Note on lifting group actions in fiber bundles

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1 Introduction.

Lifting problem of group actions in fiber bundles is discussed by several authors. In the case of torus bundles, Hattori and Yoshida [H-Y] gave a sufficient and necessary condition and classified such liftings in terms of equivarinat cohomology. Lashof, May and Segal [L-M-S] extended their result to the case of pricipal bundles with compact abelian structure groups (see also [M]). Roughly speaking, the problem is summarized as follows:

What is the relation between G-equivariant objects over a G-space X and objects over $X_G := EG \times_G X$?

Their answer is the following

Theorem([H-Y],[L-M-S]). Let G be a compact Lie group acting on X and H a compact abelian Lie group. There is a one-to-one correspondence between the set of equivalence classes of G-equivariant principal H-bundles over X and the set of equivalence classes of principal H-bundles over X_G.

The purpose of this paper is to give an another approach to this problem and consider a similar question for Galois coverings. Lifting the G-action on X to a principal bundle is equivalent to find a splitting homomorphism of a certain extension of G and the problem is reduced to Theorem (2.1). In the course of this reduction, we shall show the existence of a fixed point in the moduli space of all connections on S^1-bundles under the above assumption (see Lemma (3.2)). This lemma is valid certainly only in smooth category, but there is another way due to Jean Lannes, which is also valid in continuous category. The author is grateful to Professor Jean Lannes for helpful discussion.

2 Preliminaries and Statement of Results.

First of all, we recall some facts on classifying spaces. Suppose we have an exact sequence of topological (or Lie) groups

$$1 \to H \to \widehat{G} \to G \to 1.$$

Then we have a principal G-bundle

$$G \to BH \to B\widehat{G}.$$

Thus G acts on BH and $(EG \times BH)/G$ is homotopically equivalent to $B\widehat{G}$. Hence we have a fibration $BH \to B\widehat{G} \to BG$. We can generalize this construction as follows. $\widehat{G}$ acts on EG factoring through the homomorphism $\widehat{G} \to G$. For any $\widehat{G}$-action on EH which is an extension of the principal H-action, the diagonal action of $\widehat{G}$ on $EG \times EH$ is free and we get $B\widehat{G} = (EG \times EH)/\widehat{G} = (EG \times BH)/G$ and the fibration $BH \to B\widehat{G} \to BG$.

We shall show the following theorem in §6.

Theorem (2.1). (1). Let H be a finite group. The following two conditions are equivalent.
(i) $1 \to H \to \widehat{G} \to G \to 1$ has a splitting.
(ii) $BH \to B\widehat{G} \to BG$ has a cross section.
(2). Let H be a circle group S^1 and G a compact Lie group. The following two conditions are equivalent.
(i) A central extension $0 \to H \to \widehat{G} \to G \to 1$ is a split exact sequence.
(ii) $BH \to B\widehat{G} \to BG$ has a cross section.

More precisely, there is a ono-to-one correspondence between splittings in (i) and homotopy classes of cross sections in (ii).

Next we recall the following

Fact (2.2)([A-B I]). Let $P \to X$ be a principal H-bundle, and $\mathcal{G}(P)$ the gauge transformation group of P. Then $B\mathcal{G}(P) \cong \mathrm{Map}_P(X, BH)$, where

$\mathrm{Map}_P(X, BH)$ denotes the space of classifying mappings of $P \to X$. More precisely, we have a universal principal fibration

$$\mathcal{G}(P) \to \mathrm{Map}^H(P, EH) \to \mathrm{Map}_P(X, BH).$$

Here $\mathrm{Map}^H(P, EH)$ denotes the space of H-equivariant mappings from P to EH.

Remark. In this result, H need not be a finite group or S^1.

Let $P \to X$ be an S^1-bundle, and G a compact Lie group acting on X. If the G-action lifts to P commuting with the principal S^1-action, we get an S^1-bundle over X_G by the Borel construction

$$P_G := EG \times_G P \to X_G := EG \times_G X.$$

The problem is the converse. Actually there is a theorem.

Theorem (2.3)([H-Y],[L-M-S]). There is a one-to-one correspondence between the following two objects.
(1) isomorphism classes of G-equivariant S^1-bundles over X.
(2) isomorphism classes of S^1-bundles over X_G.

