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BURNSIDE RINGS FOR REAL 2-REPRESENTATION THEORY: THE LINEAR THEORY

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Abstract. This paper is a fundamental study of the Real 2-representation theory of 2-groups. It also contains many new results in the ordinary (non-Real) case. Our framework relies on a 2-equivariant Morita bicategory, where a novel construction of induction is introduced. We identify the Grothendieck ring of Real 2-representations as a Real variant of the Burnside ring of the fundamental group of the 2-group and study the Real categorical character theory. This paper unifies two previous lines of inquiry, the approach to 2-representation theory via Morita theory and Burnside rings, initiated by the first author and Wendland, and the Real 2-representation theory of 2-groups, as studied by the second author.

Contents

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Introduction

The notion of a 2-representation of a 2-group is introduced and studied by Barrett and Mackaay [5], Crane and Yetter [13] and Elgueta [14], amongst many others. In this setting, a 2-group acts coherently by autoequivalences of a category or, more generally, of an object of a target bicategory. An important target bicategory is $2\mathsf{Vect}_{\mathbb{K}}$, the bicategory of Kapranov–Voevodsky 2-vector spaces over a field $\mathbb{K}$, a natural categorification of the category of finite dimensional vector spaces. In this example, or in more general linear settings, there is a character theory of 2-representations, discovered independently by Bartlett [6] and Ganter and Kapranov [16]. This character theory can be seen as a concrete instance of the theory of secondary traces, as studied by Töen and Vezzosi [34, 35] and Ben-Zvi and Nadler [8]. The theory of 2-representations, with its character theory, appears naturally in many areas of mathematics, including topological gauge theory [6, 36] and equivariant elliptic cohomology [21, 16, 25]. It is indispensable in traditional representation theory, for example, through its relation to conjectures of Lusztig [10] and McKay [22], as explained in [31], or via the topological field theoretic approach to representations of algebraic groups [7].

One weakness of 2-character theory is that, in general, it cannot distinguish equivalence classes of 2-representations. This issue is resolved by Rumynin and Wendland [31] who, under mild assumptions, describe the Grothendieck ring of 2-representations of an essentially finite 2-group on $2\mathsf{Vect}_{\mathbb{K}}$ in terms of a generalized Burnside ring [18]. The mark homomorphisms of this Burnside ring not only distinguish equivalence classes of 2-representations but also recover the 2-characters. The perspective taken in [31] is that of Morita theory, so that 2-groups are represented on the Morita bicategory of algebras, bimodules and intertwiners, instead of on $2\mathsf{Vect}_{\mathbb{K}}$. This perspective is amenable to explicit calculations and constructions.

In this paper we develop a Morita-theoretic approach to Real 2-representation theory. A number of our results are new and interesting already for ordinary 2-representations. The word *Real* is capitalized, following Atiyah [3, 1], where he distinguishes “real” (objects defined over $\mathbb{R}$) and “Real” (objects with an involution). For instance, a Real vector space is a complex vector space together with an anti-linear involution.

The Real 2-representation theory of 2-groups is introduced and investigated in [37] as a categorification of the Real representation theory of groups, as studied by Atiyah and Segal [2] and Karoubi [23] in the form of equivariant KR-theory. There are two distinct notions of a Real 2-representation. In this paper we focus on *linear* Real 2-representations, in which the target bicategory is endowed with an involution which is contravariant on 2-morphisms. A second notion of an *anti-linear* Real 2-representation, related to Hermitian Morita theory [20], requires the target bicategory to be linear and endowed with an involution which is fully covariant but anti-linear on 2-morphisms. We hope to treat the anti-linear theory in consequent work. The character theory of (projective) Real 2-representations of finite groups is also studied in [37]. Real 2-representations, and their characters, appear naturally in unoriented topological field theory and orientifold string and M-theory [38] and, conjecturally, in Real variants of equivariant elliptic cohomology [37].

Let us now describe assiduously the content of the present paper, stating concisely the main theorems.

In Chapter 1 we introduce notation and set out our vision of the subject. Let $\mathfrak{G} = (G_2 \xrightarrow{\partial} G_1)$ be a crossed module with a $\mathbb{Z}_2$-grading, that is, a group homomorphism $\pi : G_1 \to \mathbb{Z}_2$ which satisfies $\mathrm{im}(\partial) \leq \ker(\pi)$. We allow π to be trivial: in this case our results belong to the ordinary (non-Real) theory. Associated to $\mathfrak{G}$ is a $\mathbb{Z}_2$-graded 2-group $\widetilde{\mathfrak{G}}$ whose action on bicategories is our primary interest. For the introduction we assume that G_1 is finite since our results are cleanest under this assumption.

We start Chapter 2 by defining weak $\mathfrak{G}$-algebras, the central notion of this paper. This is an associative $\mathbb{K}$-algebra A together with a projective action of G_1 by algebra automorphisms and anti-automorphisms, according to the grading π, and a projective group homomorphism $G_2 \to A^{\times}$ which satisfy a number of coherence conditions. Compactly, it is an instance of Noohi's weak crossed module homomorphisms [27], special cases of which play a key role in [31]. A weak $\mathfrak{G}$-algebra in which the projective homomorphisms are, in fact, genuine homomorphisms is called strict. In Section 2.3 we construct various Morita bicategories which are 2-equivariant for $\widetilde{\mathfrak{G}}$ and fit into the following diagram of subbicategories:

$$\begin{array}{ccccc} \mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}} & \hookrightarrow & \mathfrak{G}\text{-}RW\mathcal{A}\mathsf{lg}_{\mathbb{K}} & \hookrightarrow & \mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}_{\mathbb{K}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}} & \hookrightarrow & \mathfrak{G}\text{-}RW\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}} & \hookrightarrow & \mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}. \end{array}$$

The strictness of the $\mathfrak{G}$-algebras decreases from left to right and the superscript "fd" indicated the fully dualizable subbicategory. A key technical notion, realizability of a twisted 2-cocycle for the group G_1, is introduced in Section 2.4. The following *strictification* theorem is the main result of Chapter 2.

Theorem (Theorem 2.4 and Proposition 2.5). *Let $A \in \mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}_{\mathbb{K}}$ be an N-weak $\mathfrak{G}$-algebra.*

(i) If A is split semisimple, then there exists a strict (split semisimple) $\mathfrak{G}$-algebra $B \in \mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$ which is $\mathfrak{G}$-Morita equivalent to A.

(ii) In general, there exist a $\mathbb{Z}_2$-graded crossed module $\mathfrak{H}$ and a strict $\mathfrak{H}$-algebra $B \in \mathfrak{H}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}$ such that $\widetilde{\mathfrak{H}} \simeq \widetilde{\mathfrak{G}}$ as $\mathbb{Z}_2$-graded 2-groups and B is $\mathfrak{H}$-Morita equivalent to A.

In Chapter 3 we define and study induction pseudofunctors. Given a crossed submodule $\mathfrak{H}$ of $\mathfrak{G}$, it is natural to expect the existence of an induction pseudofunctor

$$\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} : \mathfrak{H}\text{-}N\mathcal{A}\mathsf{lg}_{\mathbb{K}} \to \mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}_{\mathbb{K}}.$$

Using our strictification result, it suffices to construct $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$ on the subbicategories $\mathfrak{H}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}$ and $\mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}$ of strict algebras. There are three flavours of $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$, depending on the $\mathbb{Z}_2$-gradings of $\mathfrak{H}$ and $\mathfrak{G}$. We can now state the main result of Chapter 3.

Theorem (Theorems 3.4 and 3.6).

(i) There exists an induction pseudofunctor $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} : \mathfrak{H}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}} \to \mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}$.

(ii) If the index $|G_2 : H_2|$ is finite and the characteristic of $\mathbb{K}$ does not divide $|G_2 : H_2|$, then the above pseudofunctor restricts to a pseudofunctor $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} : \mathfrak{H}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}} \to \mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$.

Our construction of $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$ is direct and explicit, illustrating the power of the Morita-theoretic approach to 2-representation theory. An important technical result is a Maschke-type theorem for induced $\mathfrak{G}$-algebras, Proposition 3.3, asserting that $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$ preserves separability. When $\mathfrak{G}$ is trivially graded, we prove that, under certain assumptions, $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$ is both left and right biadjoint to the restriction pseudofunctor $\mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}}$ (Proposition 3.5). The analogous question in the $\mathbb{Z}_2$-graded setting is more subtle (Problem 3.7). In Section 3.4 we describe the monoidal behaviour of $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$.

In Chapter 4 we return to Real 2-representations on $2\mathsf{Vect}_{\mathbb{K}}$. We construct a local biequivalence $\mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}} \to \mathsf{RRep}_{2\mathsf{Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}})$ (Proposition 4.2) that enables us to prove the first of the two main results of the paper.

Theorem (Theorem 4.6)**.** *If $\mathbb{K}$ is separably closed, then $\mathsf{RRep}_{2\mathsf{Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}})$ is biequivalent to $\mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$.*

As a consequence, we give a Morita-theoretic classification of Real 2-representations of $\widetilde{\mathfrak{G}}$ (Corollary 4.7). The resulting structure theorem for Real 2-representations (Theorem 4.8) yields a Real generalization of known results [29, 14, 16].

In Chapter 5 we describe the Grothendieck ring of $\mathsf{RRep}_{2\mathsf{Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}})$, proving the second main result of the paper.

Theorem (Theorem 5.1)**.** *The Grothendieck ring $K_0(\mathsf{RRep}_{2\mathsf{Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}}))$ is isomorphic to the generalized Burnside ring $\mathbb{B}_{\mathbb{Z}}^{\Phi}(\mathfrak{G})$, where Φ is the functor of "Real one dimensional 2-representations".*

The isomorphism is explicit and compatible with the corresponding ordinary result [31].

Finally, in Chapter 6 we turn to the character theory of Real 2-representations. We define the categorical character and 2-character of a Real 2-representation of an arbitrary $\mathbb{Z}_2$-graded 2-group $\widetilde{\mathfrak{G}}$. Our approach is geometric, being formulated in terms of various kinds of loop spaces of $\mathfrak{G}$. One feature of our approach is that it works directly with the 2-group (or its crossed module model) and applies uniformly to the ordinary and Real cases. Moreover, its form immediately suggests a generalization to the n-categorical and projective cases. In the finite case, we relate Real 2-characters to mark homomorphisms of the generalized Burnside ring in Corollary 6.4. At its core, this is a result about 2-characters of certain induced Real 2-representations (cf. Theorem 6.2 and Corollary 6.3). This provides a 2-group generalization of the corresponding result for (Real projective) 2-representations of finite groups [16, 37]. We expect these results to be representation theoretic analogues of Hopkins–Kuhn–Ravenel character theory [21, 26] of 2-equivariant elliptic cohomology (see Problem 6.5).

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1. Crossed modules and 2-groups

1.1. **$\mathbb{Z}_2$-graded crossed modules.** A crossed module $\mathfrak{G} = (G_2 \xrightarrow{\partial} G_1)$ consists of groups G_1 and G_2, an action $G_1 \times G_2 \to G_2$, $(g, x) \mapsto {}^g x$, of G_1 on G_2 by group automorphisms and a homomorphism $\partial : G_2 \to G_1$. This data is required to satisfy

$$\partial({}^g x) = g\partial(x)g^{-1}, \qquad {}^{\partial(y)}x = yxy^{-1}$$

for all g, x, y. Here, and for the rest of the paper, we use letters f, g, h for elements of G_1, x, y, z for elements of G_2 and a, b, c for elements of some algebra A.

A group G defines a crossed module $(\{e\} \xrightarrow{e} G)$, which we continue to denote by G. Denote by $\mathbb{Z}_2$ the multiplicative group $\{+1, -1\}$.

Definition. *A $\mathbb{Z}_2$-graded crossed module is a crossed module morphism $\pi : \mathfrak{G} \to \mathbb{Z}_2$.*

Explicitly, π is the data of a group homomorphism $\pi : G_1 \to \mathbb{Z}_2$ which satisfies $\mathrm{im}(\partial) \leq \ker(\pi)$. The kernel of $\pi : \mathfrak{G} \to \mathbb{Z}_2$ is the crossed submodule $\mathfrak{G}_0 = (G_2 \xrightarrow{\partial} G_0)$, where $G_0 = \ker(\pi : G_1 \to \mathbb{Z}_2)$. We call $\mathfrak{G}_0$ the ungraded crossed module of $\mathfrak{G}$. We say that $\mathfrak{G}$ is trivially graded if $\mathfrak{G}_0 = \mathfrak{G}$.

1.2. **Crossed modules of generalized automorphisms.** Fix a ground field $\mathbb{K}$. Let A be a $\mathbb{K}$-algebra, always assumed to be associative and unital. The group of units of A is $A^\times$. The centre of A is $Z(A)$. The assignment of an algebra to its opposite extends to an involution $(-)^{\mathrm{op}} : \mathsf{ALG}_{\mathbb{K}} \to \mathsf{ALG}_{\mathbb{K}}$ of the category of $\mathbb{K}$-algebras and unital algebra morphisms. Given $\epsilon \in \mathbb{Z}_2$, define ${}^{\epsilon}(-) : \mathsf{ALG}_{\mathbb{K}} \to \mathsf{ALG}_{\mathbb{K}}$ so that ${}^{+1}(-) = \mathrm{id}_{\mathsf{ALG}_{\mathbb{K}}}$ and ${}^{-1}(-) = (-)^{\mathrm{op}}$.

Let $\mathrm{Aut}^{\mathrm{gen}}(A)$ be the set of all algebra isomorphisms of the form $A \to A$ or $A^{\mathrm{op}} \to A$. Define $\pi : \mathrm{Aut}^{\mathrm{gen}}(A) \to \mathbb{Z}_2$ so that $g \in \mathrm{Aut}^{\mathrm{gen}}(A)$ is an algebra homomorphism ${}^{\pi(g)}A \to A$. We consider $\mathrm{Aut}^{\mathrm{gen}}(A)$ as a group with multiplication

$$g \cdot h = g \circ {}^{\pi(g)}h.$$

Since $(-)^{\mathrm{op}}$ acts trivially on morphisms (viewed as set maps), this is the usual composition of morphisms. The map π makes $\mathrm{Aut}^{\mathrm{gen}}(A)$ into a $\mathbb{Z}_2$-graded group with ungraded subgroup $\mathrm{Aut}(A)$.

Definition. *The crossed module of generalized automorphisms of A is*

$$\mathrm{AUT}^{\mathrm{gen}}(A) = \big(A^\times \xrightarrow{\partial} \mathrm{Aut}^{\mathrm{gen}}(A)\big),$$

where $\mathrm{Aut}^{\mathrm{gen}}(A)$ acts on $A^\times$ by ${}^g x = g(x^{\pi(g)})$ and $\partial(x)$ is the inner automorphism $a \mapsto xax^{-1}$.

To see that, for example, the first axiom of a crossed module holds, note that $\partial({}^g x)(a) = g(x^{\pi(g)})ag(x^{-\pi(g)})$ while

$$(g\partial(x)g^{-1})(a) = g(x^{\pi(g)}g^{-1}(a)x^{-\pi(g)}) = g(x^{\pi(g)})ag(x^{-\pi(g)}),$$

as required. The $\mathbb{Z}_2$-grading of $\mathrm{Aut}^{\mathrm{gen}}(A)$ induces a $\mathbb{Z}_2$-grading of $\mathrm{AUT}^{\mathrm{gen}}(A)$, the ungraded crossed module of which is $\mathrm{AUT}(A) := (A^\times \xrightarrow{\partial} \mathrm{Aut}(A))$.

1.3. **2-groups.** For categorical background, we refer the reader to Bénabou [9]. For a detailed introduction to 2-groups, see Baez and Lauda [4].

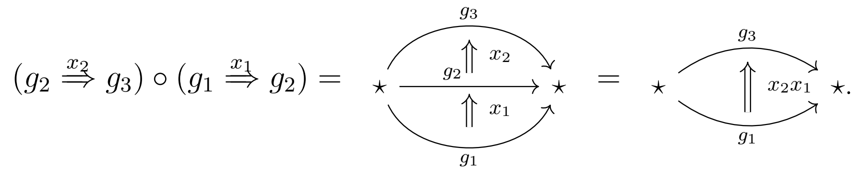
In this paper, we use the term 2-group for what is called a weak 2-group in [4], namely, a bicategory $\mathcal{G}$ with a single object in which all 1-morphisms are equivalences and all 2-morphisms are isomorphisms. Morphisms of 2-groups are pseudofunctors. Note that $\mathcal{G}$ can be seen as a monoidal groupoid in which each endofunctor $g \otimes - :$

$\mathcal{G} \to \mathcal{G}$, $g \in \mathcal{G}$, is an equivalence. We switch freely between these two perspectives. A 2-group is called strict if its underlying bicategory is a (strict) 2-category. Every 2-group is equivalent to a strict 2-group. A group can be thought of as a groupoid and, hence, as a 2-group in a canonical way.

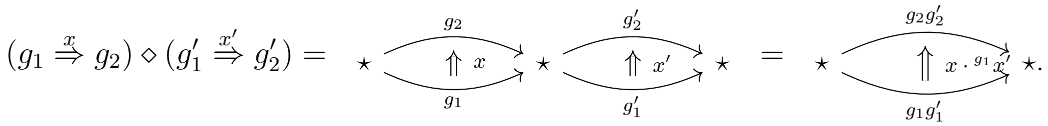
There is a well-known equivalence $\widetilde{(-)}$ from the category of crossed modules and strict crossed module morphisms to the category of strict, small 2-groups and 2-functors [12], [27, §3.3]. This functor assigns to a crossed module $\mathfrak{G}$ the 2-group $\widetilde{\mathfrak{G}}$ with object $\star$, with 1-morphisms $1\mathrm{End}_{\widetilde{\mathfrak{G}}}(\star) = G_1$ and 2-morphisms

$$2\mathrm{Hom}_{\widetilde{\mathfrak{G}}}(g_1, g_2) = \{x \in G_2 \mid \partial(x) g_1 = g_2\}.$$

The vertical composition law in $\widetilde{\mathfrak{G}}$ is illustrated by the diagram



The horizontal composition law is illustrated by



The equivalence class of the 2-group $\widetilde{\mathfrak{G}}$ is determined by the quadruple $(\pi_1(\mathfrak{G}), \pi_2(\mathfrak{G}), \alpha, [\theta])$ where α is an action of $\pi_1(\mathfrak{G}) := \mathrm{coker}(\partial)$ on $\pi_2(\mathfrak{G}) := \ker(\partial)$ and $[\theta] \in H^3(\pi_1(\mathfrak{G}), \pi_2(\mathfrak{G}))$ is the Sinh cohomology class [4, Theorem 8.3.7] (cf. [14, 30]).

A $\mathbb{Z}_2$-graded 2-group is a 2-group morphism $\pi : \mathcal{G} \to \mathbb{Z}_2$. The ungraded 2-group of $\mathcal{G}$ is the locally full subbicategory $\mathcal{G}_0$ on 1-morphisms in $\ker(\pi)$. We also write $\widetilde{(-)}$ for the induced equivalence between the categories of $\mathbb{Z}_2$-graded crossed modules and $\mathbb{Z}_2$-graded groups.

1.4. **Generalized automorphism 2-groups.** Given a bicategory $\mathcal{V}$, denote by $\mathcal{V}^{\mathrm{co}}$ the bicategory obtained from $\mathcal{V}$ by reversing its 2-cells.

We recall a construction of [37, §3.3]. Let $\mathcal{V}$ be a bicategory with weak duality involution, in the sense of Shulman [32, §2]. This is the data of a pseudofunctor $(-)^\circ : \mathcal{V}^{\mathrm{co}} \to \mathcal{V}$, a pseudonatural adjoint equivalence $\mu : 1_{\mathcal{V}} \Rightarrow (-)^\circ \circ ((-)^\circ)^{\mathrm{co}}$ and additional higher coherence data which we do not recall here. Let $V \in \mathcal{V}$. The collection of all equivalences of the form $V \to V$ or $V^\circ \to V$, together with the 2-isomorphisms between them, assembles to a $\mathbb{Z}_2$-graded 2-group $1\mathrm{Aut}^{\mathrm{gen}}_{\mathcal{V}}(V)$, called the generalized automorphism 2-group of V. The monoidal structure $\otimes$ is defined on objects by

$$f_2 \otimes f_1 = f_2 \circ ({}^{\pi(f_2)}f_1 \circ \mu_V^{\delta_{\pi(f_2),\pi(f_1),-1}}),$$

where $\pi(f) \in \mathbb{Z}_2$ is such that $f : {}^{\pi(f)}V \to V$, the symbol ${}^{\pi(f)}(-)$ determines the application of $(-)^\circ$ and

$$\delta_{\pi(f_2),\pi(f_1),-1} = \begin{cases} +1 & \text{if } \pi(f_2) = \pi(f_1) = -1, \\ 0 & \text{otherwise,} \end{cases}$$

while μ_V^0 means that the map μ_V is omitted. The definition of $\otimes$ on morphisms is similar. We omit the definition of the associator, which uses the higher coherence data.

The ungraded 2-group of $1\mathrm{Aut}^{\mathrm{gen}}_{\mathcal{V}}(V)$ is $1\mathrm{Aut}_{\mathcal{V}}(V)$, the weak automorphism 2-group of V, as defined in [4, §8].

1.5. **Real 2-representation theory.** Following [37, §3], we recall the basic definitions of the Real 2-representation theory of 2-groups.

The bicategory of 2-representations of a 2-group $\mathcal{G}$ on a bicategory $\mathcal{V}$ is

$$\mathsf{Rep}_{\mathcal{V}}(\mathcal{G}) = 1\mathrm{Hom}_{\mathsf{Bicat}}(\mathcal{G}, \mathcal{V}),$$

consisting of pseudofunctors, pseudonatural transformations and modifications. A 2-representation of $\mathcal{G}$ is, thus, the datum of an object $V \in \mathcal{V}$ and a 2-group morphism $\mathcal{G} \to 1\mathrm{Aut}_{\mathcal{V}}(V)$. For detailed studies of 2-representation theory, the reader is referred to [5, 14, 6].

Now let $\mathcal{G}$ be a $\mathbb{Z}_2$-graded 2-group. Let $\mathcal{V}$ be a bicategory with weak duality involution. The bicategory of Real 2-representations of $\mathcal{G}$ on $\mathcal{V}$ is

$$\mathsf{RRep}_{\mathcal{V}}(\mathcal{G}) = 1\mathrm{Hom}_{\mathsf{Bicat}_{\mathsf{con}}}(\mathcal{G}, \mathcal{V}).$$

Here $\mathsf{Bicat}_{\mathsf{con}}$ is the tricategory of bicategories with contravariance. We regard $\mathcal{G}$ and $\mathcal{V}$ as bicategories with contravariance [37, §§1.2, 3.3]. The ingredients of $\mathsf{RRep}_{\mathcal{V}}(\mathcal{G})$ are as introduced in [32, §4]:

- objects – contravariance preserving pseudofunctors,
- 1-morphisms – pseudonatural transformations respecting contravariance,
- 2-morphisms – modifications respecting contravariance.

In particular, all 2-morphisms are isomorphisms. In concrete terms, a Real 2-representation of $\mathcal{G}$ on $V \in \mathcal{V}$ is a morphism $\mathcal{G} \to 1\mathrm{Aut}^{\mathrm{gen}}_{\mathcal{V}}(V)$ of $\mathbb{Z}_2$-graded 2-groups.

A symmetric monoidal structure on $\mathcal{V}$ (which commutes with the weak duality involution) induces symmetric monoidal structures on $\mathsf{Rep}_{\mathcal{V}}(\mathcal{G})$ and $\mathsf{RRep}_{\mathcal{V}}(\mathcal{G})$. These monoidal structures are compatible in the sense that the restriction pseudofunctor

$$\mathrm{Res}^{\mathcal{G}}_{\mathcal{G}_0} : \mathsf{RRep}_{\mathcal{V}}(\mathcal{G}) \to \mathsf{Rep}_{\mathcal{V}}(\mathcal{G}_0)$$

is monoidal.

Example. Let Cat be the 2-category of small categories. The assignment of a category to its opposite extends to a strict duality involution $(-)^{\mathrm{op}} : \mathsf{Cat}^{\mathrm{co}} \to \mathsf{Cat}$. A Real 2-representation of a $\mathbb{Z}_2$-graded group G is the data of a category $\mathcal{C}$, equivalences

$$\rho(g) : {}^{\pi(g)}\mathcal{C} \to \mathcal{C}, \qquad g \in G$$

and composition natural isomorphisms

$$\rho_{g_2,g_1} : \rho(g_2) \circ {}^{\pi(g_2)}\rho(g_1) \Longrightarrow \rho(g_2 g_1), \qquad g_i \in G.$$

This data is required to satisfy the associativity constraints

$$\rho_{g_3g_2,g_1} \diamond \left(\rho_{g_3,g_2} \circ {}^{\pi(g_3g_2)}\rho(g_1)\right) = \rho_{g_3,g_2g_1} \diamond \left(\rho(g_3) \circ {}^{\pi(g_3)}\rho^{\pi(g_3)}_{g_2,g_1}\right), \qquad g_i \in G.$$

◁

Example. Let $2\mathsf{Vect}_{\mathbb{K}}$ be the bicategory of finite dimensional Kapranov–Voevodsky 2-vector spaces over $\mathbb{K}$. We use the semi-skeletal model [16, §2.2], [31, §1]. Objects are $[n]$, $n \in \mathbb{Z}_{\geq 0}$. A 1-morphism $[n] \to [m]$ is an $m \times n$ matrix $A = (A_{ij})$ of finite dimensional vector spaces over $\mathbb{K}$. A 2-morphism $u : A \Rightarrow B$ is a collection of $\mathbb{K}$-linear maps $(u_{ij} : A_{ij} \to B_{ij})$. We omit the definition of the various compositions. The bicategory $2\mathsf{Vect}_{\mathbb{K}}$ has a natural symmetric monoidal structure. A weak duality involution on $2\mathsf{Vect}_{\mathbb{K}}$ is defined as follows. The pseudofunctor $(-)^{\circ} : 2\mathsf{Vect}_{\mathbb{K}}^{\mathrm{co}} \to 2\mathsf{Vect}_{\mathbb{K}}$ is given on objects, 1-morphisms and 2-morphisms by

$$[n]^{\circ} = [n], \qquad (A_{ij})^{\circ} = (A_{ij}^{\vee}), \qquad (u_{ij})^{\circ} = (u_{ij}^{\vee}),$$

respectively, where $(-)^{\vee}$ is the $\mathbb{K}$-linear duality functor on the category $\mathsf{Vect}_{\mathbb{K}}$ of finite dimensional vector spaces. The remaining data for the duality involution is induced from the evaluation isomorphism $1_{\mathsf{Vect}_{\mathbb{K}}} \simeq (-)^{\vee} \circ ((-)^{\vee})^{\mathrm{op}}$. Equivalence classes of Real 2-representations of finite groups on $2\mathsf{Vect}_{\mathbb{K}}$ are classified in [37, §5.3]. ◁

1.6. **Real 2-modules.** Let $2\mathcal{M}\mathsf{od}_{\mathbb{K}}$ be the 2-category of 2-modules over $\mathbb{K}$, as defined in [31, §1] (where it is denoted 2-$\mathcal{M}od^{\mathbb{K}}$). Objects are $\mathsf{Vect}_{\mathbb{K}}$-module categories that are $\mathsf{Vect}_{\mathbb{K}}$-module equivalent to A-Mod for some $\mathbb{K}$-algebra A. We do not recall the definitions of 1- and 2-morphisms. A 2-representation of a 2-group $\mathcal{G}$ on $2\mathcal{M}\mathsf{od}_{\mathbb{K}}$ is called a 2-module over $\mathcal{G}$.

