Introduction to Affine Differential Geometry

Part I

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Preface

This is Part I of the lecture notes: Introduction to Affine Differential Geometry. It is intended as a brief introduction to classical affine differential geometry, namely, geometry of nondegenerate hypersurfaces in an affine space for which the fundamental group (in the sense of the Erlangen Program of F. Klein) is the group of equiaffine (= special affine) transformations.

When I became interested in the subject, my first aim was to understand just what it was basically all about. In these notes, I present my way of understanding this geometry from the point of view prevalent in differential geometry today. Though concise, I hope they will give the reader a self-contained comprehensible introduction. It is my intention to continue to Part II and possibly Part III in which I would like to present more results within the framework of classical affine differential geometry as well as developments made in a more general approach to the geometry of affine immersions.

I started the study of the subject at Max-Planck-Institut für Mathematik, Bonn, in 1982, and continued the subsequent research in collaboration with Ulrich Pinkall, currently at Technische Universität Berlin, through my several visits to Bonn and Berlin during the last several years. These notes, Part I, are based on my lectures and discussions at MPI, TU Berlin, Brown University and the University of Granada.

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1. Equiaffine structure on a nondegenerate hypersurface

Let $f: M^n \to R^{n+1}$ be a hypersurface immersed in the affine space R^{n+1}. To develop the equiaffine theory for M^n we assume that R^{n+1} is provided with an equiaffine structure, that is, it has a fixed volume element ω which is parallel relative to the usual flat affine connection D in R^{n+1}.

We are interested in introducing in M^n an equiaffine structure (∇, θ), where ∇ is a torsion-free affine connection and θ is a volume element such that $\nabla\theta = 0$. We shall henceforth assume that R^{n+1} is oriented so that $\omega > 0$ and that M^n is also oriented.

We first develop a local theory. We choose an arbitrary transversal vector field ξ in a neighborhood U in M^n so that we have

(1.1) $T_{f(x)}(M) = f_*(T_x(M)) + \text{Span}(\xi_x)$ at each $x \in U$

in such a way that the orientation of M^n followed by ξ coincides with the orientation of R^{n+1}. Let X and Y be vector fields in U. We may decompose $D_X f_*(Y)$ according to (1,1) and write

(1.2) $D_X f_*(Y) = f_*(T_x(M)) + h(X,Y)\xi$ at each point $x \in U$.

Just as in the classical theory of hypersurfaces in Euclidean space, we can verify that ∇ is a torsion-free affine connection in U, h is a tensor field which defines a symmetric bilinear form on each tangent space $T_x(M)$.

We call ∇ the <u>induced affine connection</u> and h the <u>affine fundamental form</u> corresponding to ξ.

We also decompose $D_X\xi$ as follows:

(1.3) $D_X\xi = - f_*(SX) + \tau(X)\xi$,

where S is a tensor field of type (1,1), called the <u>shape operator</u>, and τ is a 1-form, called the <u>transversal connection form</u>.

Now we define the induced volume element θ in U by setting

(1.4) $\theta(X_1, \ldots, X_n) = \omega(X_1, \ldots, X_n, \xi)$

and hope to achieve the property $\nabla \theta = 0$. We have

<u>Lemma</u> 1.1. $\nabla_X \theta = \tau(X)\theta$ <u>for every</u> $X \in T_X(M)$.

<u>Proof</u>. We have

$$
\begin{aligned}
&(\nabla_X\theta)(X_1, X_2, \ldots, X_n)\\
&= X\,\theta(X_1, X_2, \ldots, Xn) - \theta(\nabla_X X_1, X_2, \ldots, Xn) - \cdots\\
&\qquad - \theta(X_1, \ldots, Xn-1, \nabla_X X_n)\\
&= X\,\omega(X_1, X_2, \ldots, Xn, \xi) - \omega(\nabla_X X_1, X_2, \ldots, Xn, \xi) - \cdots\\
&\qquad - \omega(X_1, \ldots, Xn-1, \nabla_X X_n, \xi)\\
&= X\,\omega(X_1, X_2, \ldots, Xn, \xi) - \omega(D_X X_1, X_2, \ldots, Xn, \xi) - \cdots\\
&\qquad - \omega(X_1, \ldots, Xn-1, D_X X_n, \xi)\\
&= (D_X\omega)(X_1, X_2, \ldots, Xn, \xi) + \omega(X_1, X_2, \ldots, Xn, D_X\xi)\\
&= \theta(X)\theta(X_1, X_2, \ldots, Xn),
\end{aligned}
$$

where we have used $D\omega = 0$ and $D_X\omega = \tau(X)\omega$. □

Thus the property $\tau = 0$, that is, $D_X\xi$ is tangent to M^n, is crucial. We shall see that under a certain nondegeneracy condition on M^n we may choose ξ with this property and, indeed, with an additional property, which will make its choice unique. For this purpose, we have

<u>Lemma</u> 1.2. <u>If we choose another transversal vector field</u>

$\bar{\xi} = Z + \varphi\,\xi$, <u>where</u> $\varphi > 0$,

<u>then for the corresponding objects we have</u>

(i) $h = \varphi\,\bar{h}$

(ii) $\nabla_X Y = \bar{\nabla}_X Y + h(X,Y)Z$

(iii) $\tau = \bar{\tau} - d\ln\varphi - h(\,.\,,Z)/\varphi$,

<u>where</u> $h(\,.\,,Z)$ <u>is a</u> 1-<u>form whose value o</u>n X <u>is</u> $h(X,Z)$.

Proof. Straightforward verification. □

It follows from (i) that h is determined up to a scalar function $\varphi > 0$. In particular, whether h is degenerate or nondegenerate depends only on M^n and not on the choice of ξ. If h is nondegenerate at every point, we say that M^n is *nondegenerate*.

Lemma 1.3. *Let* M^n *be nondegenerate. If* ξ *is a transversal vector field and* φ *an arbitrary scalar function* > 0, *then there is a vector field Z on* M^n *such that for* $\bar{\xi} = Z + \varphi\xi$ *the transversal connection form* $\bar{\tau}$ *is* 0.

Proof. Since h is nondegenerate, we can find Z in each $T_x(M)$ such that

$$h(X,Z) = -\varphi\tau(X) - (d\varphi)(X)$$

for every $X \in T_x(M)$. By (iii) of Lemma 1.2, we have $\bar{\tau} = 0$. □

Remark. *If two transversal vector fields* ξ *and* $\bar{\xi}$ *are such that* $\tau = \bar{\tau}$ *and* $\theta = \bar{\theta}$, *then* $\xi = \bar{\xi}$. In fact, $\theta = \bar{\theta}$ implies $\varphi = 1$. (iii) of Lemma 1.2 implies that $Z = 0$.

In order to determine ξ uniquely for nondegenerate M^n, we consider one more condition. Let ν be the volume element associated to the metric h: If $\{X_1, \ldots, X_n\}$ is an oriented orthonormal basis in $T_x(M)$ for the nondegenerate metric h, then $\nu(X_1, \ldots, X_n) = 1$.

The condition we now wish to impose is that two volume elements θ and ν determined by a choice of ξ coincide. To study this condition, we define a function H_ξ as follows.

Choose a basis $\{X_1, \ldots, X_n\}$ such that $\theta(X_1, \ldots, X_n) = 1$ and set

$$h_{ij} = h(X_i, X_j)$$

and

$$H_\xi = \text{determinant of the matrix } [h_{ij}].$$

It is easily verified that H_ξ is independent of the choice of

$\{X_1, \ldots, X_n\}$ subject to $\theta(X_1, \ldots, X_n) = 1$.

Lemma 1.4. $\theta = \nu$ *if and only if the absolute value of* H_ξ *is equal to* 1.

Proof. Choose $\{X_1, \ldots, X_n\}$ as above. Suppose $\overline{X}_i = \sum a^j_i X_j$, $1 \leq i \leq n$, are orthonormal relative to h, say, $h(\overline{X}_i, \overline{X}_j) = \varepsilon_i \delta_{ij}$, where $\varepsilon_i = -1$ for $1 \leq i \leq p$, $\varepsilon_j = -1$ for $p+1 \leq j \leq n$.
Then we have

$^t A H A = [\varepsilon_i \delta_{ij}]$ so that $\det A = |H_\xi|^{-1/2}$ (assuming $\det A > 0$).
From

$$1 = \nu(\overline{X}_1, \ldots, \overline{X}_n) = (\det A)\, \nu(X_1, \ldots, X_n)$$

we get

$$\nu(X_1, \ldots, X_n) = 1/(\det A) = |H_\xi|^{1/2}.$$

Thus $\nu(X_1, \ldots, X_n) = 1$, that is, $\nu = \theta$ if and only if $|H_\xi| = 1$. □

Lemma 1.5. *For a change of transversal vector fields* $\overline{\xi} = Z + \varphi\xi$ *as in Lemma* 2, write $H = H_\xi$, $\overline{H} = H_{\overline{\xi}}$. *Then*

(i) $\overline{H} = H/\varphi^{n+2}$

(ii) $h/|H|^{1/(n+2)} = \overline{h}/|\overline{H}|^{1/(n+2)}$.

