ and $\iota(a) = (a, 0)$ and $\pi(a_1, a_2) = a_2$. One checks easily that the holomorphicity of ι and π (the fact that they commute with the connections) forces $c = 0, a = z'$ and $d = z''$. Now by induction it follows that the connections obtained by successive extensions are of the desired form.

Conversely, by induction suppose that every connection of the desired form on $\mathcal{A}_\theta^{n-1}$ can be obtained as an iterated extension of modules of the form $E_1^{z'}$. Let A be an upper triangular matrix in $M_n(\mathcal{A}_\theta)$ whose diagonal entries are in $\mathbb{C}$, *i.e.,* A is of the form

$$\begin{pmatrix} z' & b_2 & \cdots & b_n \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{pmatrix},$$

where $A' \in M_{n-1}(\mathcal{A}_\theta)$ is also of the prescribed type and $b_2, \ldots, b_n \in \mathcal{A}_\theta$. A routine calculation then shows that

$$0 \longrightarrow (\mathcal{A}_\theta, \nabla_{z'}) \xrightarrow{\iota} (\mathcal{A}_\theta^n, \delta + A) \xrightarrow{\pi} (\mathcal{A}_\theta^{n-1}, \delta + A') \longrightarrow 0$$

with $\iota(a) = (a, 0, \ldots, 0)$ and $\pi(a_1, a_2, \ldots, a_n) = (a_2, \ldots, a_n)$ is a holomorphic extension in $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$. Hence $(\mathcal{A}_\theta^n, \delta + A)$ belongs to $\mathrm{FrVect}(\mathbb{T}_\theta^\tau)$. $\square$

Remark 12. In the exact sequence in Eqn. (6), if b in $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is non-zero and $a, d \in \mathbb{C}$ such that $a \neq d$, then the matrix can be diagonalized by the change of basis matrix $\begin{pmatrix} 1 & \frac{-b}{d-a} \\ 0 & 1 \end{pmatrix}$ and hence the sequence splits.

Remark 13. Given any matrix $A \in M_n(\mathbb{C})$, with respect to a suitable basis one can reduce it to its Jordan canonical form (it is also upper triangular with diagonal entries in $\mathbb{C}$). Therefore, $\mathrm{FrVect}(\mathbb{T}_\theta^\tau)$ contains all objects of the form $(\mathcal{A}_\theta^n, \delta + A)$, where $A \in M_n(\mathbb{C})$ with respect to a basis.

As we will see later (Proposition 18), each object (M, σ, ∇) in $\mathcal{B}_q^\tau$ is isomorphic to an object, whose matrix of the connection is a constant matrix. This can be accomplished via a change of basis of M. Combining this with the above remark, we conclude that – at the level of objects – the image of ψ_* lies inside $\mathrm{FrVect}(\mathbb{T}_\theta^\tau)$.

2.2 The effect on K-theory

We infer from Eqn. (5) that the K-theory (by that we mean the Grothendieck group, *i.e.,* the free abelian group generated by the isomorphism classes of objects modulo the relations coming from all exact sequences) of $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ is isomorphic to that of $D^b(X_\tau)$ via the Polishchuk–Schwarz equivalence $\mathcal{S}_\tau$. One knows that $K_0(D^b(X_\tau)) \cong K_0(\mathrm{Coh}(X_\tau)) = K_0(X_\tau) = \mathrm{Pic}(X_\tau) \oplus \mathbb{Z}$. The composition of the functors ψ_* followed by $\mathcal{S}_\tau$ induces a homomorphism between $K_0(\mathcal{B}_q^\tau)$ and $K_0(\mathsf{Vect}(\mathbb{T}_\theta^\tau)) = \mathrm{Pic}(X_\tau) \oplus \mathbb{Z}$. One observes that applying ψ_* one obtains only elements in $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ whose underlying $\mathcal{A}_\theta$-modules are free. It is known that for $E \in \mathsf{Vect}(\mathbb{T}_\theta^\tau)$, $\mathrm{rk}\mathcal{S}_\tau(E) = -\deg(E)$ and $\deg\mathcal{S}_\tau(E) = \mathrm{rk}(E)$. The degree of the modules, which are free, is known to be zero. Hence the composition of the two functors sends every element in $\mathcal{B}_q^\tau$ to a torsion sheaf on X_τ. One can check that $\mathcal{O}(\mathbb{C}^*)$ equipped with the connection $\delta + z'$, where $z' \in \mathbb{C}$, gets mapped to the standard holomorphic bundle $E_1^{z'}$ as explained after Remark 10. From part (c) of Proposition 3.7 of [27] we know that $\mathcal{S}_\tau(E_1^{z'})$ is $\mathcal{O}_{-z'}$ (up to a shift in the derived category), which is the structure sheaf of the point $-z'$ mod $(\mathbb{Z} + \tau\mathbb{Z})$ in X_τ. All modules of the form $(\mathcal{O}(\mathbb{C}^*), \sigma, \delta + z')$ with $z' \in \mathbb{C}$ are endomorphism simple, *i.e.,* $\mathrm{End}(\mathcal{O}(\mathbb{C}^*), \sigma, \delta + z') = \mathbb{C}$. Indeed, ignoring the equivariance condition and the connection, $\mathrm{End}(\mathcal{O}(\mathbb{C}^*)) = \mathcal{O}(\mathbb{C}^*)$ and the equivariance condition says that $\sigma(mf) = \sigma(m)f$. However, by definition $\sigma(mf) = \sigma(m)\alpha(f)$ whence $\alpha(f) = f$ implying $f \in \mathbb{C}$. This module is mapped to $(\mathcal{A}_\theta, \delta_\tau + 2\pi i z') = E_1^{z'}$, which in turn is mapped to the endomorphism simple object $\mathcal{O}_{-z'}$ in $\mathcal{C}^{\theta,\tau}$. It is known that, in fact, the Grothendieck group of any nonsingular curve C is isomorphic to $\mathrm{Pic}(C)\oplus\mathbb{Z}$. In this identification the contribution to $\mathbb{Z}$ comes from the rank of the coherent sheaf, whereas $\mathrm{Pic}(C)$ can be regarded as the contribution from the torsion part (actually from the determinant bundle of the sheaf, which may be identified with a torsion sheaf via a Fourier-Mukai transform). Since we obtain only torsion sheaves, the image of the induced map on K-theory lies inside $\mathrm{Pic}(X_\tau)$.