Remark. Hattori and Yoshida treated the case of principal torus bundles and Lashof, May and Segal treated the case of principal bundles with compact abelian groups.

In the case of Galois covering spaces, we shall show the following

Theorem (2.4). Let Γ be a discrete group. There is a one-to-one correspondence between the following two objects.
(1) equivalence classes of G-equivariant Γ-bundles over X.
(2) equivalence classes of Γ-bundles over X_G.

Theorem (2.5). Let Γ be a discrete group and $\widetilde{X} \to X$ a Γ-covering space. There is a one-to-one correspondence between the following two objects.
(1) G-actions on $\widetilde{X}$ which covers the G-action on X.

(2) isomorphism classes of covering spaces over X_G whose restriction to X is $\widetilde{X} \to X$.

Remark. In Theorem (2.5), G-actions on $\widetilde{X}$ need not to commute with Γ-action and covering spaces over X_G need not be Galois Γ-covering spaces.

Theorem (2.3) and Theorem (2.4) imply the theorem for compact abelian structure groups (Theorem in Introduction).

3 Proof of Theorem (2.3).

If $P \to X$ is a G-equivariant S^1-bundle, we get $P_G \to X_G$ by the Borel construction. We will show the following

Claim (3.1). If $P \to X$ extends to an S^1-bundle $\widetilde{P} \to X_G$, then the G-action on X lifts to P.

We assume that X is a manifold and $P \to X$ is a smooth principal S^1-bundle. In fact, we can get an exact sequence (3.3) for $P \to X$ in topological category (due to Jean Lannes), and the rest of the proof continues in the same way. Let $\mathcal{B}(P)$ be the set of gauge equivalence classes of all connections on P. Then G acts on $\mathcal{B}(P)$. We shall show the following lemma in §5.

Lemma (3.2). If $P \to X$ extends to an S^1-bundle $\widetilde{P} \to X_G$, then the fixed point set $\mathcal{B}(P)^G$ is not empty.

Fixing a connection ∇ which represents a fixed point in $\mathcal{B}(P)^G$, we get the following extension of G by S^1

$$0 \to S^1 \to \widehat{G} \to G \to 1, \tag{3.3}$$

where $\widehat{G}$ consists of all bundle automorphisms of P which covers some $g \in G$ action on X and preserve the connection ∇. The exact sequence (3.3) is split exact if and only if the G-action lifts to P as bundle automorphisms.

By Theorem (2.1), it is sufficient to show Claim (3.1) that the corresponding fibration

$$BS^1 \to B\widehat{G} \to BG$$

has a cross section. Since S^1 is a subgroup of $\mathcal{G}(P)$ consisting of gauge transformations which preserve the connection ∇, BS^1 is considered as $E\mathcal{G}(P)/S^1$, namely we have a universal S^1-fibration

$$S^1 \to \mathrm{Map}^{S^1}(P, ES^1) \to \mathrm{Map}^{S^1}(P, ES^1)/S^1.$$

$\widehat{G}$ acts on P from the left, hence $\widehat{G}$ acts on $\mathrm{Map}^{S^1}(P, ES^1)$ from the right. This action decends to a G-action on $\mathrm{Map}^{S^1}(P, ES^1)/S^1$. Thus $B\widehat{G}$ is represented by

$$\{EG \times \mathrm{Map}^{S^1}(P, ES^1)\}/\widehat{G} = \{EG \times (\mathrm{Map}^{S^1}(P, ES^1)/S^1)\}/G.$$

Since $\widetilde{P} \to X_G$ is an S^1-bundle, we have a classifying mapping

$$\begin{tikzcd}
\widetilde{P} \arrow[r, "\overline{\varphi}"] \arrow[d] & ES^1 \arrow[d] \\
X_G \arrow[r, "\varphi"] & BS^1.
\end{tikzcd}$$

$\overline{\varphi}$ defines a cross section of

$$B\widehat{G} = \{EG \times (\mathrm{Map}^{S^1}(P, ES^1)/S^1\}/\widehat{G} \to BG.$$

Hence Theorem (2.1) yields the conclusion.

Next, we proceed to the one-to-one correspondence . If $\widetilde{P}$ and $\widetilde{P}'$ are distinct S^1-bundles over X_G, then their classifying mappings are not homotopic. Hence again the conclusion follows from Theorem (2.1).