Proof. We know $h = \varphi\, \overline{h}$. Choose $\{X_1, \ldots, X_n\}$ with $\theta(X_1, \ldots, X_n) = 1$ so that $H = \det[h(X_i, X_j)]$. We have

$$\overline{\theta}(X_1, \ldots, X_n) = \omega(X_1, \ldots, X_n, \overline{\xi}) = \varphi\, \theta(X_1, \ldots, X_n).$$

Write $\overline{X}_1 = X_1/\varphi$, $\overline{X}_2 = X_2, \ldots, \overline{X}_n = X_n$. Since $\overline{\theta}(\overline{X}_1, \ldots, \overline{X}_n) = 1$, we have

$$\overline{H} = \det[\overline{h}(\overline{X}_i, \overline{X}_j)] = \varphi^{-n} \det[h(\overline{X}_i, \overline{X}_j)] = \varphi^{-(n+2)} H,$$

which proves (i). (ii) follows from $h = \varphi\overline{h}$ and (i). □

From (ii) in Lemma 1.5 we have a uniquely defined form $\hat{h} = h/|H|^{1/(n+2)}$, which is called the *affine metric*.

We now have

Theorem 1.1. Let M^n be a nondegenerate hypersurface. Then there is a unique transversal vector field ξ such that

(i) the transversal connection form is 0;

and

(ii) the induced volume element coincides with the volume element for the affine fundamental form.

We may also replace (ii) by

(ii a) the affine fundamental form coincides with the affine metric.

or

(ii b) the induced volume element coincides with the volume element of the affine metric.

Proof. Start with any transversal vector field ξ and compute $H = H_\xi$.
With $\varphi = |H|^{1/(n+2)}$ let $\bar{\xi} = \varphi\,\xi + Z$. By Lemma 1.3 we can choose Z so that the transversal connection form for $\bar{\xi}$ is 0. By Lemma 1.5, (i), we have $\bar{H} = H/|H|$ so that $|\bar{H}| = 1$, which means that the induced volume element $\bar{\theta}$ coincides with the volume element $\bar{\nu}$ for the fundamental form $\bar{h}$ for $\bar{\xi}$.

By lemma 1.5, (ii), we see that $\bar{h}$ coincides with the affine metric $\hat{h}$. Hence $\bar{\nu}$ coincides with the volume element $\hat{\nu}$ for the affine metric.

We have shown the existence of a transversal vector field which satisfies (i), (ii), (ii a), and (ii b).

To show the uniqueness part of the theorem, let ξ be a transversal vector field satisfying i) and ii). Then, by Lemma 1.4, $|H_\xi| = 1$. Thus $h = \hat{h}$ and $\theta = \nu = \hat{\nu}$. Thus any two transversal vector fields satisfying ii) must have the same induced volume elements. If they both satisfy (i), they must coincide as we know from the Remark following Lemma 1.3.

The uniqueness of a transversal vector field satisfying (i) and (iia), or (i) and (iib), is also obvious from what we said in the above. □

The unique transversal vector field in Theorem 1.1 is called the affine normal. For this unique choice, we have the induced connection ∇ and the induced volume element θ (equal to the volume element $\hat{\nu}$ of the affine metric $\hat{h}$), which together defines a natural equiaffine structure on M^n. The affine fundamental form is the same as the affine metric. The approach in this section was sketched in [N].

2. Fundamental equations

Let M^n be a nondegenerate hypersurface immersed in R^{n+1} and let ξ be the affine normal (whose unique existence we have established in Theorem 1.1). For this choice, we have the induced connection ∇, the affine fundamental form h, which coincides with the affine metric $\hat{h}$, the shape operator S, the induced volume element θ which coincides with the volume element of the affine metric.

We have the following set of fundamental equations for these objects:

Equation of Gauss: The curvature tensor R of ∇ is given by

(2.1) $$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY.$$

Equation of Codazzi for h:

(2.2) $$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z).$$

Thus we get a symmetric trilinear form $C(X,Y,Z) = (\nabla_X h)(Y,Z)$, which is called the cubic form for M^n.

Equation of Codazzi for S:

(2.3) $$(\nabla_X S)(Y) = (\nabla_Y S)(X)$$

Equation of Ricci:

(2.4) $h(SX,Y) = h(X,SY)$

We make some observations.

1. _If_ $R = 0$, _then_ $S = 0$. The converse is obvious.

In fact, let $X \neq 0$. If $h(X,X) \neq 0$, we may assume $|h(X,X)| = 1$ and take Z such that $|h(Z,Z)| = 1$ and $h(X,Z) = 0$. Now take $Y = Z$ in the equation of Gauss:

$$0 = h(Z,Z)SX - h(X,Z)SY = h(Z,Z)SX \quad \text{thus} \quad SX = 0.$$

If $h(X,X) = 0$, take Y such that $h(X,Y) = 1$ and $h(Y,Y) = 0$, Setting $Z = X$ In the equation of Gauss we get $h(Y,X)SX = 0$ so $SX = 0$ again.

2. More generally, S is determined uniquely by h and R. This can be proved by similar arguments as 1 (or assuming there is another S' satisfying the equation of Gauss, apply the argument in 1 to $S - S'$).

3. If $S = 0$, the affine normals are parallel to each other in R^{n+1}. M^n is called an _improper affine hypersphere_.

4. Suppose $S = \lambda I$, where λ is a function and I the identity transformation. (We say that M^n is affine umbilical) Then Codazzi's equation for S implies that λ is a constant function. If $\lambda \neq 0$, M^n is called a _proper affine hypersphere_. All the lines from the points of M^n in the direction of ξ meet at one point, the center of the proper affine hypersphere.

5. The Ricci tensor for ∇ is given by

$$\mathrm{Ric}(Y,Z) = (\mathrm{trace}\, S)h(Y,Z) - h(SY,Z).$$

If $\mathrm{Ric} = 0$, this equation implies $S = 0$ and, consequently, $R = 0$ by 1.

$H = \mathrm{trace}\, S/n$ is called the _affine mean curvature_. $K = \det S$ is called the _affine (Gauss-Kronecker) curvature_.

6. Let $\dim M \geq 3$. Then the equation of Gauss and the equation of Codazzi for h imply the equation of Codazzi for S. To prove this, take ∇_W of both sides of the equation of Gauss and write down the

second Bianchi identity:

$$(\nabla_W R)(X,Y) + (\nabla_X R)(Y,W) + (\nabla_Y R)(W,X) = 0.$$

After some cancellation by using Codazzi's equation for h we get

$$h(Z,X)\{(\nabla_Y S)(W) - (\nabla_W S)(Y)\}$$
$$+ h(Z,Y)\{(\nabla_W S)X - (\nabla_X S)W\}$$
$$+ h(Z,W)\{(\nabla_X S)Y - (\nabla_Y S)X\} = 0.$$

Now given X and Y, choose Z such that $h(X,Z) = h(Y,Z) = 0$ and $h(Z,Z) = \pm 1$ and let $W = Z$ in the above equation. We get $(\nabla_X S)(Y) - (\nabla_Y S)(X) = 0$.

3 Graph of a function

Let $x^{n+1} = F(x^1, \ldots, x^n)$ be a function on a domain G in R^n and consider the graph immersion

(3.1) $f:(x^1, \ldots, x^n) \in G \to (x^1, \ldots, x^n, F(x^1, \ldots, x^n)) \in R^{n+1}$.

We want to find the affine normal for this hypersurface under the condition

(3.2) $\det [F_{ij}] \neq 0$,

where $F_{ij} = \partial^2 F/\partial x^i \partial x^j$.

We start with an obvious choice of a transversal field $\xi = (0, \ldots, 0, 1)$. We have

$$f_*(\partial/\partial x^j) = (0, \ldots, 1, \ldots, F_j),$$

where 1 appears as the j-th component and $F_j = \partial F/\partial x^j$. Thus

$$D_{\partial/\partial x^i} f_*(\partial/\partial x^j) = F_{ij}\, \xi,$$

that is,

(3.3) $\nabla_{\partial/\partial x^i} (\partial/\partial x^j) = 0$ and $h(\partial/\partial x^i, \partial/\partial x^j) = F_{ij}$.

