# Invariants of Legendrian and Transverse Knots in the Standard Contact Space


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## 1. Introduction

**1.1** The standard contact structure in 3-space, which arises from the identification of $\mathbb{R}^3$ with the manifold of 1-jets of smooth real functions of one variable, naturally distinguishes two major classes of smooth immersed spatial curves: *Legendrian* curves, which are integral curves of the contact distribution, that is everywhere tangent to this distribution, and *transverse* curves, which are nowhere tangent to it. Closed embedded Legendrian and transverse curves are called Legendrian and transverse knots. Theories of Legendrian and transverse knots, which are clearly related to each other, are parallel to the classical knot theory in space.

Legendrian and transverse knots have become very popular in contact geometry since the seminal work of Bennequin [Be], published in 1983. For Legendrian knots one introduces two integer-valued Legendrian isotopy invariants. The first measures the rotation of an (oriented) knot with respect to the contact distribution; we call it the Maslov number. The second one, which we call the Bennequin number, is defined as the contact self-linking number of the knot. (Exact definitions of these and subsequent notions see in Section 2.) Transverse knots have no Maslov numbers, but also have Bennequin numbers. The main achievement of Bennequin's paper consists in two inequalities for these numbers (see Theorem 2.3 below), which imply, in particular, that the Bennequin number of a topologically unknotted Legendrian knot must be always negative. In turn, this gives rise to a construction of an exotic contact structure in $\mathbb{R}^3$ (or rather to a proof, that some previously known contact structures in $\mathbb{R}^3$ are not diffeomorphic to the standard one).

Bennequin and Maslov numbers may be also used for distinguishing Legendrian or transverse isotopy classes of knots within a topological isotopy class. It is very easy to show, that any topological knot is isotopic to (actually is $C^0$ approximated by) both Legendrian and transverse knots. It is equally easy to construct topologically isotopic Legendrian or transverse knots with different Bennequin and Maslov (in the Legendrian case) numbers. Since no other specifically Legendrian or transverse invariants of knots have been found so far, one may expect, that topologically isotopic Legendrian knots with equal Bennequin and Maslov numbers are Legendrian isotopic, and similarly for transverse knots. The results of this article may be regarded as a confirmation of this conjecture.

**1.2.** First of all we prove, that Bennequin and Maslov numbers are the only specifically contact invariants of Legendrian and transverse knots at the level of Grothendieck groups. Namely, we consider commutative semigroups (with respect to the connected summations) of Legendrian and transverse knots and prove, that their Grothendieck groups are isomorphic to the Grothendieck group of topological knots plus $\mathbb{Z} \oplus \mathbb{Z}$ (Bennequin and Maslov) in the Legendrian case, and plus $\mathbb{Z}$ (Bennequin) in the transverse case.

**1.3.** Another result of the article concerns *finite order invariants* of Legendrian and transverse knots, similar to finite order invariants of topological knots in the sense of Vassiliev. Very informally speaking, finite order invariants may be effectively calculated by considering patterns of certain kinds in the knot diagram. For example, all the coefficients of the Conway polynomial have finite order, while, say, the genus of a knot (which may be effectively estimated, but not effectively calculated from the knot diagram) has infinite order.

THEOREM 1.1. *Topologically isotopic Legendrian knots with equal Bennequin and Maslov numbers cannot be distinguished by finite order Legendrian knot invariants.*

In other words, such knots, if they are not Legendrian isotopic, can be distinguished only by infinite order invariants, which demonstrates the difficulty of the problem.

A similar result holds for transverse knots. We also describe the spaces of finite order invariants of Legendrian and transverse knots (Theorems 4.6 and 5.5).

**1.4.** Of other results of this paper we mention here one, which cannot be considered genuinely new. This is Theorem 2.4, which gives an upper estimate of the Bennequin number of a transverse knot in terms of its HOMFLY polynomial. This result is obtain by a combination of simple lemmas proved by Bennequin and some results of the topological knots theory, known since as early as 1985. Still this estimate does not seem to be known in contact geometry; at least we were able to deduce from it the answer to a problem, known since Bennequin's paper (to find the maximal value of the Bennequin number of a transverse or Legendrian realization of the mirror trefoil knot).

**1.5. Acknowledgements.** We greatly benefited from discussing the subject of this work with S. Chmutov, Ya. Eliashberg, V. Ginzburg and M. Polyak; we are very grateful to them all. The second author acknowledges the hospitality of MSRI and MPIM; he was supported in part by an NSF grant DMS-9402732.

## 2. Definitions and known results.

**2.1.** A *contact structure* on a 3-dimensional manifold $M$ is a continuous field $\{C_x \subset T_xM | x \in M\}$ of tangent 2-dimensional planes, locally defined by a differential 1-form $\alpha$ with non-vanishing $\alpha \wedge d\alpha$. A manifold with a contact structure possesses a canonical orientation (determined by the volume form $\alpha \wedge d\alpha$). A contact structure is *orientable* if and only if it is globally given by such 1-form. For an oriented contact structure the determining 1-form is unique up to a positive functional factor.

A contact structure is called *parallelizable* (*parallelized*) if the 2-dimensional vector bundle $\{C_x\}$ over $M$ is trivial (trivialized). A parallelizable contact structure is orientable; if $M$ is connected, then any orientation of a parallelized contact structure is either compatible or incompatible with the parallelization. If $H^1(M; \mathbb{Z}) = 0$, then any contact structure on $M$ is orientable, and any oriented parallelizable contact structure on $M$ possesses a unique parallelization, compatible with the orientation. If $H^2(M; \mathbb{Z}) = 0$, then any orientable contact structure on $M$ is parallelizable.

2.2. Let $M$ be a 3-dimensional manifold with a contact structure $C$. A regular (immersed) curve in $M$ may be *transverse* or *tangent* to $C$; in the latter case it is also called *Legendrian*. If $C$ is oriented, then all transverse curves possess canonical orientations. Let $\gamma$ be an oriented Legendrian curve and $C$ be oriented; push $\gamma$ slightly in the directions of the unit positive and negative normal vectors (with respect to a smooth Riemannian structure on $M$) to $\gamma$ in $C_x$; we will get two new curves, $\gamma_+$ and $\gamma_-$. The curves $\gamma_+$ and $\gamma_-$ are transverse to $C$ and therefore have natural orientations. The natural orientation of $\gamma_+$ corresponds to the orientation of $\gamma$, the natural orientation of $\gamma_-$ is opposite to the orientation of $\gamma$. If $\gamma$ is transverse to $C$ and $C$ is parallelized, then we slightly push $\gamma$ in the direction of the first of the two coordinate vectors in $C_x$; the resulting curve is denoted by $\gamma'$.

A *framing* of a curve $\gamma$ is a homotopy class of nonvanishing transverse vector fields along $\gamma$. The preceding paragraph shows that if $C$ is orientable then every Legendrian curve gets a natural framing, and if $C$ is parallelized then so does every transverse curve.

2.3. Let $M$ and $C$ be as above, and let $\gamma$ be a closed regular curve in $M$. If $\gamma$ is Legendrian, and $C$ is parallelized, then we can parametrize $\gamma$ and count the number of revolutions of $\dot\gamma$ with respect to the chosen frames in $C_x$ when $x$ traverses $\gamma$ in the positive direction. This number does not depend on the parametrization, and if $H^1(M;\mathbf{Z}) = 0$, then it depends only on the orientations of $\gamma$ and $C$: it changes sign if either of these orientations is reversed. We denote this number by $\mu(\gamma)$ and call it the *Maslov number* or the *rotation number* of $\gamma$. (It is important, that this number is defined for immersed curves $\gamma$, not only for embedded ones.)

If an embedded Legendrian curve $\gamma$ is homologous to 0, and $C$ is orientable, then the linking number $\ell(\gamma, \gamma_+)$ is defined. This number is denoted by $\beta(\gamma)$ and is called the *Bennequin number* of $\gamma$. It does not depend on the orientation of either $C$ or $\gamma$. Similarly if $\gamma$ is transverse and homologous to 0 and $C$ is parallelized, then the linking number $\ell(\gamma, \gamma')$ is denoted by $\beta(\gamma)$ and is called the Bennequin number of $\gamma$ (these numbers are particular cases of the self-linking number of a framed knot).

PROPOSITION 2.1 (see [Be]). *If $\gamma$ is Legendrian and homologous to 0 and $C$ is parallelized, then*

$$\beta(\gamma_+) = \beta(\gamma) + \mu(\gamma),$$
$$\beta(\gamma_-) = \beta(\gamma) - \mu(\gamma).$$

2.4. The contact structure in $\mathbf{R}^3$, defined by the form $\alpha = y\,dx - dz$ is called *standard*. This contact structure is the canonical one in the space of 1-jets of functions on the line: $x$ is a point in the line, $z$ is the value of a function at $x$ and $y$ is the value of the derivative. The standard contact structure is extended to a contact structure in $S^3$, which is also called standard.

From now on, when we speak of transverse or Legendrian curves, we mean, unless the opposite is specified, transverse or Legendrian curves in $\mathbf{R}^3$ with this contact structure. Closed embedded transverse (Legendrian) curves are called transverse (Legendrian) knots. A transverse (Legendrian) isotopy between transverse (Legendrian) knots is an isotopy via transverse (Legendrian) knots. Usual isotopies will be called topological isotopies.

Obviously transversely (Legendrian) isotopic transverse (Legendrian) knots are topologically isotopic and have equal Bennequin numbers, and oriented Legendrian isotopic oriented Legendrian knots have equal Maslov numbers. One of the most exciting problems in the theory of transverse (Legendrian) knots is the following

QUESTION. Are topologically isotopic transverse (oriented Legendrian) knots with equal Bennequin numbers (Bennequin and Maslov numbers) transversely (oriented Legendrian) isotopic?

The answer is affirmative for topologically trivial knots:

THEOREM 2.2 (Eliashberg, [El]). *Transverse (oriented Legendrian) knots with equal Bennequin numbers (Bennequin and Maslov numbers), which are topologically isotopic to trivial knots, are transversely (oriented Legendrian) isotopic.*

In the general case the answer is not known. The results of this article provide an evidence, that the answer might be affirmative in the general case.

EXAMPLE. Let $\gamma$ be a Legendrian knot with zero Maslov number, and let $\gamma'$ be its image under the diffeomorphism $f(z,y,z) = (x,-y,-z)$. Since $f$ is a rotation of space, $\gamma'$ is topologically isotopic to $\gamma$; also $\mu(\gamma') = -\mu(\gamma) = 0$ and $\beta(\gamma') = \beta(\gamma)$. Moreover, $f$ is a contactomorphism, but it is not isotopic to the identity through contactomorhisms because it changes the sign of the contact form. Problem: are $\gamma$ and $\gamma'$ Legendrian isotopic?

REMARK. There exists an example in a contact manifold different from $\mathbf{R}^3$ (namely, in the connected sum of two copies of $S^1 \times S^2$ with a natural contact structure) of two topologically isotopic knots which are homologous to 0, have equal Bennequin and Maslov numbers, but are not Legendrian isotopic (see [Fr]).

**2.5.** Along with transverse and Legendrian knots one considers *long* transverse and Legendrian knots. A long Legendrian knot is a Legendrian embedding $\varphi: \mathbf{R} \to \mathbf{R}^3$, such that $\varphi(t) = (t,0,0)$ for $|t| > C$ with some $C$. A long transverse knot is a transverse embedding $\varphi: \mathbf{R} \to \mathbf{R}^3$, such that $\varphi(t) = (0,0,t)$ for $|t| > C$. Long transverse and Legendrian knots may be transversely and Legendrian isotopic, the isotopy being assumed identical outside a bounded domain. Long transverse and Legendrian knots also have Bennequin and Maslov numbers. To define them we remark, that long transverse and Legendrian knots may be regarded as usual transverse and Legendrian knots in $S^3$ (see the remark in the beginning of 2.4), which have Bennequin and Maslov numbers. By definition, the Maslov number of a long Legendrian knot is equal to that of the corresponding spherical knot, while the Bennequin number of a long transverse or Legendrian knot is equal to that of the corresponding spherical knot plus one. The above Question is valid for long transverse and Legendrian knots, as well as Eliashberg's result and the rest of the discussion.

**2.6.** Another important question concerns possible values of Bennequin and Maslov numbers for transverse and Legendrian knots within a given topological isotopy class. An easy fact is that for any transverse knot $\gamma$ the number $\beta(\gamma)$ is odd and for any Legendrian knot $\gamma$ the number $\beta(\gamma) + \mu(\gamma)$ is odd. On the contrary, the following estimate remains probably the most difficult theorem of the theory. In particular it shows that not every framed knot can be realized as a Legendrian or a transverse knot with its natural framing.

THEOREM 2.3 (Bennequin [Be]). *For any transverse knot* $\gamma$

$$\beta(\gamma) \leq -\chi(S),$$

*where $S$ is an arbitrary connected Seifert surface for $\gamma$, that is a compact embedded smooth surface in $\mathbb{R}^3$ with the boundary $\gamma$. Consequently, for a Legendrian knot $\gamma$*

$$\beta(\gamma) + |\mu(\gamma)| \leq -\chi(S).$$

Here is another estimate.

THEOREM 2.4 (Bennequin [Be] + Franks, Williams [F-W], and Morton [Mo]). *For any transverse knot $\gamma$*

$$\beta(\gamma) < e(\gamma),$$

*where $e(\gamma)$ is the lower degree in $v$ of the LYMPH-TOFU* polynomial of $\gamma$. Consequently for a Legendrian knot $\gamma$*

$$\beta(\gamma) + |\mu(\gamma)| < e(\gamma).$$

The LYMPH-TOFU (aka HOMFLY) polynomial

$$P_\gamma(v,z) = \sum_{k=e(\gamma)}^{E(\gamma)} a_k(z) v^k$$

is defined by the recurrent formula

$$1/v\, P_{(\times)} - v\, P_{(\times)} = z\, P_{(\asymp)}.$$

Notice that in this article, and, in particular, in the above formula, knot diagrams are mirror images of knot diagrams generally accepted in the classical knot theory. The reason for this is explained in 3.1 below (see Remark there).