Proposition 14. *The map induced by $\mathcal{S}_\tau \circ \psi_*$ between the K-theories of $\mathcal{B}_q^\tau$ and $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ gives a surjection from $K_0(\mathcal{B}_q^\tau)$ to $\mathrm{Pic}(X_\tau)$.*

Proof. The divisor class group of X_τ is the free abelian group generated by the points of X_τ modulo the principal divisors, which is also isomorphic to $\mathrm{Pic}(X_\tau)$. The class of each point $z' \in X_\tau$ of the divisor class group can be identified with the class of the torsion sheaf $\mathcal{O}_{z'}$ corresponding to the line bundle $\mathcal{O}(z') \in \mathrm{Pic}^1(X_\tau)$ and they generate $\mathrm{Pic}(X_\tau)$ as a group. By the above argument $\mathcal{O}_{z'}$ is obtained by applying the functor $\mathcal{S}_\tau \circ \psi_*$ to the element $(\mathcal{O}(\mathbb{C}^*), \sigma, \delta - z')$ of $\mathcal{B}_q^\tau$. Thus one obtains a surjection onto the generating set of $\mathrm{Pic}(X_\tau)$ from which the assertion follows. $\square$

Remark 15. From Proposition 2.1 of [27] we know that the images of $(\mathcal{O}(\mathbb{C}^*), \sigma, \delta + z_1')$ and $(\mathcal{O}(\mathbb{C}^*), \sigma, \delta + z_2')$ under ψ_* are isomorphic if and only if $z_1' \equiv z_2' \bmod (\mathbb{Z} + \tau\mathbb{Z})$. More generally, abbreviating the module $(\mathcal{O}(\mathbb{C}^*), \sigma, \delta + z')$ by $M_{z'}$, one can also rephrase the linear equivalence relation of the divisor class group to conclude that an element of the form $\sum n_i[M_{-z_i'}]$ maps to zero at the level of K-theory whenever $\sum n_i = 0$ and $\sum n_i z_i' \in (\mathbb{Z} + \tau\mathbb{Z})$. However, some of them actually represent the trivial class in the K-theory of $\mathcal{B}_q^\tau$, as we will see in the next section (see Corollary 21).

Although the image of $\mathcal{B}_q^\tau$ gives only the free modules in $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$, it has the interesting property of being a Tannakian category, as we will explore in the next section. Let us end this section by summarising the relations between $\mathcal{B}_q^\tau$ and the categories $\mathcal{B}^\tau$, $\mathcal{B}_q$, $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$:

$$\begin{tikzcd} & \mathcal{B}_q^\tau \arrow[dl] \arrow[d, "\psi_*"] \arrow[dr] & \\ \mathcal{B}_q & \mathsf{Vect}(\mathbb{T}_\theta^\tau) & \mathcal{B}^\tau \end{tikzcd}$$

where the two diagonal arrows are the forgetful functors discussed before. All of these functors are faithful and exact (but not injective on objects).

3 The Tannakian formalism and the equivariant Riemann–Hilbert correspondence

We will now analyse further the structure of $\mathcal{B}_q^\tau$ and define a tensor product on it. Our main result is that this – together with a fibre functor – makes $\mathcal{B}_q^\tau$ a Tannakian category. Via an equivariant version of the Riemann–Hilbert correspondence on $\mathbb{C}^*$, we determine the corresponding affine group scheme.

3.1 Preliminaries on Tannakian categories

We briefly recall the notion of a Tannakian category. For more details, we refer the reader to the original works [32, 11, 12] (see also Appendix B of [29]).

Let $\mathcal{C}$ be an k-linear abelian category, for a field k. Then $\mathcal{C}$ is a neutral Tannakian category over k if

1. The category $\mathcal{C}$ is a *tensor category*. In other words, there is a tensor product: for every pair of objects X, Y there is an object $X \otimes Y$. The tensor product is commutative ($X \otimes Y \simeq Y \otimes X$) and associative ($X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$) and there is a unit object 1 (such that $X \otimes 1 \simeq 1 \otimes X \simeq X$). The above isomorphisms are supposed to be functorial.

2. $\mathcal{C}$ is a *rigid* tensor category: there exists a duality $\vee : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$, satisfying

   - For any object X in $\mathcal{C}$, the functor $\_ \otimes X^\vee$ is left adjoint to $\_ \otimes X$, and the functor $X^\vee \otimes \_$ is right adjoint to $X \otimes \_$.
   - There is an evaluation morphism $\epsilon : X \otimes X^\vee \to 1$ and a unit morphism $\eta : 1 \to X^\vee \otimes X$ such that $(\epsilon \otimes 1) \circ (1 \otimes \eta) = 1_X$ and $(1 \otimes \epsilon) \circ (\eta \otimes 1) = 1_{X^\vee}$.

3. An isomorphism between $\mathrm{End}(1)$ and k is given.

4. There is a fibre functor $\omega : \mathcal{C} \to \mathsf{Vect}_k$ to the category of k-vector spaces: this is a k-linear, faithful, exact functor that commutes with tensor products.

An important result is that every Tannakian category is equivalent to the category of finite dimensional linear representations of an affine group scheme H over k. This equivalence is established by ω and the group scheme H is given as the functor of automorphisms of the fibre functor ω which is defined as follows.