In this paper we use a variation of this set-up. Let $\mathcal{A}\mathsf{LG}_{\mathbb{K}}$ be the Morita bicategory:

- objects – $\mathbb{K}$-algebras,
- 1-morphisms $A \to B$ – B-A-bimodules,
- 2-morphisms – bimodule intertwiners.

The composition of 1-morphisms is the tensor product of bimodules. Let $\mathcal{A}\mathsf{lg}_{\mathbb{K}}$ be the subbicategory of finite dimensional algebras, finite dimensional bimodules and intertwiners. Let also $\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$ be the fully dualizable subbicategory of $\mathcal{A}\mathsf{LG}_{\mathbb{K}}$ or, equivalently, the full subbicategory of $\mathcal{A}\mathsf{lg}_{\mathbb{K}}$ spanned by separable algebras. Tensor product of algebras over $\mathbb{K}$ induces symmetric monoidal structures on $\mathcal{A}\mathsf{LG}_{\mathbb{K}}$, $\mathcal{A}\mathsf{lg}_{\mathbb{K}}$ and $\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$.

The 2-representation theories of $\mathcal{G}$ on $2\mathcal{M}\mathsf{od}_{\mathbb{K}}$ and $\mathcal{A}\mathsf{LG}_{\mathbb{K}}$ are equivalent. This follows from the fact that, by Morita theory, equivalences in $2\mathcal{M}\mathsf{od}_{\mathbb{K}}$ can be represented by bimodules. The 2-representation theories of $\mathcal{G}$ on $\mathcal{A}\mathsf{lg}_{\mathbb{K}}$ and $\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$ can therefore be thought of as the finite dimensional and separable 2-module theories of $\mathcal{G}$, respectively.

Recall that each $\mathbb{K}$-algebra morphism $\phi : A \to B$ defines restriction functors from B-modules to A-modules. If M is a right B-module, then M_{ϕ} is a right A-module, equal to M as an abelian group and with right A-module structure

$$m \cdot a = m\phi(a), \qquad m \in M_{\phi},\, a \in A.$$

Similarly, a left B-module N determines a left A-module ${}_{\phi}N$. Starting with the identity bimodule ${}_{B}B_{B}$, we get a *representable* B-A-bimodule B_{ϕ}. We use the representable bimodules to embed $\mathsf{ALG}_{\mathbb{K}}$ as a locally discrete subbicategory of $\mathcal{A}\mathsf{LG}_{\mathbb{K}}$. Also relevant is the locally full subbicategory $\mathcal{A}\mathsf{LG}_{\mathbb{K}}^{\mathrm{rep}}$ of $\mathcal{A}\mathsf{LG}_{\mathbb{K}}$ on representable 1-morphisms. We record the following result for later use.

Lemma 1.1. *Let ϕ and ψ be $\mathbb{K}$-algebra isomorphisms $A \to B$. Then the map*

$$\Upsilon_{\phi}^{\psi} : \{b \in B^{\times} \mid \phi = \partial(b)\psi\} \to 2\mathrm{Hom}_{\mathcal{A}\mathsf{LG}_{\mathbb{K}}}(B_{\phi}, B_{\psi})$$

which sends an element $b \in B^{\times}$ to the map given by right multiplication by b is a bijection onto the subset of 2-isomorphisms.

As explicated in [24, Theorem 5.1], the bicategory $\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$ admits a weak duality involution. The pseudofunctor $(-)^{\circ} : (\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}})^{\mathrm{co}} \to \mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$ is defined on objects by $A^{\circ} = A^{\mathrm{op}}$, on a 1-morphism $M : A \to B$ by $M^{\circ} = \mathrm{Hom}_{\text{Mod-}A}(M, A)$ (viewing the A-B-bimodule on the right hand side as a B^{op}-A^{op}-bimodule), and on a 2-morphism $\phi : M \Rightarrow N$ by $\phi^{\circ} = (-) \circ \phi$. The additional coherence data for $(-)^{\circ}$ and its lift to a weak duality involution are constructed using the separability of algebras and the finite dimensionality of bimodules involved.

Definition. *A Real 2-module over a $\mathbb{Z}_2$-graded 2-group $\mathcal{G}$ is a Real 2-representation of $\mathcal{G}$ on $\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$.*

There is a canonical pseudofunctor $2\mathsf{Vect}_{\mathbb{K}} \to \mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$ which sends the object $[n]$ to the algebra $\mathbb{K}^n$. If $\mathbb{K}$ is separably closed, then this is a monoidal biequivalence which, with $2\mathsf{Vect}_{\mathbb{K}}$ equipped with the weak duality involution of Section 1.5, lifts to a duality biequivalence. An explicit construction of this lifting is given in [24, Theorem 6.3]. In particular, this allows us to conclude that the bicategories of Real 2-representations of $\mathcal{G}$ on $2\mathsf{Vect}_{\mathbb{K}}$ and $\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$ are monoidally biequivalent.

2. Actions of crossed modules on algebras

2.1. Weak $\mathfrak{G}$-algebras.

Let $\mathfrak{H}$ be a crossed module. A weak $\mathfrak{H}$-algebra is a $\mathbb{K}$-algebra A together with a weak morphism of crossed modules $\omega_A : \mathfrak{H} \to \mathrm{AUT}(A)$. There are (at least) two different notions of a weak morphism of crossed modules in the literature, namely those introduced by Noohi[1] [28, Definition 8.4] (N-weak for short) and Rumynin–Wendland [31, §2] (RW-weak). The latter is a particular case of the former.

Let now $\mathfrak{G}$ be a $\mathbb{Z}_2$-graded crossed module. Influenced by the ordinary case, we introduce the following definition.

Definition. *An N-weak $\mathfrak{G}$-algebra is a $\mathbb{K}$-algebra A together with a weak morphism of $\mathbb{Z}_2$-graded crossed modules $\omega_A : \mathfrak{G} \to \mathrm{AUT}^{\mathrm{gen}}(A)$. Explicitly, ω_A is the data of*

- *(i) a function $\omega_3 : G_1 \times G_1 \to A^\times$ that restricts to the identity on $G_1 \times \{e_{G_1}\}$ and $\{e_{G_1}\} \times G_1$,*
- *(ii) a unital $\partial_*\omega_3$-projective $\mathbb{Z}_2$-graded group homomorphism $\omega_1 : G_1 \to \mathrm{Aut}^{\mathrm{gen}}(A)$, that is, a pointed map over $\mathbb{Z}_2$ which satisfies*
$$\omega_1(g_2 g_1) = \partial(\omega_3(g_2, g_1))\omega_1(g_2)\omega_1(g_1), \qquad g_i \in G_1 \tag{1}$$
and
- *(iii) a unital $\partial^*\omega_3$-projective group homomorphism $\omega_2 : G_2 \to A^\times$, that is, a pointed map which satisfies*
$$\omega_2(x_2 x_1) = \omega_3(\partial x_2, \partial x_1)\omega_2(x_2)\omega_2(x_1), \qquad x_i \in G_2. \tag{2}$$

This data is required to satisfy
$$\omega_1 \circ \partial = \partial \circ \omega_2, \tag{3}$$
the non-abelian 2-cocycle condition
$$\omega_3(g_3 g_2, g_1)\omega_3(g_3, g_2) = \omega_3(g_3, g_2 g_1) \cdot {}^{\omega_1(g_3)}\omega_3(g_2, g_1), \qquad g_i \in G_1 \tag{4}$$
and the equivariance condition
$$\omega_2({}^g x) = \omega_3(g\partial x, g^{-1})\omega_3(g, \partial x) \cdot {}^{\omega_1(g)}\omega_2(x)\omega_3(g, g^{-1})^{-1}, \qquad g \in G_1,\ x \in G_2. \tag{5}$$

We write $Z^2(G_1, A^\times_\pi)$ for the set of normalized functions ω_3 which satisfy equation (4). The subscript π indicates that the action of $G_1 \setminus G_0$ on $A^\times$ involves inversion.

We utilize the following (partial) strictifications of the previous definition.

Definition.
- *(i) An RW-weak $\mathfrak{G}$-algebra is an N-weak $\mathfrak{G}$-algebra A in which ω_3 factors through $Z(A)^\times \leq A^\times$.*
- *(ii) A (strict) $\mathfrak{G}$-algebra is an N-weak $\mathfrak{G}$-algebra with trivial ω_3.*

[1] An unrelated definition appears in [27, Definition 8.4], which was later revised in [28, Definition 8.4]. The revised definition also seems to contain a typo; compare with equation (1).

Said differently, an RW-weak $\mathfrak{G}$-algebra is an N-weak $\mathfrak{G}$-algebra in which ω_1 is a $\mathbb{Z}_2$-graded group homomorphism.

The ungraded and graded notions of weak algebras are compatible in the sense that the restriction of an N-weak (resp. RW-weak, strict) $\mathfrak{G}$-algebra ω_A to $\mathfrak{G}_0$ is an N-weak (resp. RW-weak, strict) $\mathfrak{G}_0$-algebra $\omega_A : \mathfrak{G}_0 \to \mathrm{AUT}(A)$.

Definition. *A strict morphism of N-weak $\mathfrak{G}$-algebras $\phi : A \to B$ is a unital algebra morphism which makes the diagrams*

$$\begin{array}{ccc} {}^{\pi(g)}A & \xrightarrow{\pi(g)\phi} & {}^{\pi(g)}B \\ {\scriptstyle \omega_{A,1}(g)}\downarrow & & \downarrow{\scriptstyle \omega_{B,1}(g)} \\ A & \xrightarrow[\phi]{} & B \end{array} \quad , \qquad \begin{array}{ccc} & G_2 & \\ {\scriptstyle \omega_{A,2}}\swarrow & & \searrow{\scriptstyle \omega_{B,2}} \\ A^{\times} & \xrightarrow[\phi_{|A^{\times}}]{} & B^{\times} \end{array}$$

commute for all $g \in G_1$.

It follows directly from the definition that the equality $\phi_* \omega_{A,3} = \omega_{B,3}$ holds.

Let $\mathfrak{H}$ and $\mathfrak{K}$ be crossed modules. Noohi's weak crossed module morphisms are defined so that there is a biequivalence

$$\mathrm{Hom}_{\mathsf{CM}}(\mathfrak{H}, \mathfrak{K}) \simeq \mathrm{Hom}_{\mathsf{Bicat}}(\widetilde{\mathfrak{H}}, \widetilde{\mathfrak{K}}),$$

where the left-hand side is the bicategory of weak crossed module morphisms, transformations and modifications. Strictly speaking, the above biequivalence is not proved in [27] and so we do not use it in the remainder of the paper; see, however [27, Proposition 8.1] and Proposition 4.1 below. Under the above biequivalence, strict crossed module morphisms correspond to strict 2-functors. This, together with the following lemma, explains the categorical meaning of weak and strict $\mathfrak{H}$-algebras.

Lemma 2.1. *For any $\mathbb{K}$-algebra A, there is a biequivalence $\widetilde{\mathrm{AUT}(A)} \simeq 1\mathrm{Aut}_{\mathcal{A}\mathsf{LG}_{\mathbb{K}}^{\mathrm{rep}}}(A)$.*

Proof. This can be proved in the same way as [31, Proposition 2.2]. ∎

Similarly, if A is separable, then one can show that $\mathrm{AUT}^{\mathrm{gen}}(A)$ models the $\mathbb{Z}_2$-graded 2-group $1\mathrm{Aut}^{\mathrm{gen}}_{\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd,rep}}}(A)$. We therefore obtain an analogous categorical interpretation of (weak) $\mathfrak{G}$-algebras.

2.2. **Equivariant objects.** We introduce the notion of an equivariant object of a Real 2-representation on Cat. This clarify some of the constructions which follow.

Let ρ be a Real 2-representation of a $\mathbb{Z}_2$-graded group G on a category $\mathcal{C}$. An equivariant object of ρ is a pair (t, α) consisting of an object $t \in \mathcal{C}$ and isomorphisms $\alpha_g : \rho(g)(t) \to t$, $g \in G$, which make the diagrams[2]

$$\begin{array}{ccc} \rho(g_2 g_1)(t) & \xleftarrow{\rho_{g_2,g_1,t}} & \rho(g_2)(\rho(g_1)(t)) \\ {\scriptstyle \alpha_{g_2 g_1}}\downarrow & & \downarrow{\scriptstyle \rho(g_2)(\alpha_{g_1}^{\pi(g_2)})} \\ t & \xleftarrow[\alpha_{g_2}]{} & \rho(g_2)(t) \end{array} \quad , \qquad g_i \in G$$

[2]For notational simplicity, we omit the symbol $(-)^{\mathrm{op}}$ in this diagram.

commute. A morphism of equivariant objects $\phi : (t, \alpha) \to (s, \beta)$ is an isomorphism $\phi : t \to s$ which makes the diagrams

$$\begin{array}{ccc} \rho(g)(t) & \xrightarrow{\rho(g)(\phi^{\pi(g)})} & \rho(g)(s) \\ {\scriptstyle \alpha_g}\downarrow & & \downarrow{\scriptstyle \beta_g} \\ t & \xrightarrow[\phi]{} & s \end{array}, \qquad g \in G$$

commute. This defines a groupoid of equivariant objects of ρ.

When G is trivially graded, there are two possible definitions. We need not require the map ϕ to be an isomorphism. In this case the definition reduces to the standard category of equivariant objects of ρ.

2.3. **Morita bicategories of weak $\mathfrak{G}$-algebras.** In this section we introduce a more flexible notion of a morphism of N-weak $\mathfrak{G}$-algebras. We begin with some preliminary material. We use left and right twists of bimodules by (anti-)automorphisms; see Section 1.6. Note that $(-)^{\circ}$ is defined with the separability assumption.

Lemma 2.2. *Let A and B be $\mathbb{K}$-algebras. For each B-A-bimodule M and $\psi \in \mathrm{Aut}(A)$, $\phi \in \mathrm{Aut}(B)$, there is a B^{op}-A^{op}-bimodule isomorphism*

$$({}_{\phi}M_{\psi})^{\circ} \xrightarrow{\sim} {}_{\phi^{\mathrm{op}}}(M^{\circ})_{\psi^{\mathrm{op}}}, \qquad \mathfrak{m} \mapsto \big(m \mapsto \psi(\mathfrak{m}(m))\big).$$

Let $\mathfrak{G}$ be a $\mathbb{Z}_2$-graded crossed module. Let A and B be separable N-weak $\mathfrak{G}$-algebras. The category $1\mathrm{Hom}_{\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}}(A, B)$ of 1-morphisms of the underlying $\mathbb{K}$-algebras inherits the structure of a Real 2-representation λ of G_1. Explicitly, an element $g \in G_1$ acts by the functor $\lambda(g)$ which sends a B-A-bimodule M to[3]

$$\lambda(g)(M) = {}_{\omega_{B,1}(g)^{-1}}\left({}^{\pi(g)}M\right)_{\omega_{A,1}(g)^{-1}}.$$

On morphisms $\lambda(g)$ acts as ${}^{\pi(g)}(-)$; the (anti-)automorphism twists $\omega_{?,1}(g)^{-1}$ act trivially. In particular, $\lambda(g)$ is contravariant precisely when $\pi(g) = -1$. The component at M of the coherence natural transformation

$$\lambda_{g_2,g_1} : \lambda(g_2) \circ {}^{\pi(g_2)}\lambda(g_1) \Longrightarrow \lambda(g_2 g_1), \qquad g_i \in G,$$

when viewed as a $\mathbb{K}$-linear map $\lambda_{g_2,g_1,M} : {}^{\pi(g_2g_1)}M \to M$, is given by left multiplication by ${}^{\omega_{B,1}(g_2g_1)^{-1}}\omega_{B,3}(g_2, g_1)^{-1}$ and right multiplication by ${}^{\omega_{A,1}(g_2g_1)^{-1}}\omega_{A,3}(g_2, g_1)$, which we write as $\partial({}^{\omega_{?,1}(g_2g_1)^{-1}}\omega_{?,3}(g_2, g_1)^{-1})$. Implicit in this description of $\lambda_{g_2,g_1,M}$ is, when $\pi(g_2) = -1$, the use of Lemma 2.2 and, when $\pi(g_1) = \pi(g_2) = -1$, the use of the evaluation isomorphism $\mathrm{ev}_M : M \to M^{\circ\circ}$.

Let us unpack the datum of an equivariant object (M, ω_M) of $1\mathrm{Hom}_{\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}}(A, B)$. First, we have a B-A-bimodule M. Second, for each $g \in G_1$, we have a $\mathbb{K}$-linear isomorphism $\omega_M(g) : {}^{\pi(g)}M \to M$ which satisfies

$$\omega_M(g)(bma) = \omega_{B,1}(g)(b)\omega_M(g)(m)\omega_{A,1}(g)(a), \qquad a \in {}^{\pi(g)}A,\ m \in {}^{\pi(g)}M,\ b \in {}^{\pi(g)}B. \tag{6}$$

Moreover, these isomorphisms are required to satisfy

$$\omega_M(g_2g_1) = \partial(\omega_{?,3}(g_2, g_1)) \circ \omega_M(g_2) \circ {}^{\pi(g_2)}\omega_M(g_1)^{\pi(g_2)}, \qquad g_i \in G. \tag{7}$$

[3]For readability (and unlike Lemma 2.2), we henceforth omit the notation $(-)^{\mathrm{op}}$ on morphisms.

A 1-morphism $\phi : (M, \omega_M) \to (N, \omega_N)$ is a B-A-bimodule isomorphism which is G_1-equivariant, in the sense that the diagrams

$$\begin{array}{ccc} \lambda(g)(M) & \xrightarrow{\;{}^{\pi(g)}\phi^{\pi(g)}\;} & \lambda(g)(N) \\ {\scriptstyle \omega_M(g)}\downarrow & & \downarrow{\scriptstyle \omega_N(g)} \\ M & \xrightarrow[\phi]{} & N \end{array} \quad , \qquad g \in G_1$$

commute.

Using the above notation, we define a bicategory $\mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$:

- objects – separable N-weak $\mathfrak{G}$-algebras,
- 1-morphism category $1\mathrm{Hom}_{\mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}}(A, B)$ – the full subcategory of the equivariant objects of $1\mathrm{Hom}_{\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}}(A, B)$ spanned by pairs (M, ω_M) which, in addition, satisfy $\omega_M(e_{G_1}) = \mathrm{id}_M$ and
$$\omega_M(\partial x) = \partial(\omega_{?,2}(x)), \qquad x \in G_2. \tag{8}$$

The horizontal composition of 1-morphisms and the associativity data of $\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$ extend naturally to $\mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$.

Denote by $\mathfrak{G}\text{-}RW\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$ and $\mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$ the full subbicategories of $\mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$ spanned by RW-weak and strict $\mathfrak{G}$-algebras, respectively.

Example. A strict morphism $\phi : A \to B$ of separable N-weak $\mathfrak{G}$-algebras defines in a canonical way a 1-morphism $B_\phi : A \to B$ in $\mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$. ◁

Remarks. (i) A more conceptual definition of $1\mathrm{Hom}_{\mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}}(A, B)$ is as the equivariant groupoid of $1\mathrm{Hom}_{\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}}(A, B)$, viewed as a Real 2-representation of $\widetilde{\mathfrak{G}}$. We opt to avoid defining equivariant objects for 2-groups.

(ii) When $\mathfrak{G}$ is trivially graded, we do not require the existence of evaluation isomorphisms. We can therefore define a larger bicategory $\mathfrak{G}\text{-}N\mathcal{A}\mathsf{LG}_{\mathbb{K}}$. In this way we connect with the $\mathfrak{G}$-equivariant Morita contexts of [31, §2].

Definition. *Equivalence in the bicategory $\mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$ is called $\mathfrak{G}$-Morita equivalence.*

We describe some additional structures on $\mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$. Let A and B be N-weak $\mathfrak{G}$-algebras. Then the direct sum $A \oplus B$ has an obvious N-weak $\mathfrak{G}$-algebra structure $A \boxplus B$. Similarly, the tensor product $A \otimes_{\mathbb{K}} B$ is an N-weak $\mathfrak{G}$-algebra $A \boxtimes B$ with structure maps $\omega_{A\boxtimes B,i} = \omega_{A,i} \otimes \omega_{B,i}$, $i = 1, 2, 3$. This extends to a symmetric monoidal structure $\boxtimes$ on $\mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$. Both $\boxplus$ and $\boxtimes$ restrict to $\mathfrak{G}\text{-}RW\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$ and $\mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$.

Finally, given a strict $\mathfrak{G}$-algebra A, its dual $\mathfrak{G}$-algebra $A^\vee$ (see [31, §3]) is defined so that its underlying $\mathbb{K}$-algebra is A^{op} and its structure maps are
$$\omega_{A^\vee,1}(g) = \omega_{A,1}(g)^{\mathrm{op}}, \qquad \omega_{A^\vee,2}(x) = \omega_{A,2}(x)^{-1}.$$
An interested reader can generalize the construction of $A^\vee$ to the case of an N-weak $\mathfrak{G}$-algebra A.

2.4. **Realizability of twisted 2-cocycles.** Let G be a $\mathbb{Z}_2$-graded group. Fix an integer $t \geq 1$ and let $A = \mathbb{K}^t$ with RW-weak G-algebra structure ω_A. Then $\omega_{A,1} : G \to \mathrm{Aut}^{\mathrm{gen}}(A) \cong \mathbb{S}_t \times \mathbb{Z}_2$ is a group homomorphism. The datum ω_A determines a Real 2-representation λ of G on A-Mod_{fd}, the category of finite dimensional left A-modules, considered as an object of the 2-category Cat with duality involution $(-)^{\mathrm{op}}$. The action functors are defined on objects by

$$\lambda(g)(M) = {}_{\omega_{A,1}(g)^{-1}}\left({}^{\pi(g)}M\right), \qquad g \in G$$

and the coherence natural isomorphisms

$$\lambda_{g_2,g_1} : \lambda(g_2) \circ {}^{\pi(g_2)}\lambda(g_1) \Longrightarrow \lambda(g_2 g_1), \qquad g_i \in G$$

are left multiplication by ${}^{\omega_{A,1}(g_2g_1)^{-1}}\omega_{A,3}(g_2,g_1)^{-1}$ (cf. Section 2.3).

Definition. *A realization of $\omega_{A,3} \in Z^2(G, A^{\times}_{\pi})$ is an equivariant object (M,α) of λ whose underlying A-module M is faithful.*

It is convenient to interpret the equivariant object (M,α) as the pair $(M, \mathfrak{p}_M)$, where $\mathfrak{p}_M(g) \coloneqq \alpha_g({}_{\omega_1(g)}M)$. Then, for each $g \in G$, the $\mathbb{K}$-linear map $\mathfrak{p}_M(g) : {}^{\pi(g)}M \to M$ is required to satisfy $\mathfrak{p}_M(g)(am) = \omega_{A,1}(g)(a)\mathfrak{p}_M(g)(m)$. The coherence constraints on $\mathfrak{p}_M$ read

$$\mathfrak{p}_M(g_2g_1) = \omega_{A,3}(g_2,g_1)\mathfrak{p}_M(g_2) \circ {}^{\pi(g_2)}\mathfrak{p}_M(g_1)^{\pi(g_2)} \circ \mathrm{ev}_M^{\delta_{\pi(g_2),\pi(g_1),-1}}, \qquad g_i \in G.$$

In particular, for trivially graded G, we recover realizability as introduced in [31, §3].

When G is trivially graded, $(M, \mathfrak{p}_M)$ is simply a module over the skew group algebra $A\sharp_{\omega_A}G$. Using this perspective, we see that if G is finite, then $\omega_{A,3} \in Z^2(G, A^{\times}_{\pi})$ is realizable. For example, we can take M to be the left regular representation of $A\sharp_{\omega_A}G$. We can then conclude realizability in the $\mathbb{Z}_2$-graded case by taking the hyperbolic representation on the left regular representation of $A\sharp_{\omega_A}G_0$.

The following result is used repeatedly in the remainder of the paper.

Proposition 2.3. *A realization $(M, \mathfrak{p}_M)$ of $\omega_{A,3} \in Z^2(G, A^{\times}_{\pi})$ determines a $\mathbb{Z}_2$-graded group homomorphism $\mathfrak{a} : G \to \mathrm{Aut}^{\mathrm{gen}}(\mathrm{End}_A(M))$, $g \mapsto \mathfrak{a}_g$, by the formula*

$$\mathfrak{a}_g(\phi) = \mathfrak{p}_M(g) \circ {}^{\pi(g)}\phi \circ \mathfrak{p}_M(g)^{-1}, \qquad g \in G,\ \phi \in \mathrm{End}_A(M)$$

where ${}^{\pi(g)}(-)$ determines the application of A-linear duality $(-)^{\circ}$.