Although both ingredients of Theorem 2.4 has been known for at least 10 years, we were unable to trace this theorem in the literature on contact geometry. Nevertheless its proof is much more simple and elementary than that of Theorem 2.3. Namely, Bennequin remarks that any transverse knot is transversely isotopic to a closed braid centered around a fixed transverse circle (a fiber of the Hopf fibration) and calculates the number $\beta$ for a closed braid $\gamma$:

$$\beta(\gamma) = c - n,$$

where $n$ is the number of strings and $c$ is the image of $\gamma$ with respect to the canonical homomorphism $B(n) \to \mathbb{Z} = B(n)/[B(n), B(n)]$; this homomorphism takes each of the canonical generators of the braid group $B(n)$ into 1 (both results do not depend on the complicated technique, used by Bennequin in the proof of Theorem 2.3.) Franks, Williams and Morton prove that for any closed braid $\gamma$

$$c - n > e(\gamma).$$

* For Lickorish, Yetter, Millet, Przytycki, Hoste, Traczyk, Ocneanu, Freyd and Unknown further discoverers – see [B-N].

Surprisingly in many cases the simpler Theorem 2.4 provides a stronger estimate, than Theorem 2.3. For example for transverse trefoil knots $\gamma$ both theorems yield the same inequality $\beta(\gamma) \leq +1$, but for transverse mirror trefoil knot $\gamma$ Theorem 2.3 (which is insensitive to the mirroring) gives again $\beta(\gamma) \leq +1$ while Theorem 2.4 gives the inequality $\beta(\gamma) \leq -5$, which is actually the best possible, as a simple construction shows (see 3.4 below). (Compare the discussion on Bennequin invariants for trefoil knots in [Be] and [El].)

In general it is not known whether Theorem 2.4 gives the best possible estimate for the Bennequin number of transverse knots. For the Bennequin number of Legendrian knots it does not, for in a recent preprint [Ka] it is shown that for Legendrian mirror trefoil knots the Bennequin number is never greater than $-6$. Kanda's work contains also a generalization of this example (see 3.4 below). We would like to mention in this respect the papers by L. Rudolph [Ru 1-4]. In particular, it follows from these works that the Bennequin number of a Legendrian mirror trefoil knot is always $\leq -5$ (and similar estimates hold for other knots considered by Kanda).

It is not known (at least, to us), whether Theorem 2.4 implies Theorem 2.3 in general case. What is important, both theorems imply, that the Bennequin invariant of a topologically unknotted transverse or Legendrian knot is always negative, which gives rise to a simple construction of an exotic contact structure in $\mathbf{R}^3$ ([Be]). Hence the existence of such structures, which is the most striking result of Bennequin's paper, has actually a short and elementary proof.

## 3. Knot diagrams.

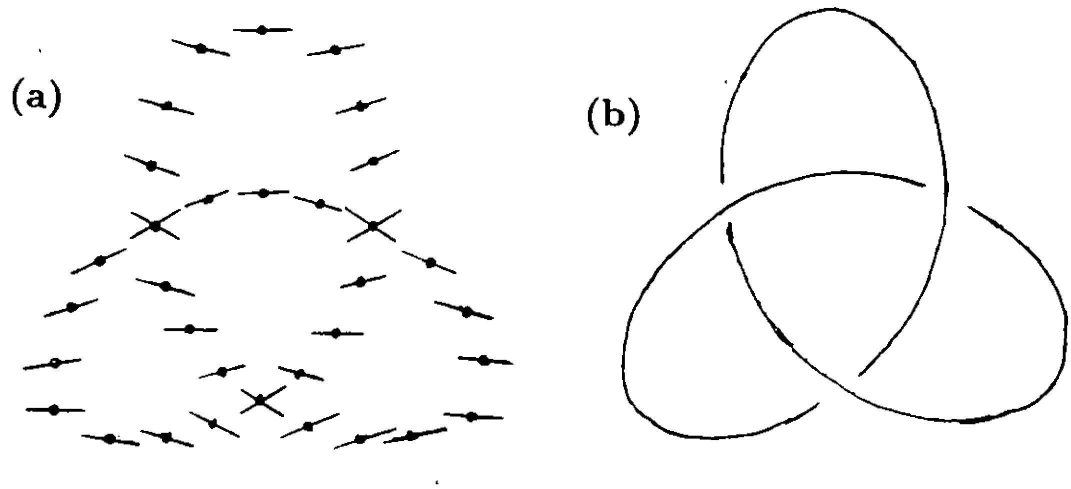


Fig. 1

**3.1.** Transverse and Legendrian knots are conveniently presented by projections into the plane $(x, z)$. (For Legendrian knots the projection onto the plane $(x, y)$ is also relevant – see 3.6 below.) Identify a point $(x, y, z) \in \mathbf{R}^3$ with a point $(x, z) \in \mathbf{R}^2$ furnished with a fixed direction of a non-vertical (unoriented) straight line through this point with the slope $y$. Then a curve in $\mathbf{R}^3$ is a one-parameter family of points with directions in $\mathbf{R}^2$, that is a curve in the plane, furnished with a field of unoriented non-vertical directions. For example, Fig. 1(a) pictures a mirror trefoil knot. If we are interested only in the isotopy classes of knots, and the projection of the knot in the plane has only double crossings, then we do not need to specify the directions, we need only to indicate at each crossing, which of the strands is the upper one, and which is the lower one (see Fig. 1(b)).

REMARK. In the classical knot theory knots are usually projected onto the plane $(x, y)$ rather than $(x, z)$. For this reason our diagrams are the mirror images of those accepted in the classical knot literature, and experts in knots theory would identify the knot in Fig.1(b) as a trefoil knot rather than a mirror trefoil knot.

**3.2.** A curve in $\mathbf{R}^3$ is transverse if and only if the corresponding curve in $\mathbf{R}^2$ is never tangent to the chosen directions along itself (the curve on Fig. 1(a) is not transverse). Again, the transversely isotopic class of a transverse knot is determined by a simplified picture like Fig. 1(b), but the existence of a family of non-vertical transverse directions along the curve imposes the following two conditions on the projected curve which is supposed to be regular and free of triple points (the former due to the fact that a transverse curve is never tangent to the $y$-direction which lies in the contact plane):

(i) the curve possesses on orientation with respect to which all its vertical tangents are directed downward (this is the canonical orientation of the transverse curve – see 2.2);

(ii) if at the crossing point the upward vertical direction lies inside the angle, formed by the two oriented tangents to the strands, then the strand directed to the right is higher, than the strand directed to the left.

Fig. 2.

The two forbidden fragments of a transverse knot diagram are presented on Fig. 2. Fig. 3(a) and (b) show, respectively, transverse trefoil knot and transverse mirror trefoil knot.

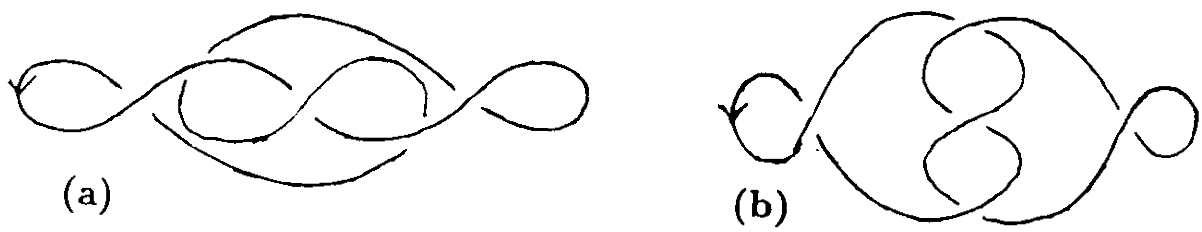

Fig. 3.

Transverse isotopy between transverse knots is represented by a deformation of the diagram with admissible modifications (moves) shown on Fig. 4.

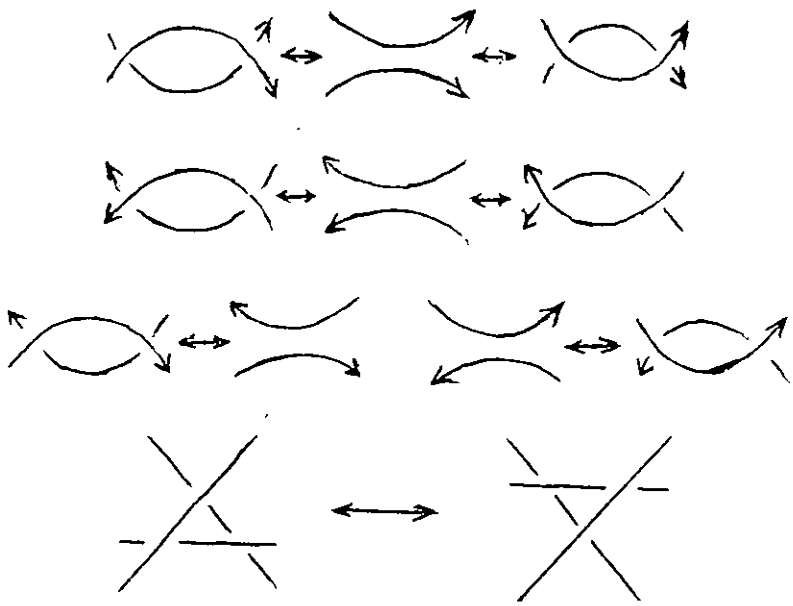

Fig. 4.

**3.3.** While a generic regular curve (in particular, generic transverse curve) in $\mathbf{R}^3$ has a regular projection onto the $(x, z)$-plane, the projection of a generic Legendrian curve onto the $(x, z)$-plane has isolated critical points (because all the planes of the standard contact structure in $\mathbf{R}^3$ are parallel to the $y$-axis). Hence the projection of a generic Legendrian curve may have cusps. On the other hand, the curve in $\mathbf{R}^3$ is Legendrian if and only if the corresponding planar curve is everywhere tangent to the field of directions; in particular, the field is determined by the curve. Hence the diagram of a Legendrian knot (in the $(x, z)$-plane) is simply a closed smooth curve with cusps, but without vertical tangents. Self- tangencies are also forbidden (they correspond to self-intersections of the curves in $\mathbf{R}^3$). To convert this diagram into a diagram of a topological knot of the same topological isotopy type one needs only to round the cusps and to make the strand with the greater slope overcross at each double point.

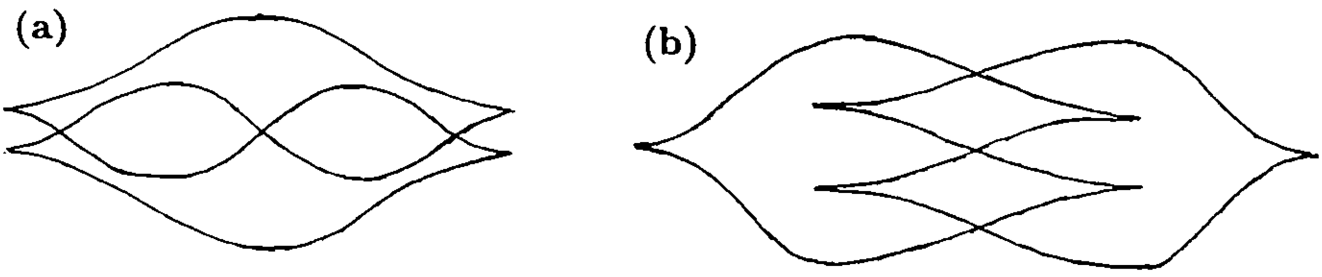

Fig. 5.

For example, Fig. 5(a) and (b) show a Legendrian trefoil knot and a Legendrian mirror trefoil knot, and Fig. 6(a) and (b) show the corresponding diagrams of topological knots.

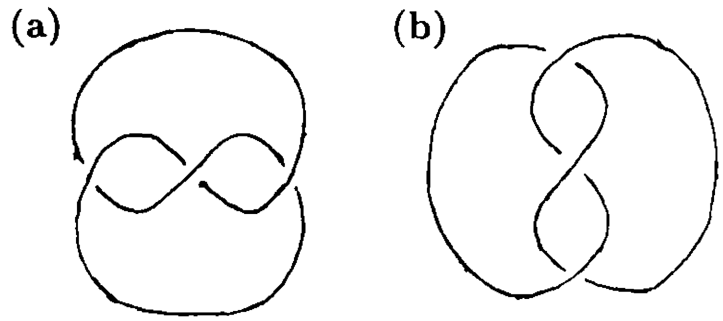


Fig. 6.

A Legendrian isotopy between Legendrian knots is represented by a deformation of the diagram with admissible modifications (moves) shown on Fig. 7 (cf. [Sw]).

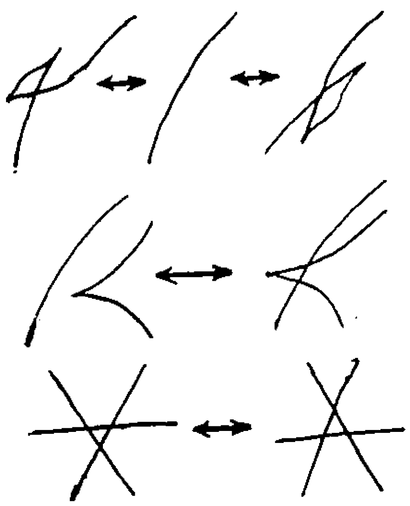

Fig. 7.