Also,

$$D_{\partial/\partial x^i}\, \xi = 0 \quad \text{so} \quad \tau = 0.$$

We see that f defines a nondegenerate hypersurface if and only if $\det [F_{ij}] \neq 0$. We now find $H = H_\xi$ as follows. Since we have

$$\theta(\partial/\partial x^1, \ldots, \partial/\partial x^n) = \omega(f_*(\partial/\partial x^1), \ldots, f_*(\partial/\partial x^n), \xi) = 1,$$

we get

$$h_{ij} = h(\partial/\partial x^i, \partial/\partial x^j) = F_{ij}$$

so that

(3.4) $\quad H = \det(h_{ij})$ i.e. Hessian of F.

We now want to find a vector field Z such that

$$\partial\, |H|^{1/(n+2)}/\partial x^i + h(X, Z) = 0 \quad \text{for all vector fields } X.$$

Write $Z = \sum Z^i(\partial/\partial x^j)$ and take $X = \partial/\partial x^i$. Then

$$\partial\, |H|^{1/(n+2)}/\partial x^i + \sum h_{ij} Z^j = 0.$$

Let $[F^{ij}]$ be the inverse matrix of $[F^{ij}]$. Then we get

$$Z^k = -\sum F^{ki}(\partial\, |H|^{1/(n+2)}/\partial x^i)$$

The affine normal is then

(3.5) $$\xi = -\sum_{k,i}(F^{ki}\, \partial\, |H|^{1/(n+2)})/\partial x^i)(0, \ldots, 1, \ldots, F_k) + |H|^{1/(n+2)}(0, \ldots, 0, 1).$$

<u>Example</u>. For the graph of $x^{n+1} = \sum a_{ij} x^i x^j$, where $[a_{ij}]$ is a constant matrix with nonzero determinant H, we have $\xi = |H|^{-(n+2)}\xi$. Thus the shape operator S is zero and the induced connection ∇ is flat.

<u>Remark</u>. It is a theorem of Jörgens [J] that <u>if</u> $F(x^1, \ldots, x^n)$ <u>is a differentiable function on the whole</u> R^n <u>such that</u> $\det[F_{ij}]$ <u>is a positive constant, then</u> F <u>is a quadratic function</u>. This theorem has interesting applications in the theory of surfaces (cf. [Sp] p.165, p. 390).

There is a generalization of this result from the point of view of affine differential geometry by Calabi in [Ca 1]. See also [Sp] for other applications of Jörgens' theorem.

4. Cubic form and apolarity

Let M be a nondegenerate hypersurface in R^{n+1}. We now consider tHe Levi-civita connection $\hat{\nabla}$ for the affine metric h and study the difference between $\hat{\nabla}$ and the induced connection ∇. We denote the difference tensor by K:

(4.1) $$K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y$$

and also write

(4.2) $$K_X(Y) = K(X,Y), \quad K_X = \nabla_X - \hat{\nabla}_X$$

which is symmetric in X and Y.

Proposition 4.1. K_X _corresponds to_ $-(1/2)(\nabla_X h)$ _relative to the metric_ h, _that is_,

(4.3) $$h(K_X Y, Z) = -(1/2)(\nabla_X h)(Y,Z).$$

Proof. We apply the derivation $\nabla_X = \hat{\nabla}_X + K_X$ on h and obtain $\nabla_X h = K_X h$. Thus we have

(4.4) $$(\nabla_X h)(Y,Z) = (K_X h)(Y,Z) = -h(K_X Y, Z) - h(Y, K_X Z).$$

Here $(\nabla_X h)(Y,Z)$ is symmetric in X, Y and Z as we know, and $h(K_X Y, Z)$ is symmetric in X and Y. It follows that $h(Y, K_X Z)$ is symmetric in X and Y, as well as in X and Z, namely, in X, Y and Z. From (1) we get

$$(\nabla_X h)(Y,Z) = -2\, h(K_X Y, Z). \qquad \square$$

Corollary. _The induced connection_ ∇ _and the Levi-Civita connection_ $\hat{\nabla}$ _for the affine metric coincide with each other if and only if_ $K = 0$, _that is, if and only if_ $\nabla h = 0$.

We'll see later that this is the case if and only if M is a quadratic hypersurface.

Proposition 4.2. (_apolarity_)

(4.5) trace $K_X = 0$ _for every tangent vector_ X.

Proof. Applying the derivation $\nabla_X = \hat{\nabla}_X + K_X$ to the volume element $\theta = \nu$ we obtain

$$0 = \nabla_X \theta = (\hat{\nabla}_X + K_X)\nu = K_X \nu,$$

which implies trace $K_X = 0$. □

Remark. In terms of the index notation for tensors, we write $h = (h_{ij})$, $(h^{ij}) = (h_{ij})^{-1}$, and $\nabla h = (h_{ijk})$. Then trace $K_X = 0$ can be written as

$$\Sigma_{i,j} h^{ij} h_{ijk} = 0.$$

5. Some more equations

We may further investigate the relationship between the curvature tensor R of the induced connection ∇ and the curvature tensor $\hat{R}$ of the Levi-Civita connection $\hat{\nabla}$ for the affine metric h.

Proposition 5.1.

$$(5.1) \qquad R(X,Y) = \hat{R}(X,Y) + (\hat{\nabla}_X K)_Y - (\hat{\nabla}_Y K)_X + [K_X, K_Y]$$
$$= \hat{R}(X,Y) + (\nabla_X K)_Y - (\nabla_Y K)_X.$$

$$(5.2) \qquad \hat{R}(X,Y)Z = (1/2)\{h(Y,Z)SX - h(X,Z)SY + h(SY,Z)X - h(SX,Z)Y\} - [K_X, K_Y]Z$$

Proof. To obtain (5.1) we compute $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ by using (4.2) and note

$$(\hat{\nabla}_X K)_Y = \hat{\nabla}_X K_Y - K_Y \hat{\nabla}_X - K_{\hat{\nabla}_X Y}$$

as well as

$$(\nabla_X K)_Y = (\hat{\nabla}_X)_Y + (K_X \cdot K)_Y = (\hat{\nabla}_X K)_Y + K_X K_Y - K_{K_X Y}.$$

From (5.2) and the equation of Gauss we have

$$h(\hat{R}(X,Y)Z,W) = h(Y,Z)h(SX,W) - h(X,Z)h(SY,W) + h((\hat{\nabla}_Y K)_X Z, W) - h((\hat{\nabla}_X K)_Y Z, W) - h([K_X, K_Y]Z, W).$$

Alternating this equation in Z and W and observing that K_X and $(\hat{\nabla}_Y K)_X = \hat{\nabla}_Y(K_X) - K_{\hat{\nabla}_Y X}$ are symmetric operators relative to h, we obtain

$$2\, h(\hat{R}(X,Y)Z,W)$$
$$= h(Y,Z)h(SX,W) - h(X,Z)h(SY,W) + h(X,W)h(SY,Z) - h(Y,W)h(SX,Z) - 2h([K_X,K_Y]Z,W).$$

This leads to (5.2). □

Remark. (5.2) is the same equation as (13), p. 136 in [Sch].

Proposition 5.2. The Ricci tensor for the affine metric h is given by

$$\widehat{Ric}\,(Y,Z) = (1/2)\{h(Y,Z)\,\mathrm{trace}\,S + (n-2)h(SY,Z)\} + \mathrm{trace}\,(K_Y K_Z), \tag{5.3}$$

where $\mathrm{trace}(K_Y K_Z) = h(K_Y, K_Z)$ (inner product extending h to the tensor space of type (1,1)).

Proof. Take trace $\{X \to \hat{R}(X,Y)Z\}$ using (5.1) and noting

$$\mathrm{trace}\,\{X \to h(X,Z)SY\} = h(SY,Z)$$

$$\mathrm{trace}\,\{X \to h(SX,Z)Y = h(X,SZ)Y\} = h(Y,SZ) = h(SY,Z)$$

$$\mathrm{trace}\,\{X \to [K_Y,K_X]\,Z\} = \mathrm{trace}\,\{X \to K_Y K_X Z\} - \mathrm{trace}\{X \to K_X K_Y Z\} = \mathrm{trace}\,(K_Y K_X),$$

because

$$K_X K_Y Z = K_{K_Y Z} X \quad \text{and} \quad \mathrm{trace}\,K_{K_Y Z} = 0 \quad \text{(by apolarity)}.$$

Remark. Our formula (5.3) is the same as (2.22) in Schneider's paper [Schn] and formula (3.18) in Calabi's paper [Ca 2].