Proof. Since $\mathfrak{p}_M(g)^{-1}(am) = \omega_{A,1}(g)^{-1}(a)\mathfrak{p}_M(g)^{-1}(m)$, the map $\mathfrak{a}_g(\phi) : M \to M$ is again A-linear. It is clear that $\mathfrak{a}_g \in \mathrm{Aut}(\mathrm{End}_A(M))$ when $\pi(g) = 1$. If $\pi(g) = -1$, then

$$\mathfrak{a}_g(\phi_2 \circ \phi_1) \;=\; \mathfrak{p}_M(g) \circ (\phi_2 \circ \phi_1)^{\circ} \circ \mathfrak{p}_M(g)^{-1} = \mathfrak{a}_g(\phi_1) \circ \mathfrak{a}_g(\phi_2).$$

Hence, we indeed have a map $\mathfrak{a} : G \to \mathrm{Aut}^{\mathrm{gen}}(\mathrm{End}_A(M))$. To verify that $\mathfrak{a}$ is a group homomorphism, suppose, for instance, that $\pi(g_1) = \pi(g_2) = -1$. We compute

$$\begin{aligned}
\mathfrak{a}_{g_2}(\mathfrak{a}_{g_1}(\phi)) &= \mathfrak{p}_M(g_2) \circ \mathfrak{p}_M(g_1)^{-\circ} \circ \phi^{\circ\circ} \circ \mathfrak{p}_M(g_1)^{\circ} \circ \mathfrak{p}_M(g_2)^{-1} \\
&= \omega_{A,3}(g_2,g_1)^{-1}\mathfrak{p}_M(g_2g_1) \circ \mathrm{ev}_M^{-1} \circ \phi^{\circ\circ} \circ \mathrm{ev}_M \circ \mathfrak{p}_M(g_2g_1)^{-1}\omega_{A,3}(g_2,g_1) \\
&= \mathfrak{p}_M(g_2g_1) \circ \phi \circ \mathfrak{p}_M(g_2g_1)^{-1} = \mathfrak{a}_{g_2g_1}(\phi).
\end{aligned}$$

The other cases are similar. □

Remark. This section can be retold for N-weak G-algebras, but we do not require that level of generality.

2.5. **Strictification of weak $\mathfrak{G}$-algebras.** In this section we prove that, subject to a realizability condition, every split semisimple N-weak $\mathfrak{G}$-algebra is $\mathfrak{G}$-Morita equivalent to a strict $\mathfrak{G}$-algebra. When the $\mathbb{Z}_2$-grading of $\mathfrak{G}$ is trivial this result is known [31, Corollary 3.3], although the proof there contains a gap. We provide a complete proof in this section, which covers also the $\mathbb{Z}_2$-graded generalization. Our proof is more conceptual and is different from the proof in [31], even in the ordinary case.

Let A be a split semisimple N-weak $\mathfrak{G}$-algebra. The Artin–Wedderburn theorem asserts that there is a $\mathbb{K}$-algebra decomposition

$$A \simeq \bigoplus_{n \geq 1} M_n(\mathbb{K})^{\oplus t_n}.$$

Each summand $A_n := M_n(\mathbb{K})^{\oplus t_n}$ is an N-weak $\mathfrak{G}$-subalgebra and $A \simeq \boxplus_{n\geq 1} A_n$. Let us consider the case $A = M_n(\mathbb{K})^{\oplus t}$. There are isomorphisms

$$\mathrm{Aut}(A) \simeq PGL_n(\mathbb{K})^{\times t} \rtimes \mathbb{S}_t, \qquad \mathrm{Aut}^{\mathrm{gen}}(A) \simeq \mathrm{Aut}(A) \rtimes \mathbb{Z}_2.$$

There is a canonical choice of a generator $s \in \mathbb{Z}_2 \leq \mathrm{Aut}^{\mathrm{gen}}(A)$: the transposition $s(a_1, \dots, a_t) = (a_1^T, \dots, a_t^T)$. Its action on $\mathrm{Aut}(A)$ is the inverse transpose:

$${}^{s}\big((a_1, \dots, a_t), \tau\big) = \big((a_1^{-1})^T, \dots, (a_t^{-1})^T), \tau\big).$$

The composition

$$\sigma : G_1 \xrightarrow{\omega_{A,1}} \mathrm{Aut}^{\mathrm{gen}}(A) \twoheadrightarrow \mathbb{S}_t$$

is a group homomorphism. Indeed, $\omega_{A,1}$ fails to be a homomorphism by conjugation by elements of $GL_n(\mathbb{K})^{\times t}$, which is not seen at the level of permutations. Choose a lift Λ of $\omega_{A,1}$ along the quotient $GL_n(\mathbb{K})^{\times t} \rtimes \mathbb{S}_t \to PGL_n(\mathbb{K})^{\times t} \rtimes \mathbb{S}_t$. This determines a function $\mu_\Lambda : G_1 \times G_1 \to (\mathbb{K}^\times)^t$ via the equation

$$\Lambda(g_2 g_1) = \mu_\Lambda(g_2, g_1) \omega_{A,3}(g_2, g_1) \Lambda(g_2) ({}^{\pi(g_2)}\Lambda(g_1)^{\pi(g_2)}).$$

Using the 2-cocycle condition on $\omega_{A,3}$, we find that $\mu_\Lambda \in Z^2(G_1, (\mathbb{K}^\times)^t_\pi)$, where $\mathbb{K}^t$ is viewed as a G_1-algebra via σ. It is straightforward to verify that different choices of Λ lead to cohomologous 2-cocycles. In this way, we attach a cohomology class $[\mu_n] \in H^2(G_1, (\mathbb{K}^\times)^{t_n}_\pi)$ to $M_n(\mathbb{K})^{\oplus t_n} \trianglelefteq A$.

Theorem 2.4. *Let $\mathfrak{G}$ be a $\mathbb{Z}_2$-graded crossed module, A a split semisimple N-weak $\mathfrak{G}$-algebra. If each cohomology class $[\mu_n] \in H^2(G_1, (\mathbb{K}^\times)^{t_n}_\pi)$, $n \geq 1$, is realizable, then there exists a $\mathfrak{G}$-algebra B which is $\mathfrak{G}$-Morita equivalent to A.*

Proof. By the discussion preceding the theorem, it suffices to consider the case $A = M_n(\mathbb{K})^{\oplus t}$. Fix a lift Λ of $\omega_{A,1}$ with associated cocycle $\mu = \mu_\Lambda \in Z^2(G_1, (\mathbb{K}^\times)^t_\pi)$. The definition of Λ implies the equality

$$\omega_{A,1}(g)(a) = \Lambda(g)({}^{\pi(g)}a)\Lambda(g)^{-1}, \qquad g \in G_1, \ a \in A.$$

Equation (3) implies that $\omega_{A,2}$ satisfies

$$\Lambda(\partial x) \circ a \circ \Lambda(\partial x)^{-1} = \omega_{A,2}(x) a \omega_{A,2}(x)^{-1}, \qquad x \in G_2, \ a \in A.$$

Hence, there exists a function $\gamma : G_2 \to (\mathbb{K}^\times)^t$ such that

$$\omega_{A,2}(x) = \gamma(x)\Lambda(\partial x), \qquad x \in G_2$$

which, by equation (2), satisfies

$$\gamma(x_2 x_1) = \mu(\partial x_2, \partial x_1)^{-1} \gamma(x_2)\gamma(x_1). \tag{9}$$

By the realizability assumption, there exists a μ^{-1}-projective Real representation (U, η_U) of G_1. Put $B = \mathrm{End}_{\mathbb{K}^t}(U)$ with the G_1-action of Proposition 2.3. Define

$\omega_{B,2} : G_2 \to \mathrm{Aut}_{\mathbb{K}^t}(U)$ by $\omega_{B,2}(x) = \gamma(x)^{-1}\eta_U(\partial x)$. Equation (9) implies that this makes B into a $\mathfrak{G}$-algebra.

Let $V = (\mathbb{K}^n)^{\oplus t}$, viewed as a Real representation of G_1 via Λ. Let $M = V \otimes_{\mathbb{K}} U$ with A-$B^\vee$-bimodule structure $a \cdot v \otimes u \cdot b = av \otimes bu$. Here $B^\vee$ denotes the $\mathfrak{G}$-algebra dual to B; see Section 2.3. For each $g \in G_1$, define a $\mathbb{K}$-linear map $\omega_M : {}^{\pi(g)}M \to M$ by

$$\omega_M(g)(v \otimes u) = \Lambda(g)(v) \otimes \eta_U(g)(u).$$

Then we have

$$\begin{aligned} \omega_M(g)(a \cdot v \otimes u \cdot b) &= \Lambda(g)({}^{\pi(g)}av) \otimes \eta_U(g)({}^{\pi(g)}bu) \\ &= \Lambda(g)({}^{\pi(g)}a)\Lambda(g)^{-1}\Lambda(g)(v) \otimes \eta_U(g)({}^{\pi(g)}a)\eta_U(g)^{-1}\eta_U(g)(u) \\ &= \omega_{A,1}(g)(a) \cdot \omega_M(g)(v \otimes u) \cdot \omega_{B,1}(g)(b), \end{aligned}$$

so that equation (6) is satisfied. Moreover, since $\omega_{B,3}$ is trivial,

$$\partial(\omega_{?,3}(g_2, g_1))\omega_M(g_2)({}^{\pi(g_2)}\omega_M(g_1)^{\pi(g_2)}(v \otimes u))$$

is equal to

$$\begin{aligned} &\omega_{A,3}(g_2, g_1)\Lambda(g_2)({}^{\pi(g_2)}\Lambda(g_1)^{\pi(g_2)}(v)) \otimes \eta_U(g_2)({}^{\pi(g_2)}\eta_U(g_1)^{\pi(g_2)}(u)) \\ = \ &\mu(g_2, g_1)^{-1}\Lambda(g_2g_1)(v) \otimes \mu(g_2, g_1)\eta_U(g_2g_1)(u) \\ = \ &\Lambda(g_2g_1)(v) \otimes \eta_U(g_2g_1)(u) = \omega_M(g_2g_1)(v \otimes u), \end{aligned}$$

so that equation (7) is satisfied. Finally, we have

$$\omega_M(\partial x)(v \otimes u) = \Lambda(\partial x)(v) \otimes \eta_U(\partial x)(u) = \gamma(x)^{-1}\omega_{A,2}(x)(v) \otimes \eta_U(\partial x)(u),$$

so that equation (8) is satisfied. Hence $(M, \omega_M) : B^\vee \to A$ is a 1-morphism in $\mathfrak{G}$-$N\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$. The bimodule M is clearly an equivalence in $\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$. This implies that (M, ω_M) is also an equivalence. □

2.6. **Strictification of 2-groups.** The following proposition is an alternative version of Theorem 2.4, where the strictification alters the crossed module representing the 2-group, instead of altering the algebra.

Proposition 2.5. *Let $\mathfrak{G}$ be a $\mathbb{Z}_2$-graded crossed module and (A, ω) an N-weak $\mathfrak{G}$-algebra. There exists a $\mathbb{Z}_2$-graded crossed module $\mathfrak{H}$, a strict homomorphism of $\mathbb{Z}_2$-graded crossed modules $\varphi : \mathfrak{H} \to \mathfrak{G}$ and a strict $\mathfrak{H}$-algebra structure $\psi : \mathfrak{H} \to \mathrm{AUT}^{\mathrm{gen}}(A)$ such that*

- *(i) the induced homomorphism of 2-groups $\widetilde{\varphi} : \widetilde{\mathfrak{H}} \to \widetilde{\mathfrak{G}}$ is an equivalence, and*
- *(ii) the N-weak $\mathfrak{H}$-algebra $(A, \omega\varphi)$ and the strict $\mathfrak{H}$-algebra (A, ψ) are $\mathfrak{H}$-Morita equivalent.*

Proof. Define constituents H_i, $i = 1, 2$ of the crossed module $\mathfrak{H}$ as extensions of G_i by the multiplicative group $A^\times$, using the systems of factors that arise from the N-weak structure:

$$1 \to A^\times \to H_i \xrightarrow{\varphi_i} G_i \to 1.$$

The homomorphism φ is (φ_1, φ_2), i.e., $\varphi|_{H_i} = \varphi_i$. Writing $H_i = A^\times \times G_i$ as sets, we can express the homomorphism $\psi = (\psi_1, \psi_2)$ as

$$\psi_1(a, g) = \mathrm{Ad}_A(a)\omega_1(g), \qquad \psi_2(a, x) = a\omega_2(x), \tag{10}$$

where, as usual, $a, b \in A^\times$, $g, h \in G_1$ and $x, y \in G_2$. The rest of the proof is devoted to verifying the technical details.

1) Groups H_i: the multiplications for H_1 and H_2 are slight variations of the standard product defined by the system of factors:

$$(a,g)\cdot(b,h) \coloneqq (a\omega_1(g)(b)\omega_3(g,h)^{-1}, gh), \quad (a,x)\cdot(b,y) \coloneqq (a({}^{\omega_2(x)}b)\omega_3(\partial x,\partial y)^{-1}, xy). \tag{11}$$

Associativity follows from the cocycle condition. The inverses are

$$(a,g)^{-1} = (\omega_3(g^{-1},g)\omega_1(g^{-1})(a^{-1}), g^{-1}), \quad (a,x)^{-1} = (\omega_3(\partial x^{-1},\partial x)({}^{\omega_2(x^{-1})}(a^{-1})), x^{-1}).$$

2) Action of H_1 on H_2: define the action by

$${}^{(a,g)}(b,x) \coloneqq (a\omega_1(g)(b)\omega_3(g,\partial x)^{-1}\omega_1(g\partial x)(\omega_3(g^{-1},g))\omega_3(g\partial x,g^{-1})^{-1}({}^{\omega_2({}^g x)}(a^{-1})), {}^g x). \tag{12}$$

We have derived this formula by assuming that G_2 is a normal subgroup of G_1 and computing $((a,g)\cdot(b,x))\cdot(a,g)^{-1}$. Observe that the evaluation of $(a,g)\cdot((b,x)\cdot(a,g)^{-1})$ produces the same formula (after a longer calculation, utilizing the cocycle condition). For clarity we simplify the notation by $\omega=\omega_3$ and dropping ω_1 and ω_2:

$${}^{(a,g)}(b,x) = (a\,({}^g b)\,\omega(g,\partial x)^{-1}\,({}^{g\partial x}\omega(g^{-1},g))\,\omega(g\partial x,g^{-1})^{-1}\,({}^{gx}(a^{-1})), {}^g x).$$

We need to verify that each element of H_1 acts by a group endomorphism. It suffices to verify this for elements of G_i and $A^\times$ separately. Eight separate verifications are required but it is easier due to the complexity of equation (12). Here we show one of them, leaving the remainder to an interested reader:

$$\begin{aligned}{}^{(a,e)}(1,x)\cdot{}^{(a,e)}(b,e) = (a({}^x(a^{-1})),x)(aba^{-1},e) = (a({}^x(a^{-1}))^x(aba^{-1}),x) = \\ (a({}^x(b))^x(a^{-1}),x) ={}^{(a,e)}({}^x b,x) = {}^{(a,e)}((1,x)\cdot(b,e)).\end{aligned}$$

Further we need to verify that it is actually an action. Again, verifying this for the elements of G_i and $A^\times$ separately requires eight separate verifications. We perform just one of them as an illustration. We need to show that

$${}^{(1,g)}({}^{(a,e)}(1,x)) = ({}^g a\,{}^g({}^x(a^{-1}))\omega(g,\partial x)^{-1\,g\partial x}\omega(g^{-1},g)\omega(g\partial x,g^{-1})^{-1}, {}^g x)$$

is equal to

$${}^{(1,g)(a,e)}(1,x) = {}^{({}^g a,g)}(1,x) = ({}^g a\omega(g,\partial x)^{-1\,g\partial x}\omega(g^{-1},g)\omega(g\partial x,g^{-1})^{-1\,gx}({}^g(a^{-1})), {}^g x).$$

Using ${}^g({}^h a) = \omega(g,h)^{-1\,gh}a\,\omega(g,h)$ (equation (2)), we are left to verify that

$$\underline{\omega(g,\partial x)^{-1}}{}^{g\partial x}(a^{-1})\underline{\omega(g,\partial x)\omega(g,\partial x)^{-1}}{}^{g\partial x}\omega(g^{-1},g)\omega(g\partial x,g^{-1})^{-1}$$

is equal to

$$\underline{\omega(g,\partial x)^{-1}}\;\underline{{}^{g\partial x}\omega(g^{-1},g)\omega(g\partial x,g^{-1})^{-1}\omega(g\partial(x)g^{-1},g)^{-1}}{}^{g\partial x}(a^{-1})\omega(g\partial(x)g^{-1},g).$$

We can cancel all underlined parts, for instance, the three terms in the second expression is equation (4) with $g_3 = g\partial(x)$, $g_2 = g^{-1}$ and $g_1 = g$. Hence, we need to see that the equality

$${}^{g\partial x}(a^{-1})^{g\partial x}\omega(g^{-1},g)\omega(g\partial x,g^{-1})^{-1} \stackrel{?}{=} {}^{g\partial x}(a^{-1})\omega(g\partial(x)g^{-1},g)$$

holds. Again this follows from equation (4) with $g_3 = g\partial(x)$, $g_2 = g^{-1}$ and $g_1 = g$.

3) Differential for $\mathfrak{H}$ and the Peiffer identity: define the differential by

$$\partial(a,x) \coloneqq (a,\partial x).$$

The similarities in the definitions of the products in H_1 and H_2 (formula (11)) ensure that ∂ is a group homomorphism. The Peiffer identity holds automatically, due to the way we have derived formula (12) in **2)**.

4) Homomorphism φ: define $\varphi_i : H_i \to G_i$ by

$$\varphi_1(a,g) = g, \qquad \varphi_2(a,x) = x.$$

It is easy to see that $\varphi = (\varphi_1, \varphi_2)$ is a strict homomorphism of crossed modules. Notice that $\pi_i(\mathfrak{H}) = \{e\} \times \pi_i(\mathfrak{G})$, $i = 1, 2$, and $\pi_i(\varphi) : \pi_i(\mathfrak{H}) \to \pi_i(\mathfrak{G})$ are identities.

5) Grading: if $\pi : G_1 \to \mathbb{Z}_2$ is the grading on $\mathfrak{G}$, a grading on $\mathfrak{H}$ is given by $\pi \circ \varphi_1 : (a,g) \mapsto \pi(g)$. Clearly, $\varphi = (\varphi_1, \varphi_2)$ is a homomorphism of graded crossed modules.

6) Homomorphism ψ: it is defined above (formula (10)). Let us verify that ψ_2 is a homomorphism:

$$\psi_2((a,x)(b,y)) = \psi_2((a\,{}^x b\omega_1(\partial x, \partial y)^{-1}, xy)) = a\,{}^x b\omega_1(\partial x, \partial y)^{-1}\omega_2(xy) =$$
$$a\omega_2(x)b\omega_2(x)^{-1}\omega_2(x)\omega_2(y) = a\omega_2(x)b\omega_2(y) = \psi_2(a,x)\psi_2(b,y).$$

The verification for ψ_1 is similar. If $\pi(g) = 1$, it is identical. If $\pi(g) = -1$, we denote $\circ^{-1}$ by $\bullet$. The key observation is $\omega_1(g) \bullet \omega_1(g) = \mathrm{id}_A$ so that

$$\psi_1((a,g)(b,h)) = \mathrm{Ad}(a) \circ \mathrm{Ad}({}^g b) \circ \mathrm{Ad}(\omega_1(g,h)^{-1}) \circ \omega_1(gh) = \mathrm{Ad}(a)\circ$$
$$\omega_1(g) \bullet \mathrm{Ad}(b) \circ \omega_1(g^{-1}) \bullet \omega_1(g) \bullet \omega_1(h) = \mathrm{Ad}(a) \circ \omega_1(g) \bullet \mathrm{Ad}(b) \circ \omega_1(h),$$

which is equal to $\psi_1(a,g)\psi_1(b,h)$.

7) Morita-equivalence: It suffices to observe that the identity bimodule $(M = {}_A A_A, \theta_M)$ with

$$\theta_M(a,g)(m) = a\omega_1(g)(m)$$

yields the Morita equivalence of $\mathfrak{H}$-algebras $(A, \omega\varphi) \to (A, \psi)$. Let us carry out the necessary verifications:

$$\theta_M(a,g)(bmc) = a\omega_1(g)(bmc) = a\omega_1(g)(b)\omega_1(g)(m)\omega_1(g)(c) =$$
$$\mathrm{Ad}_A(a)(\omega_1(g)(b))a\omega_1(g)(m)\omega_1(g)(c) = \psi_1(a,g)(b)\theta_M(a,g)(m)\omega_1(\varphi_1(a,g))(c)$$

and

$$\theta_M\big((a,g)(b,h)\big)(m) = a\,{}^g b\omega_3(g,h)^{-1}\omega_1(gh)(m) =$$
$$a\,{}^g b\omega_3(g,h)^{-1}\mathrm{Ad}\big(\omega_3(g,h)\big)\big(\omega_1(g)(\omega_1(h)(m))\big) = a\,{}^g b\omega_1(g)\big(\omega_1(h)(m)\big)\omega_3(g,h)^{-1}$$
$$= \partial\big(\omega_{?,3}((a,g),(b,h))\big)\big(\theta_M(a,g)\big(\theta_M(b,h)(m)\big)\big).$$

$\square$

Proposition 2.5 reduces all computations with Real 2-modules of a 2-group $\mathcal{G}$ to manipulations with strict $\mathfrak{G}$-algebras, albeit for different crossed modules. For instance, take two 2-modules V and W. A straightforward variation of Proposition 2.5 yields a crossed module $\mathfrak{G}$ and strict $\mathfrak{G}$-algebras A and B that realize V and W. This allows us to define $V^\vee$, $V \boxplus W$ and $V \boxtimes W$ as in the end of Section 2.3.

Problem 2.6. *Characterize those crossed modules $\mathfrak{G}$ such that any N-weak $\mathfrak{G}$-algebra is $\mathfrak{G}$-Morita-equivalent to a strict $\mathfrak{G}$-algebra. Characterize those 2-groups $\mathcal{G}$ which admit a crossed module realization $\mathcal{G} \cong \widetilde{\mathfrak{G}}$ such that $\mathfrak{G}$ satisfies the property in the previous sentence.*

3. Induction of $\mathfrak{G}$-algebras

In this section we define and study induction for $\mathfrak{G}$-algebras, in both the ordinary and $\mathbb{Z}_2$-graded settings. In view of Theorem 2.4, Proposition 2.5 and our later applications to Real 2-representation theory, we restrict our attention to strict algebras.

3.1. **The ordinary case.** Let $\mathfrak{G}$ be a crossed module with a crossed submodule $\mathfrak{H}$. There is an associated restriction pseudofunctor $\mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}} : \mathfrak{G}\text{-}\mathcal{A}\mathsf{LG}_{\mathbb{K}} \to \mathfrak{H}\text{-}\mathcal{A}\mathsf{LG}_{\mathbb{K}}$. The goal of this section is to define an induction pseudofunctor $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} : \mathfrak{H}\text{-}\mathcal{A}\mathsf{LG}_{\mathbb{K}} \to \mathfrak{G}\text{-}\mathcal{A}\mathsf{LG}_{\mathbb{K}}$. The $\mathbb{Z}_2$-graded case is treated in Section 3.4. We work under the following finiteness assumption:

each G_2-orbit on G_1/H_1 is finite.

Our construction generalizes the known definition (at the level of objects) in the case $H_2 = G_2$ and $|G_1 : H_1| < \infty$ [31, §3].

Let A be an $\mathfrak{H}$-algebra. We define a $\mathfrak{G}$-algebra $\widetilde{A} = \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} A$ as follows. Fix a left transversal $\mathcal{T}$ to H_1 in G_1. For each $t \in \mathcal{T}$, denote by $\mathbb{K}G_{2,t}$ the group algebra of G_2 with right $\mathbb{K}H_2$-module structure

$$x \cdot z = x({}^t z), \qquad x \in G_2,\ z \in H_2.$$

As a vector space, set

$$\widetilde{A} = \prod_{t \in \mathcal{T}} \mathbb{K}G_{2,t} \otimes_{\mathbb{K}H_2} A.$$

Explicitly, the tensor relations in $\widetilde{A}$ read

$$[x({}^t z) \otimes a]_t = [x \otimes \omega_{A,2}(z) a]_t, \qquad x \in G_2,\ z \in H_2,\ a \in A$$

where $[-]_t$ denotes an element of the t^{th} factor of $\widetilde{A}$. Let δ_{g_2,g_1} be the H_1-coset delta function: given $g_1, g_2 \in G_1$,

$$\delta_{g_2,g_1} = \begin{cases} 1 & \text{if } g_1 H_1 = g_2 H_1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.1. *The formula*

$$[x_2 \otimes a_2]_{t_2} \cdot [x_1 \otimes a_1]_{t_1} = \delta_{\partial(x_1^{-1})t_2, t_1} [x_2 x_1 \otimes \omega_{A,1}(t_1^{-1}\partial(x_1^{-1})t_2)(a_2) a_1]_{t_1} \tag{13}$$

defines an associative algebra structure on $\widetilde{A}$ with identity $1_{\tilde{A}} = ([e_{G_2} \otimes 1_A]_t)_{t \in \mathcal{T}}$.

Proof. To begin, observe that if $\delta_{\partial(x_1^{-1})t_2, t_1}$ is non-zero, then the argument $t_1^{-1}\partial(x_1^{-1})t_2$ of $\omega_{A,1}$ lies in H_1. The right hand side of equation (13) is therefore well-defined. Our finiteness assumption ensures that, for each $t_1 \in \mathcal{T}$, the function $\delta_{\partial(x_1^{-1})t_2, t_1}$ is non-zero for only finitely many $t_2 \in \mathcal{T}$. Hence, only finite sums appear in the calculation of the product of two arbitrary elements of $\widetilde{A}$.