The transformation $\gamma \mapsto \gamma_{\pm}$ (see 2.2) approximating a Legendrian curve with two opposite oriented transverse curves may be easily performed on diagrams. We choose an orientation of the Legendrian curve, and then replace crossings and cusps as shown on Fig. 8.

**3.4.** The Bennequin and Maslov numbers of transverse and Legendrian knots are easily calculated from the diagram data.

For a transverse knot diagram we call crossings positive or negative, if they are marked as such on Fig. 9. The Bennequin number of a transverse knot is equal to the number of positive crossings minus the number of negative crossings. In particular, the Bennequin numbers of transverse trefoil knots, shown on Fig. 3(a) and (b) are equal respectively to $+1$ and $-5$ (compare the discussion in 2.6, after Theorem 2.4).

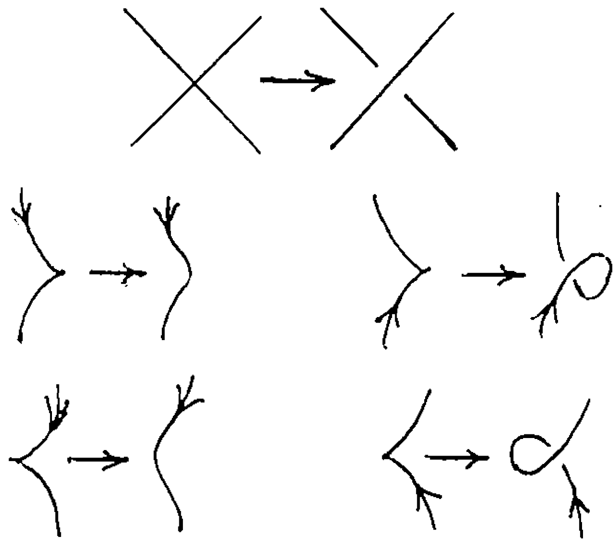

Fig. 8.

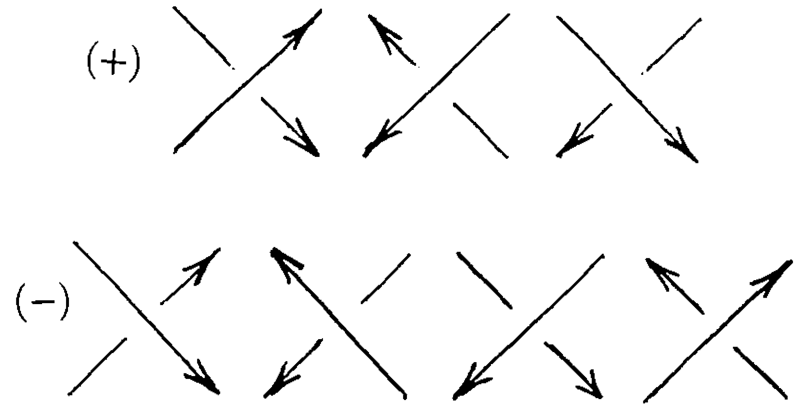

Fig. 9

For a Legendrian knot diagram we call a crossing positive (negative) if with respect to some orientation of the knot the two strands cross the vertical line in the same direction (in the opposite directions). The Bennequin number of a Legendrian knot is equal to the number of positive crossings minus the number of negative crossings minus half the number of cusps. The Maslov number of a Legendrian knot is equal to half the number of cusps passed downward minus half the number of cusps passes upward. For example, the Bennequin numbers of the Legendrian knots on Fig. 5(a) and (b) are equal respectively to $+1$ and $-6$, and their Maslov numbers are equal to 0 and $\pm 1$ (depending on the orientation).

The following generalization of trefoil knots is considered by Kanda ([Ka]). The two Legendrian knots in Fig.10(a) and (b) (which coincide with Fig. 5(a) and (b) when $n = 0$) are also topologically mirror to each other. Their Bennequin numbers are equal to $+1$

and $-6n-6$ correspondingly, their Maslov numbers are 0 and $\pm 1$. The lower degree of LYMPH-TOFU polynomials for these knots are equal to $+2$ and $-6n-4$. Kanda proves that the above Bennequin numbers are the greatest possible within the topological isotopy types. Hence for the knot in Fig. 10(b) Theorem 2.4 does not give the best possible estimate of the Bennequin number. For transverse knots of the same topological type the estimate of Theorem 2.4 is still the best possible (compare with the discussion in 2.6).

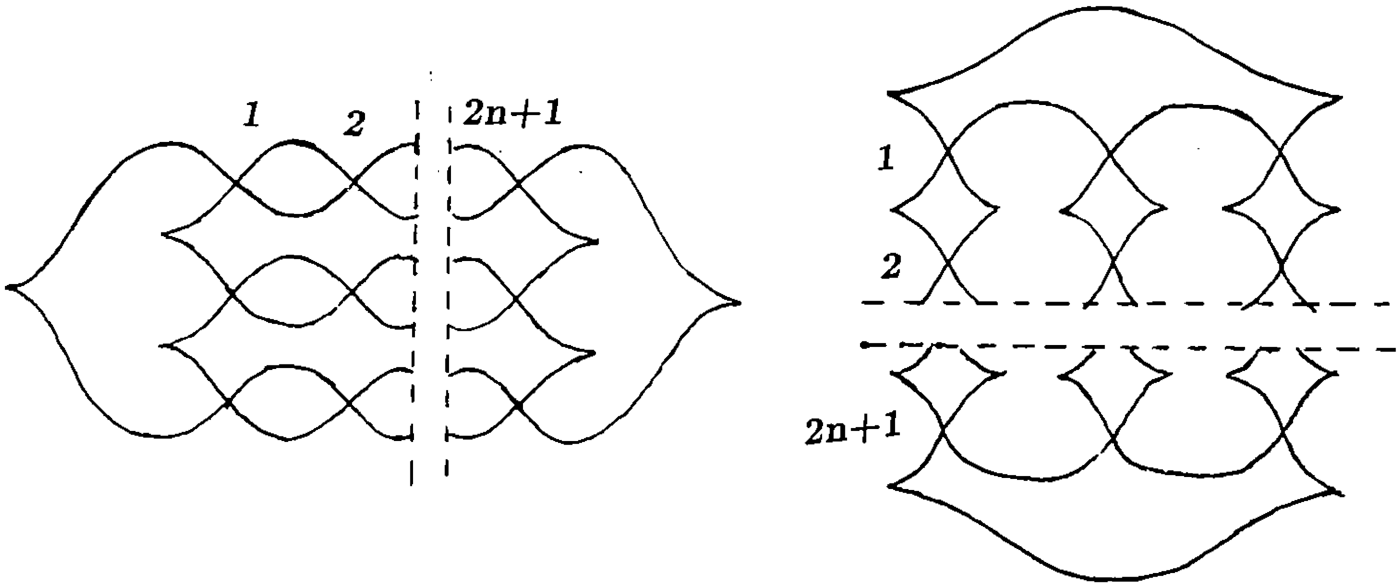


Fig. 10

**3.5.** Long transverse knots are presented by diagrams which are obtained from the $z$ axis by replacing a finite interval with a curve satisfying the conditions of 3.2; these diagrams are oriented downward. Diagrams of long Legendrian knots are obtained from the $x$ axis by replacing a finite interval with a smooth curve with cusps and without vertical tangents. Transverse and Legendrian isotopies of long transverse and Legendrian knots are performed according to the rules, outlined in 3.2 and 3.3. Bennequin and Maslov numbers are calculated as in 3.4.

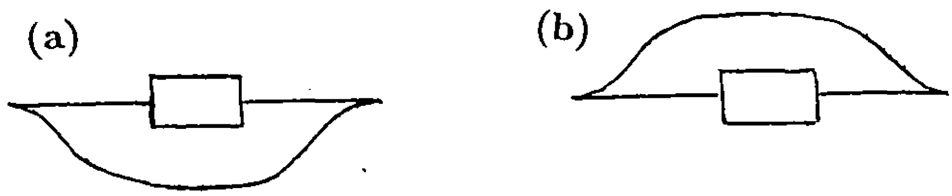


Fig. 11

There is a Legendrian isotopy invariant construction which makes a long Legendrian knot into a usual ("short") Legendrian knot. This construction may be described equivalently by either of Fig. 11 (a), (b) (the long knot is supposed to coincide with the $x$ axis outside the box; a Legendrian isotopy between the two knots is presented on Fig. 12; the slope of the arc $AB$ on Fig. 12 is assumed to be less, than the minimal slope of the knot in the box).

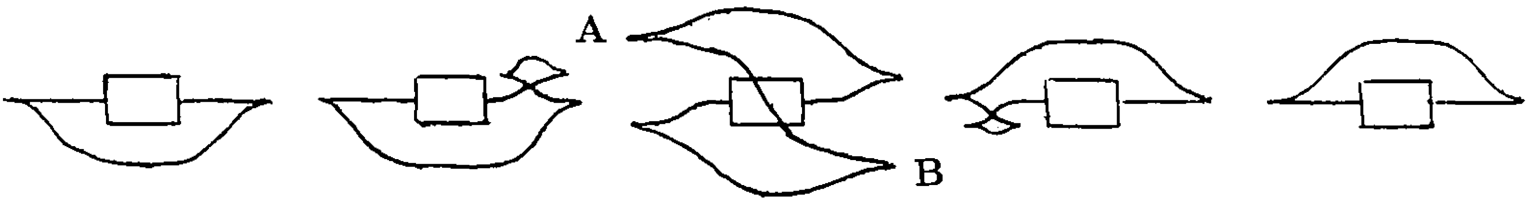


Fig. 12

Any short Legendrian knot may be obtained by this construction from a long Legendrian knot (see Fig. 13). We do not know, if this construction provides a $1-1$ correspondence between Legendrian isotopy classes of short and long Legendrian knots.

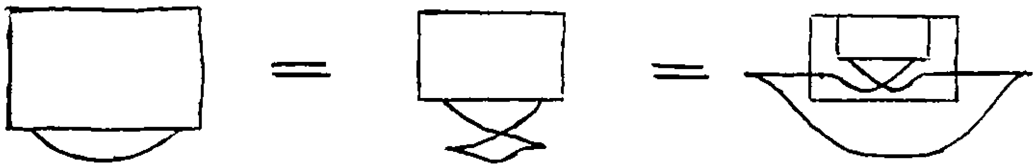

Fig. 13

**3.6.** Another possibility to draw Legendrian curves is to project them onto the $(x, y)$ plane. This projection is a smooth curve, and the missing $z$ coordinate is the integral $\int y dx$ along the curve. This shows, first, that a curve in the plane $(x, y)$ is the projection of infinitely many different Legendrian curves in $\mathbb{R}^3$, which differ from each other by translations in the direction of the $z$ axis. Second, the projection of a closed curve has to bound zero (signed) area. Hence, a Legendrian knot diagram in the $(x, y)$ plane is a smooth closed curve bounding zero area and such that the two loops into which the curve is cut by any crossing point bound non-zero area each (this is the non-self-intersecting condition).

The Maslov number of a Legendrian knot is the usual rotation number of its $(x, y)$ diagram, the Bennequin number is the number of crossings, counted with certain signs (we leave to the reader to formulate the sign rule).

For long Legendrian knots their $(x, y)$ diagrams are smooth curves, which are obtained from the $x$ axis by replacing a finite interval with a smooth curve cobounding a zero area with this interval. The non-self-intersecting condition reads as above.

## 4. The Grothendieck group and finite order invariants of Legendrian knots

**4.1.** Denote by # the connected summation (concatenation) of long Legendrian knots. This operation makes the set of Legendrian isotopy classes of long Legendrian knots into a semigroup, which we denote by $\mathcal{L}$.

LEMMA 4.1. *The semigroup $\mathcal{L}$ is commutative.*

PROOF (due to V. Ginzburg). Use projections of long Legendrian knots onto the $(x, y)$ plane. Take two long Legendrian knots $\gamma_1, \gamma_2$ and make $\gamma_2$ very small. Form the sum $\gamma_1 \# \gamma_2$ and pull $\gamma_2$ through $\gamma_1$ (Fig. 14). To adjust the area we make a small bump on the line at a distance from the knots.

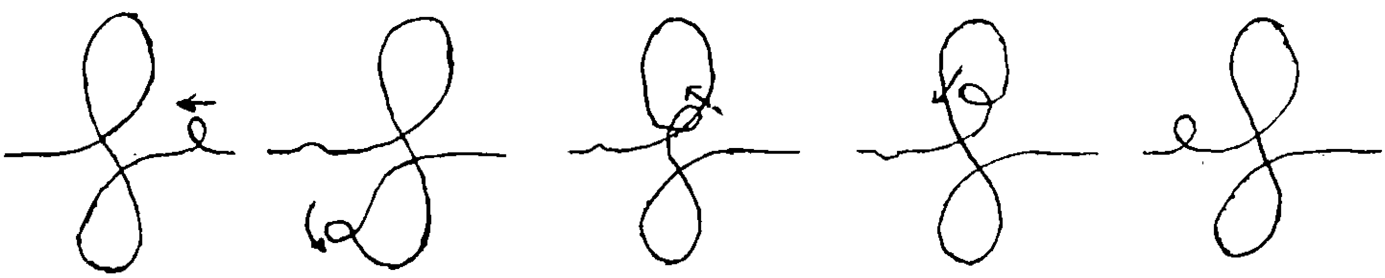

Fig. 14

Let $\mathcal{L}_0$ be the set of Legendrian isotopy classes of oriented usual ("short") Legendrian knots. Then for any $\gamma \in \mathcal{L}$ and $\gamma_0 \in \mathcal{L}_0$ we can define $\gamma_0 \# \gamma \in \mathcal{L}_0$. To do this we insert a small copy of $\gamma$ (oriented in the direction of increasing $x$) into $\gamma_0$. This operation is well defined because we can pull $\gamma$ through $\gamma_0$ as in the proof above.