Proposition 5.3. The scalar curvature $\rho = \Sigma\, h^{ij}\hat{R}_{ij}$ of the affine metric is given by

$$\hat{\rho} = (n-1)\,\mathrm{trace}\,S + h(K,K) = n(n-1)H + J \tag{5.4}$$

Proof. Immediate from (5.3). □

Remark. $H = \mathrm{trace}\,S/n$ is the affine mean curvature, as already defined. $J = h(K,K)$ is called the Pick invariant. (5.4) is the same as Schneider's formula on line 2, p.404 of [Schn] and Calabi's (3.19) in [Ca 2]. Also see Blaschke's book [Bla], p.158.

For $n = 2$, $\widehat{Ric}(Y,Z) = \kappa\, h(Y,Z)$, where κ is the curvature for h.

Thus we get

(5.5) $\quad \mathcal{K} = 2H + J,$

which is essentially the same as (14), p. 136 of [Sch].

We derive one more equation.

Let L be the bilinear symmetric form defined by

(5.6) $\quad L(X,Z) = \mathrm{trace}\,\{Y \to (\hat{\nabla}_Y K)(X,Z)\}.$

We want to prove

Proposition 5.4.

(5.7) $\quad L(X,Z) = (n/2)\{h(X,Z)H - h(SX,Z)\}.$.

Proof. Go back to the equation in the proof of Proposition 5.1:

$$h(\hat{R}(X,Y)Z,W) = h(Y,Z)h(SX,W) - h(X,Z)h(SY,W) + h((\hat{\nabla}_Y K)_X Z,W) - h((\hat{\nabla}_X K)_Y Z,W) - h([K_X,K_Y]Z,W) .$$

Adding this equation and the equation obtained by interchanging Z and W, we obtain

$$0 = 2h((\hat{\nabla}_Y K)_X Z,W) - 2h(\hat{\nabla}_X K)_Y Z,W) + h(Y,Z)h(SX,W) - h(X,Z)h(SY,W) + h(Y,W)h(SX,Z) - h(X,W)h(SY,Z).$$

Eliminating W we write

$$2\,(\hat{\nabla}_Y K)_X Z + h(Y,Z)SX + h(SX,Z)Y = 2\,(\hat{\nabla}_X K)_Y Z + h(X,Z)SY + h(SY,Z)X.$$

Now by taking the trace of the mapping $Y \to$ the above, we get (5.7) by virtue of $\mathrm{trace}\{Y \to (\hat{\nabla}_X K)_Y Z\} = 0$, which can be established as follows.

From $K_Y Z = K_Z Y$ we have $(\hat{\nabla}_X K)_Y Z = (\hat{\nabla}_X K)_Z Y$. Hence

$$\mathrm{trace}\,\{Y \to (\hat{\nabla}_X K)_Y Z\} = \mathrm{trace}\,(\hat{\nabla}_X K)_Z = \mathrm{trace}(\hat{\nabla}_X K)_Z - \mathrm{trace}\,K_{\hat{\nabla}_X Z} = X\,(\mathrm{trace}\,K_Z) = 0$$

using apolarity twice. □

Remark. (5.7) is the same as (2.24) in [Schn]. The tensor L is the same as $c_{ij} = \nabla_s T^s{}_{ij}$ in (15), p. 136, in [Sch]. Observe that

(5.4) determines H uniquely from h and K, and (5.7) determines S. Thus h and $C = \nabla h$ determine H and S uniquely.

6. Theorem of Pick and Berwald

We now prove the following classical result [Ber] for a nondegenerate hypersurface .

Theorem 6.1. *Let* $f: M^n \to R^{n+1}$ *be a nondegenerate hypersurface. Assume that the cubic form is identically zero. Then* $f(M^n)$ *lies in a quadratic hypersurface.*

Proof. We first show that f is umbilical, that is, $S = \rho I$, where ρ is a constant. Since $\nabla h = 0$, we get $R(X,Y) \cdot h = 0$ for any $X, Y \in T_x(M)$, where $R(X,Y)$ acts as a derivation. Thus $h(R(X,Y)Y,Y) = 0$. Using the equation of Gauss we obtain

$$h(Y,Y)h(SX,Y) = h(X,Y)h(SY,Y).$$

Let $\{X_1, \ldots, X_n\}$ be an orthonormal basis for the affine metric in $T_X(M)$: $h(X_i,X_j) = \varepsilon_i\delta_{ij}$, where $\varepsilon_i = \pm 1$. Then

$$h(X_j,X_j)h(SX_i,X_j) = h(Xi,Xj)h(SX_jX_j) = 0$$

and $$h(SX_i,X_j) = 0 \quad \text{for} \quad i \neq j.$$

It follows tat there are scalars ρ_i such that $SX_i = \rho_i X_i$, $1 \le i \le n$.

We now show that all ρ_i's are equal. Let $i \neq j$. Then $X_i + 2X_j$ is non-null relative to h. We may normalize it and extend it to an orthonormal basis in $T_X(M)$. From what we have shown, we have $SZ = \rho Z$ for some scalar ρ. On the other hand, we have $SZ = S(X_i + 2X_j) = \rho_i X_i + 2\rho_j X_j = \rho(X_i + 2X_j)$. From linear independence we get $\rho_i = \rho_j$.

Now we can write $S = \rho I$ on M^n, where ρ is a scalar function. From Codazzi's equation for S, we conclude that ρ is a constant function.

We now define a tensor field g of type (0,2) along the immersion f as follows. For each $x \in M^n$, g_x is a bilinear symmetric function on $T_{f(x)}(R^{n+1})$ determined by

$$g(f_*X, f_*Y) = h(X,Y) \quad \text{for } X,Y \in T_x(M^n)$$
$$(6.1) \qquad g(f_*X, \xi) = 0, \quad \text{where } \xi \text{ is the affine normal}$$
$$g(\xi,\xi) = \rho.$$

We now prove that g is parallel in R^{n+1}, that is,

$$(6.2) \qquad X\, g(U,V) = g(D_XU,V) + g(U,D_XV)$$

for any $X \in T_x(M^n)$ and for any vector fields U and V along f.
Consider three cases:

<u>Case</u> (i): $U = f_*Y$, $V = f_*Z$, where Y and Z are vector fields on M^n.
Then

$$Xg(U,V) = X\, h(Y,Z) = h(\nabla_XY,Z) + h(Y,\nabla_XZ)$$
$$g(D_XU,V) = g(D_Xf_*Y, f_*Z) = g(f_*(\nabla_XY) + h(X,Y)\xi, f_*Z)$$
$$= h(\nabla_XY,Z)$$

and

$$g(U,D_XV) = h(Y,\nabla_XZ)$$

so (6.2) is valid.

Case (ii): $U = f_*(Y)$, $V = \xi$. Then $Xg(U,V) = 0$,

$$g(D_XU,\xi) = g(f_*(\nabla_XY) + h(X,Y)\xi, \xi) = h(X,Y)\rho$$

and

$$g(U,D_X\xi) = g(U, -f_*(SX)) = -g(f_*(Y), f_*(\rho X)) = -\rho\, h(Y,X)$$

so that (6.2) is valid.

Case (iii): $U = V = \xi$. We have

$$X\, g(U,V) = X\rho = 0 \text{ and } g(D_X\xi,\xi) = g(-f_*(SX),\xi) = 0.$$

Next we define a 1-form λ along f by setting

$$(6.3) \qquad \lambda(f_*X) = g(f_*X, f(x)) \quad \text{for } X \in T_x(M^n)$$
$$\lambda(\xi) = g(\xi, f(x)) + 1,$$

where $f(x)$ denotes the position vector of the image point $f(x)$.

Again we show that λ is parallel in R^{n+1}. If Y is a vector field on

M^n, then

$$X(\lambda(f_*Y)) = X(g(f_*Y, f(x))$$
$$= g(D_X(f_*Y), f(x)) + g(f_*Y,\ f_*X)$$

$$= g(f_*(\nabla_X Y), f(x)) + h(X,Y)g(\xi, f(x)) + h(X,Y)$$
$$\lambda(D_X(f_*Y)) = \lambda(f_*(\nabla_X Y)) + h(X,Y)\xi)$$
$$= g(f_*(\nabla_X Y), f(x)) + h(X,Y)\{g(\xi, f(x)) + 1\}$$

so

$$(D_X\lambda)(f_*Y) = X(\lambda(f_*Y, f(x)) - \lambda(D_X(f_*Y)) = 0.$$

Similarly,

$$(D_X\lambda)(\xi) = X(\lambda(\xi)) - \lambda(D_X\xi) = X(g(\xi, f(x)) + 1) - \lambda(-f_*(SX))$$
$$= g(D_X\xi, f(x)) + g(\xi, f_*X) + g(f_*(SX), f(x))$$
$$= - g(\rho X, f(x)) + g(\rho f_*(X), f(x)) = 0.$$

Thus λ is parallel in R^{n+1}. This means that λ is given by a covector a (in the dual space R^{n+1} of the vector space R^{n+1}), that is, $\lambda(U) = \langle U, a\rangle$ for any vector in R^{n+1}. We may find an affine function ψ on R^{n+1} such that $d\psi = \lambda$. We may also assume that $\psi(f(x_0)) = \varphi(f(x_0))$ at a point x_0 in M^n, where φ is defined by $\varphi(p) = g(f(p), f(p))/2$, $p \in R^{n+1}$.