4 Proof of Theorem (2.4) and (2.5).

Proof of Theorem (2.4). For a G-equivariant Γ-bundle, we get a Γ-bundle Y over X_G by the Borel construction. We shall show the converse.

Let Λ be the center of the opposite group Γ^{op} of Γ, then Λ is identified with the gauge transformation group of the Γ-bundle $\widetilde{X} \to X$. Thus we get the following exact sequence

$$1 \to \Lambda \to \widehat{G} \to G \to 1,$$

where $\widehat{G}$ is the group consisting of all bundle automorphisms of $\widetilde{X} \to X$ which cover some $g \in G$ action on X. By Fact (2.2), $B\Lambda$ is represented by $\mathrm{Map}_{\widetilde{X}}(X, B\Gamma)$ and G acts on $\mathrm{Map}_{\widetilde{X}}(X, B\Gamma)$ from the right through the G-action on X from the left. Thus we get $B\widehat{G} = \{EG \times \mathrm{Map}_{\widetilde{X}}(X, B\Gamma)\}/G$. The classifying mapping of $Y \to X_G$ defines a cross section of $B\widehat{G} \to BG$. The rest of the proof continues as in the one of Theorem (2.3).

Proof of Theorem (2.5). If the G-action lifts to $\widetilde{X}$, which may not commute with covering transformations, we get a covering space Y over X_G, which may not be a Γ-Galois covering space, by the Borel construction. We shall show the converse.

As in the proof of Theorem (2.4), there is the following exact sequence

$$1 \to \Gamma^{\mathrm{op}} \to \widehat{G} \to G \to 1,$$

where $\widehat{G}$ is the group consisting of all self mappings of $\widetilde{X}$ which cover some $g \in G$ action on X and Γ^{op} is the group of covering transformations, the multiplication of which is composition of mappings.

Claim (4.1). $B\Gamma^{\mathrm{op}}$ is represented by $\mathrm{Map}_{\widetilde{X}}(X, B\Gamma)/(\Gamma^{\mathrm{op}}/\Lambda)$.

First of all, we define the $(\Gamma^{\mathrm{op}}/\Lambda)$-action on $\mathrm{Map}_{\widetilde{X}}(X, B\Gamma)$. By the Milnor construction, we can consider $E\Gamma$ as the infinite join of Γ and $B\Gamma$ as the quotient of $E\Gamma$ by the diagonal Γ-action from the right. In this model, we have a natural homomorphsim $B : \mathrm{Aut}(\Gamma) \to \mathrm{Homeo}(B\Gamma)$. More precisely, for a homomorphism $\alpha : \Gamma \to \Gamma'$, we have the following commutative diagram

$$\begin{tikzcd}
E\Gamma \arrow[r, "E\alpha"] \arrow[d] & E\Gamma' \arrow[d] \\
B\Gamma \arrow[r, "B\alpha"] & B\Gamma'
\end{tikzcd}$$

and

$$E\alpha(x \cdot \gamma) = E\alpha(x) \cdot \alpha(\gamma) \qquad \text{for } \gamma \in \Gamma,\ x \in E\Gamma.$$

$Ad(\gamma)$ denotes the inner automorphism of Γ by γ, and we get the following commutative diagram

$$\begin{tikzcd}
E\Gamma \arrow[r, "EAd(\gamma)"] \arrow[d] & E\Gamma \arrow[r, "R(\gamma)"] \arrow[d] & E\Gamma \arrow[d] \\
B\Gamma \arrow[r, "BAd(\gamma)"] & B\Gamma \arrow[r, "id"] & B\Gamma
\end{tikzcd}$$

where $R(\gamma)$ is the γ-action from the right.

Remark. $R : \Gamma \to \Gamma^{\mathrm{op}}$ is an anti-isomorphism.

Subclaim (4.2). $\psi(\gamma) := R(\gamma) \circ EAd(\gamma)$ is Γ-equivariant.

Proof.

$$\begin{aligned}
\psi(\gamma)(R(\gamma')x) &= R(\gamma) \circ EAd(\gamma) \circ R(\gamma')(x) \\
&= R(\gamma) \circ R(Ad(\gamma)\gamma') \circ EAd(\gamma)(x) \\
&= R(\gamma') \circ R(\gamma) \circ EAd(\gamma)(x) \\
&= R(\gamma') \circ \psi(\gamma)(x). \quad \square
\end{aligned}$$

Thus we have the Γ-action on $E\Gamma$ from the left, hence the Γ^{op}-action on $E\Gamma$ from the right, which induces the Γ^{op}-action on $\mathrm{Map}^{\Gamma}(\widetilde{X}, E\Gamma)$ from the right.