**4.2.** From now on we always consider diagrams of Legendrian knots in the plane $(x, z)$.

Define a *zigzag* to be a long Legendrian knot whose diagram has no crossings. Zigzags form a subsemigroup $\mathcal{Z}$ of the semigroup of long Legendrian knots. The *canonical zigzag* $\zeta(p, q)$ $(p \geq 0, q \geq 0)$ is defined by Fig. 15. The next statement follows from [El] or [Sw].

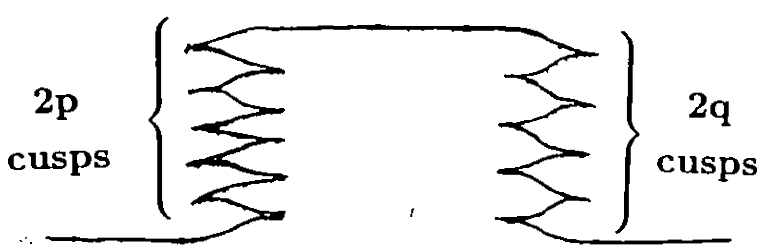


Fig. 15

LEMMA 4.2. *Any zigzag $\zeta$ is Legendrian isotopic to the canonical zigzag $\zeta(p,q)$ with*

$$p = -\frac{\beta(\zeta)+\mu(\zeta)}{2},\; q = \frac{\mu(\zeta)-\beta(\zeta)}{2}.$$

*In particular, $\mathcal{Z}$ is isomorphic to the additive semigroup $\{(x,y) \in \mathbf{Z} \oplus \mathbf{Z} \mid x + |y| \le 0, x + y\,\text{is even}\}$, the isomorphism being $\zeta \mapsto (\beta(\zeta), \mu(\zeta))$.*

PROOF. Induction in the number of cusps. Orient the knot from left to right; then we can distinguish between descending and ascending cusps. Since $\zeta$ is non-self-intersecting, it must have two consecutive descending or two consecutive ascending cusps (if descending and ascending cusps alternate, then $\zeta$ is a spiral-shaped curve as on Fig. 16 and it cannot be a long knot).

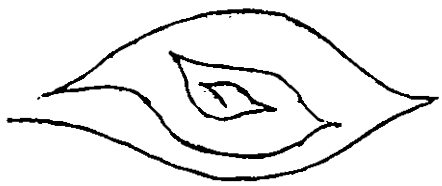

Fig. 16

If we locate a pair of descending cusps, we pull it along the knot to the right (it is shown on Fig. 17 and 18, how our pair passes a descending cusp and an ascending cusp); or if we have a pair of ascending cusps, we pull it to the left. In the end we get either $\zeta(1,0)\#\zeta'$ or $\zeta'\#\zeta(0,1)$, where $\zeta'$ is a zigzag, which has less cusps, than $\zeta$. By the induction hypothesis, $\zeta' = \zeta(p',q')$ with some $p'$ and $q'$. Obviously, $\zeta(1,0)\#\zeta(p',q') = \zeta(p'+1,q)$ and $\zeta(p',q')\#\zeta(0,1) = \zeta(p',q+1)$, which completes the proof.

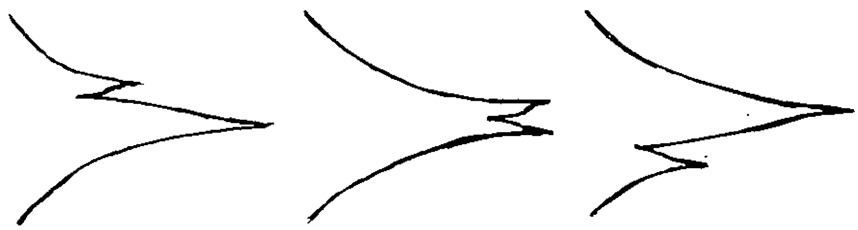

Fig. 17

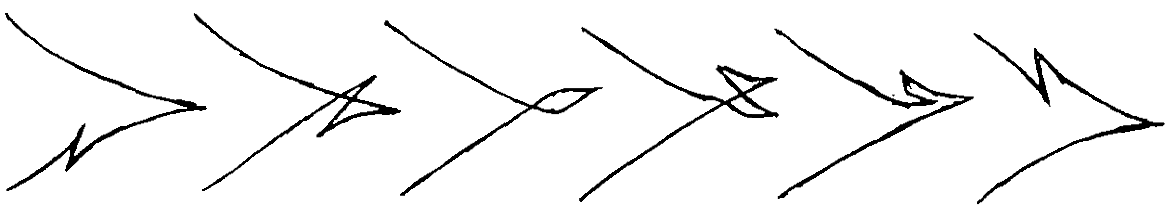

Fig. 18

LEMMA 4.3. *Let $\gamma$ be an arbitrary long Legendrian knot (which we orient from left to right) or oriented Legendrian knot. Let $\gamma'$ be obtained from $\gamma$ by adding in some $p+q$ places $p$ pairs of ascending cusps and $q$ pairs of descending cusps. Then $\gamma' = \gamma\#\zeta(p,q)$.*

PROOF. Pull all the pairs (as in the previous proof) to the right if $\gamma$ is long and to an arbitrarily chosen place if $\gamma$ is short. We will get $\gamma' = \gamma\#\zeta$, where $\zeta$ is a zigzag; according to Lemma 4.2, $\zeta = \zeta(p,q)$.

**4.3.** Let $\mathcal{GL}$ be the Grothendieck group of the semigroup of long Legendrian knots, and $\mathcal{GK}$ be that of long topological knots.

THEOREM 4.4. *$\mathcal{GL} = \mathcal{GK} \oplus \mathbb{Z} \oplus \mathbb{Z}$, the isomorphism being the direct sum of the forgetting homomorphism $\mathcal{GL} \to \mathcal{GK}$, half of the sum of the Bennequin number and the Maslov number, and half of the difference of these numbers.*

*Moreover, if $\gamma_1$ and $\gamma_2$ are two long Legendrian knots, or two oriented Legendrian knots, which are topologically isotopic, then there exist two zigzags $\zeta_1$ and $\zeta_2$, such that $\gamma_1\#\zeta_1$ is Legendrian isotopic to $\gamma_2\#\zeta_2$. More precisely, if $p$ and $q$ are sufficiently large, then $\gamma_1\#\zeta(p,q)$ is Legendrian isotopic to $\gamma_2\#\zeta(p',q')$, where*

$$p' = p + \frac{\mu(\gamma_2)-\mu(\gamma_1)}{2} + \frac{\beta(\gamma_2)-\beta(\gamma_1)}{2},$$
$$q' = q - \frac{\mu(\gamma_2)-\mu(\gamma_1)}{2} + \frac{\beta(\gamma_2)-\beta(\gamma_1)}{2}.$$

PROOF. Take long Legendrian knots $\gamma_1, \gamma_2$ replace their diagrams with diagrams $\Gamma_1, \Gamma_2$ of topological knots (see 3.3) and consider a (smooth) topological isotopy $\gamma_t$ ($1 \le t \le 2$) between $\gamma_1$ and $\gamma_2$. Let $\Gamma_t$ be the diagram of the knot $\gamma_t$. We say that a value $t$ of the parameter is generic if (i) the projection $\gamma_t \to \Gamma_t$ is regular, $\Gamma_t$ has no (ii) triple points and (iii) self-tangencies, (iv) neither of the tangents to strands of $\Gamma_t$ at self-crossings is vertical, and (v) the tangents to $\Gamma_t$ at its inflection points are not vertical. We may suppose that the parameter has only finitely many non-generic values, and for each of the latter precisely one of the conditions (i)–(v) is violated, and this violation is generic (that is $\Gamma_t$ has either precisely one generic cusp, or precisely one transverse triple point, or so on). Decompose the interval $[1,2]$ into pieces $[t_i, t_{i+1}]$, such that all singular values of parameter become interior points of different pieces, and put $\Gamma'_i = \Gamma_{t_i}$. We make $\Gamma'_i$ into a diagram of a long Legendrian knot $\gamma'_i$ by modifying points with vertical tangents and crossings as shown on Fig. 19. Notice, that a crossing, in which the overpass strand has smaller slope (we call such crossings "wrong") may be modified in two different ways; for each wrong crossing we arbitrarily choose one of the two. We claim, that for any $i$ the Legendrian knots $\gamma_i\#\zeta(p_i,q_i)$, $\gamma_{i+1}\#\zeta(p'_i,q'_i)$ with appropriate $p_i, q_i, p'_i, q'_i$ are Legendrian isotopic; this obviously implies the theorem.

Everywhere below in the proof we use implicitly Lemma 4.3, which allows us to move pairs of cusps from $\zeta(p,q)$ along the knot to any place where they are needed, and, when necessary, to collect them back into $\zeta(p',q')$.

First we remark that the Legendrian isotopy type of the knot $\gamma'_i\#\zeta(p,q)$ with large $p$ and $q$ does not depend (possibly up to a change of $p$ and $q$) on the choice of the modifications of wrong crossings of $\Gamma'_i$; the Legendrian isotopy is shown on Fig. 20.

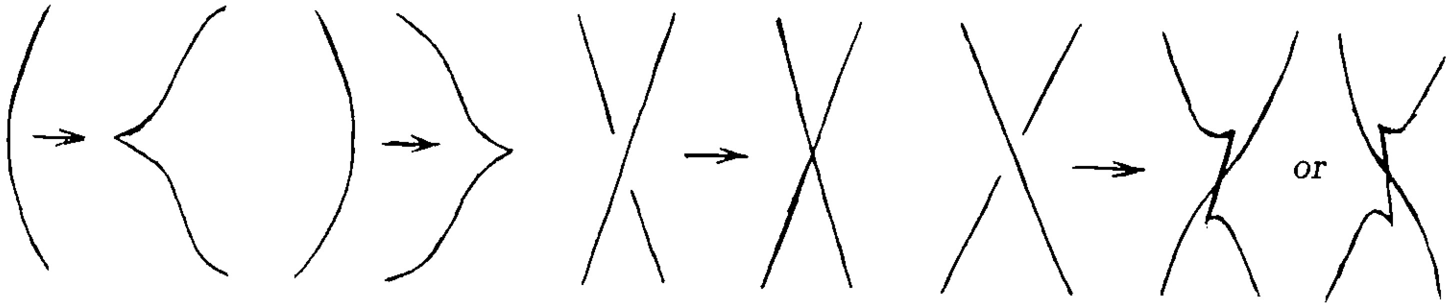


Fig. 19

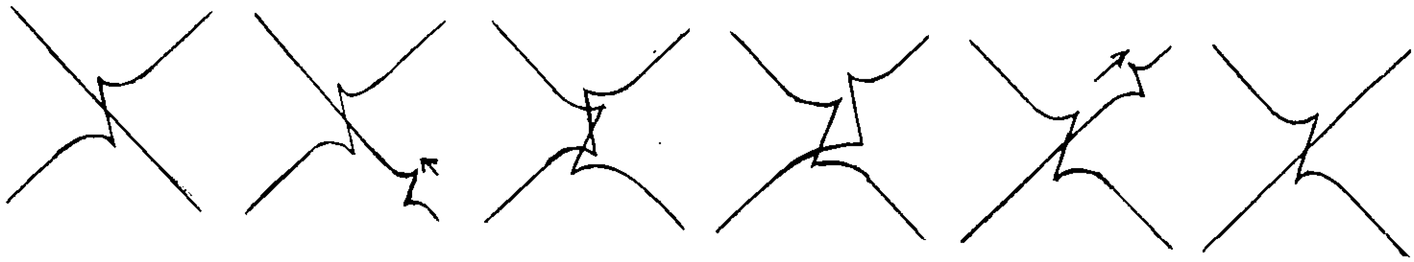

Fig. 20

If there is no singular parameter values between $t_i$ and $t_{i+1}$, then our claim is proved by the preceding remark. Otherwise the diagrams $\Gamma_i, \Gamma_{i+1}$ differ either by one of the Reidemeister moves, or by a change of the position of the diagram with respect to the vertical direction. Each of these moves has several versions depending on the crossings being right or wrong and on the direction of convexity at vertical tangency points. The total number of different types of modifications which make the diagrams $\Gamma_i, \Gamma_{i+1}$ into each other equals 11 (Fig. 21). Their Legendrian versions, some of which use extra zigzags are shown on Fig. 22 (modification *R1(b)*, *V1(b)* and *V2(b–d)* are similar to *R1*(a), *V1*(a) and *V2*(a), and are not shown).

REMARK. It is a classical result that every knot uniquely decomposes into a finite number of nondecomposable components with respect to the connected summation. We do not know whether such a result holds for long Legendrian (or transverse) knots. In view of Theorem 4.4 such a result would imply the affirmative answer to the Question in 2.4.

**4.4.** A *Legendrian knot invariant* is a complex-valued function on the set of Legendrian isotopy classes of oriented Legendrian knots. There exists a canonical way (familiar from the theory of Vassiliev knot invariants) to extend such a function to oriented closed regular Legendrian curves with transverse double points (and without other singularities).