Definition 16. *Let $(\mathcal{C}, \omega)$ be a Tannakian category. The affine group scheme of automorphisms $\mathrm{Aut}^\otimes(\omega)$ of the fibre functor ω is determined as a functor from the category of k-algebras to the category of groups as follows. If R is a k-algebra, then an element σ of $\mathrm{Aut}^\otimes(\omega)(R)$ is given by a collection of elements $\{\sigma(X)\}_X$ with X running over the collection of all objects of $X \in \mathcal{C}$. Each $\sigma(X)$ is an R-linear automorphism of $\omega(X) \otimes_k R$ such that the following hold:*

1. *$\sigma(1) = \mathrm{id}_R$.*
2. *For every morphism $f : X \to Y$ we have that $(\mathrm{id}_R \otimes \omega(f) \circ \sigma(X) = \sigma(Y) \circ (\mathrm{id}_R \otimes \omega(f))$.*
3. *$\sigma(X \otimes Y) = \sigma(X) \otimes \sigma(Y)$.*

3.2 The Tannakian category structure on $\mathcal{B}_q^\tau$

In order not to lose the reader in notational complexities, we generalize a little and let (R, δ) be a differential (commutative) ring that carries an action σ of a group G. Let $\mathsf{Mod}^{G,\delta}(R)$ denote the category consisting of free G-equivariant differential R-modules. Recall that a differential R-module is an R-module equipped with a map $\nabla : M \to M$ – a connection – that satisfies the Leibniz rule:

$$\nabla(m \cdot r) = \nabla(m) \cdot r + m \cdot \delta(r).$$

Moreover, G-equivariance means that there is an action σ of G such that

$$\sigma_g(m \cdot r) = \sigma_g(m) \cdot \alpha_g(r),$$
$$\nabla(\sigma_g(m)) = \sigma_g(\nabla(m)).$$

We will group the objects in the category $\mathsf{Mod}^{G,\delta}(R)$ into a triple (M, σ, ∇) and denote the morphisms that respect all the structures by $\mathrm{Hom}_R^{G,\delta}(M, N)$.

Proposition 17. *The category* $\mathsf{Mod}^{G,\delta}(R)$ *is a rigid tensor category with the tensor product given by*

$$(M, \sigma, \nabla) \otimes (N, \sigma', \nabla') = (M \otimes_{\mathcal{O}(\mathbb{C}^*)} N,\ \sigma \otimes \sigma',\ \nabla \otimes 1 + 1 \otimes \nabla')$$

for any two objects (M, σ, ∇) *and* (N, σ', ∇') *in* $\mathsf{Mod}^{G,\delta}(R)$.

Proof. We start by checking that the tensor product is commutative. First of all, since R is a commutative ring, the 'tensor flip' that maps $M \otimes_{\mathbb{C}} N \to N \otimes_{\mathbb{C}} M$ factorizes to a bijective map of R-modules from $M \otimes_R N$ to $N \otimes_R M$. One also checks that it intertwines the actions $\sigma \otimes \sigma'$ and $\sigma' \otimes \sigma$ and the two connections.

The duality is given as follows, for an object $(M = V \otimes R, \sigma, \nabla)$, V a vector space, we define its dual object $(M^\vee, \sigma^\vee, \nabla^\vee)$ as follows. Define an R-module by,

$$M^\vee := \mathrm{Hom}_R(M, R),$$

with $r \in R$ acting on $f \in M^\vee$ by $(f \cdot r)(m) = f(m) \cdot r = f(m \cdot r)$. It can be equipped with a dual action $\sigma^\vee$ of G by setting for $f \in M^\vee$,

$$\sigma^\vee(f) = \alpha \circ f \circ \sigma^{-1}.$$

One can check that $\sigma^\vee(f)$ is again R-linear:

$$\sigma^\vee(f)(m \cdot r) = \alpha \circ f\left(\sigma^{-1}(m) \cdot \alpha^{-1}(r)\right) = \alpha \circ f \circ \sigma^{-1}(m) \cdot r = \left(\sigma^\vee(f) \cdot r\right)(m)$$

Moreover, the action of R on $M^\vee$ is equivariant with respect to $\sigma^\vee$:

$$\sigma^\vee(f \cdot r)(m) = \alpha \circ (f \cdot r)\left(\sigma^{-1}(m)\right) = \alpha\left(f(\sigma^{-1}(m)) \cdot r\right) = \alpha \circ f \circ \sigma^{-1}(m) \cdot \alpha(r).$$

A dual connection $\nabla^\vee$ is defined by

$$\nabla^\vee(f) = \delta \circ f - f \circ \nabla,$$

which indeed satisfies the Leibniz rule

$$\nabla^\vee(f \cdot r)(m) = \delta(f(m)) \cdot r + f(m) \cdot \delta(r) - f(\nabla(m)) \cdot r = \left(\nabla^\vee(f) \cdot r\right)(m) + (f \cdot \delta(r))(m),$$

and is $\sigma^\vee$-invariant:

$$\sigma^\vee\left(\nabla^\vee(f)\right) = \alpha \circ (\delta \circ f) \circ \sigma^{-1} - \alpha \circ (f \circ \nabla) \circ \sigma^{-1} = \delta \circ (\alpha \circ f \circ \sigma^{-1}) - (\alpha \circ f \circ \sigma^{-1}) \circ \nabla,$$

since α and σ commute with δ and ∇, respectively.