Subclaim (4.3). Λ acts on $\mathrm{Map}_{\widetilde{X}}(X, B\Gamma)$ trivially.

Proof. It is obvious, since Λ is the center of Γ^{op}. $\square$

Proof of Claim (4.1). Since Γ^{op} acts on $\mathrm{Map}^{\Gamma}(\widetilde{X}, E\Gamma)$ freely, we get a universal Γ^{op}-bundle

$$\Gamma^{\mathrm{op}} \to \mathrm{Map}^{\Gamma}(\widetilde{X}, E\Gamma) \to \mathrm{Map}_{\widetilde{X}}(X, B\Gamma)/(\Gamma^{\mathrm{op}}/\Lambda). \quad \square$$

We proceed to the definition of the G-action on $\mathrm{Map}_{\widetilde{X}}(X, B\Gamma)/(\Gamma^{\mathrm{op}}/\Lambda)$. $\widehat{G}$ acts on Γ^{op} by conjugation, which we denote $\rho : \widehat{G} \to \mathrm{Aut}(\Gamma^{\mathrm{op}})$. Through conjugation by the anti-isomorphism $R : \Gamma \to \Gamma^{\mathrm{op}}$, we also have a homomorphsim $\rho^{*} : \widehat{G} \to Aut(\Gamma)$. We define the $\widehat{G}$-action from the right as follows:

$\hat{g} \in \widehat{G}$ maps

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\hat{f}"] \arrow[d] & E\Gamma \arrow[d] \\
X \arrow[r, "f"] & B\Gamma
\end{tikzcd}$$

to

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\hat{g}"] \arrow[d] & \widetilde{X} \arrow[r, "\hat{f}"] \arrow[d] & E\Gamma \arrow[r, "E\rho^*(\hat{g}^{-1})"] \arrow[d] & E\Gamma \arrow[d] \\
X \arrow[r, "g"] & X \arrow[r, "f"] & B\Gamma \arrow[r, "B\rho^*(\hat{g}^{-1})"] & B\Gamma.
\end{tikzcd}$$

Claim (4.4). $E\rho^*(\hat{g}^{-1}) \circ \hat{f} \circ \hat{g} : \widetilde{X} \to E\Gamma$ is Γ-equivariant.

Proof. Since $\widehat{G}$ acts on Γ^{op} by conjugation, we have

$$\hat{g} \circ R(\gamma)(x) = R(\rho^*(\hat{g})\gamma) \circ \hat{g}(x) \text{ for } x \in \widetilde{X}.$$

Thus we get

$$\begin{aligned}
E\rho^*(\hat{g}^{-1}) \circ \hat{f} \circ \hat{g} \circ R(\gamma)(x) &= E\rho^*(\hat{g}^{-1}) \circ \hat{f} \circ R(\rho^*(\hat{g})\gamma) \circ \hat{g}(x) \\
&= E\rho^*(\hat{g}^{-1}) \circ R(\rho^*(\hat{g})\gamma) \circ \hat{f} \circ \hat{g}(x) \\
&= R(\rho^*(\hat{g}^{-1}) \circ \rho^*(\hat{g})(\gamma)) \circ E\rho^*(\hat{g}^{-1}) \circ \hat{f} \circ \hat{g}(x) \\
&= R(\gamma) \circ E\rho^*(\hat{g}^{-1}) \circ \hat{f} \circ \hat{g}(x). \ \square
\end{aligned}$$

Claim (4.5). The above $\widehat{G}$-action is an extension of the principal Γ^{op}-action on $\mathrm{Map}^{\Gamma}(\widetilde{X}, E\Gamma)$.