To be well-defined, equation (13) must respect the tensor relations in $\widetilde{A}$. For $z \in H_2$, we have

$$[x_2 \cdot z \otimes a_2]_{t_2} \cdot [x_1 \otimes a_1]_{t_1} = \delta_{\partial(x_1^{-1})t_2, t_1} [x_2({}^{t_2} z) x_1 \otimes \omega_{A,1}(t_1^{-1}\partial(x_1^{-1})t_2)(a_2) a_1]_{t_1}. \tag{14}$$

The crossed module axioms for $\mathfrak{G}$ give $x_2({}^{t_2} z) x_1 = x_2 x_1 ({}^{\partial(x_1^{-1})t_2} z)$. Write $\partial(x_1^{-1})t_2 = t' h$ for $t' \in \mathcal{T}$ and $h \in H_1$. If the product (14) is non-zero, then $t' = t_1$. In this case, ${}^{\partial(x_1^{-1})t_2} z = {}^{t_1}({}^h z)$ and, since $\mathfrak{H}$ is a crossed submodule of $\mathfrak{G}$, we have ${}^h z \in H_2$. Noting that $t_1^{-1}\partial(x_1^{-1})t_2 = h$, the product (14) becomes

$$\delta_{\partial(x_1^{-1})t_2, t_1} [x_2 x_1 \otimes \omega_{A,2}({}^h z) \omega_{A,1}(h)(a_2) a_1]_{t_1}.$$

The axiom (5) gives $\omega_{A,2}({}^h z) = \omega_{A,1}(h)(\omega_{A,2}(z))$, so that (14) can be written as

$$\delta_{\partial(x_1^{-1})t_2, t_1} [x_2 x_1 \otimes \omega_{A,1}(h)(\omega_{A,2}(z) a_2) a_1]_{t_1}.$$

This is plainly equal to $[x_2 \otimes \omega_{A,2}(z) a_2]_{t_2} \cdot [x_1 \otimes a_1]_{t_1}$, as required.

Similarly, for each $z \in H_2$, we have

$$[x_2 \otimes a_2]_{t_2} \cdot [x_1({}^{t_1}z) \otimes a_1]_{t_1} = \delta_{\partial(({}^{t_1}z)^{-1}x_1^{-1})t_2,t_1}[x_2x_1({}^{t_1}z) \otimes \omega_{A,1}(t_1^{-1}\partial(x_1^{-1})t_2)(a_2)a_1]_{t_1}$$

and

$$[x_2 \otimes a_2]_{t_2} \cdot [x_1 \otimes \omega_{A,2}(z)a_1]_{t_1} = \delta_{\partial(x_1^{-1})t_2,t_1}[x_2x_1 \otimes \omega_{A,1}(t_1^{-1}\partial(x_1^{-1})t_2)(a_2)\omega_{A,2}(z)a_1]_{t_1}.$$

Using that $\partial(({}^{t_1}z)^{-1}x_1^{-1}) = t_1\partial(z^{-1})t_1^{-1}\partial(x_1^{-1})$, a short calculation shows that the δ-functions appearing in the two products are equal. Since

$$[x_2x_1({}^{t_1}z) \otimes \omega_{A,1}(t_1^{-1}\partial(x_1^{-1})t_2)(a_2)a_1]_{t_1} = [x_2x_1 \otimes \omega_{A,2}(z)\omega_{A,1}(t_1^{-1}\partial(x_1^{-1})t_2)(a_2)a_1]_{t_1},$$

we see that the products are indeed equal.

We omit the verification of associativity and the identity property. □

Next, we define a $\mathfrak{G}$-algebra structure on $\widetilde{A}$.

Proposition 3.2. *The maps*

$$\omega_{\tilde{A},1}(g)([x \otimes a]_t) = [{}^gx \otimes \omega_{A,1}(h)(a)]_{t'}, \qquad g \in G_1$$

where $gt = t'h$ for $t' \in \mathcal{T}$ and $h \in H_2$, and

$$\omega_{\tilde{A},2}(x) = (\, [x \otimes 1_A]_t\,)_{t \in \mathcal{T}}, \qquad x \in G_2$$

supply $\widetilde{A}$ with the structure of a $\mathfrak{G}$-algebra.

Proof. To begin, we verify that $\omega_{\tilde{A},1}(g)$ is an algebra homomorphism:

$$\begin{aligned}\omega_{\tilde{A},1}(g)\left([x_2 \otimes a_2]_{t_2} \cdot [x_1 \otimes a_1]_{t_1}\right) &= \delta_{\partial(x_1^{-1})t_2,t_1}\omega_{\tilde{A},1}(g)[x_2x_1 \otimes \omega_{A,1}(h)(a_2)a_1]_{t_1} \\ &= \delta_{\partial(x_1^{-1})t_2,t_1}[{}^g(x_2x_1) \otimes \omega_{A,1}(h_1h)(a_2)\omega_{A,1}(h_1)(a_1)]_{t_1'}.\end{aligned}$$

Here $gt_i = t_i'h_i$ for $t_i' \in \mathcal{T}$ and $h_i \in H_1$ and we have written h for $t_1^{-1}\partial(x_1^{-1})t_2$. On the other hand, $\omega_{\tilde{A},1}(g)([x_2 \otimes a_2]_{t_2}) \cdot (\omega_{\tilde{A},1}(g)[x_1 \otimes a_1]_{t_1})$ is equal to

$$\begin{aligned}&\delta_{\partial({}^gx_1^{-1})gt_2,gt_1}[{}^gx_2{}^gx_1 \otimes \omega_{A,1}(t_1'^{-1}\partial({}^gx_1^{-1})t_2'h_2)(a_2)\omega_{A,1}(h_1)(a_1)]_{t_1'} \\ =\;& \delta_{\partial(x_1^{-1})t_2,t_1}[{}^g(x_2x_1) \otimes \omega_{A,1}(h_1hh_2^{-1})(\omega_{A,1}(h_2)(a_2)\omega_{A,1}(h_1)(a_1)]_{t_1'} \\ =\;& \delta_{\partial(x_1^{-1})t_2,t_1}[{}^g(x_2x_1) \otimes \omega_{A,1}(h_1h)(a_2)\omega_{A,1}(h_1)(a_1)]_{t_1'},\end{aligned}$$

as required. To verify that $\omega_{\tilde{A},1}$ is a group homomorphism, we compute

$$\omega_{\tilde{A},1}(g_2)\left(\omega_{\tilde{A},1}(g_1)(\,[x \otimes a]_t\,)\right) = [{}^{g_2}({}^{g_1}x) \otimes \omega_{A,1}(h_2)(\omega_{A,1}(h_1)(a))]_{t_2'} = \omega_{\tilde{A},1}(g_2g_1)[x \otimes a]_t.$$

To verify that $\omega_{\tilde{A},2}$ is a group homomorphism, we compute

$$\omega_{\tilde{A},2}(x_2) \cdot \omega_{\tilde{A},2}(x_1) = (\,\delta_{\partial(x_1^{-1})t_2,t_1}[x_2x_1 \otimes 1_A]_{t_1}\,)_{t_1 \in \mathcal{T}} = (\,[x_2x_1 \otimes 1_A]_{t_1}\,)_{t_1 \in \mathcal{T}} = \omega_{\tilde{A},2}(x_2x_1).$$

To verify equation (3), we compute

$$\omega_{\tilde{A},1}(g)(\omega_{\tilde{A},2}(x)) = \omega_{\tilde{A},1}(g)(\,[x \otimes 1_A]_t\,)_{t \in \mathcal{T}} = (\,[{}^gx \otimes 1_A]_t\,)_{t \in \mathcal{T}} = \omega_{\tilde{A},2}({}^gx).$$

Finally, to verify equation (5), we compute

$$\omega_{\tilde{A},1}(\partial x_2)([x_1 \otimes a_1]_{t_1}) = [{}^{\partial x_2}x_1 \otimes \omega_{A,1}(h)(a_1)]_{t_1'},$$

where $\partial(x_2)t_1 = t_1'h$, while

$$\begin{aligned}&\omega_{\tilde{A},2}(x_2) \cdot [x_1 \otimes a_1]_{t_1} \cdot \omega_{\tilde{A},2}(x_2)^{-1} = (\,[x_2 \otimes 1_A]_t\,)_{t \in \mathcal{T}} \cdot (x_1 \otimes a_1)_{t_1} \cdot (\,[x_2^{-1} \otimes 1_A]_t\,)_{t \in \mathcal{T}} \\ =\;& (\,\delta_{\partial(x_2)t_1,t}[x_2x_1 \otimes \omega_{A,1}(t^{-1}\partial(x_2)t)(a)]_t\,)_{t \in \mathcal{T}} = [x_2x_1x_2^{-1} \otimes \omega_{A,1}(t^{-1}\partial(x_2)t_1)(a_1)]_{t_1'}.\end{aligned}$$

It follows that $t = t_1'$ and hence $t^{-1}\partial(x_2)t_1 = h$. This completes the proof. □

We complete the construction of $\operatorname{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$ as a pseudofunctor in the following section.

3.2. **A Maschke-type theorem for induced $\mathfrak{G}$-algebras.** In order to ensure that induction transforms 2-representations into 2-representations, we need the following version of Maschke's Theorem for the $\mathbb{K}$-algebra $\widetilde{A}$.

Proposition 3.3. *Let A be an $\mathfrak{H}$-algebra. Assume that both indices $|G_1 : H_1|$ and $|G_2 : H_2|$ are finite.*

(i) If A is finite dimensional over $\mathbb{K}$, then so too is $\widetilde{A}$.
(ii) Assume that $|G_2 : H_2|$ is not divisible by the characteristic of $\mathbb{K}$. If A is separable, then so too is $\widetilde{A}$.

Proof. The first statement follows immediately from the definition of $\widetilde{A}$.

Turning to the second statement, let $w = \sum_j a_j \otimes b_j \in A \otimes_{\mathbb{K}} A^{\mathrm{op}}$ be a separability idempotent for A. Pick left transversals $\mathcal{T}$ to H_1 in G_1 and $\mathcal{X}$ to H_2 in G_2. For each $x \in \mathcal{X}$ and $t \in \mathcal{T}$, there exists a unique $t_x \in \mathcal{T}$ such that $\delta_{t\partial x, t_x} = 1$. Consider the element

$$\mathfrak{w} = \sum_{x \in \mathcal{X}, t \in \mathcal{T}, j} [{}^t x \otimes a_j]_t \otimes [e \otimes b_j]_t [{}^t x^{-1} \otimes 1]_{t_x} \in \widetilde{A} \otimes_{\mathbb{K}} \widetilde{A}^{\mathrm{op}},$$

where the multiplication is in $\widetilde{A}$. We claim that $\tilde{\mathfrak{w}} = |G_2 : H_2|^{-1} \mathfrak{w}$ is a separability idempotent for $\widetilde{A}$. First, we show that $\tilde{\mathfrak{w}}$ is sent to $1_{\tilde{A}}$ under the multiplication map $\widetilde{A} \otimes_{\mathbb{K}} \widetilde{A}^{\mathrm{op}} \to \widetilde{A}$:

$$\sum_{x,t,j} [{}^t x \otimes a_j]_t [e \otimes b_j]_t [{}^t x^{-1} \otimes 1]_{t_x} = \sum_{x,t,j} [{}^t x \otimes a_j b_j]_t [{}^t x^{-1} \otimes 1]_{t_x} =$$
$$\sum_{x,t} [{}^t x \otimes 1]_t [{}^t x^{-1} \otimes 1]_{t_x} = |G_2 : H_2| \sum_t \delta_{\partial({}^t x) t, t_x} [e \otimes 1]_{t_x} = |G_2 : H_2| 1_{\tilde{A}}.$$

In the final equality we used that $\partial({}^t x) t = t \partial(x)$. To verify that $\tilde{\mathfrak{w}}$ is $\widetilde{A}$-central, it is useful to write

$$\mathfrak{w} = \sum_{x,t,j} [{}^t x \otimes 1]_t [e_{G_2} \otimes a_j]_t \otimes [e \otimes b_j]_t [{}^t x^{-1} \otimes 1]_{t_x} = \sum_{x,t} [{}^t x \otimes 1]_t w_t [{}^t x^{-1} \otimes 1]_{t_x},$$

where w_t denotes w considered as an element of $A \otimes_{\mathbb{K}} A^{\mathrm{op}}$ in degree $t \in \mathcal{T}$. It suffices to check that $\mathfrak{a}\tilde{\mathfrak{w}} = \tilde{\mathfrak{w}}\mathfrak{a}$ when $\mathfrak{a} \in \widetilde{A}$ is of one of the following two forms:

$$\mathfrak{a} = [e \otimes \gamma]_s \qquad \text{or} \qquad \mathfrak{a} = [y \otimes 1]_s.$$

In the first case, pick $q = q(x,s) \in \mathcal{T}$ such that $s = q_x$. Observe that for each $t \in \mathcal{T}$, the equalities $\delta_{t\partial x,s} = \delta_{\partial({}^tx^{-1})s,t} = 1$ hold if and only if $t = q$. We compute

$$\begin{aligned}
[e\otimes\gamma]_s\mathfrak{w} &= \sum_{x,t}[e\otimes\gamma]_s[{}^tx\otimes 1]_t w_t[{}^tx^{-1}\otimes 1]_{t_x}\\
&= \sum_{x,t}\delta_{\partial({}^tx^{-1})s,t}[{}^tx\otimes\omega_{A,1}(t^{-1}\partial({}^tx^{-1})s)(\gamma)]_t w_t[{}^tx^{-1}\otimes 1]_{t_x}\\
&= \sum_{x}[{}^qx\otimes\omega_{A,1}(\partial(x^{-1})q^{-1}s)(\gamma)]_q w_q[{}^qx^{-1}\otimes 1]_{q_x}\\
&= \sum_{x}[{}^qx\otimes 1]_q[e\otimes\omega_{A,1}(\partial(x^{-1})q^{-1}s)(\gamma)]_q w_q[{}^qx^{-1}\otimes 1]_{q_x}\\
&= \sum_{x}[{}^qx\otimes 1]_q w_q[e\otimes\omega_{A,1}(\partial(x^{-1})q^{-1}s)(\gamma)]_q[{}^qx^{-1}\otimes 1]_{q_x}\\
&= \sum_{x}[{}^qx\otimes 1]_q w_q[{}^qx^{-1}\otimes\gamma]_{q_x} = \sum_{x}[{}^qx\otimes 1]_q w_q[{}^qx^{-1}\otimes 1]_{q_x}[e\otimes\gamma]_{q_x}\\
&= \sum_{x,t}\delta_{\partial({}^tx^{-1})s,t}[{}^tx\otimes 1]_t w_t[{}^tx^{-1}\otimes 1]_{t_x}[e\otimes\gamma]_s = \mathfrak{w}[e\otimes\gamma]_s.
\end{aligned}$$

For the second case, we rewrite $y\,{}^tx = {}^tx_y\,{}^th_y$ with $x_y = x_y(t) \in \mathcal{X}$ and $h_y = h_y(t) \in H_2$ and compute

$$\begin{aligned}
[y\otimes 1]_s\mathfrak{w} &= \sum_{x,t}[y\otimes 1]_s[{}^tx\otimes 1]_t w_t[{}^tx^{-1}\otimes 1]_{t_x}\\
&= \sum_{x,t}\delta_{\partial({}^tx^{-1})s,t}[y\,{}^tx\otimes 1]_t w_t[{}^tx^{-1}\otimes 1]_{t_x} = \sum_{x}[\,{}^qx_y\otimes\omega_{A,1}(h_y)]_q w_q[{}^qx^{-1}\otimes 1]_s\\
&= \sum_{x}[\,{}^qx_y\otimes 1]_q[e\otimes\omega_{A,1}(h_y)]_q w_q[{}^qx^{-1}\otimes 1]_s = \sum_{x}[\,{}^qx_y\otimes 1]_q w_q[{}^qh_y\otimes 1]_q[{}^qx^{-1}\otimes 1]_s\\
&= \sum_{x}\delta_{\partial({}^qx)q,s}[\,{}^qx_y\otimes 1]_q w_q[{}^qh_y{}^qx^{-1}\otimes 1]_s = \sum_{x}[\,{}^qx_y\otimes 1]_q w_q[\,{}^qx_y^{-1}\,y\otimes 1]_s.
\end{aligned}$$

Notice that we have used the equality $\delta_{\partial({}^qx)q,s} = 1$. Similarly, we have

$$\mathfrak{w}[y\otimes 1]_s = \sum_{z,t}[{}^tz\otimes 1]_t w_t[{}^tz^{-1}\otimes 1]_{t_z}[y\otimes 1]_s = \sum_{z,t}\delta_{\partial(y^{-1})t_z,s}[{}^tz\otimes 1]_t w_t[{}^tz^{-1}y\otimes 1]_s\,.$$

Since both z and x_y run over the set $\mathcal{X}$, it remains to put $z = x_y$ and verify that the δ-function in the final sum is non-zero if (and hence only if) $q = t$:

$$\delta_{\partial(y^{-1})q_z,s} = \delta_{\partial(y^{-1})q\partial z,q\partial x} = \delta_{q\partial(z)q^{-1}qh_y,\partial(y)q\partial(x)q^{-1}q} = \delta_{\partial({}^qz\,{}^qh_y)q,\partial(y\,{}^qx)q} = 1.$$

This completes the proof. □

We can now complete the construction of $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$.

Theorem 3.4. *(i) The assignment* $A \mapsto \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}A$ *extends to a pseudofunctor* $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} : \mathfrak{H}\text{-}\mathcal{A}\mathsf{LG}_{\mathbb{K}} \to \mathfrak{G}\text{-}\mathcal{A}\mathsf{LG}_{\mathbb{K}}$.

(ii) If both indices $|G_1 : H_1|$ *and* $|G_2 : H_2|$ *are finite, then* $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$ *restricts to a pseudofunctor* $\mathfrak{H}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}} \to \mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}$. *If, moreover,* $|G_2 : H_2|$ *is not divisible by the characteristic of* $\mathbb{K}$, *then* $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$ *restricts to a pseudofunctor* $\mathfrak{H}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}} \to \mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$.

Proof. Let $M : A \to B$ be a 1-morphism in $\mathfrak{H}$-$\mathcal{A}\mathsf{LG}_{\mathbb{K}}$. Define $\widetilde{M} = \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} M$ to be

$$\widetilde{M} = \prod_{t \in \mathcal{T}} \mathbb{K}G_{2,t} \otimes_{\mathbb{K}H_2} M,$$

where the tensor relations read

$$[x({}^t z) \otimes m]_t = [x \otimes \omega_M(\partial z)(m)]_t, \qquad x \in G_2,\ z \in H_2,\ m \in M.$$

The left $\widetilde{B}$-$\widetilde{A}$-bimodule structure of $\widetilde{M}$ is defined by

$$[x_2 \otimes b]_{t_2} \cdot [x_1 \otimes m]_{t_1} = \delta_{\partial(x_1^{-1})t_2, t_1} [x_2 x_1 \otimes \omega_{B,1}(t_1^{-1}\partial(x_1^{-1})t_2)(b) m]_{t_1}$$

and

$$[x_2 \otimes m]_{t_2} \cdot [x_1 \otimes a]_{t_1} = \delta_{\partial(x_1^{-1})t_2, t_1} [x_2 x_1 \otimes \omega_M(t_1^{-1}\partial(x_1^{-1})t_2)(m) a]_{t_1}.$$

The structure maps for $\widetilde{M}$ are defined by

$$\omega_{\tilde{M}}(g)([x \otimes m]_t)_{t \in \mathcal{T}} = ([{}^g x \otimes \omega_M(h)(m)]_{t'})_{t' \in \mathcal{T}},$$

where $gt = t'h$ for $h \in H_1$. To verify equation (8), we note that

$$\omega_{\tilde{M}}(x_2)([x_1 \otimes m]_t) = [{}^{\partial(x_2)} x_1 \otimes \omega_M(h)(m)]_{t'},$$

where $\partial(x_2)t = t'h$, while

$$\omega_{\tilde{B},2}(x_2) \cdot [x_1 \otimes m]_t \cdot \omega_{\tilde{A},2}(x_2)^{-1} = [x_2 x_1 \otimes m]_t \cdot \omega_{\tilde{A},2}(x_2)^{-1} = [x_2 x_1 x_2^{-1} \otimes \omega_M({t'}^{-1}\partial(x_2)t)(m)]_t.$$

Given a 2-morphism $\phi : M \Rightarrow N$ in $\mathfrak{H}$-$\mathcal{A}\mathsf{LG}_{\mathbb{K}}$, define $\tilde{\phi} = \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} \phi$ by

$$\tilde{\phi}([x \otimes m]_t) = [x \otimes \phi(m)]_t.$$

This is left $\widetilde{B}$-linear because

$$\begin{aligned} \tilde{\phi}([x_2 \otimes b]_{t_2} \cdot [x_1 \otimes m]_{t_1}) &= \delta_{\partial(x_1^{-1})t_2, t_1} [x_2 x_1 \otimes \phi(\omega_{B,1}(t_1^{-1}\partial(x_1^{-1})t_2)(b) m)]_{t_1} \\ &= \delta_{\partial(x_1^{-1})t_2, t_1} [x_2 x_1 \otimes \omega_{B,1}(t_1^{-1}\partial(x_1^{-1})t_2)(b) \phi(m)]_{t_1}, \end{aligned}$$

which is clearly equal to $[x_2 \otimes b]_{t_2} \cdot [x_1 \otimes \phi(m)]_{t_1}$. Similarly, it is right $\widetilde{A}$-linear. The H_1-equivariance of ϕ implies the G_1-equivariance of $\tilde{\phi}$.

The 2-isomorphisms relating compositions of 1-morphisms are induced by those of $\mathcal{A}\mathsf{LG}_{\mathbb{K}}$. It follows directly from the definitions that $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$ strictly preserves identity 1-morphisms. This completes the construction of $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$.

The second statement now follows from Proposition 3.3 and the observation that $\widetilde{M}$ is finite dimensional if M is so. □

3.3. **Induction as a biadjunction.** In this section, we prove the biadjointness of $\mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}}$ and $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$ in two, in a sense, opposite situations. For the notion of a (left or right) biadjunction between pseudofunctors, see [19, Definition 2.1].

Proposition 3.5. *If either $H_2 = G_2$ or $H_1 = G_1$, then $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$ is right biadjoint to $\mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}}$. This statement holds for pseudofunctors between ?-$\mathcal{A}\mathsf{LG}_{\mathbb{K}}$ or, with the assumptions of Theorem 3.4(ii), between ?-$\mathcal{A}\mathsf{lg}_{\mathbb{K}}$ or ?-$\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$.*

Proof. Suppose that $H_2 = G_2$. To begin, we need to define pseudonatural transformations

$$\epsilon : \mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}} \circ \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} \Longrightarrow \mathrm{id}_{\mathfrak{H}\text{-}\mathcal{A}\mathsf{LG}_{\mathbb{K}}}, \qquad \eta : \mathrm{id}_{\mathfrak{G}\text{-}\mathcal{A}\mathsf{LG}_{\mathbb{K}}} \Longrightarrow \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} \circ \mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}}.$$

Write $t_e \in \mathcal{T}$ for the representative of the identity coset. Given an $\mathfrak{H}$-algebra A, we claim that the $\mathbb{K}$-linear map

$$\epsilon_A : \mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}} \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} A \to A, \qquad [a]_t \mapsto \delta_{t,t_e} \omega_{A,1}(t)(a)$$

is a strict $\mathfrak{H}$-algebra morphism. Firstly, ϵ_A is a $\mathbb{K}$-algebra morphism:

$$\epsilon_A([a_2]_{t_2} \cdot [a_1]_{t_1}) = \delta_{t_2,t_1}\epsilon_A([a_2a_1]_{t_1}) = \delta_{t_2,t_1}\delta_{t_1,t_e}\omega_{A,1}(t_1)(a_2a_1)$$

is equal to

$$\epsilon_A([a_2]_{t_2}) \cdot \epsilon_A([a_1]_{t_1}) = \delta_{t_2,t_e}\delta_{t_1,t_e}\omega_{A,1}(t_2)(a_2)\omega_{A,1}(t_1)(a_1).$$

Clearly ϵ_A preserves multiplicative identities. Moreover, ϵ_A is H_1-equivariant:

$$\epsilon_A\left(\omega_{\tilde{A},1}(h)([a]_t)\right) = \epsilon_A\left([\omega_{A,1}(h')(a)]_{t'}\right) = \delta_{t',t_e}\omega_{A,1}(t'h')(a),$$

where $ht = t'h'$, is equal to

$$\omega_{A,1}(h)(\epsilon_A([a]_t)) \quad = \quad \delta_{t,t_e}\omega_{A,1}(ht_e)(a).$$

Finally, ϵ_A is $\omega_{?,2}$-compatible: for each $z \in H_2$, we have

$$\epsilon_A(\omega_{\tilde{A},2}(z)) = \epsilon_A(([\omega_{A,2}({}^{t^{-1}}z)]_t)_{t\in\mathcal{T}}) = \omega_{A,1}(t_e)(\omega_{A,2}({}^{t_e^{-1}}z)) = \omega_{A,2}(z).$$

We henceforth interpret ϵ_A as a representable 1-morphism $\mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}}\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}A \to A$ in $\mathfrak{H}$-$\mathcal{A}\mathsf{LG}_{\mathbb{K}}$. Given a 1-morphism $M : A \to B$ in $\mathfrak{H}$-$\mathcal{A}\mathsf{LG}_{\mathbb{K}}$, define

$$\epsilon_M : B \otimes_{\tilde{B}} \widetilde{M} \Longrightarrow M \otimes_A A_{\epsilon_A} \simeq M_{\epsilon_A}, \qquad b \otimes [m]_t \mapsto \delta_{t,t_e} b\omega_M(t)(m).$$

Calculations similar to those above show that ϵ_M is indeed a 2-morphism in $\mathfrak{H}$-$\mathcal{A}\mathsf{LG}_{\mathbb{K}}$ and that $\{\epsilon_A\}_A$ and $\{\epsilon_M\}_M$ satisfy the coherence conditions required to define the pseudonatural transformation ϵ.

