GRAPHS ATTACHED TO CERTAIN COMPLEX HYPERBOLIC DISCRETE REFLECTION GROUPS

by

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§0. Introduction

In the last few years, several discrete (complex) reflection groups acting on the unit ball

$$B = \{(x_1,x_2) \in \mathbb{C}^2 \mid |x_1|^2 + |x_2|^2 < 1\}$$

were found and studied. Picard ([11]), Terada ([13]) and Deligne-Mostow ([3]) studied the monodromy groups of the Appell's hypergeomtric differential equations and found 27 groups. Call these as PTDM groups. Hirzebruch ([5]) and Höfer ([7]) calculated Chern numbers of some orbifolds attached to some line arrangements on the projective plane and found several groups, which will be called HH groups. Mostow ([10])

constructed his "remarkable" polyhedra in B and found groups generated by three reflections described by triangular Coxeter graphs. PTDM groups and HH groups are also reflection groups. In this paper, I shall introduce new convention for graphs and shall give pentagonal graphs for all the PTDM groups, which are generated by five reflections. For some HH groups, I shall give graphs which describe how the mirrors of reflections are situated in B and what kinds of Fuchsian groups are acting on the mirrors.

The author would like to thank to Max-Planck-Institut für Mathematik, especially to professor Hirzebruch, where this paper was written for hospitality and excellent working conditions and to professor Im Hof with whom he had valuable discussions in the boat.

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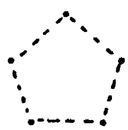
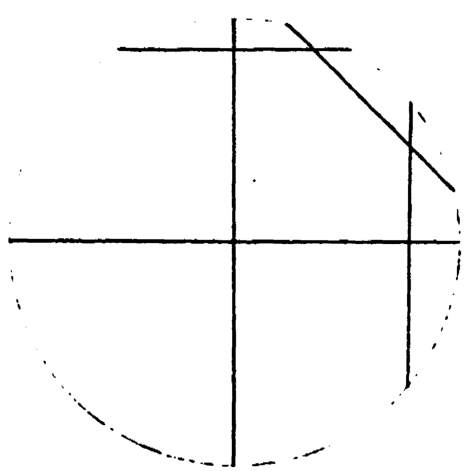
§1. Discrete reflection groups on H_2

Although discrete reflection groups acting on the hyperbolic space are far from being classified, recently, Im Hof ([8]) classified the "complete orthoschemes". I omitt the details and shall give the Coxeter graphs of two dimensional complete orthoschemes.

	two real parameters
•----•—p—•----•	$3 \leq p \leq \infty$, one real parameter
•—p—•—q—•	$3 \leq p,\ q \leq \infty,\ \frac{1}{p} + \frac{1}{q} < \frac{1}{2}$

Here the symbol •----• stands for two reflections which admit a common perpendicular. For example, the arrangement of the mirrors of generating reflections of the first group is as follows :



Two lines, if they meet, are perpendicular.

§2. Complex hyperbolic plane

Let V be a three dimensional vector space over $\mathbb{C}$, on which we fix a Hermitian form H of signature $(+,+,-)$. The unitary group of H operates on the projective space $\mathbb{P}_2 = V - \{0\}/\mathbb{C}^\times$; we denote the resulting group on $\mathbb{P}_2$ by $PU(H)$. Set $\langle p,q \rangle = H(p,q)$ and

$$V^- = \{p \in V \mid \langle p,p \rangle < 0\},$$

$$V^0 = \{p \in V \mid \langle p,p \rangle = 0\},$$

$$V^+ = \{p \in V \mid \langle p,p \rangle > 0\},$$

$$\pi : V - \{0\} \to \mathbb{P}_2 \quad \text{canonical map},$$

$$B = \mathbb{C}H_2 = \pi(V^-) \quad \text{ball},$$

$$\mathrm{Aut}(B) = \text{the restriction of } PU(H) \text{ to } B.$$

We regard B as a Riemannian manifold equiped with the Aut(B)-invariant metric. Let us choose basis of V so that

$$H = -|x_0|^2 + |x_1|^2 + |x_2|^2,$$

then B is the unit ball in §0. The isotropy subgroup of Aut(B) at the origin is the unitary group $U(2)$. The ball has two kinds of totally geodesic real two dimensional surfaces :

$$H_2 = \pi\{(1,x_1,x_2) \in B \mid x_1,x_2 \in \mathbb{R}\},$$

$$\mathbb{C}H_1 = \pi\{(x_0,x_1,x_2) \in B \mid x_1 = 0\},$$

that is, two dimensional real hyperbolic space and the one dimensional complex hyperbolic space. In the sequel we shall see that the ball B looks like an amalgamation of the both.