Proof. It is easy to see that

$$\rho(R(\gamma))R(\xi) = R(\gamma^{-1} \cdot \xi \cdot \gamma)$$

and by the definition

$$\rho^*(\hat{g})(\xi) = R^{-1} \circ \rho(\hat{g}) \circ R(\xi).$$

Therefore we get $Ad(\gamma) = \rho^*(R(\gamma^{-1}))$, hence

$$\begin{aligned}
E\rho^*(R(\gamma)^{-1}) \circ \hat{f} \circ R(\gamma) &= EAd(\gamma) \circ \hat{f} \circ R(\gamma) \\
&= EAd(\gamma) \circ R(\gamma) \circ \hat{f} \\
&= R(\gamma) \circ EAd(\gamma) \circ \hat{f} \ . \square
\end{aligned}$$

Since the diagonal action of $\widehat{G}$ on $EG \times \mathrm{Map}^{\Gamma}(\widetilde{X}, E\Gamma)$ is free, we have a universal $\widehat{G}$-bundle

$$\widehat{G} \to EG \times \mathrm{Map}^{\Gamma}(\widetilde{X}, E\Gamma) \to \{EG \times \mathrm{Map}^{\Gamma}(\widetilde{X}, E\Gamma)\}/\widehat{G}.$$

Since $\{EG \times \mathrm{Map}^{\Gamma}(\widetilde{X}, E\Gamma)\}/\widehat{G} = \{EG \times \mathrm{Map}_{\widetilde{X}}(X, B\Gamma)/(\Gamma^{\mathrm{op}}/\Lambda)\}/G$, we get a fiber bundle

$$\begin{tikzcd}
\mathrm{Map}_{\widetilde{X}}(X, B\Gamma)/(\Gamma^{\mathrm{op}}/\Lambda) \arrow[r] \arrow[d, no head, "\wr"'] & \{EG \times \mathrm{Map}_{\widetilde{X}}(X, B\Gamma)/(\Gamma^{\mathrm{op}}/\Lambda)\}/G \arrow[r] \arrow[d, no head, "\wr"'] & BG \arrow[d, equal] \\
B\Gamma^{\mathrm{op}} \arrow[r] & B\widehat{G} \arrow[r] & BG\,.
\end{tikzcd}$$

Although $Y \to X_G$ is not a Galois Γ-covering space, we can get a "family of classifying mappings" to a certain $B\Gamma$-bundle over BG.

$\mathcal{F}(\Gamma) \to BG$ denotes the bundle of covering transformation groups of a family of covering spaces $Y \to X_G$ parametrized by BG, namely $\mathcal{F}(\Gamma)|_p$ is the covering transformation group of $Y|_p \to X_G|_p$ for $p \in BG$. Then we have a family of universal bundles $\mathcal{F}(E\Gamma) \to \mathcal{F}(B\Gamma)(\to BG)$. By the standard obstruction theoretical argument, we get a family of classifying mappings

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & \mathcal{F}(E\Gamma) \arrow[d] \\
X_G \arrow[r] \arrow[d] & \mathcal{F}(B\Gamma) \arrow[d] \\
BG \arrow[r, equal] & BG.
\end{tikzcd}$$

Remark that the ambiguity of identification between the covering transformation group of $\widetilde{X} \to X$ and Γ is inner automorphisms of Γ. Thus the family of classifying mappings determines a cross section of

$$B\widehat{G} = \{EG \times \mathrm{Map}_{\widetilde{X}}(X, B\Gamma)/(\Gamma^{\mathrm{op}}/\Lambda)\}G \to BG.$$

The rest of the proof continues in a similar way as in Theorem (2.3).

Remark. A cross section of $B\widehat{G} \to BG$ is also given in the following way. $X_G \to BG$ is a fiber bundle associated to the universal bundle $EG \to BG$ and the principal bundle associated to $Y \to BG$ is a $\widehat{G}$-bundle. The classifying mapping of this $\widehat{G}$-bundle gives a cross section of $B\widehat{G} \to BG$. This argument also works for proof of Theorem (2.3) and (2.4).