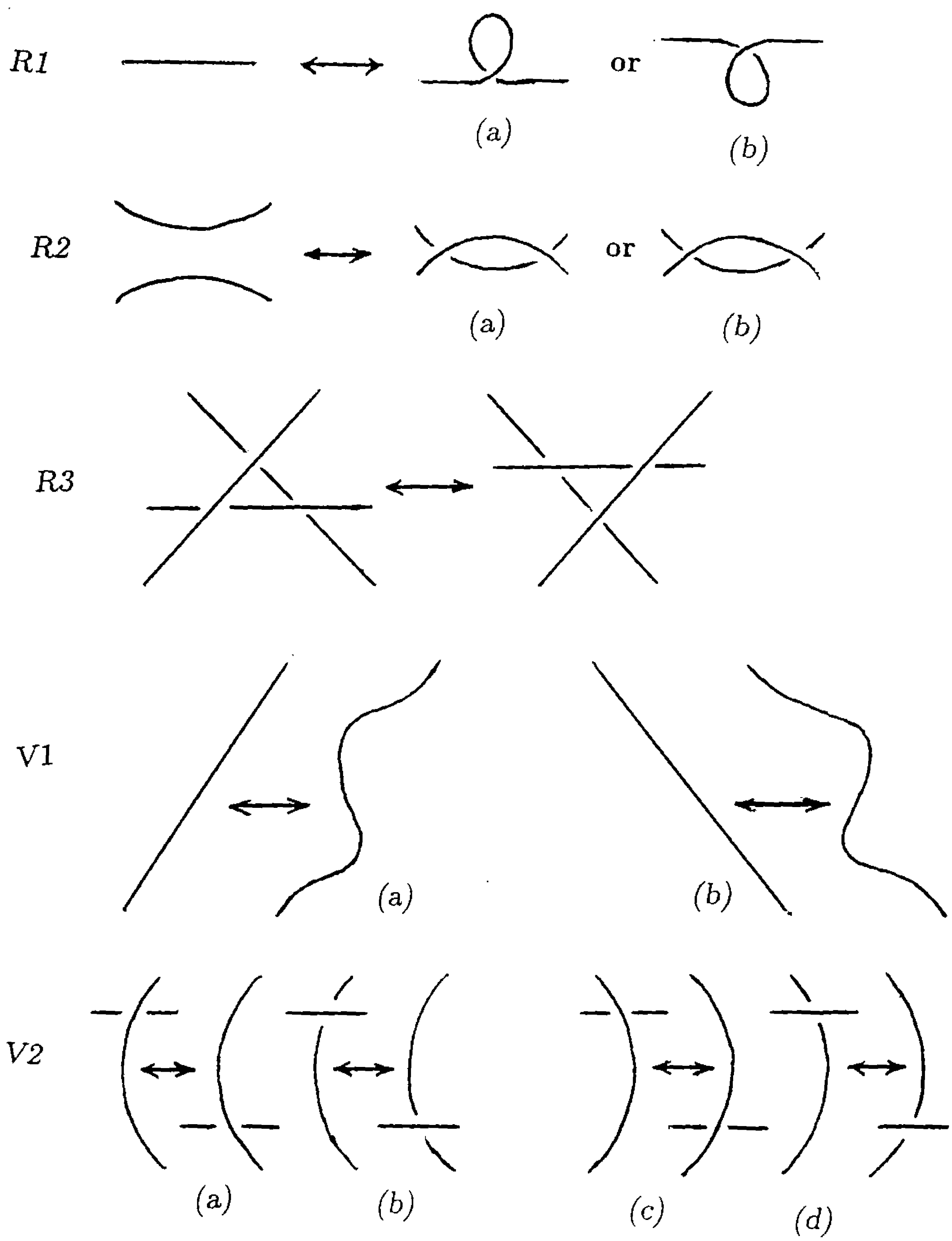


Fig. 21

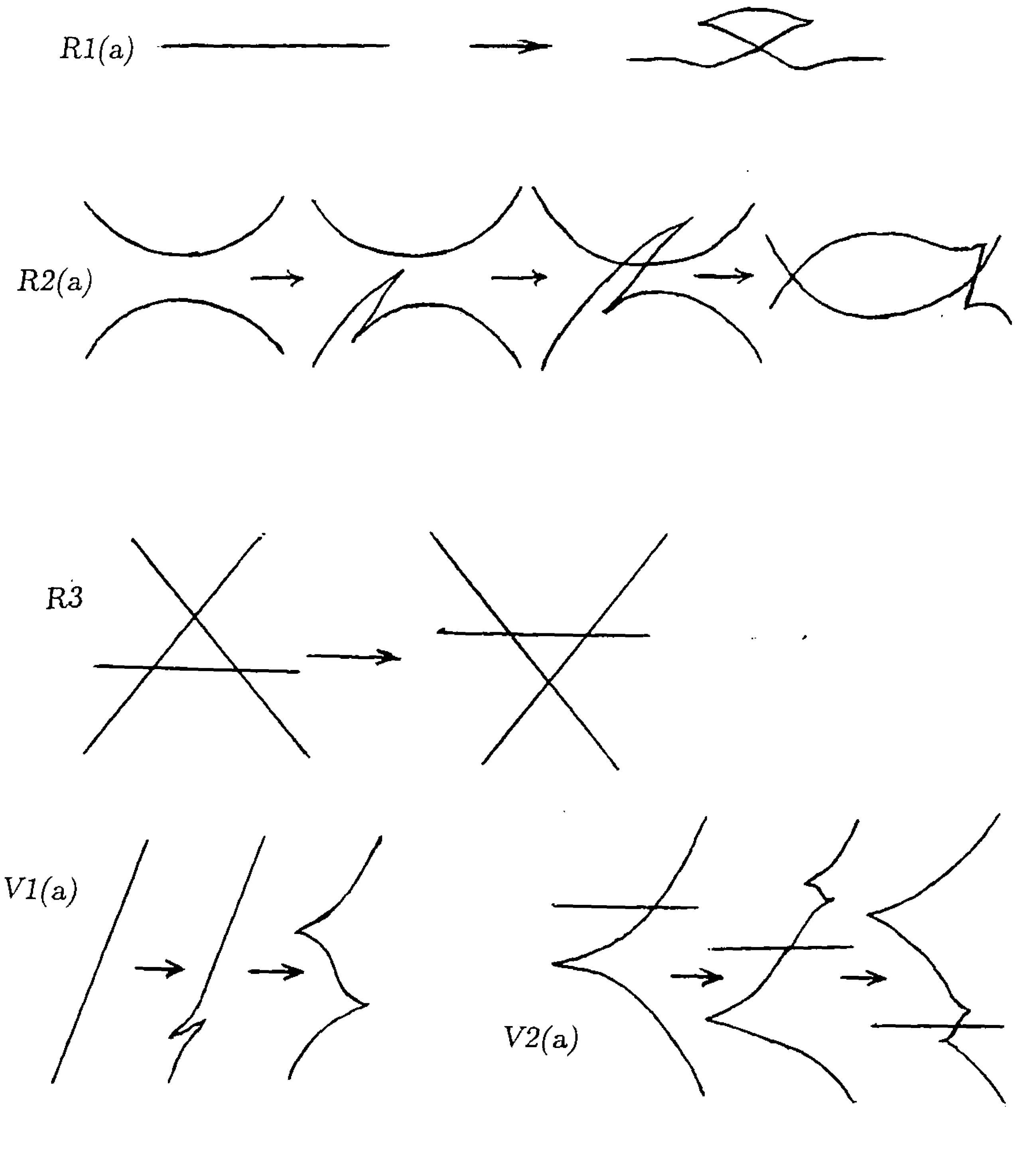


Fig. 22

Let $\gamma$ be an oriented regular spatial curve with a transverse double point $x$. We call a resolution of $\gamma$ positive if the tangent vector to the first strand, the tangent vector to the second strand and the vector from the second strand to the first one form a positive frame (this does not depend on the order of the strands). In other words, if $\gamma$ is a Legendrian curve with only one self-intersection, then the positive resolution is the resolution with the greater Bennequin number. (Four possible diagrams of double points of a Legendrian curve and their positive and negative resolutions are shown on Fig. 23.)

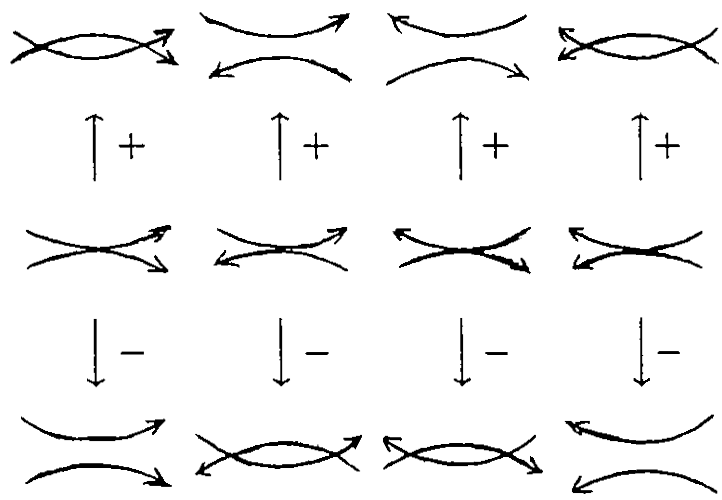


Fig. 23

Let $f$ be a Legendrian knot invariant. Take a closed oriented Legendrian curve $\gamma$ with $m$ transverse double points, and suppose, that for closed oriented Legendrian curves with less than $m$ double points the value of $f$ has been already defined. Choose a double point and resolve it in two different ways. Let $\gamma_+$ be the curve obtained by the positive resolution, and $\gamma_-$ be the curve obtained by the negative resolution. Then we put $f(\gamma) = f(\gamma_+) - f(\gamma_-)$.

DEFINITION. A Legendrian knot invariant has order $\leq n$ if it vanishes on any closed oriented Legendrian curve with at least $n+1$ double points.

Invariants of long Legendrian knots are handled in the same way. We call them *long Legendrian knot invariants*. The last definition also applies to long Legendrian knot invariants.

EXAMPLE. The Maslov number is an order 0 invariant. This is essentially the only order 0 invariant (see [Gr]). The Bennequin number is an order 1 invariant. There are essentially no other order $\leq 1$ invariants (see Corollary 4.7 below).

Notice that since the Maslov number does not change under self-intersections of Legendrian curves, the concept of finite order invariants still makes sense in the space of (long or short) Legendrian knots with a certain fixed Maslov number.

4.5. THEOREM 4.5. *No finite order invariant can distinguish between long Legendrian knots or oriented Legendrian knots, which are topologically isotopic and have equal Bennequin and Maslov numbers.*

PROOF. The cases of short and long Legendrian knots being completely similar, we will consider only the case of long Legendrian knots.

Let $L$ be the complex vector space, spanned by Legendrian isotopy classes of long Legendrian knots. Long Legendrian knot invariants are linear functionals on $L$. The operation $\#$ makes $L$ into a commutative associative algebra with the identity element $1 =$ the straight line.

For an immersed long Legendrian curve with $n$ transverse double points and without other singularities we denote by $d_n\gamma$ the linear combination of the $2^n$ possible resolutions of $\gamma$ with the coefficients $\pm 1$ depending on the parity of the number of negatively resolved double points. Denote by $L_n$ the subspace of $L$ spanned by all vectors of the form $d_n\gamma$. Obviously,

$$L = L_0 \supset L_1 \supset \dots, \text{ and } L_p \# L_q \subset L_{p+q}.$$

It is clear also, that an invariant $f$ has order $\leq n$ if and only if $f|L_{n+1} \equiv 0$. We call two vectors from $L$ $n$- equivalent if their difference belongs to $L_{n+1}$.

Let $\zeta = \zeta(1,1)$ be an elementary zigzag. Since $\zeta$ may be connected with 1 by a regular homotopy passing through one self-intersection (Fig. 24), $1 - \zeta \in L_1$. Hence $(1-\zeta)^{n+1} \in L_{n+1}$. Put

$$\omega_n = \frac{1-(1-\zeta)^{n+1}}{\zeta} = \sum_{i=0}^{n} (-1)^i \binom{n+1}{i+1} \zeta^i.$$

Then $\zeta \# \omega_n$ is $n$-equivalent to 1.

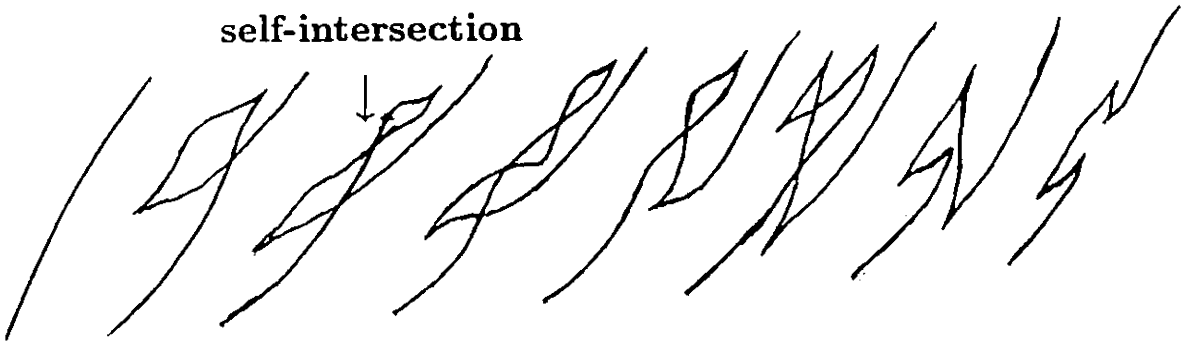


Fig. 24

Let $\gamma_1, \gamma_2$ be two topologically isotopic long Legendrian knots with equal Maslov and Bennequin numbers. Then according to Theorem 4.4 $\gamma_1 \# \zeta^p = \gamma_2 \# \zeta^p$ for sufficiently large $p$ (here $=$ means *Legendrian isotopic*). Hence

$$\gamma_1 \# \zeta^p \# \omega_n^p = \gamma_2 \# \zeta^p \# \omega_n^p,$$

that is

$$\gamma_1 \# (1-(1-\zeta)^{n+1})^p = \gamma_2 \# (1-(1-\zeta)^{n+1})^p$$

which implies, that $\gamma_1 - \gamma_2 \in L_{n+1}$, and $\gamma_2$ cannot be distinguished from $\gamma_1$ by invariants of order $\leq n$.