Now

$$(d\varphi)(X) = X\, g(f(x), f(x))/2 = g(f_*X, f(x)) = \lambda(f_*X) = (d\psi)(f_*X)$$

so $d\varphi = d\psi$. Hence $\varphi\circ f = \psi\circ f$ on M^n. This means that $f(M^n)$ lies in a quadratic hypersurface. □

<u>Remark</u> 1. For any affine coordinate system we may write

$$\varphi(u) = \Sigma_{i,j}\, a_{ij}\, u^i u^j, \qquad \psi(u) = 2\, \Sigma\, a_i u^i + b,$$

so $\varphi = \psi$ is an equation for a quadratic hypersurface.

<u>Remark</u> 2. Theorem 6.1 is generalized in [NP 2].

7. Conormal immersions

Let $f\colon M \to R^{n+1}$ be a nondegenerate hypersurface with affine normal ξ. We denote by R_{n+1} the vector space dual to the vector space R^{n+1} underlying the affine space R^{n+1}. We define a mapping $v\colon M \to R_{n+1} - \{o\}$ as follows.

For each $x \in M$, v_x is an element of R_{n+1} naturally identified with an element in the dual space of $T_x(R^{n+1})$ such that

(7.1) $\quad v_x(f_*Y) = 0$ for $Y \in T_x(M)$ and $v_x(\xi_x) = 1$.

We call v the <u>affine conormal</u>. Denoting by D the usual flat affine connection in R_{n+1}, we have

(7.2) $\quad (D_Y v)(\xi) = 0 \quad$ and $\quad (D_Y v)(f_*X) = -h(Y,X)$

for all $x, Y \in T_x(M)$.

Indeed, from $v(\xi) = 1$ we get

$$0 = Y(v(\xi)) = (D_Y v)(\xi) + v(D_Y \xi) = (D_Y v)(\xi) + v(-f_*(SY))$$
$$= (D_Y v)(\xi) .$$

Also, from $v(f_*(X)) = 0$, where X is any vector field, we get

$$0 = Y(v(f_*(X)) = (D_Y v)(f_*(X)) + v(D_Y f_*(X))$$
$$= (D_Y v)(f_*(X)) + v(f_*(\nabla_Y X) + h(Y,X)\xi)$$
$$= (D_Y v)(f_*(X)) + h(Y,X).$$

<u>Lemma</u>. <u>The conormal mapping</u> v <u>is an immersion of</u> M <u>into</u> $R_{n+1} - \{0\}$.

<u>Proof</u>. Note that $v_*(Y) = D_Y(v)$ for any $Y \in T_x(M)$. Thus if $v_*(Y) = 0$, then by (7.2) we have $h(Y,X) = 0$ for every $x \in T_x(M)$. Since h is nondegenerate, we get $Y = 0$. □

For each $x \in M$, v_x is transversal to the hypersurface $v(M)$, because $v_x(\xi_x) = 1$, but $v_*(X) = D_X v$ (with $X \in T_x(M)$) satisfies $D_X(v)(\xi) = 0$. Thus we now consider $v\colon M \to R_{n+1} - \{0\}$ a central hypersurface by taking v as a transversal vector field:

$T_{f(x)}(R^{n+1}) = v_*(T_x(M)) + \{v_x\}$. We write

(7.3) $D_X(v_*(Y)) = v_*(\nabla^*{}_X Y) + h^*(X,Y)\, v,$

where ∇^* is the induced affine connection on M by v and h^* is the affine fundamental form for v. (Here $h*$ is allowed to be degenerate.)

<u>Proposition</u> 7.1. <u>We have</u>

(7.4) $h^*(X,Y) = h(SX,Y)$ <u>for all</u> $X, Y \in T_x(M)$.

(7.5) $X\, h(Y,Z) = h(\nabla_X Y, Z) + h(Y, \nabla^*{}_X Z)$ <u>for any vector fields</u> Y <u>and</u> Z <u>on</u> M <u>and</u> $X \in Tx(M)$. (We express this property by saying that ∇^* is <u>conjugate</u> to ∇ relative to h.)

(7.6) $\hat{\nabla}_X Y = 2(\nabla_X Y + \nabla^*{}_X Y)$ <u>for all vector fields</u> X <u>and</u> Y <u>on</u> M.

<u>Proof</u>. From $(v_*Y)(\xi) = 0$ we obtain

$$0 = X((v_*Y)(\xi)) = (D_X(v_*Y))(\xi) + (v_*Y)(D_X\xi).$$

Since $(v_*Y)(D_X\xi) = (v_*Y)(-f_*(SX)) = -h(Y,SX)$ by (7.2), we get

$$(D_X(v_*Y))(\xi) = h(Y,SX).$$

On the other hand, from (7.3) and (7.2) we get

$$(D_X(v_*Y))(\xi) = h^*(X,Y).$$

Thus we have (7.4)

To prove (7.5) we start with $v_*(Y)(f_*Z) = -h(Y,Z)$ as in (7.2). We get

$$X(v_*(Y)(f_*Z)) = (D_X v_*(Y))(f_*Z) + v_*(Y)(D_X(f_*Z)).$$

Here

$$(D_X v_*(Y))(f_*Z) = (v_*(\nabla^*{}_X Y))(f_*Z) + h^*(X,Y)v(f_*Z)$$
$$= -h(\nabla^*{}_X Y, Z) \quad \text{by (7.2)}$$

and

$$v_*(Y)(D_X(f_*Z)) = (v_*(Y))[f_*(\nabla_X Z) + h(X,Z)\xi] = -h(Y, \nabla_X Z).$$

From these we obtain $X\, h(Y,Z) = h(\nabla^*{}_X Y, Z) + h(\nabla_X Z, Y)$.

Interchanging Y and Z we get (7.5).

Using (7.5) we get

$$(\nabla_X h)(Y,Z) = Xh(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z)$$
$$= h(\nabla^*{}_X Y,Z) - h(\nabla_X Y,Z).$$

On the other hand, we have

$$(\nabla_X h)(Y,Z) = -2h(K_X Y,Z) = -2h(\nabla_X Y - \hat{\nabla}_X Y,Z).$$

It follows that $\nabla^*{}_X Y - \nabla_X Y = -2(\nabla_X Y - \hat{\nabla}_X Y)$, which implies (7.6). □

Remark. (7.5) and (7.6) appear as (21), p.127, and (28), p.129 in [Sch]. They call ∇ the affine connection of the first kind and ∇^* the affine connection of the second kind. ∇ and ∇^* coincide if and only if $\nabla = \hat{\nabla}$, that is, if and only if $\nabla h = 0$. We already know that this implies that M is a quadratic hypersurface.

We now discuss a geometric application of affine conormal to the question of shadow boundary. For the sake of simplicity we discuss surfaces in R^3.

Let M be a nondegenerate surface imbedded in R^3. A curve x_t on M is said to be a shadow boundary for a parallel lighting in the direction of a vector a if the line through each point x_t in the direction of a is tangent to M at x_t, so that the cylinder through the curve x_t with generators parallel to a is tangent to M along x_t. We now prove

Proposition 7.1. *Let* M *be a nondegenerate surface imbedded in* R^3. *A curve* x_t *on* M *is a shadow boundary if and only if it is a pre-geodesic relative to the connection* ∇^*.

Proof. A curve x_t is a pre-geodesic for ∇^* if and only if $\nabla^*{}_t\vec{x}_t = \varphi_t\vec{x}_t$. (Geometrically, it means that the the tangent line field is parallel along the curve. In this case, we may reparametrize the curve so that we have $\nabla^*{}_t\vec{x}_t=0$.)

Now using the affine conormal v we set $v_t = v(x_t)$ so that $v_*(\vec{x}_t)$

$= dv/dt$. Then we get

$(d/dt)(dv/dt) = v_*(\nabla^*_t \vec{x}_t) - h^*(\vec{x}_t, \vec{x}_t)v_t = \varphi_t\, dv/dt + \psi_t\, v_t$,

where $\psi_t = -h^*(\vec{x}_t, \vec{x}_t)$. Thus we get a second-order linear differential equation

(7.7) $\quad d^2v/dt^2 = \varphi_t\, dv/dt + \psi_t\, v_t$.