Note that since $M = V \otimes R$, we can identify,

$$M^\vee \simeq \mathrm{Hom}_R(V \otimes R, R) \simeq \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \otimes R \simeq V^* \otimes R,$$

from which it follows that $M^{\vee\vee} \simeq M$. Indeed, one checks that the induced map respects the extra (G, δ)-structure:

$$\sigma^{\vee\vee}(m)(f) = \alpha \circ m \circ \left(\sigma^\vee\right)^{-1}(f) = \alpha \circ m \circ (\alpha^{-1} \circ f \circ \sigma) = f(\sigma(m))$$
$$\nabla^{\vee\vee}(m)(f) = (\delta \circ m)(f) - m \circ \nabla^\vee(f) = \delta(f(m)) - m \circ (\delta \circ f) + f(\nabla(m)) = f(\nabla(m)).$$

for all $m \in M, f \in M^\vee$. In addition, it allows one to prove that the association

$$\phi \in \mathrm{Hom}_R^{G,\delta}(N_1, N_2 \otimes_R M^\vee) \mapsto \tilde{\phi} \in \mathrm{Hom}_R^{G,\delta}(N_1 \otimes_R M, N_2)$$
$$\tilde{\phi}(n_1 \otimes m) := \phi(n_1)(m) \in N_2.$$

induces an isomorphism. Again, it is enough to show that this map is both G-equivariant and δ-invariant, which is left as an exercise.
In a similar way, one proves that

$$\mathrm{Hom}_R^{G,\delta}(N_1 \otimes_R M^\vee, N_2) \simeq \mathrm{Hom}_R^{G,\delta}(N_1, N_2 \otimes_R M).$$

Finally, there is an evaluation morphism and a unit morphism given in terms of a basis $\{e_i\}$ of V and its dual $\{\hat{e}_i\}$ of V^* by

$$\epsilon(m \otimes f) = f(m), \qquad \eta(1_R) = \hat{e}_i \otimes e_i,$$

that satisfy the required properties. □

Let us now return to the category $\mathcal{B}_q^\tau$ of Definition 4. It is not difficult to see that the above tensor product respects the regular singularity condition in the definition of $\mathcal{B}_q^\tau$. Hence this becomes a rigid tensor category as well. We would like to show that it is in fact a Tannakian category by constructing a fibre functor to $\mathsf{Vect}_{\mathbb{C}}$. The following observations turn out to be essential in what follows.

Via a series of changes of basis, it is possible to bring the matrix A in the form of a constant matrix with all eigenvalues in the same transversal of $\tau\mathbb{Z}$. In other words, its eigenvalues never differ by an integer multiple of τ. Before we explain how this can be achieved, recall that a *transversal* to $\tau\mathbb{Z}$ in $\mathbb{C}$ is the image of a section of the projection map $\mathbb{C} \to \mathbb{C}/\tau\mathbb{Z}$ (*e.g.,* the strip $0 \leq \Re(z/\tau) < 1$). We follow the argument of Section 17 in [35]. Let $A(z) = A_0 + A_1 z + \cdots$ be a matrix with holomorphic entries. We first bring the constant term A_0 in Jordan canonical form via a constant change of basis matrix. Subsequently, we can bring all the eigenvalues of A_0 in the same transversal of $\tau\mathbb{Z}$ by the so-called *shearing transformations.* Let us consider the case of a 2×2 matrix $A(z)$ and write

$$A(z) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix},$$

with $a = a_1 z + a_2 z^2 + \cdots$ and similarly b, c and d. Let us suppose that $\lambda_1 - \lambda_2 = k\tau$ for some positive integer k. The change of basis is given by the matrix $D = \mathrm{diag}\{1, z\}$ and transforms A to

$$A' = D^{-1}AD + D^{-1}\delta D = \begin{pmatrix} \lambda_1 & 0 \\ c_1 & \lambda_2 + \tau \end{pmatrix} + \begin{pmatrix} a(z) & zb(z) \\ c_2 z + c_3 z^2 + \cdots & d(z) \end{pmatrix},$$

and one readily observes that the constant term A'_0 of this matrix has eigenvalues that differ by $(k-1)\tau$. Proceeding in this way, one can transform A to a matrix that has constant term with eigenvalues in the same transversal. The generalisation to arbitrary dimensions is straightforward and can be found in Section 17.1 of *loc. cit.*

Proposition 18. *For each object in $\mathcal{B}_q^\tau$ there is an isomorphic object $(M \simeq V \otimes \mathcal{O}(\mathbb{C}^*), \sigma, \nabla)$ in $\mathcal{B}_q^\tau$ with V a vector space and*

1. *$\nabla = \delta + A$ with A a constant matrix with all eigenvalues in the same transversal of $\tau\mathbb{Z}$,*
2. *σ is given by $\sigma(v \otimes f) = Bv \otimes \alpha(f)$ for an invertible constant matrix B.*

Proof. Since M is a free $\mathcal{O}(\mathbb{C}^*)$-module, there is a vector space V such that $M \simeq V \otimes \mathcal{O}(\mathbb{C}^*)$. We show *1.* by adopting an argument from Section 5 of [35]. By the above observations, we can write the matrix of the connection as $A = A_0 + A_1 z + \cdots$, with A_0 having eigenvalues that never differ by an element of $\tau\mathbb{Z}$. We construct a matrix $P = \mathrm{I} + P_1 z + \cdots$ (P_k in $M_n(\mathbb{C})$) which solves $PA_0 = AP - \delta P$. Comparing the powers of z, we find

$$A_0 P_k - P_k(A_0 + k\mathrm{I}) = -(A_k + A_{k-1}P_1 + \cdots + A_1 P_{k-1})$$

which can be solved recursively by our assumption on the eigenvalues of A_0. This gives a formal power series expansion and we would like to show that the entries of P are in fact holomorphic functions on $\mathbb{C}^*$.

Now by Theorem 5.4 of [35] one knows that the radius of convergence of the entries of P is the same as that of the entries of A, which is infinity. Hence, $P \in M_n(\mathcal{O}(\mathbb{C}^*))$.