Similarly, given a $\mathfrak{G}$-algebra B, we claim that the $\mathbb{K}$-linear map

$$\eta_B : B \to \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}\mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}}B, \qquad b \mapsto ([\omega_{B,1}(t^{-1})(b)]_t)_{t\in\mathcal{T}}$$

is a strict $\mathfrak{G}$-algebra morphism. First, η_B is multiplicative:

$$\begin{aligned}\eta_B(b_2) \cdot \eta_B(b_1) &= ([\omega_{B,1}(t_2^{-1})(b_2)]_{t_2})_{t_2\in\mathcal{T}} \cdot ([\omega_{B,1}(t_1^{-1})(b_1)]_{t_1})_{t_1\in\mathcal{T}} \\ &= (\delta_{t_2,t_1}[\omega_{B,1}(t_1^{-1}t_2)(\omega_{B,1}(t_2^{-1})(b_2))\omega_{B,1}(t_1^{-1})(b_1)]_{t_1})_{t_1\in\mathcal{T}} \\ &= ([\omega_{B,1}(t^{-1})(b_2b_1)]_t)_{t\in\mathcal{T}} = \eta_B(b_2b_1).\end{aligned}$$

Clearly, η_B is unital. Moreover, η_B is G_1-equivariant:

$$\eta_B(\omega_{B,1}(g)(b)) \quad = \quad ([\omega_{B,1}(t^{-1}g)(b)]_t)_{t\in\mathcal{T}}$$

while

$$\omega_{\tilde{B},1}(g)(\eta_B(b)) = \omega_{\tilde{B},1}(g)([\omega_{B,1}(t^{-1})(b)]_t)_{t\in\mathcal{T}} = ([\omega_{B,1}(ht^{-1})(b)]_{t'})_{t'\in\mathcal{T}}$$

where $gt = t'h$. Finally, η_B is $\omega_{?,2}$-compatible: for $x \in G_2$, we have

$$\eta_B(\omega_{B,2}(x)) = ([\omega_{B,1}(t^{-1})(\omega_{B,2}(x))]_t)_{t\in\mathcal{T}} = ([\omega_{B,2}({}^{t^{-1}}x)]_t)_{t\in\mathcal{T}} = \omega_{\tilde{B},2}(x).$$

We interpret η_B as a 1-morphism $B \to \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}\mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}}B$ in $\mathfrak{G}$-$\mathcal{A}\mathsf{LG}_{\mathbb{K}}$. Given a 1-morphism $N : A \to B$ in $\mathfrak{G}$-$\mathcal{A}\mathsf{LG}_{\mathbb{K}}$, define the required 2-morphism by

$$\eta_B : \widetilde{B}_{\eta_B} \otimes_B N \Longrightarrow \widetilde{N} \otimes_{\tilde{A}} \widetilde{A}_{\eta_A} \simeq \widetilde{N}_{\eta_A}, \qquad [b]_t \otimes n \mapsto [b\omega_N(t^{-1})(n)]_t.$$

This data defines the pseudonatural transformation η.

It remains to define invertible zig-zag modifications

$$\Gamma : (\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} \diamond \epsilon) \circ (\eta \diamond \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}) \Longrightarrow 1_{\mathrm{Ind}}, \qquad \Lambda : 1_{\mathrm{Res}} \Longrightarrow (\epsilon \diamond \mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}}) \circ (\mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}} \diamond \eta).$$

Hence, for each $A \in \mathfrak{H}$-$\mathcal{A}\mathsf{LG}_{\mathbb{K}}$ we need to define a 2-isomorphism

$$\Gamma_A : \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}\epsilon_A \otimes_{\mathrm{IndResInd}A} \eta_{\mathrm{Ind}A} \Longrightarrow \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}A.$$

Since $\iota := \mathrm{Ind}\epsilon_A \circ \eta_{\mathrm{Ind}A}$ is the identity map, the domain bimodule $(\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} A)_\iota$ of Γ_A is the identity bimodule. After making this identification, we take Γ_A to be the identity map. For $B \in \mathfrak{G}\text{-}\mathcal{A}\mathsf{LG}_{\mathbb{K}}$, we define a 2-isomorphism

$$\Lambda_B : \mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}} B \Longrightarrow \epsilon_{\mathrm{Res}B} \otimes_{\mathrm{ResIndRes}B} \mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}} \eta_B.$$

The codomain bimodule is isomorphic to $(\mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}} B)_{\epsilon_{\mathrm{Res}B} \circ \mathrm{Res}\eta_B} \simeq \mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}} B$, and we use this isomorphism to define Λ_B. It is straightforward to verify that the above data satisfies the required coherence conditions, proving the proposition in the case $H_2 = G_2$.

The case in which $H_1 = G_1$ is similar, so we are brief. Define an algebra homomorphism

$$\epsilon'_A : A \to \mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}} \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} A, \qquad a \mapsto e_{G_2} \otimes a.$$

The G_1-equivariance of ϵ'_A is clear and it is $\omega_{?,2}$-compatible because

$$\epsilon'_A(\omega_{A,2}(z)) = e_{G_2} \otimes \epsilon'_A(\omega_{A,2}(z)) = z \otimes 1_A.$$

We henceforth interpret ϵ'_A as the 1-morphism

$$\epsilon_A = {}_{\epsilon'_A}\widetilde{A} : \mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}} \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} A \to A$$

in $\mathfrak{H}\text{-}\mathcal{A}\mathsf{LG}_{\mathbb{K}}$. Given a 1-morphism $M : A \to B$ in $\mathfrak{H}\text{-}\mathcal{A}\mathsf{LG}_{\mathbb{K}}$, define a 2-morphism

$$\epsilon_M : {}_{\epsilon'_B}\widetilde{B} \otimes_{\tilde{B}} \widetilde{M} \simeq {}_{\epsilon'_B}\widetilde{M} \Longrightarrow M \otimes_A ({}_{\epsilon'_A}\widetilde{A}), \qquad x \otimes m \mapsto m \otimes (x \otimes 1_A).$$

This defines the pseudonatural transformation ϵ.

Let

$$\eta'_B : \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} \mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}} B \to B, \qquad x \otimes b \mapsto \omega_{B,2}(x) b.$$

This map is well-defined because

$$x \cdot z \otimes b \mapsto \omega_{B,2}(xz) b = \omega_{B,2}(x) \omega_{B,2}(z) b$$

and $x \otimes \omega_{B,2}(z) b \mapsto \omega_{B,2}(x) \omega_{B,2}(z) b$. Moreover, η'_B is clearly a G_1-equivariant unital $\mathbb{K}$-algebra homomorphism and is $\omega_{?,2}$-compatible:

$$\eta'_B(\omega_{\tilde{B},2}(x)) = \eta'_B(x \otimes 1_B) = \eta'_{B,2}(x).$$

We interpret η'_B as the 1-morphism

$$\eta_B = {}_{\eta'_B} B : B \to \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} \mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}} B$$

in $\mathfrak{G}\text{-}\mathcal{A}\mathsf{LG}_{\mathbb{K}}$. Given a 1-morphism $N : A \to B$ in $\mathfrak{G}\text{-}\mathcal{A}\mathsf{LG}_{\mathbb{K}}$, define a 2-morphism

$$\eta_N : {}_{\eta'_B} B \otimes_B N \simeq {}_{\eta'_B} N \Longrightarrow \widetilde{N} \otimes_{\tilde{A}} ({}_{\eta'_A} A), \qquad n \mapsto (e_{G_2} \otimes n) \otimes 1_A.$$

Similar to the case $H_2 = G_2$, after suitable identifications, we can take the 2-isomorphisms Γ_A and Λ_B to be the respective identities. $\square$

Remarks. (i) In the setting of Proposition 3.5, one can also prove that $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$ is left biadjoint to $\mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}}$. When $H_2 = G_2$, for example, this is done by interpreting ϵ_A as a 1-morphism $A \to \mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}} \mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}} A$ in $\mathfrak{H}\text{-}\mathcal{A}\mathsf{LG}_{\mathbb{K}}$, and similarly for η_B. This is analogous to what was done in the proof of Proposition 3.5 in the case $H_1 = G_1$.

(ii) We expect that $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$ is in fact left and right biadjoint to $\mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}}$ without the assumption $H_1 = G_1$ or $H_2 = G_2$. This would categorify the left and right adjunctions between induction and restriction in the representation theory of finite groups.

3.4. **The Real case.** We extend the constructions of Sections 3.1 and 3.2 to the $\mathbb{Z}_2$-graded setting. Since the calculations are similar, we are occasionally brief.

Crossed submodules $\mathfrak{H}$ of a $\mathbb{Z}_2$-graded crossed module $\mathfrak{G}$ come in two flavours:

(i) non-trivially graded: H_1 is a non-trivially $\mathbb{Z}_2$-graded subgroup of G_1,
(ii) trivially graded: H_1 is a trivially $\mathbb{Z}_2$-graded subgroup of G_1.

There is a restriction pseudofunctor $\mathrm{Res}^{\mathfrak{G}}_{\mathfrak{H}} : \mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}} \to \mathfrak{H}\text{-}\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$. We define a pseudofunctor $\mathrm{Ind}^{\mathfrak{G}}_{\mathfrak{H}} : \mathfrak{H}\text{-}\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}} \to \mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$. To avoid confusion, we sometimes denote $\mathrm{Ind}^{\mathfrak{G}}_{\mathfrak{H}}$ by $\mathrm{RInd}^{\mathfrak{G}}_{\mathfrak{H}} A$ and $\mathrm{HInd}^{\mathfrak{G}}_{\mathfrak{H}} A$ in the case of non-trivially and trivially graded $\mathfrak{H}$, respectively, matching the notation for the Real and hyperbolic induction of [37, §7]. We work under the finiteness assumption and notation of Section 3.1.

Let A be an $\mathfrak{H}$-algebra. Define

$$\widetilde{A} = \prod_{t \in \mathcal{T}} \mathbb{K}G_{2,t} \otimes_{\mathbb{K}H_2} {}^{\pi(t)}A,$$

where ${}^{\pi(t)}A$ is A or $A^\vee$, depending on $\pi(t) \in \mathbb{Z}_2$. The $\mathbb{K}$-algebra structure of $\widetilde{A}$ is again defined by equation (13), keeping in mind that we use the multiplication of ${}^{\pi(t)}A$ in the t^{th} factor. Define $\omega_{\tilde{A},1}$ and $\omega_{\tilde{A},2}$ by

$$\omega_{\tilde{A},1}(g)([x \otimes a]_t) = [{}^g x \otimes {}^{\pi(t')}\omega_{A,1}(h)(a)]_{t'}, \qquad \omega_{\tilde{A},2}(x) = (\, [x \otimes 1_A]_t \,)_{t \in \mathcal{T}},$$

where $gt = t'h$ for $t' \in \mathcal{T}$ and $h \in H_1$.

Let us make explicit the trivially graded case. Since the morphism $\mathfrak{H} \hookrightarrow \mathfrak{G}$ factors through $\mathfrak{G}_0 \hookrightarrow \mathfrak{G}$, it suffices to consider the case $\mathfrak{H} = \mathfrak{G}_0$. The general case can then obtained as the composition

$$\mathrm{HInd}^{\mathfrak{G}}_{\mathfrak{H}} = \mathrm{HInd}^{\mathfrak{G}}_{\mathfrak{G}_0} \circ \mathrm{Ind}^{\mathfrak{G}_0}_{\mathfrak{H}},$$

where $\mathrm{Ind}^{\mathfrak{G}_0}_{\mathfrak{H}}$ is as in Theorem 3.4. Fix an element $\mathfrak{h} \in G_1 \setminus G_0$ and take $\mathcal{T} = \{e, \mathfrak{h}\}$. Then $\widetilde{A} = [A]_e \oplus [A^{\mathrm{op}}]_{\mathfrak{h}}$ with

$$\omega_{\tilde{A},1}(g)([a]_t) = [\omega_{A,1}(g')(a)]_{t'}, \qquad \omega_{\tilde{A},2}(x) = [\omega_{A,2}(x)]_e + [\omega_{A,2}({}^{\mathfrak{h}^{-1}}x)^{-1}]_{\mathfrak{h}},$$

where $gt = t'g'$ with $g' \in G_0$. For example, $g \in G_0$ acts on $[A^{\mathrm{op}}]_{\mathfrak{h}}$ by $\mathfrak{h}^{-1}g\mathfrak{h}$.

Theorem 3.6. *The above constructions define a $\mathfrak{G}$-algebra $\widetilde{A}$.*

Proof. The proof in the non-trivially graded case is similar to that of Theorem 3.6, so we focus on the trivially graded case. Let us check that $\omega_{\tilde{A},1}$ is a generalized algebra automorphism. When $\pi(g) = -1$, for example, we have

$$\begin{aligned}
\omega_{\tilde{A},1}(g)\,([a_2]_e \cdot [a_1]_e) &= \omega_{\tilde{A},1}(g)([a_2a_1]_e) \\
&= [\omega_{A,1}(\mathfrak{h}^{-1}g)(a_1)]_{\mathfrak{h}} \bullet [\omega_{A,1}(\mathfrak{h}^{-1}g)(a_2)]_{\mathfrak{h}} \\
&= \omega_{A,1}(g)([a_1]_e) \bullet \omega_{A,1}(g)([a_2]_e),
\end{aligned}$$

where $\bullet$ indicates multiplication in A^{op}, and

$$\omega_{\tilde{A},1}(g)\,([a_2]_{\mathfrak{h}} \cdot [a_1]_{\mathfrak{h}}) = [\omega_{A,1}(g\mathfrak{h})(a_1a_2)]_e = \omega_{\tilde{A},1}(g)([a_1]_{\mathfrak{h}}) \bullet \omega_{\tilde{A},1}(g)([a_2]_{\mathfrak{h}}).$$

We omit the proof that $\omega_{\tilde{A},1}$ is a group homomorphism. To see that $\omega_{\tilde{A},2}$ is a group homomorphism, we compute

$$\begin{aligned}
\omega_{\tilde{A},2}(x_2x_1) &= [\omega_{A,2}(x_2x_1)]_e + [\omega_{A,2}({}^{\mathfrak{h}^{-1}}(x_2x_1))^{-1}]_{\mathfrak{h}} \\
&= [\omega_{A,2}(x_2)\omega_{A,2}(x_1)]_e + [\omega_{A,2}({}^{\mathfrak{h}^{-1}}x_1)^{-1}\omega_{A,2}({}^{\mathfrak{h}^{-1}}x_2)^{-1}]_{\mathfrak{h}}
\end{aligned}$$

while

$$\begin{aligned}\omega_{\tilde{A},2}(x_2)\omega_{\tilde{A},2}(x_1) &= [\omega_{A,2}(x_2)]_e \cdot [\omega_{A,2}(x_1)]_e + [\omega_{A,2}({}^{\mathfrak{h}^{-1}}x_2)^{-1}]_{\mathfrak{h}} \cdot [\omega_{A,2}({}^{\mathfrak{h}^{-1}}x_1)^{-1}]_{\mathfrak{h}} \\ &= [\omega_{A,2}(x_2)\omega_{A,2}(x_1)]_e + [\omega_{A,2}({}^{\mathfrak{h}^{-1}}x_1)^{-1}\omega_{A,2}({}^{\mathfrak{h}^{-1}}x_2)^{-1}]_{\mathfrak{h}},\end{aligned}$$

as required. To verify equation (3), fix $x \in G_2$ and compute

$$\omega_{\tilde{A},1}(\partial x)([a]_{\mathfrak{h}}) = [a]_{\partial(x)\mathfrak{h}} = [a]_{\mathfrak{h}\partial({}^{\mathfrak{h}^{-1}}x)} = [\omega_{A,2}({}^{\mathfrak{h}^{-1}}x)^{-1} \bullet a \bullet \omega_{A,2}({}^{\mathfrak{h}^{-1}}x)]_{\mathfrak{h}},$$

which is equal to $\partial(\omega_{\tilde{A},2}(x))([a]_{\mathfrak{h}})$. The computation with e in place of $\mathfrak{h}$ is similar. Turning to equation (5), consider, for example, the case $\pi(g) = -1$. We have

$$\begin{aligned}\omega_{\tilde{A},1}(g)(\omega_{\tilde{A},2}(x)^{-1}) &= \omega_{\tilde{A},1}(g)([\omega_{A,2}(x)^{-1}]_e) + \omega_{\tilde{A},1}(g)([\omega_{A,2}({}^{\mathfrak{h}^{-1}}x)]_{\mathfrak{h}}) \\ &= [\omega_{A,1}(\mathfrak{h}^{-1}g)(\omega_{A,2}(x)^{-1})]_{\mathfrak{h}} + [\omega_{A,1}(g\mathfrak{h})(\omega_{A,2}({}^{\mathfrak{h}^{-1}}x))]_e \\ &= [\omega_{A,2}({}^{\mathfrak{h}^{-1}g}x)^{-1}]_{\mathfrak{h}} + [\omega_{A,2}({}^{g}x)]_e,\end{aligned}$$

which is plainly equal to $\omega_{\tilde{A},2}({}^{g}x)$. □

There is an extension of Proposition 3.5 to the $\mathbb{Z}_2$-graded setting. We briefly indicate the construction in the hyperbolic setting; the construction for $\mathrm{RInd}_{\mathfrak{H}}^{\mathfrak{G}}$ is similar. More precisely, the assignment $A \mapsto \mathrm{HInd}_{\mathfrak{G}_0}^{\mathfrak{G}}$ extends to a pseudofunctor

$$\mathrm{HInd}_{\mathfrak{G}_0}^{\mathfrak{G}} : \mathfrak{G}_0\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd},2\text{-}\simeq} \to \mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$$

with domain the maximal locally groupoidal subbicategory of $\mathfrak{G}_0\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$. Given a 1-morphism $M : A \to B$ in $\mathfrak{G}_0\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$, define $\widetilde{M} = \mathrm{HInd}_{\mathfrak{G}_0}^{\mathfrak{G}} M$ as follows. As a $\widetilde{B}$-$\widetilde{A}$-bimodule, $\widetilde{M}$ is simply $M \oplus M^{\circ}$. The structure maps of $\widetilde{M}$ are defined by

$$\omega_{\tilde{M}}(g) = \begin{pmatrix} \omega_M(g) & 0 \\ 0 & \omega_{M^{\circ}}(\mathfrak{h}^{-1}g\mathfrak{h}) \end{pmatrix}, \qquad \omega_{\tilde{M}}(f) = \begin{pmatrix} 0 & \omega_{M^{\circ}}(f\mathfrak{h}) \\ \mathrm{ev}_M \circ \omega_M(\mathfrak{h}^{-1}f) & 0 \end{pmatrix}$$

where $g \in G_0$ and $f \in G_1 \setminus G_0$ and

$$\omega_{M^{\circ}}(\mathfrak{h}^{-1}g\mathfrak{h})(\mathfrak{m})(m) = \omega_{A,1}(\mathfrak{h}^{-1}g\mathfrak{h}) \left[\mathfrak{m}(\omega_M(\mathfrak{h}^{-1}g\mathfrak{h})^{-1}(m))\right], \qquad \mathfrak{m} \in M^{\circ},\, m \in M.$$

Given a 2-isomorphism $\phi : M \Rightarrow N$ in $\mathfrak{G}_0\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$, the G_1-equivariant intertwiner $\mathrm{HInd}_{\mathfrak{G}_0}^{\mathfrak{G}}\phi$ is defined to be $\phi \oplus \phi^{-\circ}$, where

$$\phi^{-\circ}(\mathfrak{m})(n) = \mathfrak{m}(\phi^{-1}(n)), \qquad \mathfrak{m} \in M^{\circ},\, n \in N.$$

The $\mathbb{Z}_2$-graded analogue of Proposition 3.5 is more subtle.

Problem 3.7. *Investigate the adjunction properties of* $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$ *and* $\mathrm{Res}_{\mathfrak{H}}^{\mathfrak{G}}$ *when* $\mathfrak{G}$ *is non-trivially* $\mathbb{Z}_2$*-graded.*

The difficulty stems from the fact that $\mathrm{RInd}_{\mathfrak{H}}^{\mathfrak{G}}$, for example, is defined only on 2-isomorphisms of $\mathfrak{H}\text{-}\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$, while the 2-morphism components of ϵ (as in the proof of Proposition 3.5) are not 2-isomorphisms. For this reason, we expect the biadjointness properties to be more naturally formulated in the anti-linear approach to Real 2-representations.

It is useful to decompose the assignment $A \mapsto \mathrm{HInd}_{\mathfrak{G}_0}^{\mathfrak{G}} A$ into two steps. If $\mathfrak{G}$ is a $\mathbb{Z}_2$-graded crossed module, $\mathfrak{h} \in G_1 \setminus G_0$ and A is a $\mathfrak{G}_0$-algebra, denote by $\mathfrak{h} \cdot A$ the $\mathfrak{G}_0$-algebra with underlying $\mathbb{K}$-algebra A and structure maps

$$\omega_{\mathfrak{h}\cdot A,1}(g) = \omega_{A,1}(\mathfrak{h}^{-1}g\mathfrak{h}), \qquad \omega_{\mathfrak{h}\cdot A,2}(x) = \omega_{A,2}({}^{\mathfrak{h}^{-1}}x).$$

It then follows immediately from the definitions that there is a $\mathfrak{G}$-algebra isomorphism

$$\mathrm{HInd}_{\mathfrak{G}_0}^{\mathfrak{G}} A \simeq A \boxplus \mathfrak{h} \cdot A^{\vee},$$

where an element $f \in G_1 \setminus G_0$ acts on $A \boxplus \mathfrak{h} \cdot A^\vee$ by the matrix

$$\begin{pmatrix} 0 & \omega_{A,1}(f\mathfrak{h}) \\ \omega_{A,1}(\mathfrak{h}^{-1}f)^{\mathrm{op}} & 0 \end{pmatrix}.$$

This isomorphism generalizes as follows.

Lemma 3.8. *Let $\mathfrak{G}$ be a $\mathbb{Z}_2$-graded crossed module with trivially graded crossed submodule $\mathfrak{H}$. For each $\mathfrak{H}$-algebra A and $\mathfrak{h} \in H_1 \setminus H_0$, there is a $\mathfrak{G}$-algebra isomorphism*

$$\mathrm{HInd}_{\mathfrak{H}}^{\mathfrak{G}} A \simeq \mathrm{HInd}_{\mathfrak{h}\cdot\mathfrak{H}\cdot\mathfrak{h}^{-1}}^{\mathfrak{G}} \mathfrak{h} \cdot A^\vee.$$

While the pseudofunctor $\mathrm{HInd}_{\mathfrak{G}_0}^{\mathfrak{G}}$ is not monoidal, it does admit a natural enhancement to a $\mathfrak{G}$-$\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$-module pseudofunctor. We do not use the full strength of this statement, only that for each $\mathfrak{G}_0$-algebra A and $\mathfrak{G}$-algebra B, there is a $\mathfrak{G}$-algebra isomorphism

$$B \boxtimes \mathrm{HInd}_{\mathfrak{G}_0}^{\mathfrak{G}} A \simeq \mathrm{HInd}_{\mathfrak{G}_0}^{\mathfrak{G}} \left(\mathrm{Res}_{\mathfrak{G}_0}^{\mathfrak{G}} B \boxtimes A\right). \tag{15}$$

Indeed, from the perspective described above Lemma 3.8, the left and right hand sides of the desired isomorphism (15) are represented by

$$\left((B \boxtimes A) \boxplus (B \boxtimes \mathfrak{h} \cdot A^\vee), \begin{pmatrix} 0 & \omega_{B,1}(\mathfrak{h}) \otimes \omega_{A,1}(\mathfrak{h}^2) \\ \omega_{B,1}(\mathfrak{h}) \otimes \mathrm{id}_A & 0 \end{pmatrix} \right)$$

and

$$\left((B \boxtimes A) \boxplus (\mathfrak{h} \cdot B^\vee \boxtimes \mathfrak{h} \cdot A^\vee), \begin{pmatrix} 0 & \omega_{B,1}(\mathfrak{h}^2) \otimes \omega_{A,1}(\mathfrak{h}^2) \\ \mathrm{id}_{B\otimes A} & 0 \end{pmatrix} \right),$$

respectively, where we have displayed the matrices giving the action of $\mathfrak{h}$ in each case (which, together with the underlying $\mathfrak{G}_0$-algebra structure, determines the $\mathfrak{G}$-algebra structure). These pairs are equivalent via the map $\begin{pmatrix} \mathrm{id}_{B\otimes A} & 0 \\ 0 & \omega_{B,1}(\mathfrak{h})\otimes 1_A \end{pmatrix}$.

4. The classification of Real 2-representations on $2\mathsf{Vect}_{\mathbb{K}}$

We apply the results of the previous chapters to Real 2-representation theory.

4.1. From N-weak $\mathfrak{G}$-algebras to Real 2-modules.

We begin by connecting N-weak algebras to 2-representation theory. The proof of the following result can be seen as justifying (or even deriving) the definition of an N-weak algebra.

Proposition 4.1. *Let $\mathfrak{G}$ be a $\mathbb{Z}_2$-graded crossed module. Every separable N-weak $\mathfrak{G}$-algebra $\omega_A : \mathfrak{G} \to \mathrm{AUT}^{\mathrm{gen}}(A)$ induces a Real 2-module $\Theta_A : \widetilde{\mathfrak{G}} \to \mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$.*

Proof. The proof is similar to that of [31, Proposition 2.3], which treats the ordinary RW-weak case. The present setting is complicated by the fact that we work in the non-abelian group $A^\times$, instead of $Z(A)^\times$.

Set $\Theta_A(\star) = A$. For a 1-morphism $g : \star \to \star$, let $\Theta_A(g)$ be the A-${}^{\pi(g)}A$-bimodule $A_{\omega_1(g)}$. For a 2-morphism $x : g \Rightarrow \partial(x)g$, set (in the notation of Lemma 1.1)

$$\Theta_A(g, x) = \Upsilon_{\omega_1(g)}^{\omega_1(\partial(x)g)} \left((\omega_3(\partial x, g)\omega_2(x))^{-1} \right).$$

Equations (1) and (3) imply that $\Theta_A(g, x)$ is well-defined. For $g_i \in G_1$, define

$$\Theta_A(g_2, g_1) : \Theta_A(g_2) \diamond \Theta_A(g_1) \Longrightarrow \Theta_A(g_2 g_1)$$

to be $\Upsilon_{\omega_1(g_2)\circ\omega_1(g_1)}^{\omega_1(g_2 g_1)}(\omega_3(g_2, g_1)^{-1})$. Equation (1) implies that $\Theta_A(g_2, g_1)$ is well-defined.

To see that Θ_A is compatible with the vertical composition of 2-morphisms, note that $\Theta_A(\partial(x_1)g_1, x_2) \circ \Theta_A(g_1, x_1)$ is equal to $\Upsilon^{\omega_1(\partial(x_2x_1)g_1)}_{\omega_1(g_1)}(\clubsuit^{-1})$, where

$$\begin{aligned}
\clubsuit &= \omega_3(\partial x_2, \partial(x_1)g_1)\underline{\omega_2(x_2)\omega_3(\partial x_1, g_1)}\omega_2(x_1) \\
&\overset{(3)}{=} \underline{\omega_3(\partial x_2, \partial(x_1)g_1) \cdot {}^{\partial x_2}\omega_3(\partial x_1, g_1)}\omega_2(x_2)\omega_2(x_1) \\
&\overset{(4)}{=} \omega_3(\partial(x_2x_1), g_1)\underline{\omega_3(\partial x_2, \partial x_1)\omega_2(x_2)\omega_2(x_1)} \\
&\overset{(2)}{=} \omega_3(\partial(x_2x_1), g_1)\omega_2(x_2x_1).
\end{aligned}$$

By abuse of notation, we write ${}^{\partial x_2}\omega_3(\partial x_1, g_1)$ in place of ${}^{\omega_1(\partial x_2)}\omega_3(\partial x_1, g_1)$. At each stage of the calculation where we specify $\overset{(N)}{=}$, this indicates that equation (N) is applied to the underlined part of the previous expression. It follows that $\Theta_A(\partial(x_1)g_1, x_2) \circ \Theta_A(g_1, x_1) = \Theta_A(g_1, x_2x_1)$.