It follows that

$$v_t = \sigma_t\, \alpha + \rho_t\, \beta,$$

where α and β are certain constant covectors.

Take a vector $a \in R^{n+1}$ such that $\alpha(a) = \beta(a) = 0$. Since $v_t(a) = 0$ for each t, it follows that a is tangent to M at x_t.

Conversely, suppose x_t is a shadow boundary for a parallel lighting in the direction of a. Then for $v_t = v(x_t)$ we have $v_t(a) = 0$, since a is tangent to M at x_t. Then $(dv/dt)(a) = (d^2v/dt^2)(a) = 0$. Thus the covectors v_t, dv/dt and d^2v/dt^2 are linearly dependent. Thus we have equation (7.7), which implies that $\nabla^*_t \vec{x}_t = \varphi_t \vec{x}_t$, that is, x_t is a pre-geodesic for ∇^*. □

The following gives a characterization of a quadric in terms of shadow boudaries, which we state without a proof.

Proposition 7.3. _Let_ M _be a nondegenerate surface imbedded in_ R^3. _If every shadow boundary is a plane curve_, _then_ M _is a quadric_.

We shall prove here the following version of a result (Satz 3.3, (b)) due to Simon [Si 1].

We recall that two torsion-free affine connections ∇ and ∇' are said to be projectively equivalent if there is a 1-form ρ such that

(7.8) $\quad \nabla'_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X \quad$ for all vector fields X and Y.

It is known (for example, cf. [T]) that two affine connections are projectively equivalent if and only if they have the same family of curves as pregeodesics.

Proposition 7.4. Let M^n be a nondegenerate hypersurface in R^{n+1}. If the affine connections ∇ and ∇^* are projectively equivalent, then M^n is (part of) a quadric.

Proof. From (7.5) and (7.8) we obtain

$$X\, h(Y,Z) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z) + \rho(X)h(Y,Z) + \rho(Z)h(Y,X),$$

that is,

(7.9) $\quad (\nabla_X h)(Y,Z) = \rho(X)h(Y,Z) + \rho(Z)h(X,Z).$

Since the left-hand side is symmetric in X and Y by Codazzi's equation, we obtain $\rho(X)h(Y,Z) = \rho(Y)h(X,Z)$. Since h is nondegenerate, this implies $\rho(X)Y = \rho(Y)X$. This being valid for any X and Y, we conclude that $\rho = 0$. Hence $\nabla = \nabla^*$ and $\nabla h = 0$. It follows that M^n is (part of) a quadratic hypersurface. □

8. Homogeneous affine surfaces

A nondegenerate surface M imbedded in R^3 is said to be homogeneous if there is a Lie subgroup G of the group of all special affine transformations $A(3) = SL(3,R)\cdot R^3$ such that M is the orbit of a certain point by G. Homogeneous affine surfaces are classified, up to affine transformations, e.g. Chapter 12 of [G].

Here we shall describe all such surfaces together with the corresponding groups. An affine surface is elliptic or hyperbolic according as whether the affine metric h is positive definite or not.

Example 8.1. Quadrics.

i) ellipsoid: $x^2 + y^2 + z^2 = 1$, which is the orbit of ${}^t(1,0,0)$ by SO(3).

ii) one-sheeted hyperboloid: $x^2 + y^2 - z^2 = 1$, which is the orbit of $^t(1,0,0)$ by $SO^+(2,1)$.

iii) <u>two-sheeted hyperboloid</u>: $x^2 + y^2 - z^2 = -1$, which is the orbit of $^t(0,0,1)$ by $SO^+(2,1)$.

iv) <u>elliptic paraboloid</u>: $z = \frac{1}{2}(x^2 + y^2)$, which is the orbit of the origin $^t(0,0,0)$ by the group of all matrices of the form

$$\begin{bmatrix} \cos t & -\sin t & 0 & a \\ \sin t & \cos t & 0 & b \\ a \cos t + b \sin t & -a \sin t\, t + b \cos t & 1 & \frac{1}{2}(a^2 + b^2) \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

v) <u>hyperbolic paraboloid</u>: $z = \frac{1}{2}(x^2 - y^2)$, which is the orbit of the origin by the group of all matrices similar to the one above, where cos t and sin t are replaced by cosh t and sinh t and $\frac{1}{2}(a^2 + b^2)$ by $\frac{1}{2}(a^2 - b^2)$.

<u>Example</u> 8.2 <u>Elliptic surfaces</u>

i) $z = 1/xy,\ x, y > 0$, which if the orbit of the point $^t(1,1,1)$ by the group of all matrices of the form

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1/ab \end{bmatrix} \qquad a, b > 0$$

This is an affine sphere whose center is the origin.

ii) $(z^2 - 2x)^3 y^2 = -1,\ y > 0$, equivalently,

the graph $x = \frac{1}{2}(y^{-2/3} + z^2),\ y > 0$.

This is the orbit of $^t(\frac{1}{2}, 1, 0)$ by the group of all matrices of the form

$$\begin{pmatrix} a^{-2/3} & 0 & ba^{-1/3} & b^2/2 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{1/3} & b \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Example 8.3 Hyperbolic surfaces

i) $z = 1/(x^2 + y^2)$, $x > 0, y > 0$. This is the orbit of ${}^t(1,0,1)$ by the group of all matrices of the form

$$\begin{pmatrix} c\cos t & -c\sin t & 0 \\ c\sin t & c\cos t & 0 \\ 0 & 0 & c^{-2} \end{pmatrix} \qquad c > 0.$$

ii) $x = (z^2 - y^{-2/3})/8$, $y > 0$. This is the orbit of ${}^t(0,1,1)$ by the group of all matrices of th form

$$\begin{pmatrix} a^2 & 0 & ab/4 & b^2/8 \\ 0 & a^{-3} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Example 8.4. Ruled surface $z = xy - y^3/3$, called a Cayley surface, which is the orbit of the origin by the group of all matrices of the form

$$\begin{pmatrix} 1 & b & 0 & a \\ 0 & 1 & 0 & b \\ b & a & 1 & ab-b^3/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is an improper affine sphere (so the induced connection ∇ is flat with x, y as flat affine coordinates on the surface). By computation, all $h_{ijk} = 0$ except $h_{222} = -2$. Thus $\nabla h \neq 0$ but $\nabla^2 h$

$= 0$. The line $(t,0,0)$ lies on the surface as well as the lines $(t+a, b, bt + ab - b^3/3)$.

For any point $p = (x_0, y_0, z_0)$ which is not on the surface, the orbit by the same group is the set of all points (u,v,w), where

$u = x_0 + b\,y_0 + a, \quad v = y_0 + b, \quad w = b\,x_0 + a\,y_0 + z_0 + a\,b - b3/3.$

We get

$$w = u\,v - v^3/3 + [z_0 - (x_0\,y_0 - y_0{}^3/3)].$$

Thus the orbit of p is obtained from the original surface by a translation $z \to z + c$, where $c = [z_0 - (x_0\,y_0 - y_0{}^3/3)]$.

A recent result shows that a nondegenerate affine surface in R^3 which satisfies $\nabla h \neq 0$ and $\nabla^2 h = 0$ is essentially congruent to (part of) the Cayley surface by a special affine transformation of R^3. See [NP 3].

9. Laplacian of affine distance and harmonicity of the conormal mapping

Let M be a nondegenerate hypersurface imbedded in R^{n+1} and identify every point of R^{n+1} with its position vector (from a certain fixed point o).

Pick a point p in R^{n+1}. For any point $x \in M$, write

(9.1) $\qquad x - p = Z_x + \rho(x)\,\xi_x, \qquad Z_x \in T_x(M),\ \rho(x) \in R,$

where ξ is the affine normal. If we use the affine conormal v introduced in section 7, we have from (9.1)

(9.2) $\qquad \rho(x) = v(x - p).$

This number is defined as the affine distance from p to x. Fixing p in R^{n+1}, consider the affine distance $v(x - p)$ as a function on M.

Proposition 9.1. *For a given point p in* R^{n+1}, *the function* $v(x - p)$ *on* M *has an extremum at* $u \in M$ *if and only if the vector* $\overrightarrow{up}$ *is in the direction of the affine normal* ξ_u.

Proof. From (9.1) we get for any $X \in T_x(M)$

$X = D_X(x - p) = D_X Z + (X\rho)\xi + \rho D_X \xi = \nabla_X Z + h(X,Z)\xi + (X\rho)\xi - \rho SX$

and

(9.3) $\quad - X - \rho SX + \nabla_X Z = 0$ and $h(X,Z) + X\rho = 0$.