REMARK. It is proved in [Gu] that every knot $\gamma$ is $n$- invertible, that is there exists a knot $\gamma'$ such that $\gamma\#\gamma'$ is $n$-equivalent to the unknot. A similar result holds for framed knots. We do not know whether it holds for long Legendrian (or transverse) knots.

4.6. In this section we strengthen Theorem 4.5 to the effect that the space of invariants of order $\leq n$ of (long or short) Legendrian knots with a fixed Maslov number is linearly isomorphic to that of framed knots. Denote the latter space by $J^n$. We start with a description of this space – see [B-N; L-M 1,2; Go 1].

A knot, represented by a plane knot diagram, has a natural, so called, blackboard framing, provided by vectors tangent to the fibers of the projection of space to the plane. Every framed knot can be represented this way. The second and the third Reidemeister moves preserve the blackboard framing while the first one, that is, inserting or deleting a curl, change it. Notice that the blackboard framing is the natural framing of a Legendrian knot presented by its projection to the $(x, y)$ plane.

A *chord diagram* is the real line with a number of chords connecting disjoint pairs of its points; chord diagrams are considered modulo orientation-preserving diffeomorphisms of the line. A framed knot invariant $f$ is extended to singular framed knots with $n$ transverse double points according to the formula:

$$f(\times) = f(\times) - f(\times).$$

An invariant has order $\leq n$ if it vanishes on all singular framed knots with more than $n$ double points.

To such a singular knot $S^1 \to \mathbf{R}^3$ with a base point a chord diagram corresponds whose chords connect the preimages of the double points in the source $S^1$, cut at the base point to obtain the line. The invariant $f$ determines a function on chord diagrams called its *symbol* $\mathrm{Symb}_f$. The symbol of every invariant satisfies the *4-term relation* depicted in Fig. 25 (with the usual understanding that the four chord diagrams coincide outside the shown fragments).

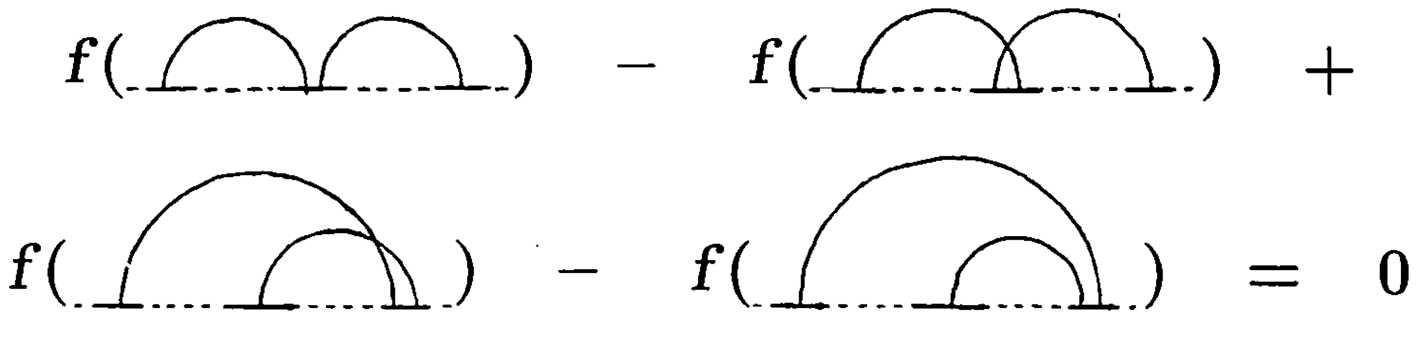


Fig. 25

The 4-term relation implies that if $d$ and $d'$ are two $n$-chord diagrams, corresponding to the same singular knot with different choices of the base points, then $f(d) = f(d')$ for every invariant $f$ of order $\leq n$. Denote the space of functions on $n$-chord diagrams subject to the 4-term relation by $W_n$.

The main result of this theory is that every element of $W_n$ is the symbol of a framed knot invariant of order $\leq n$. Thus the spaces $J^n/J^{n-1}$ and $W_n$ are isomorphic. In particular $J^n$ is finite dimensional for every $n$.

Denote by $I^n$ the space of order $\leq n$ invariants of long Legendrian knots with a certain fixed Maslov number and by $I_0^n$ the space of order $\leq n$ invariants of Legendrian knots.

THEOREM 4.6. *The spaces $I^n/I^{n-1}$ and $I_0^n/I_0^{n-1}$ are isomorphic to the space $W_n$.*

PROOF. The cases of long and closed Legendrian knots being similar we consider the former, indicating the difference between the two when necessary. Throughout the proof all Legendrian curves have a certain fixed Maslov number.

First, given a long Legendrian curve $\gamma$ (closed Legendrian curve with a base point) with a transverse double point $P$ assign a ***sign*** to $P$ according to whether the orientation of the frame, consisting of the tangent vectors to $\gamma$ at the first and at the second visit to $P$ coincides with that of the contact plane at $P$. The ***signed chord diagram*** $d(\gamma)$ of a long Legendrian curve $\gamma\colon \mathbf{R} \to \mathbf{R}^3$ with $n$ transverse double points and no other singularities is the source real line with $n$ signed chords connecting the preimages of the double points; the sign of a chord is that of the double point. Signed chord diagrams are considered modulo orientation-preserving diffeomorphisms of the line.

Let $f$ be an invariant of order $\leq n$ of long Legendrian knots. As above one extends it to Legendrian curves with $n$ transverse double points. We claim that $f$ determines a well-defined function $\mathrm{Symb}_f$ on signed chord diagrams according to the formula $\mathrm{Symb}_f(d(\gamma)) = f(\gamma)$.

To show this fix $n$ points $P_1, ..., P_n \in \mathbf{R}^3$ and an unordered pair of transversely intersecting simple oriented Legendrian arcs through each $P_i$ (in the $(x, z)$ plane these arcs are presented by $n$ disjoint pairs of quadratically tangent oriented arcs). The pair of arcs at each point $P_i$ can be ordered in two ways; altogether there are $2^n$ such orderings. Given a signed chord diagram $d$ order its chords from left to right according to the position of the left ends of the chords. Fix a long Legendrian curve $\gamma_d$ with exactly $n$ double points $P_i$ that contains all the chosen oriented arcs at these points and visits them in the order prescribed by the diagram $d$ and according to the signs of its arcs.

Let $\gamma$ be some long Legendrian curve with $n$ transverse double points whose signed chord diagram is $d$. We claim that $f(\gamma) = f(\gamma_d)$ for every invariant $f$ of order $\leq n$.

Bring the crossings of $\gamma$ by a Legendrian isotopy to the crossings of the curve $\gamma_d$ (preserving the orientation). It is sufficient to construct a regular Legendrian homotopy between $\gamma$ and $\gamma_d$ modulo the two ends and the neighborhoods of the crossings, no matter, whether this homotopy passes through additional self-intersections: since $f$ has order $\leq n$, and the curve $\gamma$ already has $n$ double points, passing through additional self-intersections would not change the value of $f$. Let us see whether such a homotopy exists.

The curves $\gamma$ and $\gamma_d$ coincide at the neighborhoods of $\pm\infty$ and $2n$ preimages of the double points. Hence they may be different only on $2n + 1$ connecting pieces. These pieces are not closed, and they have no Maslov numbers; but we can count the differences between the rotations of tangent vectors of the two curves on each of these pieces. We get $2n + 1$ integers $\mu_1, \ldots, \mu_{2n+1}$, and the above homotopy exists if and only if all these numbers equal 0. The only thing we know is that $\sum \mu_i = 0$, because this is the difference between the Maslov numbers of $\gamma$ and $\gamma_d$.

We can change the numbers $\mu_i$ in the following way. Using the homotopy, shown on Fig. 24 (it passes through one self-intersection), we can create the zigzag $\zeta(1,1)$ on any of the $2n+1$ pieces of $\gamma$. Resolve all the double points of $\gamma$ (with the zigzag) by a small finitely supported regular Legendrian homotopy, then move the ascending or descending half of the zigzag to another of the $2n+1$ pieces by a Legendrian isotopy and recreate all $n$ double point. For the new curve $\gamma'$ we will have $f(\gamma') = f(\gamma)$, because all the $2^n$ long Legendrian knots, which are obtained from $\gamma'$ by different resolutions of singularities will be Legendrian isotopic to similar knots for $\gamma$, and the values of $f$ on $\gamma$ and $\gamma'$ are defined as the same linear combinations of the values of $f$ on this resolved knots. On the other hand, for $\gamma'$ one of $\mu_i$'s in greater by 1, and one is less by 1, than for $\gamma$. Having applied this trick sufficiently many times, we will get a curve $\tilde{\gamma}$, for which $f(\tilde{\gamma}) = f(\gamma)$, and all $\mu_i$'s are zeroes. Hence also $f(\tilde{\gamma}) = f(\gamma_d)$, and $f(\gamma) = f(\gamma_d)$. Thus $\mathrm{Symb}_f$ is a well-defined function on signed chord diagrams.

Next we claim that $\mathrm{Symb}_f(d)$ does not depend on the signs of the chords of the diagram $d$. Pick a chord in $d$ and consider the corresponding double point of $\gamma_d$. The following transformations, depicted in the $(x, z)$ plane, change $\gamma_d$ only in a neighbourhood of this point – see Fig. 26 (the first equality is due to the fact that an extra double point does not change the value of $f$ on $\gamma_d$; the second and third hold by the definition of the extension of $f$ to singular knots). The signed chord diagram of the last curve differs from $d$ precisely in the sign of the chord under consideration.

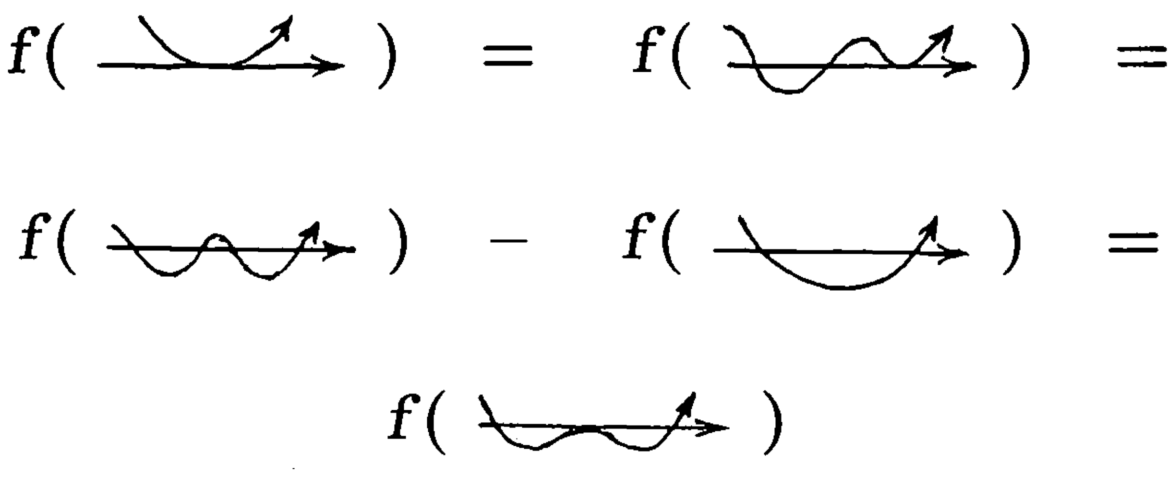

Fig. 26

Thus $\mathrm{Symb}_f$ is a function on (unsigned) chord diagrams; it also satisfies the 4-term relation. This relation follows from the equality shown in Fig. 27, which is proved by resolving the tangency point of the upper parabola in each of the four cases and collecting terms. Abusing notation we denote by $Symb : I^n/I^{n-1} \to W^n$ the linear map $f \to \mathrm{Symb}_f$. We want to show that this map is a linear isomorphism.

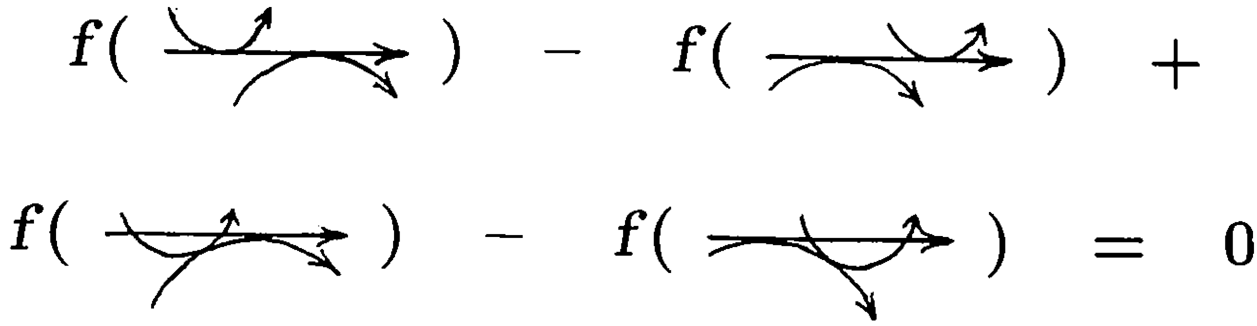

Fig. 27

Injectivity of *Symb* is easily seen: if for some invariant $f$ its symbol vanishes on all $n$-chord diagrams then $f$ vanishes on all long Legendrian knots with $n$ double points, and therefore $f \in I^{n-1}$. Let $w \in W^n$; according to the discussion, preceding Theorem 4.6, there exists a framed knot invariant $f$ of order $\leq n$ whose symbol is $w$. This invariant can be considered as an element of the space $I^n$: first close a long Legendrian knot as described in 3.5 and then evaluate $F$ on the obtained Legendrian knot (with its natural framing). The map *Symb* sends $f \in I^n$ to $w$, and surjectivity of *Symb* follows.

