On Quasirational (by Abhyankar) Singularities

S. Yu. Orevkov

NIISI (System Research Institute)
Academy of Sciences of Russia
ul. Avtozavodskaya 23
Moscow

Russia

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI / 92-62

ON QUASIRATIONAL (BY ABHYANKAR) SINGULARITIES.

S.Yu.Orevkov.

Max-Planck-Institut für Mathematik, Bonn, Germany
System Research Institute, Moskow, Russia

Let p be a singular point on a copmlex analytic surface X, and let $\sigma : \tilde{X} \to X$ be a resolution of the singularity of X at p. Abhyankar [1] called the singularity at p to be *quasirational*, if each irreducible component of the exeptional curve $E = \sigma^{-1}(p)$ is a rational curve.

Let U be a neighbourhood of 0 in $\mathbf{C}^2$, and C an analytic curve in U defined by an equation $f(x,y) = 0$. Suppose that C has an analytically irreducible singularity at 0. Given an integer g, consider a surface X in $\mathbf{C}^3$ defined by the equation $z^g + f(x,y) = 0$. In [1] is proved, that the singularity of X at the origin is quasirational if C has at the origin a single characteristic pair (m,n), and all the numbers m,n,g are pairwise coprime. It is the main result of the first part of [1], and in the second part an analogous statement is proved for any ground field.

In this short note we shall prove some generalization of the above Abhyankar's theorem (only in analytic case over $\mathbf{C}$) for arbitrary number of characteristic pairs. It is an immediate consequence of well-known facts.

Let, as above, the germ of f at the origin be analytically irreducible. Choose the coordinates so that one of the coordinate axes is tangent to C at 0, and let m and n be the orders of the restrictions of f on the coordinate axes. (When the singularity of C at 0 has only one characterictic pair, the pair is exactly (m,n).)

Proposition. *Let either m or n be coprime with g. Then the singularity of X at 0 is quasirational.*

Proof. Let S^5 and S^3 be sufficiently small spheres in $\mathbf{C}^3$ and $\mathbf{C}^2$ respectively, centered by 0 , and let $M = X \cap S^5$, $K = C \cap S^3$. Obviously, M is a 3-manifold and K is a knot in S^3. According to [4], $H_1(M,\mathbf{Q})$ contains a subgroup isomorphic to $\dot{H}_1(E,\mathbf{Q})$. Hence, vanishing of $H_1(M,\mathbf{Q})$ implies that the singularity of X at 0 is quasirational.

On the other hand, M is g-sheeted cyclic covering of S^3, branched over K (cf. [5]). Hence, by [3, p.149], $H_1(M,\mathbf{Q})$ vanishes iff Alexander polynomial $\Delta_K(t)$ has no common roots with the polynomial $t^g - 1$. Thus, to prove the proposition it suffices to show, that $\Delta_K(\omega^j) \neq 0$ for any j, where ω is a primitive g-root of unity. To prove it, we shall use an explicite formula due to Zariski [6] for the Alexander polynomial $\Delta_K(t)$ via the characteristic sequence of the singularity.

One of m, n, is coprime with g. Let it be n. It is the order of the restriction of f on one of the axes, say y. Then the Puiseux expansion of C at 0 is as follows: $x = t^n$, $y = \sum_{i\geq m} a_i t^i$. According to [2], introduce the following notation: $d_1 = n$, $m_1 = m$;

$$d_i = \gcd(d_{i-1}, m_i), \; m_i = \min\{ j \mid a_j \neq 0, d_i \nmid j \}, \; i > 1;$$

denote by h an integer such that $d_h \neq 1$, $d_{h+1} = 1$ (thus, m_i is defined for $i = 1, ..., h$, and d_i is defined for $i = 1, ..., h+1$). Let $n_i = d_i/d_{i+1}$, $i = 1, ..., h$, and let $r_1 = m_1$,

$$r_i = r_{i-1}n_{i-1} + m_i - m_{i-1}, \; i = 2, ..., h.$$

Note, that $n = n_1 ... n_h$, and hence, g is coprime with each of $n_1, ..., n_h$.

According to [6],

$$\Delta_K(t) = \frac{t-1}{t^n - 1} \prod_{i=1}^{h} \frac{t^{r_i n_i} - 1}{t^{r_i} - 1}.$$

Suppose that $\Delta_K(\omega^j) = 0$ for some j. Then ω^j is a root of some polynomial $(t^{r_i n_i} - 1)/(t^{r_i} - 1)$, i.e.

$$\omega^{j r_i n_i} = 1, \tag{1}$$

$$\omega^{j r_i} \neq 1. \tag{2}$$

Since ω is a primitive g-root of unity, (1) implies that g divides $j r_i n_i$. But g and n_i are coprime, hence, g divides $j r_i$. But it contradicts to (2). Q.E.D.

Remark 1. We proved that if g is coprime with n, then the group $H_1(M, \mathbf{Q})$ vanishes. If additionally to claim that g is coprime with each of $r_1, ..., r_h$, it can be shown that $H_1(M, \mathbf{Z})$ also vanishes.

Remark 2. In fact, the condition of vanishing of $H_1(M, \mathbf{Q})$ (or, equivalently, the condition that $\Delta_K(t)$ and $t^g - 1$ have no common root) is not only sufficient, but also necessary for quasirationality of X at 0. It follows from the fact that the graph of irreducible components of E (dual graph of E) is a tree.

Remark 3. Necessary and sufficient conditions for the quasirationality of X can be written as a numerical conditions on g and the characteristic sequence $(r_1, n_1; ...; r_h, n_h)$. The conditions are that for each $i = 1, ..., h$ at least one the two following equalities holds:

$$\gcd(g, r_i n_i) = \gcd(g, r_i) \tag{3}$$

$$\gcd(g, r_i n_i) = \gcd(g, d_i). \tag{4}$$

Remark 4. Copmlete topological description of the curve E (including the intersection matrix of irreducible components) can be obtained in terms of the branching number g and the characteristic sequence. For example, each i for which both conditions (3) and (4) do not hold, gives $\gcd(g, d_{i+1})$ irreducible components of E of genus

$$\frac{\gcd(g, r_i n_i) + \gcd(g, d_{i+1}) - \gcd(g, r_i) - \gcd(g, d_i)}{2 \gcd(g, d_{i+1})}$$

Other irreducible components of E are rational.

Remark 5. If the germ of C at 0 is not analytically irreducible, then vanishing of $H_1(M, \mathbf{Q})$ is not necessary for quasirationality of X at 0, because in this case non-zero 1-cycles may appear not only because of positive genus of a component of E, but also because the dual graph of E can be not a tree. Example: $f = x^5 + x^2y^2 + y^5$, $g = 2$.

This work was done when I stayed in Max-Planck-Institut fr Mathematik and I thank this institute for the hospitality.

REFERENCES

1. S.S.Abhyankar, *Quasiraional singularities*, Amer. J. Math. **101** (1979), 267–300.
2. S.S.Abhyankar, *Expansion technique in algebraic geometry*, Tata Inst. Fund. Res., Bombay, 1977.
3. R.H.Fox, *Topology of 3-manifolds and related topics. Proc. Univ. Georgia Inst. 1961*, Prentice Hall, 1962.
4. D.Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Publ. Math. I.H.E.S. **9** (1961), 229–246.
5. I.V.Saveliev, *Structure of the singularities of a class of complex hypersurfaces*, Math. Notes **25** (1979), no. 4, 497–503.
6. O.Zariski, *On the topology of algebroid singularities*, Amer. J. Math. **54** (1932), 453–465.