Next, the action of σ can be written as $\sigma(v \otimes f) = Bv \otimes \alpha(f)$ for some invertible matrix $B \in M_n(\mathcal{O}(\mathbb{C}^*))$ with n the dimension of V. Expressed in terms of A and B, the equivariance condition $\sigma \circ \nabla = \nabla \circ \sigma$ reads

$$\delta B + [A, B] = 0, \tag{7}$$

and as observed above, we may assume that A has constant entries and with eigenvalues that are all in the same transversal. We adopt the argument from the proof of Theorem 4.4 in [18] to show that B is in fact constant. Writing B as a Laurent series $B = \sum_{k\in\mathbb{Z}} B_k z^k$ we obtain the following relations

$$(A - \tau k\mathrm{I}_n)B_k = B_k A, \qquad k = 0, 1, \ldots$$

This implies [35, Theorem 4.1] (see also Lemma 4.6 in [18]) that $(A - \tau k\mathrm{I}_n)$ and A have at least one common eigenvalue. But since the eigenvalues of A are all in a transversal of $\tau\mathbb{Z}$ in $\mathbb{C}$, this is impossible unless $k = 0$, and we conclude that $B_k = 0$ for all $k \neq 0$. □

Our next task is to show that $\mathcal{B}_q^\tau$ is in fact a Tannakian category and compute the corresponding affine group scheme. For this, we use an equivariant version of the Riemann–Hilbert correspondence.

Theorem 19. 1. *The category $\mathcal{B}_q^\tau$ is a Tannakian category with the fibre functor given by*

$$\begin{aligned} \omega : \mathcal{B}_q^\tau &\longrightarrow \mathsf{Vect}_\mathbb{C} \\ (M, \sigma, \nabla) &\longmapsto (\ker \nabla)_z, \end{aligned}$$

mapping to the germs at a fixed point $z \in \mathbb{C}^*$ of local solutions to the differential equation $\delta f + Af = 0$, where $\nabla = \delta + A$ with respect to a suitable basis of M.

2. *The category $\mathcal{B}_q^\tau$ is equivalent to the category* $\mathsf{Rep}(\mathbb{Z}+\theta\mathbb{Z})$ *of finite dimensional representations of* $\mathbb{Z}+\theta\mathbb{Z} \simeq \mathbb{Z}^2$.

Proof. By the existence and uniqueness of local solutions of linear differential equations, there are n local solutions to the system of differential equations $\delta U = -AU$ once we have fixed the initial conditions, so that $(\ker \nabla)_z$ is an n-dimensional complex vector space. That the functor ω is faithful can be seen as follows. Suppose ϕ is a morphism between two objects (M, σ, ∇) and (M', σ', ∇') and suppose that these objects are of the form as in Proposition 18, with the eigenvalues of A, A' in the same transversal. We claim that ϕ is given by a constant matrix so that $\omega(\phi)$ mapping $(\ker \nabla)_z$ to $(\ker \nabla')_z$ coincides with ϕ. The argument is very similar to that used in the second part of the proof of Proposition 18 since compatibility of ϕ with the connections implies

$$(A' - \tau k \mathrm{I}_n)\phi_k = \phi_k A, \qquad k = 0, 1, \ldots$$

where we have written $\phi = \sum_{k\geq 0} \phi_k z^k$. An application of Theorem 4.1 in [35] then implies that A and $A' - \tau k \mathrm{I}_n$ have a common eigenvalue. This is impossible unless $k = 0$ since by assumption A and A' have eigenvalues in the same transversal. We conclude that $\phi_k = 0$ for all $k > 0$.

The general case follows by observing that Proposition 18 implies that a morphism between two objects in $\mathcal{B}_q^\tau$ can always be written as $D_2 \circ \phi \circ D_1^{-1}$ with ϕ constant as above and with D_i certain (invertible) change of basis matrices.

For *2.*, fix a transversal $\mathbf{T}$ to $\tau\mathbb{Z}$ in $\mathbb{C}$. We construct a tensor functor $\mathsf{F}_\mathbf{T} : \mathsf{Rep}(\mathbb{Z}^2) \to \mathcal{B}_q^\tau$ that is full, faithful and essentially surjective. Let ρ_1, ρ_2 be two mutually commuting representations of $\mathbb{Z}$ on a vector space V. Then we define $A \in \mathrm{End}(V)$ via $\rho_1(1) = e^{2\pi i A/\tau}$ and B as $\rho_2(1)$. By Lemma 4.5 in [18], there exists a unique matrix A such that $\rho_1(1) = e^{2\pi i A/\tau}$ with its eigenvalues in the transversal $\mathbf{T}$ and a unique matrix B' such that $B = e^{2\pi i B'}$. We set $\mathsf{F}_\mathbf{T}(V) = (M, \sigma, \nabla)$ in $\mathcal{B}_q^\tau$ by setting $M = V \otimes \mathcal{O}(\mathbb{C}^*)$, $\sigma(v \otimes f) = Bv \otimes \alpha(f)$ and finally $\nabla(v \otimes f) = Av \otimes f + v \otimes \delta f$; for a morphism $\phi \in \mathrm{Hom}(V, V')$ we simply set $\mathsf{F}_\mathbf{T}(\phi) = \phi \otimes 1$. Once again by Lemma 4.5 *ibid.* the matrices A and B' commute, whence A and $B = e^{2\pi i B'}$ commute. Thus the compatibility condition between σ and ∇ given by Eqn. (7) is satisfied. Moreover, $\mathsf{F}_\mathbf{T}(\phi)$ is compatible with σ and ∇ and thus a morphism in $\mathcal{B}_q^\tau$.

We infer from Proposition 18 that the functor $\mathsf{F}_\mathbf{T}$ is essentially surjective, since any object in $\mathcal{B}_q^\tau$ is isomorphic to an object obtained from an element in $\mathsf{Rep}(\mathbb{Z}^2)$ by the above procedure.