Consider next the horizontal multiplicativity of Θ_A. Let $g_i \in G_1$ and $x_i \in G_2$. Then[4] $\Theta_A(g_2g_1, x_2 \cdot {}^{g_2}x_1) \circ \Theta_A(g_2, g_1) = \Upsilon(\spadesuit^{-1})$, where

$$\begin{aligned}
\spadesuit &= \omega_3(\partial(x_2 \cdot {}^{g_2}x_1), g_2g_1)\underline{\omega_2(x_2 \cdot {}^{g_2}x_1)}\omega_3(g_2, g_1) \\
&\overset{(2)}{=} \omega_3(\partial(x_2 \cdot {}^{g_2}x_1), g_2g_1)\omega_3(\partial x_2, \partial({}^{g_2}x_1))\omega_2(x_2)\omega_2({}^{g_2}x_1)\omega_3(g_2, g_1).
\end{aligned}$$

Two applications of the cocycle condition (4) give

$$\begin{aligned}
&\omega_3(\partial(x_2)g_2\partial(x_1)g_2^{-1}, g_2g_1)\omega_3(\partial x_2, g_2\partial(x_1)g_2^{-1}) = \\
&\qquad \omega_3(\partial x_2, g_2\partial(x_1)g_1) \cdot {}^{\partial x_2}\left[\omega_3(g_2\partial(x_1), g_1)\omega_3(g_2\partial(x_1)g_2^{-1}, g_2) \cdot {}^{\partial({}^{g_2}x_1)}\omega_3(g_2, g_1)^{-1}\right].
\end{aligned}$$

Equation (3) implies that

$${}^{\partial x_2}\left[{}^{\partial({}^{g_2}x_1)}\omega_3(g_2, g_1)^{-1}\right]\omega_2(x_2)\omega_2({}^{g_2}x_1)\omega_3(g_2, g_1) = \omega_2(x_2)\omega_2({}^{g_2}x_1)$$

so that

$$\begin{aligned}
\spadesuit &= \omega_3(\partial x_2, g_2\partial(x_1)g_1) \cdot \underline{{}^{\partial x_2}\left[\omega_3(g_2\partial x_1, g_1)\omega_3(g_2\partial(x_1)g_2^{-1}, g_2)\right]\omega_2(x_2)}\omega_2({}^{g_2}x_1) \\
&\overset{(3)}{=} \omega_3(\partial x_2, g_2\partial(x_1)g_1)\omega_2(x_2)\omega_3(g_2\partial x_1, g_1)\underline{\omega_3(g_2\partial(x_1)g_2^{-1}, g_2)}\omega_2({}^{g_2}x_1) \\
&\overset{(4)}{=} \omega_3(\partial x_2, g_2\partial(x_1)g_1)\omega_2(x_2)\omega_3(g_2\partial x_1, g_1) \cdot {}^{g_2\partial x_1}\omega_3(g_2^{-1}, g_2)\omega_3(g_2\partial x_1, g_2^{-1})^{-1}\omega_2({}^{g_2}x_1).
\end{aligned}$$

Using equation (5) to rewrite $\omega_2({}^{g_2}x_1)$ in terms of ${}^{g_2}\omega_2(x_1)$, and then using equation (4), we find that

$$\omega_3(g_2\partial x_1, g_2^{-1})^{-1}\omega_2({}^{g_2}x_1) = \omega_3(g_2, \partial x_1) \cdot {}^{g_2}\omega_2(x_1)\omega_3(g_2, g_2^{-1})^{-1}.$$

[4]For notational simplicity, we omit the sub/superscripts of Υ for the remainder of the proof.

Continuing, we have

$$\begin{aligned}
\spadesuit \; = \; & \omega_3(\partial(x_2)g_2, \partial(x_1)g_1)\omega_3(\partial x_2, g_2) \cdot {}^{\partial x_2}\underline{\omega_3(g_2, \partial(x_1)g_1)^{-1}} \cdot \\
& \underline{{}^{\partial x_2}\omega_3(g_2 \partial x_1, g_1)}\omega_2(x_2) \cdot {}^{g_2 \partial x_1}\omega_3(g_2^{-1}, g_2)\omega_3(g_2, \partial x_1) \cdot {}^{g_2}\omega_2(x_1)\omega_3(g_2, g_2^{-1})^{-1} \\
\overset{(4)}{=} \; & \omega_3(\partial(x_2)g_2, \partial(x_1)g_1)\underline{\omega_3(\partial x_2, g_2) \cdot {}^{\partial x_2}({}^{g_2}\omega_3(\partial x_1, g_1))} \\
& \cdot {}^{\partial x_2}\omega_3(g_2, \partial x_1)^{-1}\omega_2(x_2) \cdot \underline{{}^{g_2 \partial x_1}\omega_3(g_2^{-1}, g_2)\omega_3(g_2, \partial x_1)} \cdot {}^{g_2}\omega_2(x_1)\omega_3(g_2, g_2^{-1})^{-1} \\
\overset{(1),(1)}{=} \; & \omega_3(\partial(x_2)g_2, \partial(x_1)g_1) \cdot {}^{\partial(x_2)g_2}\omega_3(\partial x_1, g_1)\omega_3(\partial x_2, g_2) \cdot \\
& \underline{{}^{\partial x_2}\omega_3(g_2, \partial x_1)^{-1}\omega_2(x_2)}\omega_3(g_2, \partial x_1) \cdot \underline{{}^{g_2}({}^{\partial x_1}\omega_3(g_2^{-1}, g_2)) \cdot {}^{g_2}\omega_2(x_1)}\omega_3(g_2, g_2^{-1})^{-1} \\
\overset{(3),(3)}{=} \; & \omega_3(\partial(x_2)g_2, \partial(x_1)g_1) \cdot {}^{\partial(x_2)g_2}\omega_3(\partial x_1, g_1)\omega_3(\partial x_2, g_2)\underline{\omega_2(x_2)} \\
& \omega_3(g_2, \partial x_1)^{-1}\omega_3(g_2, \partial x_1) \cdot \underline{{}^{g_2}(\omega_2(x_1)}\omega_3(g_2^{-1}, g_2))\omega_3(g_2, g_2^{-1})^{-1} \\
\overset{(3)}{=} \; & \omega_3(\partial(x_2)g_2, \partial(x_1)g_1) \cdot {}^{\partial(x_2)g_2}\omega_3(\partial x_1, g_1)\underline{\omega_3(\partial x_2, g_2)^{\partial x_2}({}^{g_2}(\omega_2(x_1)))} \\
& \omega_2(x_2) \cdot \underline{{}^{g_2}\omega_3(g_2^{-1}, g_2)\omega_3(g_2, g_2^{-1})^{-1}} \\
\overset{(3),(4)}{=} \; & \omega_3(\partial(x_2)g_2, \partial(x_1)g_1) \cdot {}^{\partial(x_2)g_2}\left[\omega_3(\partial(x_1), g_1)\omega_2(x_1)\right]\omega_3(\partial(x_2), g_2)\omega_2(x_2).
\end{aligned}$$

We need to show that $\Upsilon(\spadesuit^{-1})$ is equal to the composition

$$\Theta_A(\partial(x_2)g_2, \partial(x_1)g_1) \circ (\Theta_A(g_2, x_2) \diamond \Theta_A(g_1, x_1)).$$

Since $\Theta_A(g_2, x_2) \diamond \Theta_A(g_1, x_1)$ is the result of applying Υ to the inverse of

$${}^{\partial(x_2)g_2}\left[\omega_3(\partial x_1, g_1)\omega_2(x_1)\right]\omega_3(\partial x_2, g_2)\omega_2(x_2),$$

the desired equality follows.

The pentagon identity holds by equation (4); details are left to the reader. □

Proposition 4.2. *The assignment $A \mapsto \Theta_A$ of Proposition 4.1 extends to a locally fully faithful pseudofunctor* $\Theta : \mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}} \to \mathrm{Hom}_{\mathsf{Bicat}_{\mathsf{con}}}(\widetilde{\mathfrak{G}}, \mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}})$.

Proof. After writing out the explicit description of 1- and 2-morphisms in the bicategory $1\mathrm{Hom}_{\mathsf{Bicat}_{\mathsf{con}}}(\widetilde{\mathfrak{G}}, \mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}})$ (see Section 1.5) and comparing them to the definition of those of $\mathfrak{G}\text{-}N\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}$, the statement is clear. □

Remarks. (i) More conceptually, Θ_A can be described as the composition of the structure map $\widetilde{\omega}_A : \widetilde{\mathfrak{G}} \to \widetilde{\mathrm{AUT}^{\mathrm{gen}}}(A)$ (constructed as in the proof of Proposition 4.1) with

$$\widetilde{\mathrm{AUT}^{\mathrm{gen}}}(A) \simeq 1\mathrm{Aut}^{\mathrm{gen}}_{\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd,rep}}}(A) \hookrightarrow 1\mathrm{Aut}^{\mathrm{gen}}_{\mathcal{A}\mathsf{lg}_{\mathbb{K}}^{\mathrm{fd}}}(A).$$

(ii) When $\mathfrak{G}$ is trivially graded, the proof of Proposition 4.1 yields a 2-module $\Theta_A : \widetilde{\mathfrak{G}} \to \mathcal{A}\mathsf{LG}_{\mathbb{K}}$ without any separability assumption on A. Compare [31, Proposition 2.3].

4.2. **Morita bicategories and 2-representations on $2\mathsf{Vect}_{\mathbb{K}}$.** Permutation actions are at the heart of Real 2-representations on $2\mathsf{Vect}_{\mathbb{K}}$, as we now review.

Proposition 4.3. *A Real 2-representation of $\widetilde{\mathfrak{G}}$ on $2\mathsf{Vect}_{\mathbb{K}}$ of dimension t determines a homomorphism $\sigma : G_1 \to \mathbb{S}_t$ which factors through the quotient $G_1 \to \pi_1(\mathfrak{G})$ and a cocycle $\mu \in Z^2(G_1, (\mathbb{K}^\times)^t_\pi)$, where $\mathbb{K}^t$ is considered as a G_1-algebra via σ.*

Proof. Since G_1 is a $\mathbb{Z}_2$-graded crossed submodule of $\mathfrak{G}$, each Real 2-representation of $\widetilde{\mathfrak{G}}$ restricts to a Real 2-representation of G_1. The construction of σ and μ are then given in the proof of [37, Theorem 5.7]. There it is assumed that G_1 is finite, but this is not used in this part of the proof. It is straightforward to show that σ factors through $G_1 \to \pi_1(\mathfrak{G})$. □

Lemma 4.4 (cf. [31, Lemma 2.4]). *Let $\mathfrak{G}$ be a $\mathbb{Z}_2$-graded crossed module and let ρ be a Real 2-module over $\widetilde{\mathfrak{G}}$ which restricts to a G_1-algebra A. Then there exists an RW-weak $\mathfrak{G}$-algebra structure extending the G_1-structure of A such that ρ and Θ_A are equivalent.*

Proof. In the ordinary case this is [31, Lemma 2.4]. The definition of the RW-weak $\mathfrak{G}$-algebra and the verification of the required axioms extend verbatim to the $\mathbb{Z}_2$-graded case:

$$\omega_2(x) := \Upsilon^{-1}(\rho(e \overset{x}{\Rightarrow} \partial x)), \qquad \omega_3(f,g) := \Upsilon^{-1}(\rho(f,g)).$$

It follows from the definition of Θ_A, given in the proof of Proposition 4.1, that Θ_A is equivalent to ρ. □

Proposition 4.5. *Let ρ be a Real 2-representation of $\widetilde{\mathfrak{G}}$ on $2\mathsf{Vect}_{\mathbb{K}}$ with induced morphism $\sigma : G_1 \to \mathbb{S}_t$ and cocycle $\mu \in Z^2(G_1, (\mathbb{K}^\times)^t_\pi)$. If μ is realizable, then there exists a split semisimple RW-weak $\mathfrak{G}$-algebra A such that ρ and Θ_A are equivalent.*

Proof. Write R for $\mathbb{K}^t$, viewed as the target object of the Real 2-module associated to ρ. Let M be a realization of μ^{-1} with Peirce decomposition $M \simeq \oplus_i M_i$. Then $A = \mathrm{End}_R(M) \simeq \oplus_i \mathrm{End}_{\mathbb{K}}(M_i)$ is a split semisimple $\mathbb{K}$-algebra which, by Proposition 2.3, has a G_1-algebra structure $\mathfrak{a}$.

We first define an equivalence $F : \rho \Rightarrow \Theta_A$ of Real 2-modules over G_1. Let $F(\star) : R \to A$ be the A-R-bimodule M, its right R-module structure defined via its left R-module structure (recall that R is commutative). For each $g \in G_1$, define an A-R-bimodule intertwiner

$$F(g) : A_{\mathfrak{a}_g} \otimes_{\pi(g)A} ({}^{\pi(g)}M) \Longrightarrow M \otimes_R \rho(g)$$

as follows. Identify $M \otimes_R \rho(g)$ with $M_{\rho(g)}$ and set $F(g)(a \otimes m) = a(\mathfrak{p}_M(g)(m))$. The map is well-defined because, for each $b \in A$, we have

$$\begin{aligned} F(a\mathfrak{a}_g(b) \otimes m) &= a(\mathfrak{a}_g(b)(\mathfrak{p}_M(g)(m))) = a(\mathfrak{p}_M(g)(b(\mathfrak{p}_M(g)^{-1}(\mathfrak{p}_M(g)(m))))) \\ &= a(\mathfrak{p}_M(g)(b(m))) = F(a \otimes b(m)). \end{aligned}$$

Left A-linearity of $F(g)$ is clear. Right R-linearity follows from the computation

$$F(g)((a \otimes m)r) = a(\mathfrak{p}_M(g)(rm)) = (g \cdot r)a(\mathfrak{p}_M(g)(m)) = F(g)(a \otimes m) \cdot_{\rho(g)} r,$$

where $- \cdot_{\rho(g)} -$ indicates the $\rho(g)$-twisted right R-module structure of M.

The coherence of the assignments $g \mapsto F(g)$ amounts to the commutativity of the diagram

$$\begin{array}{ccc}
M \otimes (\rho(g_2) \circ {}^{\pi(g_2)}\rho(g_1)) & \xrightarrow{\rho_{g_2,g_1}} & M \otimes \rho(g_2 g_1) \\
\uparrow & & \uparrow {\scriptstyle F(g_2g_1)} \\
(M \otimes \rho(g_2)) \circ {}^{\pi(g_2)}\rho(g_1) & & \Theta_A(g_2g_1) \otimes {}^{\pi(g_2g_1)}M \\
{\scriptstyle F(g_2)} \uparrow & & \uparrow {\scriptstyle \Theta_A(g_2,g_1)} \\
(\Theta_A(g_2) \otimes {}^{\pi(g_2)}M) \circ {}^{\pi(g_2)}\rho(g_1) & & (\Theta_A(g_2) \circ {}^{\pi(g_2)}\Theta_A(g_1)) \otimes {}^{\pi(g_2g_1)}M \\
\uparrow & & \uparrow \\
\Theta_A(g_2) \circ {}^{\pi(g_2)}(M \otimes \rho(g_1)) & \xleftarrow[{}^{\pi(g_2)}F(g_1)]{} & \Theta_A(g_2) \circ {}^{\pi(g_2)}(\Theta_A(g_1) \otimes {}^{\pi(g_1)}M),
\end{array}$$

where all unlabelled arrows are associativity isomorphisms. By construction, the cocycle $\mu \in Z^2(G_1, R^\times_\pi)$ is determined by the 2-isomorphisms ρ_{g_2,g_1}. Since A is a strict G_1-algebra, the map $\Theta_A(g_2, g_1)$ is induced by the corresponding map in $\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$ and, in particular, does not involve a cocycle. On the other hand, "the composition" of $F(g_2)$ with ${}^{\pi(g_2)}F(g_1)$ (indicated by the dashed arrow in the diagram above) is equal to $\mu(g_2, g_1)^{-1}F(g_2g_1)$. This gives the required commutativity.

We can now apply Lemma 4.4 to conclude that we can extend the G_1-algebra structure of A to an RW-weak $\mathfrak{G}$-algebra structure such that $\rho \simeq \Theta_A$ as Real 2-modules. □

Denote by $\mathsf{RRep}^r_{2\mathsf{Vect}_{\mathbb{K}}}(\mathcal{G})$ the full subbicategory of $\mathsf{RRep}_{2\mathsf{Vect}_{\mathbb{K}}}(\mathcal{G})$ on realizable Real 2-representations of $\mathcal{G}$. It is a monoidal subbicategory.

Theorem 4.6. *Assume that $\mathbb{K}$ is separably closed. Let $\mathfrak{G}$ be a $\mathbb{Z}_2$-graded crossed module. Under the identification $\mathsf{RRep}_{2\mathsf{Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}}) \simeq 1\mathrm{Hom}_{\mathsf{Bicat}_{\mathsf{con}}}(\widetilde{\mathfrak{G}}, \mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}})$, the subbicategory $\mathsf{RRep}^r_{2\mathsf{Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}})$ is biequivalent to $\mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$.*

Proof. By Proposition 4.5, the biessential image of $\mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$ under the embedding of Proposition 4.2 can be identified with $\mathsf{RRep}^r_{2\mathsf{Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}})$. □

We can now describe equivalence classes of Real 2-representations of $\widetilde{\mathfrak{G}}$ on $2\mathsf{Vect}_{\mathbb{K}}$ in terms of $\mathfrak{G}$-algebras.

Corollary 4.7. *Assume that $\mathbb{K}$ is separably closed and that G_1 is finite. Then there is a bijection*

$$\pi_0(\mathsf{RRep}_{2\mathsf{Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}})) \simeq \{\text{separable strict } \mathfrak{G}\text{-algebras}\}/\mathfrak{G}\text{-Morita equivalence}.$$

Proof. Since G_1 is finite, each element of $Z^2(G_1, (\mathbb{K}^\times)^t_\pi)$ is realizable and we have $\mathsf{RRep}^r_{2\mathsf{Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}}) = \mathsf{RRep}_{2\mathsf{Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}})$. Theorem 4.6 then gives the desired bijection. □

Remarks. (i) Without the finiteness assumption on G_1, the following version of Corollary 4.7 still holds:

$$\pi_0(\mathsf{RRep}^r_{2\mathsf{Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}})) \simeq \{\text{separable strict } \mathfrak{G}\text{-algebras}\}/\mathfrak{G}\text{-Morita equivalence}.$$

(ii) When $\mathfrak{G}$ is trivially graded, we recover the finite dimensional results of [31, §3]. It is not clear, however, if the results involving semi-matrix algebras (used to treat non-realizable cocycles) admit a $\mathbb{Z}_2$-graded generalization.

Example. Suppose that $\mathfrak{G}$ is a $\mathbb{Z}_2$-graded crossed module in which ∂ is trivial; this corresponds to so-called split $\mathbb{Z}_2$-graded 2-groups, that is, those with trivial Sinh 3-cocycle. The set $\pi_0(\mathfrak{G}\text{-}\mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}})$ appearing in Corollary 4.7 can be described as follows. Since every $\mathfrak{G}$-algebra is $\mathfrak{G}$-Morita equivalent to an RW-weak $\mathfrak{G}$-algebra whose underlying $\mathbb{K}$-algebra is $A = \mathbb{K}^t$ for some $t \geq 1$, it suffices to restrict attention to such A. We have a homomorphism $\omega_{A,1} : G_1 \to \mathbb{S}_t$, a G_1-invariant homomorphism $\omega_{A,2} : G_2 \to (\mathbb{K}^\times)^t_\pi$ and a cocycle $\omega_{A,3} \in Z^2(G_1, (\mathbb{K}^\times)^t_\pi)$. However, different triples $\{\omega_{A,i}\}_i$ define $\mathfrak{G}$-Morita equivalent RW-weak $\mathfrak{G}$-algebras. Recall that the generalized automorphism 2-group of $\mathbb{K}^t \in \mathcal{A}\mathsf{lg}^{\mathrm{fd}}_{\mathbb{K}}$ is modelled by $\mathrm{AUT}^{\mathrm{gen}}(\mathbb{K}^t) = ((\mathbb{K}^\times)^t \xrightarrow{\partial} \mathbb{S}_t \times \mathbb{Z}_2)$. Pulling back along $\tau \in \mathbb{S}_t \times \mathbb{Z}_2$ sends ω_A to the $\mathfrak{G}$-Morita equivalent RW-weak $\mathfrak{G}$-algebra with structure maps

$$(\tau\omega_{A,1}(-)\tau^{-1}, {}^{\tau}\omega_{A,2}(-), \tau_*\omega_{A,3}(-)).$$

The remaining ambiguity is due to non-trivial G_1-module structures on the identity $\mathbb{K}^t$-bimodule, which are determined by maps $\mu : G_1 \to (\mathbb{K}^\times)^t$. This gives a $\mathfrak{G}$-Morita equivalence with the RW-weak $\mathfrak{G}$-algebra $(\omega_{A,1}, \omega_{A,2}, \omega_{A,3}d\mu)$. In this way, we obtain a $\mathbb{Z}_2$-graded generalization of the split case of [14, Theorem 5.5]. See also [37, §5.3]. ◁

We use Theorem 4.6 to model direct sums, tensor products and duals of realizable (Real) 2-modules in terms of the corresponding operations for $\mathfrak{G}$-algebras. We also define various inductions of (Real) 2-modules in the same way. For example, if Θ_A is a realizable 2-module over an non-trivially graded crossed submodule $\mathfrak{H}$ of $\mathfrak{G}$, then the Real 2-module $\mathrm{RInd}^{\mathfrak{G}}_{\mathfrak{H}}\Theta_A$ is defined to be $\Theta_{\mathrm{RInd}^{\mathfrak{G}}_{\mathfrak{H}}A}$.

4.3. **A structure theorem for Real 2-modules.** The next result plays an important role in Section 5.2.

Theorem 4.8. *Let $\mathfrak{G}$ be a $\mathbb{Z}_2$-graded crossed module, ρ a realizable Real 2-module over $\widetilde{\mathfrak{G}}$. The following statements hold.*

(i) There exist indecomposable Real 2-modules $\rho_1, \dots, \rho_n$ over $\widetilde{\mathfrak{G}}$, unique up to equivalence, such that $\rho \simeq \boxplus_{i=1}^n \rho_i$.

(ii) If ρ is indecomposable, then there exists a subgroup H of G_1 such that $\mathfrak{G}_H := (G_2 \xrightarrow{\partial} H)$ is a crossed submodule of $\mathfrak{G}$ and a one dimensional realizable 2-module ν over $\widetilde{\mathfrak{G}}_H$, Real if H is non-trivially graded, such that $\rho \simeq \mathrm{Ind}^{\widetilde{\mathfrak{G}}}_{\widetilde{\mathfrak{G}}_H}\nu$. Moreover, the pair (H, ν) is unique up to G_1-conjugation, $(H,\nu) \mapsto (gHg^{-1}, g \cdot {}^{\pi(g)}\nu)$, $g \in G_1$, and equivalence in ν.

Proof. By Theorem 4.6, it suffices to prove the statement at the level of $\mathfrak{G}$-algebras.

Let A be a split semisimple $\mathfrak{G}$-algebra. Decompose the underlying $\mathbb{K}$-algebra into simple factors, $A = \oplus_{i=1}^t A_i$. Then G_1 acts on the set $\{1, \dots, t\}$, giving an orbit decomposition $\mathcal{O}_1 \sqcup \cdots \sqcup \mathcal{O}_n$, so that each $A_{\mathcal{O}_i} := \oplus_{j \in \mathcal{O}_i} A_j$, $i = 1, \dots, n$, is an indecomposable $\mathfrak{G}$-algebra and $A \simeq \boxplus_{i=1}^n A_{\mathcal{O}_i}$. The uniqueness statement is clear.

Turning to the second statement, if A is indecomposable, then it is necessarily of the form $M_n(\mathbb{K})^{\oplus t}$ and the G_1-action on $\{1, \dots, t\}$ is transitive. Let $H \leq G_1$ be the stabilizer of $1 \in \{1, \dots, t\}$; it is trivially graded if no element of $G_1 \setminus G_0$ fixes 1, and is non-trivially graded otherwise. Set $B = M_n(\mathbb{K})$, regarded as the first summand of A. Then B inherits from A a $\mathfrak{G}_H$-algebra structure. It follows immediately from the definitions that $\mathrm{Ind}^{\mathfrak{G}}_{\mathfrak{G}_H} B \simeq A$ as $\mathfrak{G}$-algebras, where Ind denotes HInd or RInd as appropriate. It is clear that we can replace B with any $\mathfrak{G}_H$-Morita equivalent $\mathfrak{G}_H$-algebra. If we consider instead the stabilizer of a point other than 1, then H is replaced with a G_1-conjugate, say $H' = gHg^{-1}$, and B is replaced $g \cdot {}^{\pi(g)}B$. □

Remark. When $\mathfrak{G}$ is trivially graded, variants of Theorem 4.8(ii) are well-known; see, for example, [29, Theorem 3.2], [16, Proposition 7.3] and [31, Theorem 3.7].

5. The Grothendieck ring of Real 2-representations

In this section we describe the Grothendieck ring of $\mathsf{RRep}_{2\mathsf{Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}})$ in terms of Real generalized Burnside rings.

5.1. **Generalized Burnside rings.** We recall some basic material about generalized Burnside rings. The reader is referred to [18, §1] and [31, §4] for details.

Given a finite group G, let $\mathcal{S}(G)$ be the category whose objects are subgroups of G and whose morphisms are conjugations,

$$\mathrm{Hom}_{\mathcal{S}(G)}(H_1, H_2) = \{\gamma_g \mid g \in G,\ gH_1g^{-1} \leq H_2\}.$$

Morphisms are composed using the multiplication in G. We emphasize that, even if $e \neq g_1g_2^{-1}$ is in the centralizer of H_1, the morphisms $\gamma_{g_1}, \gamma_{g_2} : H_1 \to H_2$ are not identified. This is as in [31] and is in contrast to [18].