5 Proof of Lemma (3.2).

Let ω be a G-invariant 2-form representing the real first Chern class $c_1(P)_{\mathbf{R}}$. According to the de Rham model of the equivariant cohomology [A-B II], a G-invariant closed 2-form η on X extends to a closed 2-form on X_G if and only if $i(\mathbf{g})\eta \in \Omega^1(X;\mathbf{g}^*)^G$ is exact, where $i(\mathbf{g})$ is the interior product by fundamental vector fields of the G-action, i.e. there exists a $\mathbf{g}^*$-valued G-equivariant function $\mu : X \to \mathbf{g}^*$ such that $d\mu + i(\mathbf{g})\eta = 0$. Let θ be a connection on $EG \to BG$ and Ω its curvature form. Using the connection θ, we can extend η to a vertical 2-form η_G. μ extends to $\mu_G : EG \times_G X \to EG \times_{Ad^*} \mathbf{g}^*$. Then it is easy to see that $\tilde{\eta} := \eta_G + < \mu_G, \Omega >$ is a closed 2-form on X_G. Since $\mathrm{H}^2(BG;\mathbf{R}) \to \mathrm{H}^2(X_G;\mathbf{R}) \to \mathrm{H}^2(X;\mathbf{R})^G$ is an exact sequence, the real cohomology class $c_1(\tilde{P})_{\mathbf{R}}$ is represented by $\tilde{\omega}' = \tilde{\omega} + \xi$, where ξ is a horizontal 2-form coming from BG. We fix a connection $\widetilde{\nabla}$ on $\tilde{P} \to X_G$ whose curvature is $\tilde{\omega}'$. Let ∇ be the restriction of $\widetilde{\nabla}$ to X. We shall show that the gauge equivalence class of ∇ is a G-fixed point in $\mathcal{B}(P)$.

Connections ∇ and ∇' are gauge equivalent if and only if the corresponding holonomies for all loops are same. Let γ be a loop in X. For each $g \in G$, there is a loop l in BG whose holonomy with respect to θ is g. The parallel translation of γ along l defines a cylinder C in X_G whose boundaries are γ and $g \cdot \gamma$ in X. Then we get

$$hol(g \cdot \gamma) \cdot hol(\gamma)^{-1} = \exp 2\pi i \int_C \tilde{\omega}'.$$

Since $\tilde{\omega}'$ is a sum of a vertical 2-form and horizontal 2-forms, the integration on the right hand side vanishes. Hence $g^*\nabla$ and ∇ are gauge equivalent.

6 Proof of Theorem (2.1).

It is obvious that the condition (i) implies the condition (ii). We will show the converse and the one-to-one correspondence.

(1). The case that H is a discrete group.

If there is a cross section of $BH \to B\widehat{G} \to BG$, we have a splitting as H-spaces of $\Omega BH \to \Omega B\widehat{G} \to \Omega BG$, which is homotopically equivalent to the covering group $H \to \widehat{G} \to G$. Hence we have a homotopy left inverse s of $\widehat{G} \to G$, which means that $\widehat{G}$ consists of copies of G as a topological space. Since s is a splitting as H-spaces, the image of s is a subgroup of $\widehat{G}$. Thus we get a splitting. The one-to-one correspondence is clear from the above argument.

(2). The case that H is S^1.

First of all, we assume that G is connected. The following simplified argnument is due to Jean Lannes.

Note that BS^1 is a H-space and $BS^1 \to B\widehat{G} \to BG$ is a principal BS^1-bundle. Existence of a cross section of this bundle is equivalent to existence of a splitting $B\widehat{G} \to BS^1$. By applying the loop functor, we get a continuous mapping $\widehat{G} \to S^1$ which is a homotopic left inverse of the inclusion $S^1 \to \widehat{G}$. The conclusion follows from the following

Lemma (6.1). Let K be a compact connected Lie group. Then we have

$$\pi_0 \mathrm{Map}(K, S^1) \cong \mathrm{Hom_c}(K, S^1),$$

where $\mathrm{Hom_c}(K, S^1) = \{f : K \to S^1 |$ continuous homomorphism $\}$.

Proof. Since S^1 is a $K(\mathbf{Z}, 1)$ space, $\pi_0\mathrm{Map}(K, S^1)$ is isomorphic to $\mathrm{H}^1(K, \mathbf{Z}) = \mathrm{Hom}(\pi_1 K; \mathbf{Z})$. It is enough to show that $\mathrm{Hom}(\pi_1 K; \mathbf{Z}) \cong \mathrm{Hom_c}(K, S^1)$. For any compact Lie group K, there is a finite covering group $\widetilde{K}$ which is isomorphic to $K_s \times T^l$, where K_s is a 1-connected compact semi-simple Lie group and T^l is a toral group. Let Γ be the kernel of $\widetilde{K} \to K$. Then we have the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom_c}(K, S^1) \arrow[r] \arrow[d] & \mathrm{Hom_c}(\widetilde{K}, S^1) \arrow[r] \arrow[d] & \mathrm{Hom}(\Gamma, S^1) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(\pi_1 K, \mathbf{Z}) \arrow[r] & \mathrm{Hom}(\pi_1 \widetilde{K}, \mathbf{Z}) \arrow[r] & \mathrm{Ext}^1(\Gamma, \mathbf{Z})
\end{tikzcd}$$