Fullness and faithfulness of this functor can be seen as follows. Let V, V' be two vector spaces with the action of $\mathbb{Z}^2$ given by $e^{2\pi i A/\tau}, B$ and $e^{2\pi i A'/\tau}$, B' respectively. We can choose the square matrices A and A' such that their eigenvalues lie in the transversal $\mathbf{T}$. It then follows by the same reasoning as before that an element $\rho \in \mathrm{Hom}_{\mathcal{O}(\mathbb{C}^*)}^{\theta\mathbb{Z},\delta}(M, M')$ is given by a constant matrix that intertwines A, B and A', B', respectively. Hence, it is given by an element in $\mathrm{Hom}(V, V')$ that commutes with ρ_1 and ρ_2 (i.e. a morphism in $\mathsf{Rep}(\mathbb{Z}^2)$).

Finally, we show that $\mathsf{F}_\mathbf{T}$ is a tensor functor. Suppose that (V, ρ_1, ρ_2) and (V', ρ_1', ρ_2') are two objects in $\mathsf{Rep}(\mathbb{Z}^2)$; we need to show that there are natural isomorphisms $c_{V,V'} : F(V) \otimes F(V') \to F(V \otimes V')$. As before, we define the connection matrix A by setting $e^{2\pi i A/\tau} = \rho_1(1)$ and $B = \rho_2(1)$; in the same manner we define A' and B' from ρ_1' and ρ_2'. We then have

$$F(V, \rho_1, \rho_2) \otimes F(V', \rho_1', \rho_2') = \left((V \otimes \mathcal{O}(\mathbb{C}^*)) \otimes_{\mathcal{O}(\mathbb{C}^*)} (V' \otimes \mathcal{O}(\mathbb{C}^*)), \sigma \otimes \sigma', \delta + A \otimes 1 + 1 \otimes A'\right).$$

One observes that the eigenvalues of the matrix $A \otimes 1 + 1 \otimes A'$ lie possibly outside the transversal $\mathbf{T}$. However, there is a unique matrix $\tilde{A}$ with all its eigenvalues in $\mathbf{T}$ such that

$$e^{2\pi i \tilde{A}/\tau} := e^{2\pi i (A\otimes 1 + 1 \otimes A')/\tau} = e^{2\pi i A/\tau} \otimes e^{2\pi i A'/\tau} \equiv \rho_1(1) \otimes \rho_1'(1). \tag{8}$$

The procedure of associating to $A \otimes 1 + 1 \otimes A'$ the matrix $\tilde{A}$ defines the required map $c_{V,V'}$ since $\tilde{A}$ is the connection matrix that one would have obtained (via $\mathsf{F}_{\mathbf{T}}$) from $\rho_1 \otimes \rho_1'$. In fact, it follows that if $A \otimes 1 + 1 \otimes A'$ commutes with $B \otimes B' \equiv \rho_2(1) \otimes \rho_2'(1)$ then so does $\tilde{A}$. This map is natural in V and V' and the usual diagrams expressing associativity and commutativity (*cf.* for instance [12, Definition 1.8]) are satisfied. Moreover, it is bijective since an inverse can be constructed from Eqn. (8) by using the identification $\mathrm{End}_{\mathbb{C}}(V \otimes V') = \mathrm{End}_{\mathbb{C}}(V) \otimes \mathrm{End}_{\mathbb{C}}(V')$ to obtain A and A' back from $\tilde{A}$. □

Note that the choice of the transversal $\mathbf{T}$ is irrelevant since two functors $\mathsf{F}_{\mathbf{T}}$ and $\mathsf{F}_{\mathbf{T}'}$ associated to two different transversals $\mathbf{T}$ and $\mathbf{T}'$ to $\tau\mathbb{Z}$ are related via a natural transformation that is given explicitly by a shearing transformation as discussed before Proposition 18.

We observe that it is also possible to prove the above equivalence directly by means of the fibre functor ω. For this we consider the full subcategory of $\mathcal{B}_q^\tau$ such that the connection matrices have all eigenvalues in the same transversal $\mathbf{T}$. It follows from Proposition 18 that this category is equivalent to $\mathcal{B}_q^\tau$. By constructing the maps $c_{M,M'}$ very similar to those appearing in the above proof, one can show that this is an equivalence of rigid tensor categories. Moreover, the restriction of the fibre functor gives it the structure of a Tannakian category. The fibre functor induces an equivalence with $\mathsf{Rep}(\mathbb{Z}^2)$ by defining the action of $\mathbb{Z}^2$ on $(\ker \nabla)_z$ to be given by the matrices $e^{2\pi i A/\tau}$ and B. Clearly, the functor $\mathsf{F}_{\mathbf{T}}$ from the proof of Theorem 19 is the inverse to this fibre functor.

Remark 20. For any group H the category of its finite dimensional representations over $\mathbb{C}$ forms a neutral Tannakian category, which should be equivalent to the category of representations of some affine group scheme, say $\hat{H}$. The group scheme $\hat{H}$ is called the *algebraic hull* of H. Strictly speaking, the affine group scheme underlying $\mathcal{B}_q^\tau$ is the algebraic hull of $\mathbb{Z}^2$. We refer the readers to [28] for an explicit computation of the algebraic hull of $\mathbb{Z}$, which is $\mathrm{Hom}(\mathbb{C}/\mathbb{Z}, \mathbb{C}^*) \times \mathbb{G}_a$.