Suppose now that G is $\mathbb{Z}_2$-graded. The degree $\pi(H)$ of $H \in \mathcal{S}(G)$ is defined to be $+1$ if H is trivially graded and -1 otherwise. There are no morphisms in $\mathcal{S}(G)$ from an object of degree -1 to an object of degree $+1$.

Fix a functor $\Phi : \mathcal{S}(G)^{\mathrm{op}} \to \mathsf{SGrp}$ to the category of semigroups. A Φ-decorated G-set is a G-set X together with frills $\mathfrak{f}_{\mathbf{x}} \in \Phi(\mathrm{Stab}_G(\mathbf{x}))$, $\mathbf{x} \in X$, which satisfy $\Phi(\gamma_g)(\mathfrak{f}_{\mathbf{x}}) = \mathfrak{f}_{g\mathbf{x}}$ for each $g \in G$. The generalized Burnside ring $\mathbb{B}^{\Phi}(G)$ is defined to be the Grothendieck group of the category of finite Φ-decorated G-sets. An explicit description of $\mathbb{B}^{\Phi}(G)$ in our case of interest is given in Section 5.2. Given a commutative ring $\mathbb{A}$, we write $\mathbb{B}^{\Phi}_{\mathbb{A}}(G)$ for $\mathbb{B}^{\Phi}(G) \otimes_{\mathbb{Z}} \mathbb{A}$.

5.2. **The Grothendieck ring of Real 2-representations.** Let $\mathfrak{K}$ be a $\mathbb{Z}_2$-graded crossed module. Denote by $\pi_0(\mathsf{RRep}_{[1]}(\widetilde{\mathfrak{K}}))$ the abelian group of equivalence classes of Real 2-representations of $\widetilde{\mathfrak{K}}$ on $[1] \in 2\mathsf{Vect}_{\mathbb{K}}$. If $\mathfrak{K}$ is trivially graded, then this is simply $\pi_0(\mathsf{Rep}_{[1]}(\widetilde{\mathfrak{K}}))$. The group operation is the tensor product $\boxtimes$ of (Real) 2-representations.

Let $\mathfrak{G}$ be a $\mathbb{Z}_2$-graded crossed module with G_1 finite. The group $\pi_1(\mathfrak{G})$ inherits a $\mathbb{Z}_2$-grading from that of G_1. Given a subgroup $P \leq \pi_1(\mathfrak{G})$, denote by $\overline{P}$ its pre-image under the quotient map $G_1 \to \pi_1(\mathfrak{G})$. The crossed module $\mathfrak{G}_P := (G_2 \xrightarrow{\partial} \overline{P})$ is a $\mathbb{Z}_2$-graded crossed submodule of $\mathfrak{G}$; the grading trivial if and only if $P \leq \pi_1(\mathfrak{G})_0$.

Define a functor $\Phi : \mathcal{S}(\pi_1(\mathfrak{G}))^{\mathrm{op}} \to \mathsf{Ab}$ to the category of abelian groups as follows. At the level of objects, set

$$\Phi(P) = \pi_0(\mathsf{RRep}_{[1]}(\widetilde{\mathfrak{G}}_P)).$$

Let $\gamma_g : P_1 \to P_2$ be a morphism in $\mathcal{S}(\pi_1(\mathfrak{G}))$. The choice of a lift $\dot{g} \in G_1$ of $g \in \pi_1(\mathfrak{G})$ induces a strict $\mathbb{Z}_2$-graded crossed module homomorphism $\gamma_{\dot{g}} : \mathfrak{G}_{P_1} \to \mathfrak{G}_{P_2}$ by

$$\overline{P}_1 \ni p \mapsto \dot{g}p\dot{g}^{-1}, \qquad G_2 \ni x \mapsto {}^{\dot{g}}x.$$

The associated $\mathbb{Z}_2$-graded 2-group homomorphism is $\gamma_{\dot{g}} : \widetilde{\mathfrak{G}}_{P_1} \to \widetilde{\mathfrak{G}}_{P_2}$. With this notation, define $\Phi(\gamma_g)$ by

$$\Phi(\gamma_g)(\rho) = \begin{cases} {}^{\pi(g)}\rho \circ \gamma_{\dot{g}} & \text{if } \pi(P_1) = \pi(P_2), \\ \mathfrak{F}(\rho) \circ \gamma_{\dot{g}} & \text{otherwise.} \end{cases}$$

Here $\mathfrak{F} : \pi_0(\mathsf{RRep}_{[1]}(\widetilde{\mathfrak{G}}_P)) \to \pi_0(\mathsf{Rep}_{[1]}(\widetilde{\mathfrak{G}}_{P_0}))$ is the forgetful map. Well-definedness of Φ, that is, independence of the lift $\dot{g}$ of g, can be verified as in [31, Lemma 4.1].

Definition. *The Real Burnside ring of* $\mathfrak{G}$ *is* $\mathbb{B}^{\Phi}_{\mathbb{A}}(\mathfrak{G}) := \mathbb{B}^{\Phi}_{\mathbb{A}}(\pi_1(\mathfrak{G}))$.

It follows from the general theory of generalized Burnside rings that $\mathbb{B}^{\Phi}_{\mathbb{A}}(\mathfrak{G})$ is generated as an $\mathbb{A}$-module by pairs $\langle \rho, P\rangle$, where $P \leq \pi_1(\mathfrak{G})$ and $\rho \in \pi_0(\mathsf{RRep}_{[1]}(\widetilde{\mathfrak{G}}_P))$. The ring structure of $\mathbb{B}^{\Phi}_{\mathbb{A}}(\mathfrak{G})$ is determined by the formula

$$\langle \rho, P\rangle \cdot \langle \theta, Q\rangle = \sum_{PgQ \in P\backslash \pi_1(\mathfrak{G})/Q} \langle \Phi(\gamma_e : P\cap gQg^{-1} \to P)(\rho) \boxtimes \Phi(\gamma_{g^{-1}} : P \cap gQg^{-1} \to Q)(\theta), P\cap gQg^{-1}\rangle.$$

Example. We consider three illustrative cases of the product.

(i) Let $\mathfrak{h} \in \pi_1(\mathfrak{G}) \setminus \pi_1(\mathfrak{G})_0$ with lift $\dot{\mathfrak{h}} \in G_1 \setminus G_0$ and identify $\pi_1(\mathfrak{G})_0\backslash\pi_1(\mathfrak{G})/\pi_1(\mathfrak{G})_0$ with $\{e, \mathfrak{h}\}$. Then we have

$$\langle \rho, \pi_1(\mathfrak{G})_0\rangle \cdot \langle \theta, \pi_1(\mathfrak{G})_0\rangle \quad = \quad \langle \rho \boxtimes \theta, \pi_1(\mathfrak{G})_0\rangle + \langle \rho \boxtimes \dot{\mathfrak{h}} \cdot \theta^{\vee}, \pi_1(\mathfrak{G})_0\rangle,$$

where the notation $\dot{\mathfrak{h}} \cdot \theta^{\vee}$ is as in Section 3.4.

(ii) Similarly, we can identify $\pi_1(\mathfrak{G})\backslash\pi_1(\mathfrak{G})/\pi_1(\mathfrak{G})_0$ with $\{e\}$, so that

$$\langle \rho, \pi_1(\mathfrak{G})\rangle \cdot \langle \theta, \pi_1(\mathfrak{G})_0\rangle = \langle \mathrm{Res}^{\widetilde{\mathfrak{G}}}_{\widetilde{\mathfrak{G}}_0}(\rho) \boxtimes \theta, \pi_1(\mathfrak{G})_0\rangle.$$

(iii) Finally, identifying $\pi_1(\mathfrak{G})\backslash\pi_1(\mathfrak{G})/\pi_1(\mathfrak{G})$ with $\{e\}$, we have

$$\langle \rho, \pi_1(\mathfrak{G})\rangle \cdot \langle \theta, \pi_1(\mathfrak{G})\rangle = \langle \rho \boxtimes \theta, \pi_1(\mathfrak{G})\rangle.$$

$\triangleleft$

More generally, we can define the symbol $\langle \rho, P\rangle$ without the assumption that ρ is one dimensional. To do so, set $\langle \rho_1 \boxplus \rho_2, P\rangle := \langle \rho_1, P\rangle + \langle \rho_2, P\rangle$ for any two (Real) 2-representations ρ_1, ρ_2. Set also $\langle \mathrm{Ind}^{\widetilde{\mathfrak{G}}_Q}_{\widetilde{\mathfrak{G}}_P} \rho, Q\rangle := \langle \rho, P\rangle$ for subgroups $P \leq Q \leq \pi_1(\mathfrak{G})$. Here $\mathrm{Ind}^{\widetilde{\mathfrak{G}}_Q}_{\widetilde{\mathfrak{G}}_P}$ has one of three meanings, depending on the $\mathbb{Z}_2$-gradings of P and Q. Theorem 4.8 implies that these definitions are unambiguous.

For any subgroup $P \leq \pi_1(\mathfrak{G})$ and $g \in \pi_1(\mathfrak{G})$, the relation

$$\langle \rho, P\rangle = \langle \Phi(\gamma_g)(\rho), g^{-1}Pg\rangle$$

holds. In particular, when P is trivially graded and $\pi(g) = -1$, this relation becomes that of Lemma 3.8. Note that there is no reason for this relation to hold in $\mathbb{B}^{\Phi}_{\mathbb{A}}(\mathfrak{G}_0)$.

The Grothendieck group $K_0(\mathcal{V})$ of a bicategory $\mathcal{V}$ is defined to be the free abelian group generated by equivalence classes of objects of $\mathcal{V}$. If $\mathcal{V}$ is symmetric monoidal, then $K_0(\mathcal{V})$ has the structure of a commutative ring.

The (ungraded) Burnside ring of $\mathfrak{G}_0$, defined to be $\mathbb{B}^{\Phi}_{\mathbb{A}}(\mathfrak{G}_0) := \mathbb{B}^{\Phi}_{\mathbb{A}}(\pi_1(\mathfrak{G}_0))$, is shown in [31, Proposition 4.2] to be isomorphic to $K_0(\mathsf{Rep}_{\mathsf{2Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}}_0)) \otimes_{\mathbb{Z}} \mathbb{A}$. The next result gives a Real generalization.

Theorem 5.1. *Assume that* G_1 *is finite. Then the assignment* $\langle \rho, P\rangle \mapsto \mathrm{Ind}^{\widetilde{\mathfrak{G}}}_{\widetilde{\mathfrak{G}}_P} \rho$ *extends to an* $\mathbb{A}$*-algebra isomorphism* $\mathcal{I} : \mathbb{B}^{\Phi}_{\mathbb{A}}(\mathfrak{G}) \xrightarrow{\sim} K_0(\mathsf{RRep}_{\mathsf{2Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}})) \otimes_{\mathbb{Z}} \mathbb{A}$.

Proof. It follows from Theorem 4.8 that $\mathcal{I}$ is an $\mathbb{A}$-module isomorphism. We need to show that $\mathcal{I}$ is also a map of algebras.

As a first case, suppose that $P, Q \leq \pi_1(\mathfrak{G})$ are trivially graded. Let $\mathfrak{h} \in \pi_1(\mathfrak{G}) \setminus \pi_1(\mathfrak{G})_0$. Fix a complete set $\mathcal{T} \subset \pi_1(\mathfrak{G})_0$ of representatives of $P\backslash\pi_1(\mathfrak{G})_0/Q$. Then

$\mathcal{T} \sqcup \mathcal{T} \cdot \mathfrak{h}$ is a complete set of representatives of $P\backslash\pi_1(\mathfrak{G})/Q$ and we can write

$$\langle \rho, P\rangle \cdot \langle \theta, Q\rangle = \sum_{g\in\mathcal{T}} \langle \Phi(\gamma_e : P\cap gQg^{-1} \to P)(\rho) \boxtimes \Phi(\gamma_{g^{-1}} : P\cap gQg^{-1} \to Q)(\theta), P\cap gQg^{-1}\rangle +$$
$$\sum_{g\in\mathcal{T}} \langle \Phi(\gamma_e : P\cap g\mathfrak{h}Q(g\mathfrak{h})^{-1} \to P)(\rho)\boxtimes\Phi(\gamma_{(g\mathfrak{h})^{-1}} : P\cap g\mathfrak{h}Q(g\mathfrak{h})^{-1} \to Q)(\theta), P\cap g\mathfrak{h}Q(g\mathfrak{h})^{-1}\rangle.$$

Let us first interpret the right hand side as an element of the (ungraded) ring $\mathbb{B}^{\Phi}_{\mathbb{A}}(\mathfrak{G}_0)$ with product $- \cdot_0 -$. The first and second lines are then $\langle \rho, P\rangle \cdot_0 \langle \theta, Q\rangle$ and $\langle \rho, P\rangle \cdot_0 \langle \dot{\mathfrak{h}} \cdot \theta^{\vee}, \mathfrak{h}Q\mathfrak{h}^{-1}\rangle$, respectively. Under the isomorphism $\mathbb{B}^{\Phi}_{\mathbb{A}}(\mathfrak{G}_0) \simeq K_0(\mathsf{Rep}_{\mathsf{2Vect}_{\mathbb{K}}}(\widetilde{\mathfrak{G}}_0)) \otimes_{\mathbb{Z}} \mathbb{A}$, the right hand side is thus

$$\big(\mathrm{Ind}^{\widetilde{\mathfrak{G}}_0}_{\widetilde{\mathfrak{G}}_P}(\rho)\boxtimes\mathrm{Ind}^{\widetilde{\mathfrak{G}}_0}_{\widetilde{\mathfrak{G}}_Q}(\theta)\big)\boxplus\big(\mathrm{Ind}^{\widetilde{\mathfrak{G}}_0}_{\widetilde{\mathfrak{G}}_P}(\rho)\boxtimes\mathrm{Ind}^{\widetilde{\mathfrak{G}}_0}_{\widetilde{\mathfrak{G}}_{\mathfrak{h}Q\mathfrak{h}^{-1}}}(\dot{\mathfrak{h}}\cdot\theta^{\vee})\big) \simeq \mathrm{Ind}^{\widetilde{\mathfrak{G}}_0}_{\widetilde{\mathfrak{G}}_P}(\rho)\boxtimes\mathrm{Res}^{\widetilde{\mathfrak{G}}}_{\widetilde{\mathfrak{G}}_0}\big(\mathrm{HInd}^{\widetilde{\mathfrak{G}}}_{\widetilde{\mathfrak{G}}_Q}(\theta)\big).$$

Applying the map $\mathcal{I}$ then corresponds to applying hyperbolic induction. Doing so gives $\mathrm{HInd}^{\widetilde{\mathfrak{G}}}_{\widetilde{\mathfrak{G}}_P}(\rho) \boxtimes \mathrm{HInd}^{\widetilde{\mathfrak{G}}}_{\widetilde{\mathfrak{G}}_Q}(\theta)$, as required.

If instead P is non-trivially graded and Q is trivially graded, then we can choose a complete set of representatives of $P\backslash\pi_1(\mathfrak{G})/Q$ of the form $\mathcal{T} \subset \pi_1(\mathfrak{G})_0$ and

$$\langle \rho, P\rangle \cdot \langle \theta, Q\rangle = \sum_{g\in\mathcal{T}} \langle \Phi(\gamma_e : P\cap gQg^{-1} \to P)(\rho) \boxtimes \Phi(\gamma_{g^{-1}} : P\cap gQg^{-1} \to Q)(\theta), P\cap gQg^{-1}\rangle.$$

Interpreted as an element of $\mathbb{B}^{\Phi}_{\mathbb{A}}(\mathfrak{G}_0)$, the right hand side is $\langle \rho, P_0\rangle \cdot_0 \langle \theta, Q\rangle$, which corresponds to

$$\mathrm{Ind}^{\widetilde{\mathfrak{G}}_0}_{\widetilde{\mathfrak{G}}_{P_0}}\big(\mathrm{Res}^{\widetilde{\mathfrak{G}}_P}_{\widetilde{\mathfrak{G}}_{P_0}}(\rho)\big) \boxtimes \mathrm{Ind}^{\widetilde{\mathfrak{G}}_0}_{\widetilde{\mathfrak{G}}_Q}(\theta) \simeq \mathrm{Res}^{\widetilde{\mathfrak{G}}}_{\widetilde{\mathfrak{G}}_0}\big(\mathrm{RInd}^{\widetilde{\mathfrak{G}}}_{\widetilde{\mathfrak{G}}_P}(\rho)\big) \boxtimes \mathrm{Ind}^{\widetilde{\mathfrak{G}}_0}_{\widetilde{\mathfrak{G}}_Q}(\theta)$$

in $\mathbb{B}^{\Phi}_{\mathbb{A}}(\mathfrak{G}_0)$. Applying $\mathrm{HInd}^{\widetilde{\mathfrak{G}}}_{\widetilde{\mathfrak{G}}_0}$ and using the equivalence (15) then gives $\mathrm{RInd}^{\widetilde{\mathfrak{G}}}_{\widetilde{\mathfrak{G}}_P}(\rho) \boxtimes \mathrm{HInd}^{\widetilde{\mathfrak{G}}}_{\widetilde{\mathfrak{G}}_Q}(\theta)$, as required.

The case in which both P and Q are non-trivially $\mathbb{Z}_2$-graded is similar. ☐

6. Real categorical character theory

The categorical character theory of 2-representations was developed by Bartlett [6, §3] and Ganter and Kapranov [16, §4] in the case of finite groups and extended to essentially finite 2-groups by Rumynin and Wendland [31, §5]. A Real generalization for finite groups was introduced in [37, §5]. In each of these settings, there are two levels of characters, reflecting to the higher categorical nature of the representations involved. Motivated by work of Willerton [36], in this section we give a geometric formulation of this theory, simultaneously generalizing it to Real 2-representations of essentially finite 2-groups.

6.1. **Loop spaces of crossed modules.** Let G be a finite group. It is well-known (see, for example, [36]) that the loop groupoid of BG is equivalent to action groupoid $G/\!\!/G$, with G acting by conjugation, while the double loop groupoid is equivalent to the action groupoid associated to the simultaneous conjugation of commuting pairs in G. In this section we describe crossed module generalizations of these statements.

Let $\mathfrak{G}$ be a crossed module. We do not impose any finiteness conditions on $\mathfrak{G}$. Denote by $|\mathfrak{G}|$ the geometric realization of $\mathfrak{G}$. The homotopy 2-type of the free loop space

$\mathrm{Maps}(S^1, |\mathfrak{G}|)$ can be modelled by a crossed module in groupoids, which we denote by $\mathcal{L}\mathfrak{G}$. A result of Brown [11, Theorem 2.1] gives the following explicit description of $\mathcal{L}\mathfrak{G}$. The base groupoid $\mathcal{L}_{\leq 1}\mathfrak{G} = (\mathcal{L}_1\mathfrak{G} \rightrightarrows \mathcal{L}_0\mathfrak{G})$ has objects $\mathcal{L}_0\mathfrak{G} = G_1$ and morphisms $g_1 \to g_2$ given by pairs $(f, x) \in G_1 \times G_2$ which satisfy $g_2 = f\partial(x)g_1 f^{-1}$. Morphisms are composed according to the rule

$$\big(g_2 \xrightarrow{(f_2,x_2)} g_3\big) \circ \big(g_1 \xrightarrow{(f_1,x_1)} g_2\big) = g_1 \xrightarrow{(f_2 f_1, {}^{f_1^{-1}} x_2 x_1)} g_3.$$

The groupoid $\mathcal{L}_2\mathfrak{G}$ is totally disconnected. The group $\mathcal{L}_2\mathfrak{G}(g)$ sitting over $g \in \mathcal{L}_0\mathfrak{G}$ is G_2 and the restriction of the boundary functor $\partial : \mathcal{L}_2\mathfrak{G} \to \mathcal{L}_{\leq 1}\mathfrak{G}$ to $\mathcal{L}_2\mathfrak{G}(g)$ is given on morphisms by

$$\partial_g(z) = (\partial z, z^{-1}({}^g z)), \qquad z \in G_2.$$

The action of $(g_1 \xrightarrow{(f,x)} g_2) \in \mathcal{L}_{\leq 1}\mathfrak{G}$ on $z \in \mathcal{L}_2\mathfrak{G}(g)$ is defined only when $g = g_1$, in which case it is equal to ${}^f z \in \mathcal{L}_2\mathfrak{G}(g_2)$.

The fundamental groupoid $\pi_{\leq 1}(\mathcal{L}\mathfrak{G})$ of $\mathcal{L}\mathfrak{G}$ is defined to be the quotient of $\mathcal{L}_{\leq 1}\mathfrak{G}$ by the totally disconnected normal subgroupoid $\partial(\mathcal{L}_2\mathfrak{G})$. Similarly, the loop groupoid $\Lambda\pi_{\leq 1}(\mathcal{L}\mathfrak{G})$ is defined to be the groupoid of functors $B\mathbb{Z} \to \pi_{\leq 1}(\mathcal{L}\mathfrak{G})$. Concretely, objects of $\Lambda\pi_{\leq 1}(\mathcal{L}\mathfrak{G})$ are pairs (g, γ), where $g \in G_1$ and $\gamma \in \mathrm{End}_{\pi_{\leq 1}(\mathcal{L}\mathfrak{G})}(g) \simeq \mathrm{End}_{\mathcal{L}_{\leq 1}\mathfrak{G}}(g)/\mathrm{im}(\partial_g)$. A morphism $\mu : (g_1, \gamma_1) \to (g_2, \gamma_2)$ is a morphism $\mu : g_1 \to g_2$ in $\pi_{\leq 1}(\mathcal{L}\mathfrak{G})$ which satisfies $\mu\gamma_1 = \gamma_2\mu$.

Reflection of the circle S^1 defines a weak involution $\mathfrak{i} : \mathcal{L}\mathfrak{G} \to \mathcal{L}\mathfrak{G}$. Following the proof of [11, Theorem 2.1], we find that $\mathfrak{i}$ is given on $\mathcal{L}_0\mathfrak{G}$ by $\mathfrak{i}(g) = g^{-1}$, on $\mathcal{L}_1\mathfrak{G}$ by

$$\mathfrak{i}(g_1 \xrightarrow{(f,x)} g_2) = g_1^{-1} \xrightarrow{(f, {}^{g_1^{-1}} x^{-1})} g_2^{-1}$$

and on $\mathcal{L}_2\mathfrak{G}$ by $\mathfrak{i}(z) = z$.

Suppose now that we are given a $\mathbb{Z}_2$-grading $\pi : \mathfrak{G} \to \mathbb{Z}_2$. At the level of classifying spaces, π determines an equivalence class of double covers of $\mathfrak{G}$ which, as is easy to verify, is represented by the canonical map $\mathfrak{G}_0 \to \mathfrak{G}$. By choosing an element $\mathfrak{h} \in G_1 \setminus G_0$, the non-trivial deck transformation of $\mathfrak{G}_0$ can be realized as

$$G_1 \ni g \mapsto \mathfrak{h} g \mathfrak{h}^{-1}, \qquad G_2 \ni x \mapsto {}^{\mathfrak{h}}x.$$

Note that this squares to the inner automorphism determined by $\mathfrak{h}^2 \in G_0$ and, hence, is homotopic to the identity.

Similarly, π induces a grading $\mathcal{L}\mathfrak{G} \to \mathbb{Z}_2$. The associated double cover is realized by the canonical map $\mathcal{L}(\mathfrak{G}_0) \to \mathcal{L}\mathfrak{G}$ together with the non-trivial deck transformation $\sigma_{\mathfrak{h}} : \mathcal{L}(\mathfrak{G}_0) \to \mathcal{L}(\mathfrak{G}_0)$ given on $\mathcal{L}_0(\mathfrak{G}_0)$ by $\sigma_{\mathfrak{h}}(g) = \mathfrak{h} g \mathfrak{h}^{-1}$, on $\mathcal{L}_1(\mathfrak{G}_0)$ by

$$\sigma_{\mathfrak{h}}(g_1 \xrightarrow{(f,x)} g_2) = \mathfrak{h} g_1 \mathfrak{h}^{-1} \xrightarrow{(\mathfrak{h} f \mathfrak{h}^{-1}, {}^{\mathfrak{h}}x)} \mathfrak{h} g_2 \mathfrak{h}^{-1}$$

and on $\mathcal{L}_2(\mathfrak{G}_0)$ by $\sigma_{\mathfrak{h}}(z) = {}^{\mathfrak{h}}z$.

In the setting of Real representation theory, an unoriented version $\mathcal{L}^{\mathrm{ref}}\mathfrak{G}$ of the free loop space $\mathcal{L}(\mathfrak{G}_0)$ arises in a natural way; the superscript "ref" stands for reflection. This is now a $\mathbb{Z}_2$-graded crossed module in groupoids. The base groupoid $\mathcal{L}^{\mathrm{ref}}_{\leq 1}\mathfrak{G}$ has objects $\mathcal{L}^{\mathrm{ref}}_0\mathfrak{G} = G_0$ and morphisms $g_1 \to g_2$ given by pairs $(f, x) \in G_1 \times G_2$ which satisfy $g_2 = f\partial(x)g_1^{\pi(f)} f^{-1}$. The composition law is $(f_2, x_2) \circ (f_1, x_1) = (f_2 f_1, ({}^{f_1^{-1}} x_2)\tilde{x}_1)$, where

$$\tilde{x}_1 = \begin{cases} x_1 & \text{if } \pi(f_2) = +1, \\ {}^{(g_1^{-\pi(f_1)})} x_1^{-1} & \text{if } \pi(f_2) = -1. \end{cases}$$

Each group $\mathcal{L}^{\mathrm{ref}}_2\mathfrak{G}(g)$ is again G_2 and $\partial_g(z) = (\partial z, z^{-1}({}^g z))$. The action of $\mathcal{L}^{\mathrm{ref}}_{\leq 1}\mathfrak{G}$ on $\mathcal{L}^{\mathrm{ref}}_2\mathfrak{G}$ is as for $\mathcal{L}\mathfrak{G}$. There is an induced $\mathbb{Z}_2$-grading $\pi : \mathcal{L}^{\mathrm{ref}}\mathfrak{G} \to B\mathbb{Z}_2$ which records the degree of morphisms in $\mathcal{L}^{\mathrm{ref}}_{\leq 1}\mathfrak{G}$. The associated double cover is the canonical morphism $\mathcal{L}(\mathfrak{G}_0) \to \mathcal{L}^{\mathrm{ref}}\mathfrak{G}$ with the non-trivial deck transformation $\mathfrak{i}_{\mathfrak{h}} := \mathfrak{i} \circ \sigma_{\mathfrak{h}} = \sigma_{\mathfrak{h}} \circ \mathfrak{i}$, that is, the diagonal $\mathbb{Z}_2$-action induced by loop reflection and the deck transformation of $\mathfrak{G}_0$.