where the first two column homomorphisms are induced homomorphisms between fundamental groups, the last one is an isomorphism by definition, and the lower exact sequence is a consequence of the exact sequence $0 \to \pi_1 \widetilde{K} \to \pi_1 K \to \Gamma \to 0$. Since $\mathrm{Hom_c}(K_s, S^1)$ is a singleton consisting of the trivial homomorphism, the middle column homomorphism is an isomorphism. Therefore we get $\mathrm{Hom}_c(K, S^1) \cong \mathrm{Hom}(\pi_1 K, \mathbf{Z})$. $\square$

Since the set of homotopy classes of cross sections of $B\widehat{G} \to BG$ is isomorphic to $\pi_0\mathrm{Map}(BG, BS^1)$, Lemma (6.1) yields the one-to-one correspondence. Next we prove Theorem (2.1) in case that G is a finite group. Let n be the order of G and $\widetilde{G} = \{x \in \widehat{G} | x^n = 1\}$. Then we have a principal S^1-bundle $S^1 \to B\widetilde{G} \to B\widehat{G}$. Pulling back this S^1-bundle by the section $s : BG \to B\widehat{G}$, we get a principal S^1-bundle over BG and denote it $E \to BG$. Since G is a finite group, $\mathrm{H}^2(BG; \mathbf{Z})$ is a finite module. Hence there is an positive integer k such that $k \cdot c_1(E) = 0$, i.e. the k-th tensor product $E^{\otimes k}$ of the S^1-bundle E is trivial. It is easy to see that the k-th tensor product of $S^1 \to B\widetilde{G} \to B\widehat{G}$ is isomorphic to $S^1 \to B\widetilde{G}^{(k)} \to B\widehat{G}$, where $\widetilde{G}^{(k)} = \{x \in \widehat{G} | x^{nk} = 1\}$. Therefore the pull back of $S^1 \to B\widetilde{G}^{(k)} \to B\widehat{G}$ by the section $s : BG \to B\widehat{G}$ is trivial, which yields that there is a homotopic left inverse of $B\widetilde{G}^{(k)} \to BG$. $G^{(k)}$ is a central extension of G by $\mathbf{Z}/nk\mathbf{Z}$, and we get a splitting homomorphism $G \to \widetilde{G}^{(k)}$. (In fact, its image is contained in $\widetilde{G}$.) The composition of this homomorphism with the inclusion mapping $\widetilde{G}^{(k)} \to \widehat{G}$ is a desired splitting homomorphism $G \to \widehat{G}$. The one-to-one correspondence is reduced to the case that H is a finite group. $\square$

We proceed to the case of a general compact Lie group G. Let $\widehat{G}_0$ and G_0 denote the identity component of $\widehat{G}$ and G respectively and $\Gamma = G/G_0$. We have the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & S^1 \arrow[r] \arrow[d, equal] & \widehat{G}_0 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & S^1 \arrow[r] & \widehat{G} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
 & & \Gamma \arrow[r, equal] & \Gamma &
\end{tikzcd}$$

Pulling back the section $s : BG \to B\widehat{G}$ by the bundle mapping

$$\begin{tikzcd}
BS^1 \arrow[r, equal] \arrow[d] & BS^1 \arrow[d] \\
B\widehat{G}_0 \arrow[r] \arrow[d] & B\widehat{G} \arrow[d] \\
BG_0 \arrow[r] & BG ,
\end{tikzcd}$$

we get a section of $B\widehat{G}_0 \to BG_0$. Hence there is a splitting homomorphism $\phi : G_0 \to \widehat{G}_0$ of $0 \to S^1 \to \widehat{G}_0 \to G_0 \to 1$. We shall show the following lemma later.

Lemma (6.2). $\mathrm{Im}(\phi)$ is a normal subgroup of $\widehat{G}$.

Let $\widehat{\Gamma}$ be $\widehat{G}/\mathrm{Im}(\phi)$. Then we can show the following

Claim (6.3). $BS^1 \to B\widehat{\Gamma} \to B\Gamma$ admits a section, hence $0 \to S^1 \to \widehat{\Gamma} \to \Gamma \to 1$ is split exact.