As a consequence we are able to conclude that the K-theory of $\mathcal{B}_q^\tau$ is the same as that of $\mathsf{Rep}(\mathbb{Z}^2)$. An object of $\mathsf{Rep}(\mathbb{Z}^2)$ is a vector space V equipped with two commuting linear invertible endomorphisms. Using the fact that the two endomorphisms commute, *i.e.,* respect each others eigenspaces, one can always find a common eigenvector w. This gives an exact sequence $0 \longrightarrow \langle w \rangle \longrightarrow V \longrightarrow V/\langle w \rangle \longrightarrow 0$ in $\mathsf{Rep}(\mathbb{Z}^2)$. Therefore, the K-theory of $\mathsf{Rep}(\mathbb{Z}^2)$ is the free abelian group generated by the simple objects, which are one dimensional representations with two actions a and b, with $a, b \in \mathbb{C}^*$ (the actions are given by multiplication by a and b respectively). The fibre functor sends the isomorphism class of $(\mathcal{O}(\mathbb{C}^*), b, \delta + z')$ with $z' \in \mathbb{C}$ to the simple object $(\mathbb{C}, b, e^{2\pi i z'/\tau})$ in $\mathsf{Rep}(\mathbb{Z}^2)$. Note that $(\mathcal{O}(\mathbb{C}^*), b, \delta + z')$ and $(\mathcal{O}(\mathbb{C}^*), b, \delta + (z' + n\tau))$ are isomorphic via the shearing transformation by z^n. Indeed,

$$(\delta + z')z^n f = n\tau z^n f + z^n \delta f + z' z^n f = z^n \left(\delta + (z' + n\tau)\right) f$$

and their images also get identified via the exponentiation. Summarising, we obtain

Corollary 21. *The K-theory of $\mathcal{B}_q^\tau$ is the free abelian group generated by the isomorphism classes of the objects $(\mathcal{O}(\mathbb{C}^*), b, \delta + z')$ with $b \in \mathbb{C}^*$ and $z' \in \mathbb{C}/\tau\mathbb{Z}$. Under this identification, one finds that the map on K-theory induced by the functor $\mathcal{S}_\tau \circ \psi_*$ sends the class of $(\mathcal{O}(\mathbb{C}^*), b, \delta + z')$ to the divisor class of the point $-z' \in X_\tau$ and their linear combinations accordingly.*

Remark 22. One possible perspective of our work is the notion of a fundamental group of noncommutative tori. Given a noncommutative space described by its category of representations in the appropriate sense, *e.g.,* coherent sheaves, vector bundles with connections, *etc.*, it is conceivable that a description of its fundamental group can be obtained by finding a suitably defined Tannakian subcategory inside it. This philosophy stems from the original work of Nori [21, 22] in the commutative case. For a classical complex torus $X_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ the fundamental group is just the lattice $\mathbb{Z} + \tau\mathbb{Z}$. The category $\mathcal{B}_q^\tau$ describes the quotient $\mathbb{C}/(\mathbb{Z} + \theta\mathbb{Z})$ with the infinitesimal $\tau\mathbb{Z}$ action providing the complex structure. Our construction proposes $\mathbb{Z} + \theta\mathbb{Z}$ as a candidate for the fundamental group in a Tannakian setting.

Remark 23. Invoking Manin's point of view again, we may disregard the order in which the quotients are performed. Ideally one would like to perform the double quotient operation in two different orders and show that they agree even *at infinity.* Consider first $X_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and the infinitesimal action of $\theta\mathbb{Z}$ on it. It is described by the category $\mathcal{C}^{\theta,\tau}$, which is the heart of the t-structure of Example 2 on $D^b(X_\tau)$. If $g \in SL(2,\mathbb{Z})$ and g acts on τ by fractional linear transformation then $\mathcal{C}^{\theta,g\tau} \cong \mathcal{C}^{\theta,\tau}$. There is a unique cusp corresponding to the orbit of the rational numbers with respect to the modular group $SL(2,\mathbb{Z})$. This point corresponds to the nodal Weierstraß cubic E. One may consider a similar infinitesimal action of $\theta\mathbb{Z}$ in terms of t-structures on $D^b(E)$ depending on θ and their hearts as studied in [4]. On the other hand from Proposition 2 one finds that the $SL(2,\mathbb{Z})$ invariance of the categories $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$ can be proven without referring to the equivalence with $\mathcal{C}^{\theta,\tau}$. In fact, it is possible to substitute any real value (in particular rational number) for τ in δ_τ. However, δ_τ does not remain injective for rational values of τ. In fact, one can check that for each rational number p/q, the kernel of $\delta_{p/q}$ is a *-subalgebra of $\mathcal{A}_\theta$ generated by $U_1^{-q}U_2^p$.

It is still plausible that the categories $\mathsf{Vect}(\mathbb{T}_\theta^\tau)$, with $\tau \in \mathbb{Q}$ will be related to the hearts of the t-structures on $D^b(E)$ by functors similar to $\mathcal{S}_\tau$. The following observation might be a useful summary.

The action of $SL(2,\mathbb{Z})$ extends to the whole lower half plane $\mathbb{H}^-$. When adjoined with $\mathbb{P}^1(\mathbb{R})$, the quotient space contains the usual modular curve with an invisible stratum arising from the action of $SL(2,\mathbb{Z})$ on the irrationals, which has been investigated by Connes, Douglas and Schwarz in [6] and separately by Manin and Marcolli in [20]. On one hand, for a fixed θ, the action of $SL(2,\mathbb{Z})$ on $\mathbb{H}^-$ is encoded in the isomorphism $\mathcal{S}_\tau \cong \mathcal{S}_{g\tau}$ of Polishchuk–Schwarz functors. In particular, $D^b(X_\tau) \cong D^b(X_{g\tau})$ inducing $\mathcal{C}^{\theta,\tau} \cong \mathcal{C}^{\theta,g\tau}$. On the other hand, for a fixed τ, the action of $SL(2,\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{R})$ manifests itself by the action induced by the *twist functors* $\mathrm{T}_{\mathcal{O}}, \mathrm{T}_{k(p_0)} \in \mathrm{Aut}(D^b(X_\tau))$ on the t-structures, $(D^{\theta,\leqslant 0}, D^{\theta,\geqslant 0}) \mapsto (D^{g\theta,\leqslant 0}, D^{g\theta,\geqslant 0})$ up to a shift (see Proposition 2.6 [26]).