For later convenience, if $\mathfrak{G}$ is trivially graded, we take $\mathcal{L}^{\mathrm{ref}}\mathfrak{G}$ to mean $\mathcal{L}\mathfrak{G}$.

6.2. **Real categorical characters and 2-characters.** Let $\mathfrak{G}$ be a $\mathbb{Z}_2$-graded crossed module. Let ρ be a Real 2-representation of $\widetilde{\mathfrak{G}}$ on a object V of a $\mathbb{K}$-linear bicategory $\mathcal{V}$ with weak duality involution $(-)^{\circ}$. The Real categorical character of ρ is defined as follows. For each $g \in G_0$, define a vector space

$$\mathbb{T}\mathrm{r}_\rho(g) := 2\mathrm{Hom}_{\mathcal{V}}(\mathrm{id}_V, \rho(g)).$$

We assume that $\mathbb{T}\mathrm{r}_\rho(g)$ is finite dimensional over $\mathbb{K}$. We do not assign a value of $\mathbb{T}\mathrm{r}_\rho$ to elements of $G_1 \setminus G_0$. For each $(f, x) \in G_1 \times G_2$, define a linear map

$$\mathbb{T}\mathrm{r}_\rho(g; f, x) : \mathbb{T}\mathrm{r}_\rho(g) \to \mathbb{T}\mathrm{r}_\rho(f\partial(x)(g^{\pi(f)})f^{-1})$$

so that its value on $(\mathrm{id}_V \overset{u}{\Rightarrow} \rho(g)) \in \mathbb{T}\mathrm{r}_\rho(g)$ is the composition

$$\mathrm{id}_V \Rightarrow \rho(f)\circ\rho(f^{-1}) \overset{u}{\Rightarrow} \rho(f)\circ\rho(g)\circ\rho(f^{-1}) \overset{\rho(x)}{\Longrightarrow} \rho(f)\circ\rho(\partial(x)g)\circ\rho(f^{-1}) \Rightarrow \rho(f\partial(x)gf^{-1})$$

when $\pi(f) = +1$, as in [16, 31], and

$$\begin{aligned}\mathrm{id}_V &\Rightarrow \rho(f) \circ \rho(f^{-1})^{\circ} \Rightarrow \rho(f) \circ \rho(g)^{\circ} \circ \rho(g^{-1})^{\circ} \circ \rho(f^{-1})^{\circ} \overset{u^{\circ}}{\Rightarrow} \\ &\rho(f) \circ \rho(g^{-1})^{\circ} \circ \rho(f^{-1})^{\circ} \overset{(\rho(x)^{-1})^{\circ}}{\Longrightarrow} \rho(f) \circ \rho(\partial(x)g^{-1})^{\circ} \circ \rho(f^{-1})^{\circ} \Rightarrow \rho(f\partial(x)g^{-1}f^{-1})\end{aligned}$$

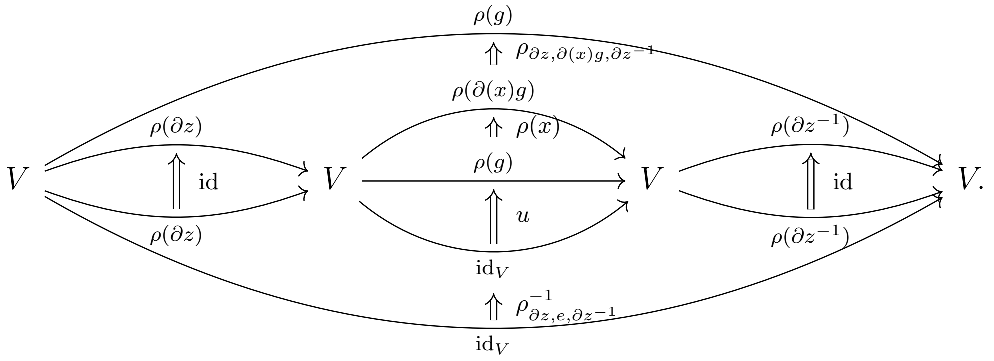
when $\pi(f) = -1$. The unlabelled maps appearing in the previous compositions are 2-isomorphisms constructed from the composition 2-isomorphisms of ρ. In the second composition $\rho(x)$ is a 2-morphism $\rho(g^{-1}) \Rightarrow \rho(\partial(x)g^{-1})$, so that $\rho(x)^{\circ} : \rho(\partial(x)g^{-1})^{\circ} \Rightarrow \rho(g^{-1})^{\circ}$.

Proposition 6.1. *The Real categorical character of ρ is a functor $\mathbb{T}\mathrm{r}_\rho : \mathcal{L}^{\mathrm{ref}}\mathfrak{G} \to \mathsf{Vect}_{\mathbb{K}}$.*

Proof. Because $\mathsf{Vect}_{\mathbb{K}}$ is a 1-category, this statement is equivalent to $\mathbb{T}\mathrm{r}_\rho$ defining a functor $\pi_{\leq 1}(\mathcal{L}^{\mathrm{ref}}\mathfrak{G}) \to \mathsf{Vect}_{\mathbb{K}}$. It is straightforward to verify that, as defined above, the structure maps assemble to a functor $\mathbb{T}\mathrm{r}_\rho : \mathcal{L}^{\mathrm{ref}}_{\leq 1}\mathfrak{G} \to \mathsf{Vect}_{\mathbb{K}}$; see [16, Proposition 4.10], [31, §5]. We need to show that this functor factors through $\pi_{\leq 1}(\mathcal{L}^{\mathrm{ref}}\mathfrak{G})$, that is, for each $z \in G_2$, the linear maps

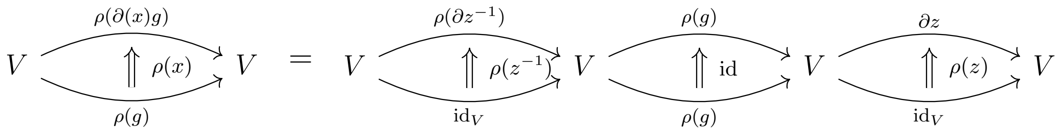
$$\mathbb{T}\mathrm{r}_\rho(g; f, x)\ ,\ \mathbb{T}\mathrm{r}_\rho(g; (f, x) \circ \partial_g(z)) : \mathbb{T}\mathrm{r}_\rho(g) \to \mathbb{T}\mathrm{r}_\rho(f\partial(x)(g^{\pi(f)})f^{-1})$$

are equal. To do so, it suffices to show that $\mathbb{T}\mathrm{r}_\rho(g; \partial z, z^{-1}({}^g z))$ is the identity endomorphism of $\mathbb{T}\mathrm{r}_\rho(g)$. This endomorphism is given by the diagram

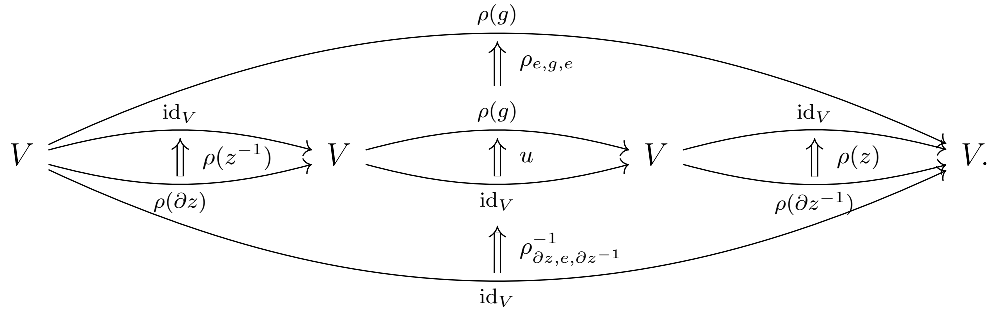


(16)

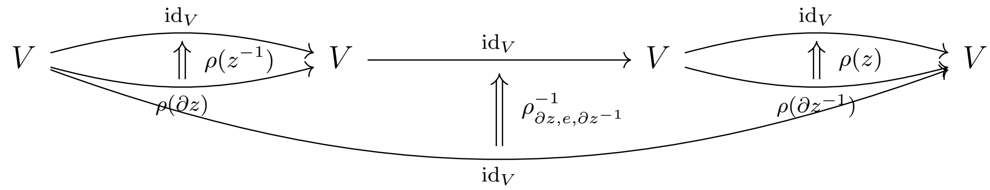
The composition rules in $\widetilde{\mathfrak{G}}$ give



so that the diagram (16) can be rewritten as



Since the diagram



is the identity 2-morphism of id_V, we conclude that (16) is equal to $u : \mathrm{id}_V \Rightarrow \rho(g)$. $\square$

Working in this geometric set-up, we define the Real 2-character χ_ρ of ρ to be the holonomy of $\mathbb{T}\mathrm{r}_\rho : \mathcal{L}^{\mathrm{ref}} \mathfrak{G} \to \mathsf{Vect}_{\mathbb{K}}$. It is therefore a locally constant function $\chi_\rho : \mathcal{L}\mathcal{L}^{\mathrm{ref}} \mathfrak{G} \to \mathbb{K}$, or equivalently, $\chi_\rho : \Lambda \pi_{\leq 1}(\mathcal{L}^{\mathrm{ref}} \mathfrak{G}) \to \mathbb{K}$. Using either of these interpretations, we find that χ_ρ is function on the set

$$\mathbb{G} = \{(g; f, x) \in G_0 \times G_1 \times G_2 \mid f \partial(x)(g^{\pi(f)}) f^{-1} = g\}$$

which is invariant under the action of $\mathrm{im}(\partial_g)$, $g \in G_0$ and the conjugation action of morphisms in $\pi_{\leq 1}(\mathcal{L}\mathfrak{G})$. When G_1 is trivially graded this gives a more precise description of the symmetries of 2-characters than previously available. For example, it refines the G_0-conjugation invariance of χ_ρ proved in [31, Proposition 5.1].

The space of locally constant functions $\mathcal{L}\mathcal{L}^{\mathrm{ref}}\mathfrak{G} \to \mathbb{K}$, that is, Real 2-class functions for $\mathfrak{G}$, is a direct sum of elliptic 2-class functions, which are supported on $\mathbb{G} \cap (G_0 \times G_0 \times G_2)$ and Klein 2-class functions, which are supported on $\mathbb{G} \cap (G_0 \times (G_1 \backslash G_0) \times G_2)$. The nomenclature stems from the appearance, modulo $\mathrm{im}(\partial)$, of the relations for the fundamental groups of the 2-torus and Klein bottle in the definition of $\mathbb{G}$. This leads to an interpretation of $\mathcal{L}\mathcal{L}^{\mathrm{ref}}\mathfrak{G}$ in terms of moduli spaces of principal $\widetilde{\mathfrak{G}}$-bundles over the torus or Klein bottle, where in the latter case a compatibility condition with the orientation double cover is imposed, cf. [38, Section 3.2]. The 2-character of the underlying 2-representation of ρ is in the elliptic sector. The Real nature of ρ is reflected in the Klein sector of χ_ρ as well as the invariance of the elliptic sector of χ_ρ under the conjugation action of odd elements of $\pi_{\leq 1}(\mathcal{L}\mathfrak{G})$.

6.3. **Induced categorical characters.** In this section we determine the form of induced Real categorical and 2-characters.

Theorem 6.2. *Let $\mathfrak{G}$ be a $\mathbb{Z}_2$-graded crossed module with crossed submodule $\mathfrak{H}$. Assume that G_1 is finite and $H_2 = G_2$. Let ρ be a 2-representation of $\widetilde{\mathfrak{H}}$ on $2\mathsf{Vect}_{\mathbb{K}}$, which is Real if $\mathfrak{H}$ is non-trivially graded. Then there is a canonical isomorphism*

$$\mathbb{T}\mathrm{r}_{\mathrm{Ind}_{\widetilde{\mathfrak{H}}}^{\widetilde{\mathfrak{G}}}\rho} \simeq \mathrm{Ind}_{\mathcal{L}^{\mathrm{ref}}\mathfrak{H}}^{\mathcal{L}^{\mathrm{ref}}\mathfrak{G}} \mathbb{T}\mathrm{r}_\rho$$

of vector bundles $\mathcal{L}^{\mathrm{ref}}\mathfrak{G} \to \mathsf{Vect}_{\mathbb{K}}$.

Proof. The proof is similar to those of [16, Theorem 7.5], [37, Theorems 7.3, 7.7], which treat the cases in which $\mathfrak{G}$ is a group, possibly $\mathbb{Z}_2$-graded. We focus on the differences.

Arguing as in the beginning of the proof of Proposition 6.1, it suffices to prove the statement at the level of vector bundles $\mathcal{L}^{\mathrm{ref}}_{\leq 1}\mathfrak{G} \to \mathsf{Vect}_{\mathbb{K}}$. A choice of representatives of the connected components of $\mathcal{L}^{\mathrm{ref}}_{\leq 1}\mathfrak{G}$ induces an equivalence of groupoids

$$\mathcal{L}^{\mathrm{ref}}_{\leq 1}\mathfrak{G} \simeq \bigsqcup_{g \in \pi_0(\mathcal{L}^{\mathrm{ref}}_{\leq 1}\mathfrak{G})} BZ_{\mathfrak{G}}(g),$$

where $Z_{\mathfrak{G}}(g) \leq G_2 \rtimes G_1$ is the stabilizer $\{(f,x) \in G_1 \times G_2 \mid f\partial(x)(g^{\pi(f)})f^{-1} = g\}$. Fix $g \in G_0$. We have a vector space isomorphism

$$\mathbb{T}\mathrm{r}_{\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}\rho}(g) \simeq \bigoplus_{\substack{t \in \mathcal{T} \\ t^{-1}gt \in H_1}} \mathbb{T}\mathrm{r}_\rho(t^{-1}gt),$$

where $\mathcal{T} \subseteq G_1$ is a left transversal to H_1. We need to describe the action of $Z_{\mathfrak{G}}(g)$ on $\mathbb{T}\mathrm{r}_{\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}\rho}(g)$. Write

$$[g]_{\mathfrak{G}} \cap H_1 = \bigsqcup_{i=1}^{n} [h_i]_{\mathfrak{H}},$$

where $[g]_{\mathfrak{G}}$ denotes the $G_2 \rtimes G_1$-orbit of g, and similarly for $[h_i]_{\mathfrak{H}}$. Let $\mathcal{T}_i = \{t \in \mathcal{T} \mid t^{-1}gt \in [h_i]_{\mathfrak{H}}\}$. For each $i \in \{1, \ldots, n\}$, fix an element $t_i \in \mathcal{T}_i$ and put $h_i = t_i^{-1}gt_i$. The assumption $H_2 = G_2$ implies that there is a canonical bijection

$$Z_{\mathfrak{G}}(h_i)/Z_{\mathfrak{H}}(h_i) \simeq Z_{G_1}(h_i)/Z_{H_1}(h_i).$$

We can therefore choose adapted representatives of $\mathcal{T}_i \subset \mathcal{T}$, as in [16, Lemma 7.7] (see also [37, Lemma 7.4]). Consider the composition

$$\phi_i : Z_{\mathfrak{H}}(h_i) \hookrightarrow Z_{\mathfrak{G}}(h_i) \xrightarrow{(f,x) \mapsto (t_i f t_i^{-1}, {}^{t_i}x)} Z_{\mathfrak{H}}(g).$$

Then we can verify that $\mathbb{T}\mathrm{r}_{\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}\rho}(g)$ is isomorphic to $\bigoplus_{i=1}^n \mathrm{Ind}_{\phi_i} \mathbb{T}\mathrm{r}_\rho(h_i)$, exactly as in the case $G_2 = \{e\}$. □

Corollary 6.3. *In the setting of Theorem 6.2, the Real 2-character of* $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}\rho$ *is*

$$\chi_{\mathrm{Ind}_{\tilde{\mathfrak{H}}}^{\tilde{\mathfrak{G}}}\rho}(g; f, x) = \frac{1}{c|H_1|} \sum_{\substack{t \in G_1 \\ t^{-1}(g,f)t \in H_0 \times H_1}} \chi_\rho(t^{-1} g^{\pi(t)} t; t^{-1} f t, {}^{t^{-1}}x),$$

where $c = 1$ *if* $\mathfrak{H}$ *is trivially graded and* $c = 2$ *otherwise.*

Proof. By Theorem 6.2, it suffices to compute the character of $\mathrm{Ind}_{\mathcal{L}^{\mathrm{ref}}\mathfrak{H}}^{\mathcal{L}^{\mathrm{ref}}\mathfrak{G}} \mathbb{T}\mathrm{r}_\rho$. This can be done using [16, Proposition 6.11], giving the claimed result. □

Considered as locally constant functions, Real 2-characters are multiplicative:

$$\chi_{\rho_1 \boxtimes \rho_2} = \chi_{\rho_1} \chi_{\rho_2}.$$

This can be proved in the same way as [31, Proposition 5.1]. See also [15, Corollary 4.2]. In particular, using the identification of Theorem 5.1, we find that each tuple $(g; f, x) \in \mathbb{G}$ defines a ring homomorphism

$$\chi(g; f, x) : \mathbb{B}_{\mathbb{Z}}^{\Phi}(\mathfrak{G}) \to \mathbb{K}, \qquad \langle \rho, P \rangle \mapsto \chi_{\mathrm{Ind}_{\tilde{\mathfrak{G}}_P}^{\tilde{\mathfrak{G}}}\rho}(g; f, x).$$

Motivated by [31, Theorem 5.2], we would like to understand this homomorphism in terms of marks. This can be done as follows.

Let R be a $\mathbb{K}$-algebra. Assume that the cardinality of $\pi_1(\mathfrak{G})$ is invertible in $\mathbb{K}$. Given $Q \in \mathcal{S}(G)$ and a semigroup homomorphism $\alpha : \Phi(Q) \to R^\times$, the associated mark homomorphism is the $\mathbb{K}$-linear map $f_Q^\alpha : \mathbb{B}_{\mathbb{K}}^{\Phi}(\mathfrak{G}) \to R$ defined by

$$f_Q^\alpha(\langle \rho, P \rangle) = \frac{1}{|P|} \sum_{\substack{g \in G_1 \\ gQg^{-1} \subset P}} \alpha(\Phi(\gamma_g : Q \to P)(\rho)). \tag{17}$$

This is in fact a morphism of R-algebras [18, §1].

Corollary 6.4. *Assume that the cardinality of* $\pi_1(\mathfrak{G})$ *is invertible in* $\mathbb{K}$*. Let* $(g; f, x) \in \mathbb{G}$ *and let* P *be the subgroup of* $\pi_1(\mathfrak{G})$ *generated by the images of* g *and* f*. Then* $\chi(g; f, x)$ *is the mark homomorphism* $\mathbb{B}_{\mathbb{Z}}^{\Phi}(\mathfrak{G}) \to \mathbb{K}$ *determined via equation* (17) *by the restriction* $\chi(g; f, x)_{|P} : \Phi(P) \to \mathbb{K}^\times$.

Proof. This follows by comparing equation (17) with the result of Corollary 6.3. □

The analogy between 2-character theory and Hopkins–Kuhn–Ravenel character theory [21] of Borel equivariant elliptic cohomology, as developed by Ganter and Kapranov [16], suggests the following problem.

Problem 6.5. *Interpret the results of Sections 6.2 and 6.3 in terms of (transchromatic) Hopkins–Kuhn–Ravenel character theory* [21, 33, 26] *of 2-equivariant elliptic cohomology. In particular, relate Theorem 6.2 and Corollary 6.3 to the relevant transfer maps.*

Remark. A 3-cocycle $\alpha \in Z^3(B\widetilde{\mathfrak{G}}, \mathbb{K}_\pi^\times)$ on the classifying space determines a $\mathbb{Z}_2$-graded central extension ${}^\alpha\widetilde{\mathfrak{G}}$ of $\widetilde{\mathfrak{G}}$ by $\mathbb{K}^\times$. An α-twisted Real 2-representation of $\widetilde{\mathfrak{G}}$ is then by definition a Real 2-representation of ${}^\alpha\widetilde{\mathfrak{G}}$ whose restriction to $\mathbb{K}^\times$ is scalar multiplication. When $\mathfrak{G}$ is a finite group, the character theory of twisted representations is studied in [17] and [37]. Together with the results of this section, this suggests an obvious candidate for the categorical character theory of twisted Real 2-representations of finite 2-groups in terms of vector bundles over $\mathcal{L}^{\mathrm{ref}}\mathfrak{G}$ which are twisted by the gerbe represented by the loop transgression of α.

References

[1] M. Atiyah. K-theory and reality. *Quart. J. Math. Oxford Ser. (2)*, 17:367–386, 1966.

[2] M. Atiyah and G. Segal. Equivariant K-theory and completion. *J. Differential Geometry*, 3:1–18, 1969.

[3] M. F. Atiyah. Bott periodicity and the index of elliptic operators. *Quart. J. Math. Oxford Ser. (2)*, 19:113–140, 1968.

[4] J. Baez and A. Lauda. Higher-dimensional algebra. V. 2-groups. *Theory Appl. Categ.*, 12:423–491, 2004.

[5] J. Barrett and M. Mackaay. Categorical representations of categorical groups. *Theory Appl. Categ.*, 16:No. 20, 529–557, 2006.

[6] B. Bartlett. The geometry of unitary 2-representations of finite groups and their 2-characters. *Appl. Categ. Structures*, 19(1):175–232, 2011.

[7] D. Ben-Zvi and D. Nadler. The character theory of a complex group. arXiv:0904.1247, 2009.

[8] D. Ben-Zvi and D. Nadler. Secondary traces. arXiv:1305.7177, 2013.

[9] J. Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. Springer, Berlin, 1967.

[10] R. Bezrukavnikov, M. Finkelberg, and V. Ostrik. Character D-modules via Drinfeld center of Harish-Chandra bimodules. *Invent. Math.*, 188(3):589–620, 2012.

[11] R. Brown. Crossed modules and the homotopy 2-type of a free loop space. arXiv:1003.5617, 2010.

[12] R. Brown and C. Spencer. G-groupoids, crossed modules and the fundamental groupoid of a topological group. *Nederl. Akad. Wetensch. Proc. Ser. A* **79**, 38(4):296–302, 1976.

[13] L. Crane and D. Yetter. Measurable categories and 2-groups. *Appl. Categ. Structures*, 13(5-6):501–516, 2005.

[14] J. Elgueta. Representation theory of 2-groups on Kapranov and Voevodsky's 2-vector spaces. *Adv. Math.*, 213(1):53–92, 2007.

[15] N. Ganter. Inner products of 2-representations. *Adv. Math.*, 285:301–351, 2015.

[16] N. Ganter and M. Kapranov. Representation and character theory in 2-categories. *Adv. Math.*, 217(5):2268–2300, 2008.

[17] N. Ganter and R. Usher. Representation and character theory of finite categorical groups. *Theory Appl. Categ.*, 31:Paper No. 21, 542–570, 2016.

[18] P. Gunnells, A. Rose, and D. Rumynin. Generalised Burnside rings, G-categories and module categories. *J. Algebra*, 358:33–50, 2012.

[19] N. Gurski. Biequivalences in tricategories. *Theory Appl. Categ.*, 26:No. 14, 349–384, 2012.

[20] A. Hahn. A Hermitian Morita theorem for algebras with anti-structure. *J. Algebra*, 93(1):215–235, 1985.

[21] M. Hopkins, N. Kuhn, and D. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.*, 13(3):553–594, 2000.

[22] I. M. Isaacs, Gunter Malle, and Gabriel Navarro. A reduction theorem for the McKay conjecture. *Invent. Math.*, 170(1):33–101, 2007.

[23] M. Karoubi. Sur la K-théorie équivariante. In *Séminaire Heidelberg-Saarbrücken-Strasbourg sur la K-théorie (1967/68)*, Lecture Notes in Mathematics, Vol. 136, pages 187–253. Springer, Berlin, 1970.

[24] J. Lorand and A. Valentino. Morita bicategories of algebras and duality involutions. arXiv:1902.04866, 2019.

[25] J. Lurie. A survey of elliptic cohomology. In *Algebraic topology*, volume 4 of *Abel Symp.*, pages 219–277. Springer, Berlin, 2009.

[26] J. Lurie. Elliptic cohomology III: Tempered cohomology. Available from the author's webpage, 2019.

[27] B. Noohi. Notes on 2-groupoids, 2-groups and crossed modules. *Homology Homotopy Appl.*, 9(1):75–106, 2007.

[28] B. Noohi. Notes on 2-groupoids, 2-groups and crossed modules. arXiv:0512106, version 3, 2008.

[29] V. Ostrik. Module categories, weak Hopf algebras and modular invariants. *Transform. Groups*, 8(2):177–206, 2003.
[30] D. Rumynin, D. Vakhrameev, and M. Westaway. Covering groups of nonconnected topological groups and 2-groups. *Comm. Alg.*, to appear. arXiv:1709.09728.
[31] D. Rumynin and A. Wendland. 2-Groups, 2-characters, and Burnside rings. *Adv. Math.*, 338:196–236, 2018.
[32] M. Shulman. Contravariance through enrichment. *Theory Appl. Categ.*, 33:Paper No. 5, 95–130, 2018.
[33] N. Stapleton. Transchromatic generalized character maps. *Algebr. Geom. Topol.*, 13(1):171–203, 2013.
[34] B. Toën and G. Vezzosi. Chern character, loop spaces and derived algebraic geometry. In *Algebraic topology*, volume 4 of *Abel Symp.*, pages 331–354. Springer, Berlin, 2009.
[35] B. Toën and G. Vezzosi. Caractères de Chern, traces équivariantes et géométrie algébrique dérivée. *Selecta Math. (N.S.)*, 21(2):449–554, 2015.
[36] S. Willerton. The twisted Drinfeld double of a finite group via gerbes and finite groupoids. *Algebr. Geom. Topol.*, 8(3):1419–1457, 2008.
[37] M. Young. Real representation theory of finite categorical groups. arXiv:1804:09053, 2018.
[38] M. Young. Orientation twisted homotopy field theories and twisted unoriented Dijkgraaf–Witten theory. *Comm. Math. Phys.*, 2019.

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