Assume that $\rho = v(x - p)$ has an extremum at $x = u$. Then $X\rho = 0$ for every $X \in T_x(M)$. From (9.3), we get $h(X,Z) = 0$. This implies that $Z = 0$ so that $u - y = \rho\,\xi$, that is, $\overrightarrow{up}$ is in the direction of ξ.

Conversely, assume that up is in the direction of ξ_u. Evaluating (9.3) at u, we conclude $Z = 0$ and $X\rho = 0$ for every $X \in T_x(M)$. Thus $v(x-p)$ has an extremum at $x = u$. □

Now we consider the affine distance from the origin o to $x \in M$, which is expressed by $\rho_x = v(x)$. We also define a vector field $x \to Z_x$ by

(9.4) $\quad x = Z_x + \rho_x \xi_x$.

Recall that the Laplacian $\Delta\,\varphi$ for any smooth function φ on M relative to a nondegenerate metric (in our case, the affine metric h) is defined by

$$\Delta\,\varphi = \mathrm{div}\,(\mathrm{grad}\,\varphi),$$

where grad φ is the vector field such that $d\,\varphi(Y) = h(\mathrm{div}\,\varphi, Y)$ for every vector field Y, and $(\mathrm{div}\,W)_x = \mathrm{trace}\,\{X \in T_x(M) \to \hat{\nabla}_X W \in T_x(M)\}$. Note, however, that in this last eauation, $\hat{\nabla}_X$ can be replaced by ∇_X by virtue of $\nabla_X W = \hat{\nabla}_X W + K_X W = \hat{\nabla}_X W + K_W X$ and the apolarity: trace $K_W = 0$.

Proposition 9.2. For the function ρ defined by (9.4) we have

(9.5) $\quad \Delta\,\rho = -n\,(1 + H\rho)$, where H is the affine mean curvature.

Proof. From the second equation in (9.3) we see that $Z = -\,\mathrm{grad}\,\rho$. From the first equation, we get $\mathrm{div}\,Z = n + \rho\,\mathrm{trace}\,S = n + n\,\rho\,H$. Thus we get (9.5). □

Remark. If we define an $(n-1)$-form $\beta = \iota(Z)\theta/n$ on M, that is,

$$\beta(X_1, \dots, X_{n-1}) = \theta(Z, X_1, \dots, X_{n-1})/n$$
$$= \omega(x, X_1, \dots, X_{n-1}, \xi)/n,$$

then

(9.6) $d\beta = (1 + H\rho)\theta.$

Theorem 9.3. If a nondegenerate hypersurface M is compact, then the affine mean curvature cannot be identically 0.

Proof. If H is identically 0, then $\Delta\rho = -n$ or, equivalently, $d\beta = \theta$. From the theorem of Stokes, we get $\int \operatorname{div} Z\, \theta = 0$, or equivalently, $\int\theta = 0$, leading to a contradiction. □

This result is found in [Ch]. For Stokes's theorem, see [KN],pp.281-3. There is another geometric quantity which is closely related to the affine mean curvature. Let $v: M \to R_{n+1}$ be the conormal mapping for a nondegenerate hypersurface $f: M \to R^{n+1}$. We want to calculate the tension field for the mapping v (relative to the affine metric h).

Let us recall the definition of the tension field for a mapping, say, φ from a manifold (M,g), where g is an arbitrary riemannian or pseudo-riemannian metric (with Levi-Civita connection ∇) into a manifold $\tilde{M}$ with a torsion-free affine connection $\tilde{\nabla}$. For vector fields X and Y on M, we consider

$$H_\varphi(X,Y) = \tilde{\nabla}_X \varphi_*(Y) - \varphi_*(\nabla_X Y),$$

which can be easily verified to be tensorial, that is, the value (in $T_{\varphi(x)}(\tilde{M})$) depends only on X_x and Y_x. This H_φ is the Hessian for φ relative to $(\nabla, \tilde{\nabla})$. Now the tension field $\tau(\varphi)$ is defined as the trace of H_φ relative to the metric g, namely, if $\{X_1, \dots, X_k, X_{k+1}, \dots, X_n\}$ is an orthonormal basis relative to g in $T_x(M)$ with $g(X_i, X_i) = \varepsilon_i = \pm 1$, then

$$\tau(\varphi)x = \Sigma_i\, \varepsilon_i H_\varphi(X_i, X_i) \in T_{\varphi(x)}.$$

This is independent of the choice of an orthonormal basis. If we take an arbitrary basis $\{X_1, \ldots, X_n\}$ and the components of g are (g_{ij}), then

$$\tau(\varphi)_x = \Sigma\, g^{ij} H_\varphi(X_i, X_j),$$

where $[g^{ij}]$ is the inverse of the matrix $[g_{ij}]$.

The mapping φ is said to be harmonic if $\tau(\varphi)$ vanishes everywhere.

Now we go back to the conormal mapping and prove the following result.

Proposition 9.4. The affine mean curvature of a nondegenerate hypersurface M in R^{n+1} is 0 if and only if the conormal mapping v: $M \to R_{n+1}$ is harmonic.

Proof. From Sections 4 and 7 we know $\nabla_X - \hat{\nabla}_X = K_X = \hat{\nabla}_X - \nabla^*_X$, where ∇ is the induced connection on M, ∇^* the conjugate connection (induced by the conormal mapping), and $\hat{\nabla}$ the Levi-Civita connection for the affine metric h. For the conormal mapping v the Hessian (relative to the flat connection D in R_{n+1} and the connection $\hat{\nabla}$) is given by

$$\begin{aligned} H_v(X,Y) &= D_X\,(v_*(Y)) - v_*(\hat{\nabla}_X Y) = D_X\,(v_*(Y)) - v_*(\nabla^*_X Y + K_X Y) \\ &= h^*(X,Y)\,v - v_*(K(X,Y)) = h(SX,Y)\,v - v_*(K(X,Y)) \end{aligned}$$

by virtue of (7.3) and (7.4). To take the trace of H_v, let $\{X_1, \ldots, X_n\}$ be any basis and let $[h_{ij}]$ and $[h^{ij}]$ be the component matrix for h and its inverse. Then

$$\begin{aligned} \Sigma_{i,j}\, h^{ij}\, h(SX_i, X_j)\, v &= \Sigma_{i,j,k}\, h^{ij}\, S^k{}_i\, h_{kj}\, v = \Sigma_{i,k}\, \delta^i{}_k\, S^k{}_i\, v \\ &= \Sigma_i\, S^i{}_i\, v = (\text{trace } S)\, v. \end{aligned}$$

We have also

$$\begin{aligned} \Sigma_{i,j}\, h^{ij}\,(K(X_i,X_j)) &= \Sigma_k(\Sigma_{i,j}\, h^{ij} K^k{}_{ij})\, X_k \\ &= \Sigma_k(\Sigma_{i,j}\, h^{ij}\, \Sigma_m\, (-\tfrac{1}{2}) h^{km}\, h_{mij}) X_k \\ &= \Sigma_k\, (\Sigma_m\, (-\tfrac{1}{2}) h^{km}\, (\Sigma_{i,j}\, h^{ij}\, h_{ijm})) X_k = 0 \end{aligned}$$

by using Propositions 4.1 and 4.2 (see the remark following

Proposition 4.2, namely, $\Sigma_{i,j} h^{ij} h_{ijm} = 0$). Hence

$$\Sigma_{i,j} h^{ij} v_*(K(X_i,X_j)) = v_*(\Sigma_{i,j} h^{ij} (K(X_i,X_j))) = 0.$$

The tension field of v is

$$\tau(v) = \text{trace } H_v = (\text{trace } S)\, v,$$

from which the assertion of Proposition 9.4 is obvious. □

We may also obtain Theorem 9.3 from Proposition 9.4 using the fact that if M is compact, then the harmonic mapping $v: M \to R_{n+1}$ must be a constant, contrary to the fact that v is an immersion.

Remark. Proposition 9.4 is in [Ca 3] for locally strictly convex affine surfaces.

10. An example: SL(n,R)/SO(n).

We denote by s(n) the vector space of all real symmetric matrices of degree n. We define a mapping $f: GL(n,R) \to s(n)$ by

(10.1) $f(A) = \frac{1}{2}A\,{}^tA$

and an action σ of GL(n,R) on s(n) by

(10.2) $\sigma(A)X = A X\,{}^tA$ for $A \in GL(n,R)$ and $X \in s(n)$.

The mapping f is equivariant in the sense that

(10.3) $f(AB) = \sigma(A)f(B)$ for $A,B \in GL(n,R)$.

It is easy to verify that the image f(GL(n,R)) coincides with the set p(n) of all positive-definite matrices in s(n). p(n) is a connected open subset of s(n). It is also known that the exponential mapping $X \to \exp X$ gives a diffeomorphism of s(n) onto p(n).