References

[1] V. Baranovsky and V. Ginzburg. Conjugacy classes in loop groups and G-bundles on elliptic curves. *Int. Math. Res. Not.* 15 (1996) 733–751.

[2] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)* , Vol. 100 of *Astérisque*, pp. 5–171. Soc. Math. France, Paris, 1982.

[3] A. Bondal and M. van den Bergh. Generators and representability of functors in commutative and noncommutative geometry. *Mosc. Math. J.* 3 (2003) 1–36.

[4] I. Burban and B. Kreußler. Derived categories of irreducible projective curves of arithmetic genus one. *Compositio Math.* 142 (2006) 1231–1262.

[5] A. Connes. C^* algèbres et géométrie différentielle. *C. R. Acad. Sci. Paris Sér. A-B* 290 (1980) A599–A604.

[6] A. Connes, M. R. Douglas and A. Schwarz. Noncommutative geometry and matrix theory: compactification on tori. *J. High Energy Phys.* 9802 (1998) 003.

[7] A. Connes and M. Marcolli. Renormalization and motivic Galois theory. *Int. Math. Res. Not.* 76 (2004) 4073–4092.

[8] A. Connes and M. Marcolli. From Physics to Number Theory via Noncommutative Geometry, Part II: Renormalization, the Riemann-Hi lbert correspondence, and motivic Galois theory. In *Frontiers in Number Theory, Physics, and Geometry II*, pp. 617–713, Springer Verlag, Berlin, 2006.

[9] A. Connes and M. A. Rieffel. Yang-Mills for non-commutative two-tori. *Contemp. Math.* 62 (1987) 237–266.

[10] P. Deligne Equations différentielle à points singuliers réguliers. Lecture Notes in Mathematics, Vol. 163, Springer, Berlin (1970).

[11] P. Deligne Catégories tannakiennes. In *Grothendieck Festschrift*, Vol. 2, pp 111–195, Birkhäuser Boston, 1990. Progress in Mathematics Vol. 87.

[12] P. Deligne and J. Milne. Tannakian categories. In *Hodge Cycles, Motives and Shimura Varieties*, pp. 101–228, 1982. Lecture Notes in Mathematics, Vol. 900.

[13] L. Fuchs and L. Salce. Modules over valuation domains. Vol 97 of *Lecture notes in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1985.

[14] J. M. Gracia-Bondía and J. C. Várilly and H. Figueroa. Elements of noncommutative geometry Birkhäuser Boston, 2001.

[15] D. Happel, I. Reiten, and S. O. Smalø. Tilting in abelian categories and quasitilted algebras. *Mem. Amer. Math. Soc.* 120 (1996).

[16] G. Harder and M. S. Narasimhan. On the cohomology groups of moduli spaces of vector bundles on curves. *Math. Ann.* 212 (1974/75) 215–248.

[17] B. Keller. Deriving DG categories. *Ann. Sci. École Norm. Sup.* (4) 27 (1) (1994) 63–102.

[18] B. Malgrange. Regular connections, after Deligne. Chapter IV of A. Borel et al., *Algebraic D-Modules.* Academic Press, Boston, 1987.

[19] Yu. I. Manin. The notion of dimension in geometry and algebra. *Bull. Amer. Math. Soc. (N.S.)* 43 (2006) 139–161.

[20] Yu. I. Manin and M. Marcolli. Continued fractions, modular symbols, and noncommutative geometry. *Selecta Math. (N.S.)* 8 (2002) 475–521.

[21] M. V. Nori. On the representations of the fundamental group. *Compositio Math.* 33 (1976) 29–41.

[22] M. V. Nori. The fundamental group-scheme. *Proc. Indian Acad. Sci. Math. Sci.* 91 (1982) 73–122.

[23] M. Pimsner and D. Voiculescu. Imbedding the irrational rotation C^*-algebra into an AF-algebra. *J. Operator Theory* 4 (1980) 201–210.

[24] A. Yu. Pirkovskiĭ. On certian homological properties of Stein algebras. *J. Math. Sci. (New York)* 95 (1999) 2690–2702.

[25] A. Polishchuk. Classification of holomorphic vector bundles on noncommutative two-tori. *Doc. Math.* 9 (2004) 163–181.

[26] A. Polishchuk. Noncommutative two-tori with real multiplication as noncommutative projective varieties. *J. Geom. Phys.* 50 (2004) 162–187.

[27] A. Polishchuk and A. Schwarz. Categories of holomorphic vector bundles on noncommutative two-tori. *Commun. Math. Phys. 236* (2003) 135 - 159.

[28] M. van der Put. Differential Galois theory, universal rings and universal groups. Differential algebra and related topics, World Sci. Publ., River Edge, NJ, (2002) 171–189.

[29] M. van der Put and M. F. Singer. Galois Theory of Linear Differential Equations. Springer Verlag, Berlin, 2003.

[30] M. A. Rieffel. C^*-algebras associated with irrational rotations. *Pacific J. Math.* 93 (1981) 415–429.

[31] M. A. Rieffel. Noncommutative tori–a case study of noncommutative differentiable manifolds. *Contemp. Math.* 105 (1990) 191–211.

[32] N. Saavedra Rivano. Catégories Tannakiennes, Vol 265 of *Lecture Notes in Mathematics*. Springer Verlag, Berlin, 1972.

[33] Y. Soibelman and V. Vologodsky. Noncommutative compactifications and elliptic curves. *Int. Math. Res. Not.* (2003) 1549–1569.

[34] R. G. Swan. Vector bundles and projective modules. *Trans. Amer. Math. Soc.*, 105 (1962) 264–277.

[35] W. Wasow. Asymptotic Expansions for Ordinary Differential Equations. Robert E. Krieger Publishing Company, New York, 1976.