COROLLARY 4.7. *The Bennequin number $\beta$ is the only invariant of order 1 of (long or short) Legendrian knots with a fixed Maslov number. All order $\leq 1$ invariants of (long or short) Legendrian knots are of the form $F(\mu) + G(\mu)\beta$ where $F$ and $G$ are some functions of the Maslov number.*

PROOF. Since there exists only one 1-chord diagram and the 4-term relation involves diagrams with $\geq 2$ chords, the space $W_1$ is 1-dimensional.

4.7. Another contact 3-fold that recently attracted much interest is the manifold $M$ of cooriented contact elements in the plane. A contact element is a 1-dimensional subspace of the tangent plane at a point; coorientation is a choice of one of the two sides of this line. Let $(x, y)$ be Cartesian coordinates in the plane and $\alpha$ the angle made by a contact element with a fixed direction. The natural contact structure in $M$ is given by the 1-form $\sin\alpha dx - \cos\alpha dy$. The manifold $M$ is diffeomorphic to the solid torus; its universal cover is the standard contact space, which is also imbedded into $M$ as the subset of non-vertical contact elements.

The projection of a Legendrian knot in $M$ to the $(x, y)$ plane is a cooriented front without "dangerous" self-tangencies; a self-tangency is dangerous if the coorientations of the tangent branches coincide. Unlike the fronts of Legendrian curves in 3-space the fronts of Legendrian curves in $M$ may have the vertical direction. Conversely a Legendrian curve in $M$ is recovered by its cooriented front. The theory of Legendrian knots in $M$ is often referred to as Arnold's $J^+$-theory ([Ar]).

A Legendrian knot in $M$ is a framed knot in the solid torus. We are confident that the analog of Theorem 4.6 holds for $M$ as well (see [Go 2] for the chord diagrammatic description of finite order invariants of framed knots in the solid torus; it is also proved in this paper that the space of finite order invariants of cusp-free immersed curves in the plane without dangerous self-tangencies is isomorphic with that of framed knots in the solid torus). This would imply, in particular, that the invariants of Legendrian knots in $M$ introduced in a recent preprint [Li] are actually topological invariants of framed knots therein.

## 5. The Grothendieck group and finite order invariants of transverse knots

**5.1.** Begin with the case of long transverse knots. The operation $\#$ of connected summation makes the set of Legendrian isotopy classes of long transverse knots into a semigroup, which we denote by $\mathcal{T}$. Obviously, $\beta$ is an additive function on $\mathcal{T}$.

LEMMA 5.1. *The semigroup $\mathcal{T}$ is commutative.*

PROOF. Consider the sum $\gamma_1\#\gamma_2$ of two long transverse knots, make $\gamma_1$ small and enclose it into the rectangular box with the entrance in the middle of the top and the exit in the middle of the bottom. Then we pull the box through $\gamma_2$ in such a way, that the directions of the sides of the box never change (see Fig. 28). (It is possible to do this, because $\gamma_2$ never goes vertically upward.)

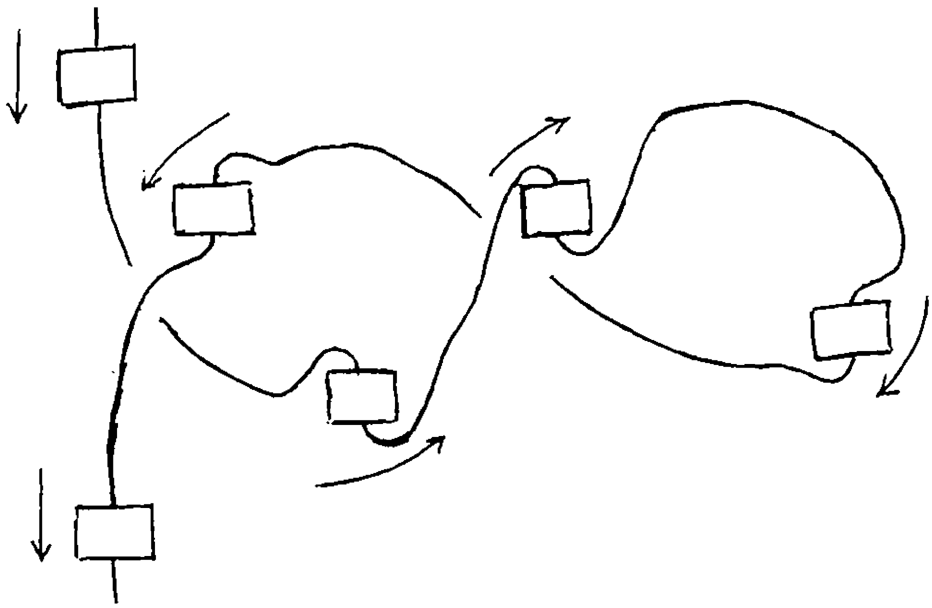

Fig. 28

The only difficulty may arise when we drag the box through a crossing: a forbidden crossing may be formed by a strand of $\gamma_2$ and a strand inside the box (see Fig. 29). To avoid this we suppose that the absolute value of the slope of ascending curves in the box is never greater than 1, and then deform the crossings of $\gamma_2$ as shown on Fig. 30 in such a way, that the slope of the curve $AB$ at the crossing is greater than $+2$, and the slope of the curve $CD$ at the crossing is less than $-2$. After this no forbidden crossings ever emerge, and we obtain an isotopy between $\gamma_1\#\gamma_2$ and $\gamma_2\#\gamma_1$.

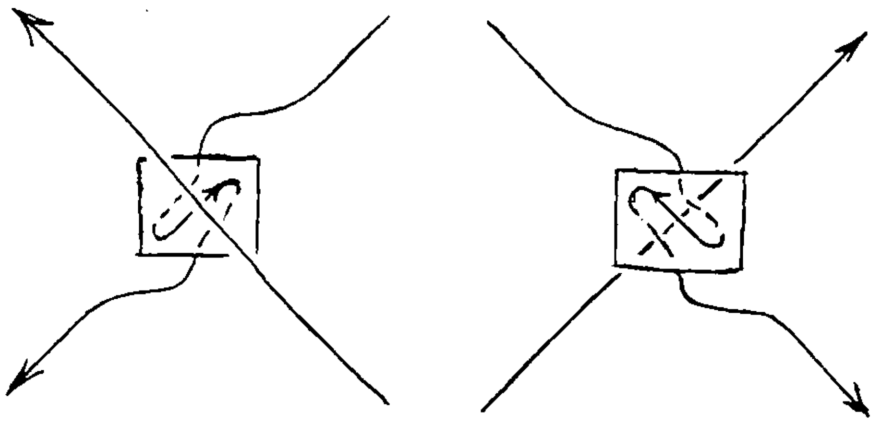

Fig. 29

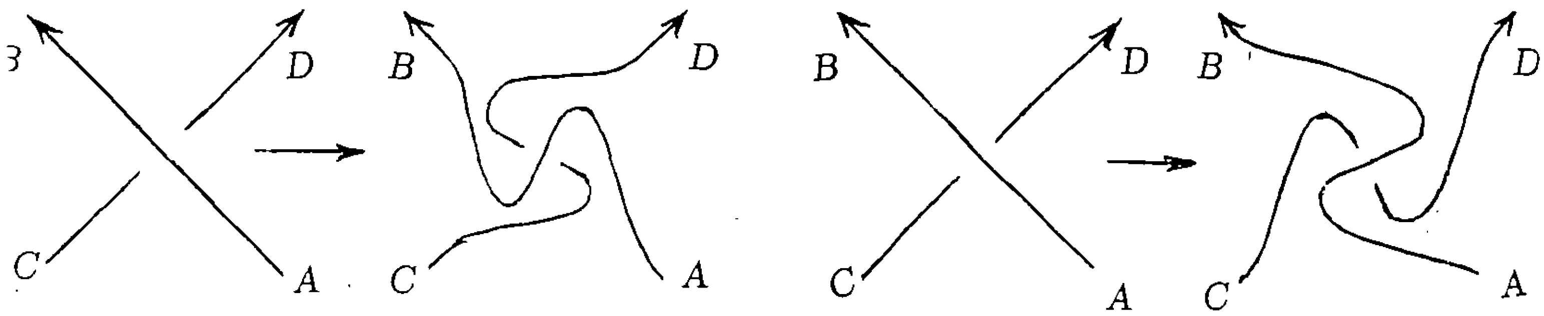


Fig. 30

**5.2.** Now we consider usual (short) transverse knots. Suppose that two short transverse knots $\gamma_1$ and $\gamma_2$ are located, correspondingly, in the halfspaces $x < 0$ and $x > 0$. Choose an extreme right point on the diagram of $\gamma_1$ and an extreme left point on the diagram of $\gamma_2$ (they may be not unique), delete (almost vertical and directed downward) neighborhoods of these points and join the ends of deleted intervals as shown on Fig. 31. We get a new transverse knot which we denote by $\gamma_1 \# \gamma_2$. Notice, that

$$\beta(\gamma_1 \# \gamma_2) = \beta(\gamma_1) + \beta(\gamma_2) + 1.$$

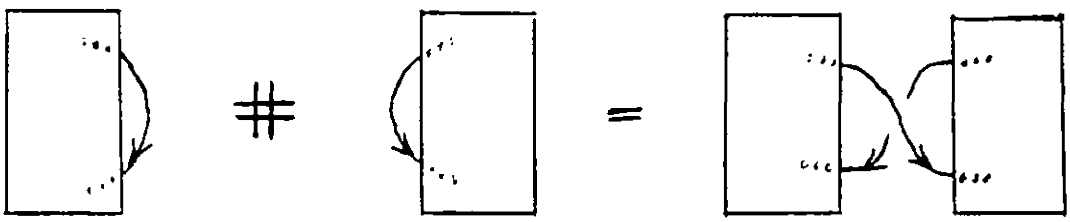

Fig. 31

PROPOSITION 5.2. *The operation $\#$ is well defined and isotopy invariant, that is if $\gamma_1$ is transversely isotopic to $\gamma_1'$ and $\gamma_2$ is transversely isotopic to $\gamma_2'$, then $\gamma_1\#\gamma_2$ is transversely isotopic to $\gamma_1'\#\gamma_2'$.*

PROOF. Shrink $\gamma_2$ and place it in a box as in 5.1. Then we transversely isotope $\gamma_1$ to $\gamma_1'$ and move the box with $\gamma_2$ to the position, where $\gamma_1'$ is to be joined with $\gamma_2'$. While doing this, we avoid forbidden crossings precisely as in 5.1. Then we open the box, expand $\gamma_2$, shrink $\gamma_1'$, close it in the box, and repeat the operation with $\gamma_2$.

With respect to the operation $\#$ the set $\mathcal{T}_0$ of transversely isotopic classes of (short) transverse knots form a semigroup. Obviously, the function $\beta - 1: \mathcal{T}_0 \to \mathbf{Z}$ is additive.

PROPOSITION 5.3. *The semigroup $\mathcal{T}_0$ is commutative.*

PROOF. Denote by $\lambda$ the transverse knot shown on Fig. 32. Obviously, $\lambda$ is the zero of $\#$: $\lambda\#\gamma = \gamma\#\lambda = \gamma$. Hence $\gamma_1\#\gamma_2 = \gamma_1\#(\gamma_2\#\lambda)$. But the last sum is transversely isotopic to the knot shown on Fig. 33. In this knot $\gamma_1$ and $\gamma_2$ are permutable as in 5.1.

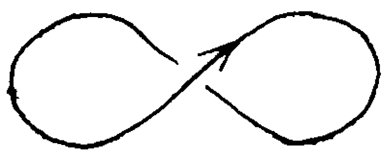

Fig. 32

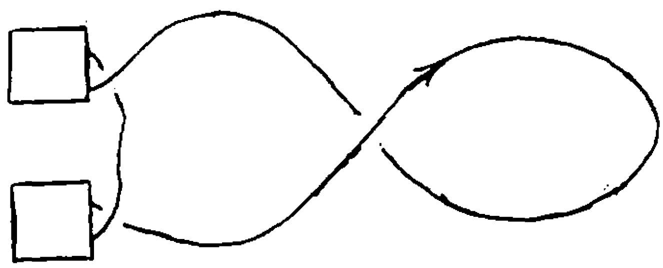

Fig. 33

There is a canonical semigroup homomorphism $\sigma: \mathcal{T} \to \mathcal{T}_0$ (Fig. 34), which is an epimorphism (Fig. 35). Probably, it is an isomorphism, but we have no proof of it. There is also an obvious multiplication $\mathcal{T} \times \mathcal{T}_0 \to \mathcal{T}_0$ with the property $\gamma\gamma_0 = \sigma(\gamma)\gamma_0$.

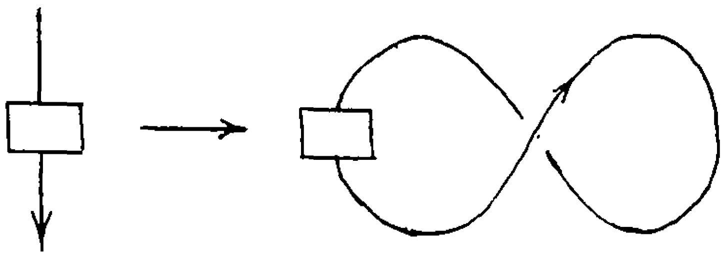

Fig. 34

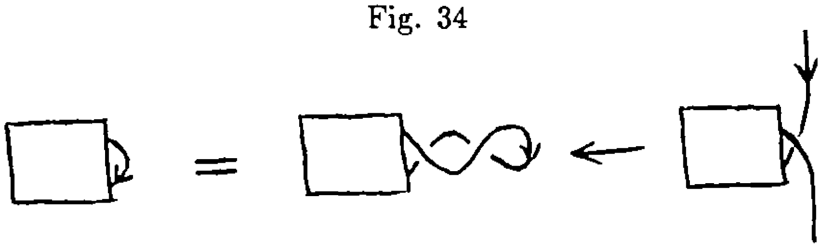

Fig. 35

5.3. The long transverse knot, which plays the role of the zigzag $\zeta(p,q)$, is the knot $\rho(p) = \underbrace{\rho\#\ldots\#\rho}_{p}$, where $\rho$ is the double loop, presented on Fig. 36. Obviously, $\beta(\rho) = -2$, and $\beta(\rho(p)) = -2p$.