Proof. We have the following diagram

$$\begin{tikzcd}
B\widehat{G}_0 \arrow[r] \arrow[d] & B\widehat{G} \arrow[r] \arrow[d, shift left] & B\widehat{\Gamma} \arrow[d] \\
BG_0 \arrow[r] & BG \arrow[r] \arrow[u, shift left, "s"'] & B\Gamma .
\end{tikzcd}$$

Since BG_0 is 1-connected, we have a section $t : B\Gamma^{(2)} \to BG$ of $BG \to B\Gamma$ over the 2-skeleton $B\Gamma^{(2)}$. Composing $s \circ t$ with the mapping $B\widehat{G} \to B\widehat{\Gamma}$, we have a section s' of $B\widehat{\Gamma} \to B\Gamma$. Since the restriction of s to BG_0 is homotopic to the mapping $BG_0 \to B\widehat{G}_0$ induced by the homomorphism ϕ and the composition of ϕ with $\widehat{G}_0 \to \widehat{G} \to \widehat{\Gamma}$ is trivial, s' extends over the 3-skeleton. Meanwhile $\pi_k(BS^1)$ vanishes for $k \geq 3$, therefore s' extends over $B\Gamma$. □

Recall the following diagram

$$\begin{tikzcd}
\widehat{G}_0 \arrow[r] \arrow[d] & \widehat{G} \arrow[r] \arrow[d] & \widehat{\Gamma} \arrow[d] \\
G_0 \arrow[r] & G \arrow[r] & \Gamma .
\end{tikzcd}$$

The pull back of the splitting homomorphism $\Gamma \to \widehat{\Gamma}$ by the homomorphism $G \to \Gamma$ gives a splitting homomorphism $G \to \widehat{G}$. □

Proof of Lemma (6.2). ϕ corresponds to a section s_0 of $B\widehat{G}_0 \to BG_0$ which is the restriction of $s : BG \to B\widehat{G}$.

$$\begin{tikzcd}
B\widehat{G} \arrow[r, phantom, "="] \arrow[d] & \{E\Gamma \times B\widehat{G}_0\}/\Gamma \arrow[d] \\
BG \arrow[r, phantom, "="] & \{E\Gamma \times BG_0\}/\Gamma
\end{tikzcd}$$

In the above diagram, Γ action on $B\widehat{G}_0$ and BG_0 is the action of $\Gamma = \widehat{G}/\widehat{G}_0 = G/G_0$ on $B\widehat{G}_0 = (E\widehat{G})/\widehat{G}_0$ and $BG_0 = (EG)/G_0$, and $\widehat{\psi}$ and ψ denote these actions respectively. Since s_0 is the restriction of s, s_0 is equivariant under $\widehat{\psi}$ and ψ, i.e. s_0 is invariant under Γ action on $\mathrm{Map}(BG_0, B\widehat{G}_0)$. On the other hand, $\widehat{G}$ and G act on $\widehat{G}_0$ and G_0 by conjugation, which induces the action on $B\widehat{G}_0$ and BG_0 (see §4), and $\widehat{\varphi}$ and φ denote these actions respectively. Then the following two commutative diagrams

$$\begin{tikzcd}
B\widehat{G}_0 \arrow[r, "\widehat{\psi}(\gamma)"] \arrow[d] & B\widehat{G}_0 \arrow[d] \\
BG_0 \arrow[r, "\psi(\gamma)"] & BG_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
B\widehat{G}_0 \arrow[r, "\widehat{\varphi}(\hat{g}^{-1})"] \arrow[d] & B\widehat{G}_0 \arrow[d] \\
BG_0 \arrow[r, "\varphi(g^{-1})"] & BG_0
\end{tikzcd}$$

are homotopically equivalent if $\hat{g}$ and g are lifts of γ with respect to homomorphisms $\widehat{G} \to \Gamma$ and $G \to \Gamma$ respectively. Hence the one-to-one correspondence statement for the connected Lie group G_0 yields that $\phi = Ad(\hat{g}) \circ \phi \circ Ad(g)^{-1}$. Hence $\mathrm{Im}(\phi)$ is a normal subgroup of $\widehat{G}$. The one-to-one correspondence follows from the one for G_0 and the one for Γ. □

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