We now restrict f and σ to the subgroup SL(n,R) of GL(n,R). The image f(SL(n,R)) coincides with $p_1(n) = \{X \in p(n);\ \det X = 1\}$, which is also equal to the orbit of the identity matrix I by the action σ(SL(n,R)). The isotropy group at I equals SO(n) so that $p_1(n) =$ SL(n,R)/SO(n). It is known that this is a symmetric homogeneous space where the involutive automorphism of $SL(n,R)$ is $A \to {}^tA^{-1}$.

Now consider $\mathbf{s}(n)$ as a vector space or a centro-affine space. It has a volume element invariant by $SL(n,R)$ acting on it by σ in (10.2). For the imbedded hypersurface $f: SL(n,R)/SO(n) \to p_1 \subset s(n)$, we take the position vectors as transversal vectors: for each $p \in SL(n,R)/SO(n)$, choose $A \in SL(n,R)$ such that $p = \pi(A)$, where π is the projection of $SL(n,R) \to SL(n,R)/SO(n)$, and let $\xi_p = 2A\, {}^tA$, regarded as position vector, which depends only on p.

For $SL(n,R)/SO(n)$, its tangent space at $p_0 = \pi(I)$ may be represented by the subspace $\mathbf{m}_0 = \{X \in \mathbf{sl}(n,R),\ {}^tX = X\}$, such that $sl(n,R) = \mathbf{m}_0 + \mathbf{o}(n)$, where $\mathbf{o}(n)$ is the Lie algebra of $SO(n)$. For each $X \in \mathbf{m}_0$, we have $f_*(X) = X$, as tangent vector at I to $\mathbf{s}(n)$. Now let $a_s = \exp(sX)$ and define the vector field $\tilde{X}$ along $\pi(a_s)$ by $\tilde{X}_s = a_s(X)$. Then $f_*(\tilde{X}) = a_s X\, {}^ta_s$ and

$$[D_s f_*(\tilde{X})]_{s=0} = 2X^2.$$

From this we get

(10.4) $h(X,X) = \mathrm{trace}\, X^2/n$ so $h(X,Y) = \mathrm{trace}\, XY/n$

and

(10.5) $\nabla_X X = 2X^2 - 2\,\mathrm{trace}\, X^2/n$ so $\nabla_X Y = 2(XY - \mathrm{trace}\,(XY)/n)$.

Since ξ, h and ∇ are invariant by the action of $SL(n,R)$, it follows that ξ is the affine normal, h in (10.4) is the expression on $\mathbf{m}_0$ of the affine metric invariant by $SL(n,R)$, and ∇ in (10.5) is the expression on $\mathbf{m}_0$ of the induced affine connection which is also invariant by $SL(n,R)$. It also follows that our hypersurface is indeed an affine hypersphere. See [Sa].

Since the linear isotropy group $Ad(SO(n))$ on $\mathbf{m}_0$ is irreducible, it follows that the natural invariant Riemannian metric for the symmetric space $SL(n,R)/SO(n)$ coincides with the invariant affine metric up to a scalar factor. In fact, the former is the restriction to $\mathbf{m}_0$ of the Killing-Cartan form of the Lie algebra $\mathbf{sl}(n,R)$, namely, 2n trace XY. This means that by adjusting the choice of ξ by a constant factor, we can make the affine metric to coincide with the natural invariant Riemannian metric on SL(n,R)/SO(n). The Levi-Civita connection of this metric is expressed on m_0 by

$\hat{\nabla}_X Y = [X,Y]/2$. In particular, $\hat{\nabla}_X X = 0$.

For n = 2, we see that $\nabla = \hat{\nabla}$. In this case, SL(2,R)/SO(2) (hyperbolic plane) is imbedded in $\mathbf{s}(2) = R^3$ as one component of the two-sheeted hyperboloid (thus a quadric).

Remark 1. We may write p(n) as the homogeneous space $GL^+(n,R)/SO(n)$, which can be provided with an invariant Riemannian metric. The natural imbedding of SL(n,R)/SO(n) into $GL^+(n,R)/SO(n)$ is equivariant and isometric. If we consider the former as an affine hypersurface in the latter, what properties does it have?

Remark 2. The space p(n) may be decomposed into the union of hypersurfaces $p_\lambda(n)$ consisting of positive-definite symmeric matrices of determinant $\lambda > 0$.

11. Affine locally symmetric hypersurfaces.

In the example of SL(n,R)/SO(n) as an affine hypersurface the affine metric, which coincides with the standard invariant Riemannian metric is locally symmetric, that is, $\hat{\nabla}\hat{R} = 0$. The following theorem concerning an affine hypersurface such that $\nabla R = 0$ has been proved in [VV].

Theorem 11.1. *Let* M *be a nondegenerate hypersurface in* R^{n+1}, $n \geq 3$. *Then* M *is affine locally symmetric, that is*, $\nabla R = 0$, *if and only if* M *is an improper affine hypersphere or a non-degenerate quadratic hypersurface*.

The essence of the proof is to show the following. A nondegenerate hypersurface M in R^{n+1} satisfies $R(X,Y)\cdot R = 0$

(namely, the derivation $R(X,Y)$ maps R into 0 for any tangent vectors X,Y) if and only if M is an affine hypersphere. Once this is established, then $\nabla R = 0$, which implies $R(X,Y)\cdot R = 0$, will give $S = \rho I$, where ρ is a constant. Then the equation of Gauss and $\nabla R = 0$ implies $\nabla h = 0$.

Remark. Every surface M in R^3 satisfies $R(X,Y).R = 0$.

References

As general and historical references in classical affine differential geometry, see [Bla], [Sch] and [Sim 2]. For a more general approach to geometry of affine immersions which are not treated in Part I, see [NP 1], [NP 2]. Other than these, the list is confined to those references that are quoted in the text for a specific purpose.

[Ber] L. Berwald, Die Grundlagen der Hyperflächen im euklidischen Raum gegenüber inhaltstreuen Affinitäten, Monatsh. für Math. 32(1922), 89-106.

[Bla]* W. Blaschke, Vorlesungen über Diffrentialgeometrie, II Affine Differentialgeometrie, Berlin, Springer, 1923.

[Ca 1] E. Calabi, Improper affine hyperspheres of convex type and a generalization of a theorem of Jörgens, Michigan Math. J. 5(1958), 105-126.

[Ca 2] E. Calabi, Hypersurfaces with maximal affinely invariant area, Amer. Math. J. 104(1982), 91-126.

[Ca 3] E. Calabi, Convex affine maximal surfaces, Results in Math. 13(1988), 199-223.

[Ch] S.S. Chern, Affine minimal hypersurfaces, Proc. US-Japan Seminar, Tokyo 1977 (pp. 17-30).

[G] H. Guggenheimer, Differential Geometry, McGraw-Hill, 1963.

[J] K. Jörgens, Über die Lösungen der Differentialgleichung $rt - s^2 = 1$, Math. Ann. 127(1954), 130-134.

[KN] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, vol. I, John Wiley, 1963.

[N] K. Nomizu, On completeness in affine differential geometry, Geom.Dedicata 20(1986), 43-49.

[NP 1] K. Nomizu and U. Pinkall, On the geometry of affine immersions, Math.Z. 195(1987), 165-178.

[NP 2] K. Nomizu and U. Pinkall, Cubic form theorem for affine immersions, Results in Math. 13(1988), 338-362.

[NP 3] K. Nomizu and U. Pinkall, Cayley surfaces in affine differential geometry, MPI preprint 88-20.

[Sa] T. Sasaki, Hyperbolic affine hyperspheres, Nagoya Math. J.77(1980), 107-123.

[Sch]* P.A. Schirokow and A.P. Schirokow, Affine Differentialgeometrie, Teubner, Leibzig, 1962.

[Schn] R. Schneider, Zur affinen differentialgeometrie im Grossen, I Math.Z. 101(1967), 375-406.

[Sim 1] U. Simon, Zur Relativgeometrie: symmetrische Zusammenhänge auf Hyperflächen, Math. Z.106(1968), 35-46.

[Sim 2]* U. Simon, Zur Entwickelung der affinen Differentialgeometrie nach Blaschke, Gesammelte Werke Blaschke, Band 4, Thales,Essen, 1985

[Sp] M. Spivak: A Comprehensive Introduction to Differential Geometry, vol.IV, Publish or Perish Inc. 1975.

[T] N. Tanaka, Projective connections and projective transformations, Nagoya Math. J. 11(1957), 1-24.

[VV] P. Verheyen and L. Verstraelen, Locally symmetric affine hypersurfaces, Proc. Amer. Math. Soc. 93(1985), 101-103.