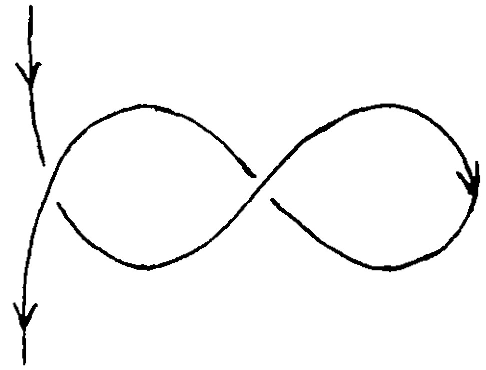

Fig. 36

Let $\mathcal{GT}$ and $\mathcal{GT}_0$ be the Grothendieck groups of the semigroups of long transverse knots and transverse knots; $\mathcal{GK}$ and $\mathcal{GK}_0$ denote those of topological long knots and topological knots.

THEOREM 5.4. *$\mathcal{GT} = \mathcal{GK} \oplus \mathbb{Z}$, the isomorphism being the direct sum of the forgetting homomorphism $\mathcal{GT} \to \mathcal{GK}$ and half of the Bennequin number. $\mathcal{GT}_0 = \mathcal{GK}_0 \oplus \mathbb{Z}$, the isomorphism being the direct sum of the forgetting homomorphism $\mathcal{GT}_0 \to \mathcal{GK}_0$ and half of the Bennequin number minus 1.*

*Moreover, if $\gamma_1$ and $\gamma_2$ are two long transverse knots or two transverse knots, then for sufficiently large p the knot $\gamma_1\#\rho(p)$ is transversely isotopic to $\gamma_2\#\rho(p')$, where*

$$p' = p + \frac{\beta(\gamma_2) - \beta(\gamma_1)}{2}.$$

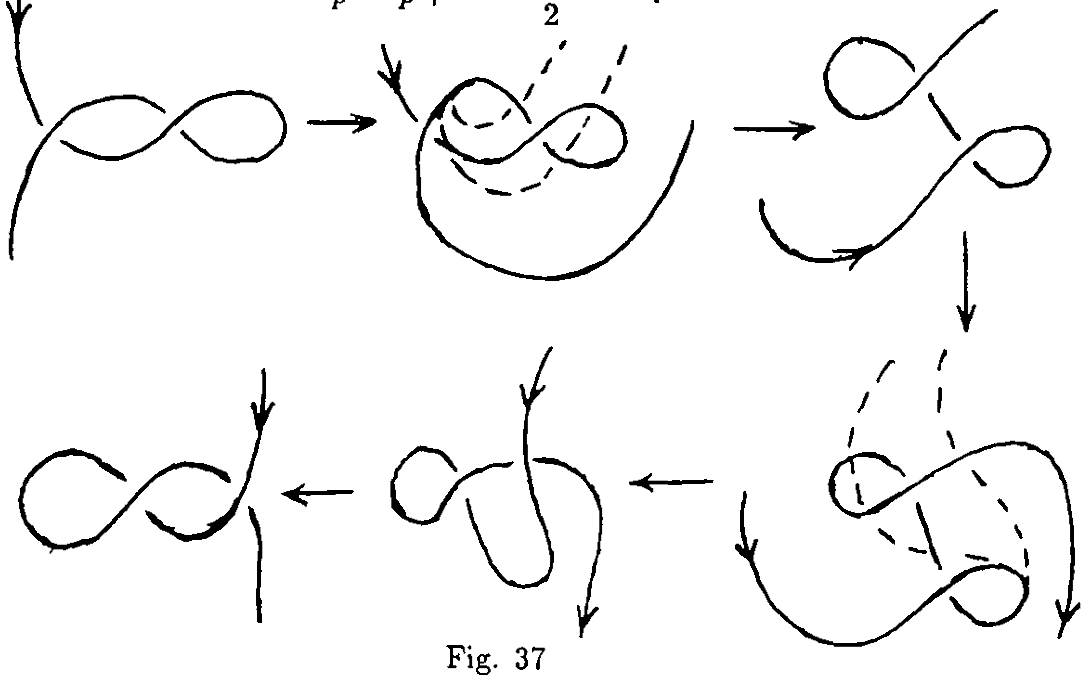

Fig. 37

PROOF (compare to the proof of Theorem 4.4). Take $\gamma_1\#\rho(p)$ with large $p$ and consider the topological isotopy of $\gamma_1$ to $\gamma_2$; the summand $\rho(p)$ allows to make this isotopy transverse. Notice, that we can move double loops from $\rho(p)$ along the knot (as in the proof of Lemma 5.1); on ascending portions of the knot we can decompose the double loop into two single loops (Fig. 37), and we can transfer the double loop from one side of the knot to the other side on the descending portions. Decompose our topological isotopy between $\gamma_1$ and $\gamma_2$ into shorter isotopies passing through at most one of the modifications $R$ and $V$ of Fig. 21; let $\Gamma_i'$ be intermediate topological knots diagrams. We make them diagrams of transverse knots $\gamma_i'$ by replacing forbidden fragments (Fig. 2) with allowed ones as shown on Fig. 38. We claim that for any $i$ the transverse knots $\gamma_i'\#\rho(p), \gamma_{i+1}'\#\rho(p')$ with appropriate $p, p'$ are transversely isotopic; his obviously implies the theorem.

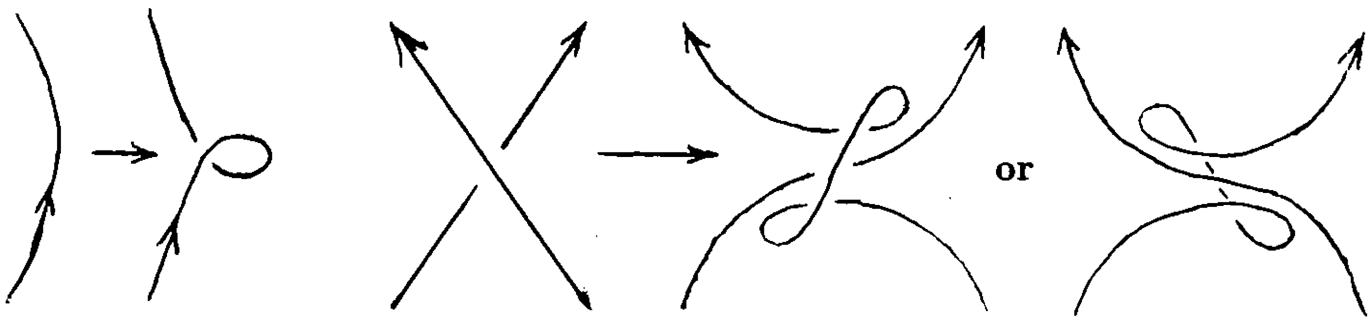


Fig. 38

First we remark that the transverse isotopy type of $\gamma_i'\#\rho(p)$ with large $p$ does not depend on the choice between two possible modifications of wrong crossings of $\Gamma_i'$ (the transverse isotopy involved is similar to the Legendrian isotopy of Fig. 20). Then we provide transverse versions of the moves of Fig. 21. In the transverse case the number of different moves is greater than that in the Legendrian case, because all the strands are oriented. But in many cases transverse moves do not differ from the topological ones; in particular, the moves *R3* and *V2* provide no specific difficulties. The transverse version of the move *V1* is the isotopy of Fig. 37, the transverse versions of the moves *R1–2* are shown on Fig. 39.

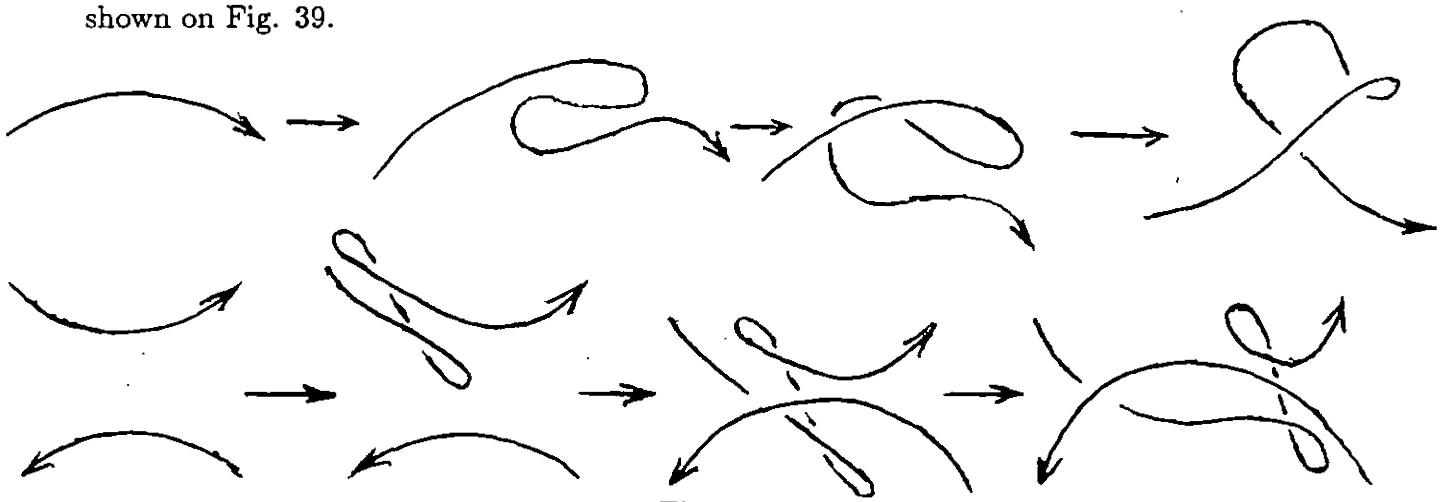

Fig. 39

**5.4.** The definition of (long or short) transverse knot invariants and of invariants of order $\leq n$ is a replica of the corresponding Legendrian definitions (see 4.4). Transverse knots have no invariants of order 0; the Bennequin number is an invariant of order 1.

THEOREM 5.5. *No finite order invariant can distinguish between long or short transverse knots, which are topologically isotopic and have equal Bennequin numbers.*

The proof is the same as that of Theorem 4.5. One needs only to replace $\zeta$ with $\rho$ and notice, that $\rho - 1$, like $\zeta - 1$, belongs to $L_1$.

**5.5.** THEOREM 5.6. *The space of transverse knot invariants of order $\leq n$ is linearly isomorphic to that of framed knots.*

The proof is similar to that of Theorem 4.6. The part of the proof concerning the partial Maslov numbers is not needed because transverse knots do not have Maslov numbers.

## *BIBLIOGRAPHY*


[Ar] V. Arnold. Topological invariants of plane curves and caustics. AMS University Lecture Series, v. 5, Providence, 1994.

[Be] D. Bennequin. Entrelacement et equations de Pfaff. Asterisque, 107-108 (1983), 83-161.

[B-N] D. Bar-Natan. On the Vassiliev knot invariants. Topology, 34 (1995), 423-472.

[El] Ya. Eliashberg. Legendrian and transversal knots in tight contact 3-manifolds. In Topological Methods in Modern Mathematics, 171-193. Publish or Perish, 1993.

[F-W] J. Franks, R. Williams. Braids and the Jones polynomial. Trans. AMS, 303 (1987), 97-108.

[Fr] M. Fraser. Legendrian knots in contact 3-manifolds. Ph. D. Thesis, Stanford, 1993.

[Go 1] V. Goryunov. Vassiliev invariants of knots in $\mathbf{R}^3$ and in a solid torus. Preprint, 1995.

[Go 2] V. Goryunov. Vassiliev type invariants in Arnold's $J^+$-theory of plane curves without direct self-tangencies. Preprint, 1995

[Gr] M. Gromov. Partial differential relations. Springer-Verlag, 1986.

[Gu] M. Gusarov. $N$-equivalent knots and invariants of finite degree. In Adv. of Soviet Math., v.18, 1994.

[Ka] Y. Kanda. On the Thurston-Bennequin invariant of Legendrian knots and non exactness of the Bennequin inequality. Preprint, 1995.

[L-M 1] T.Q.T. Le, J. Murakami. Representation of the category of tangles by Kontsevich's iterated integral. MPIM preprint, 1993.

[L-M 2] T.Q.T. Le, J. Murakami. The universal Vassiliev-Kontsevich invariant for framed oriented links. MPIM preprint, 1994.

[Li] X.-S. Lin. Invariants of Legendrian knots. Preprint, 1995.

[Mo] H. Morton. Seifert circles and knot polynomials. Math. Proc. Camb. Phil. Soc., 99 (1986), 107-109.

[Ru 1] L. Rudolph. Constructions of quasipositive knots and links, II. Contemp. Math., 35 (1984), 485-491.

[Ru 2] L. Rudolph. A congruence between link polynomials. Math. Proc. Camb. Phil. Soc., 107 (1990), 319-327.

[Ru 3] L. Rudolph. Quasipositive annuli (constructions of quasipositive knots and links, IV). J. of Knot Th. and Its Ramif., 1 (1992), 451-466.

[Ru 4] L. Rudolph. An obstruction to sliceness via contact geometry and "classical" gauge theory. Invent. math., 119 (1995), 155-163.

[Sw] J. Swiatkowski. On the isotopy of Legendrian knots. Ann. Glob. Anal. Geom., 10 (1992), 195-207.