§3. Lines in B

A vector $r \in V^+$ (resp. V^-, V^0) defines a line

$$r^\perp = \pi\{x \in V - \{0\} \mid \langle x,r\rangle = 0>$$

through (resp. outside, touching) B. An element of Aut(B) which fixes the line $r^\perp$ is represented by

$$(3.1) \qquad x \to x + (\xi - 1)\frac{\langle x,r\rangle}{\langle r,r\rangle}\, r\ , \quad |\xi| = 1, \quad r \in V^\pm,$$

$$(3.2) \qquad x \to x + \sqrt{-1}\, k \langle x,r\rangle\, r\ , \quad k \in \mathbb{R}, \quad r \in V^0.$$

In the sequel, by a line in B we mean the intersection of $\bar{B}$ and a line $r^\perp$ for some $r \in V^+$. For two distinct lines in B, represented by r_1 and r_2, introduce the invariant

$$\rho(r_1,r_2) = \frac{|\langle r_1,r_2\rangle|}{\sqrt{\langle r_1,r_1\rangle\langle r_2,r_2\rangle}}\ .$$

The two lines meet in B, meet on ∂B or admits a (unique) perpendicular line in B if and only if $\rho(r_1,r_2) < 1$, $= 1$ or > 1. We want to consider line arrangements in B. Let $\bigcirc$ denote a line in B. For two distinct lines, represented by r_1 and r_2, we symbolize as follows :

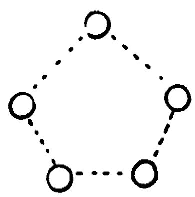
 $\rho(r_1,r_2) = 0$, perpendicular

 $0 < \rho(r_1,r_2) < 1$

 $\rho(r_1,r_2) = 1$

 $\rho(r_1,r_2) > 1$.

We are particularly interested in the following arrangements :

(3.3)  ,  ,  .

Up to Aut(B), these arrangements have two real parameters as in §1. (If one segment becomes bold faced then the arrangements fails one parameter.)

§4. Reflections in Aut(B)

Definition. A transformation A in Aut(B) is called a reflection if A is represented by (3.1) for some non-trivial root of unity ξ, say $1^{1/p}$, and $r \in V^{+}$. It is denoted by p. A reflection group in Aut(B) is a group generated by reflections.

Definition. An element $A \in$ Aut(B) is called an E-reflection if A is represented by (3.1) for some non-trivial root of unity ξ^{-}, say $1^{1/p}$, and $r \in V^{-}$. It is denoted by $-p$. An element $A \in$ Aut(B) is called a P-reflection if A is represented by (3.2) for some $k \in \mathbb{R}^{\times}$ and $r \in V^{0}$. It is denoted by ∞. In any three cases, the fixed locus of A is called the mirror of A.

Let A and B be two reflections in Aut(B) such that the group $\langle A,B \rangle$ generated by A and B is discrete. If the mirrors of A and B are situated as ○——○ then $\langle A,B \rangle$ is a unitary reflection group, which will be briefly reviewed in §5. If they are situated as ○━━○ then $\langle A,B \rangle$ is a subgroup of the parabolic subgroup of Aut(B), which we also review in §6. If they are situated as ○---○ then $\langle A,B \rangle$ acts as a Fuchsian group on the common perpendicular line in B. If it is a Fuchsian group of the first kind (i.e.

finite covolume) then it must be a triangle group. Let p and r be the order of the reflection A and B and (p,q,r) be the type of the triangle group. We shall symbolize the group $\langle A,B\rangle$ by

$$(p) \overset{2q}{- - -} (r) \qquad \frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1, \quad 2 \leq q \leq \infty.$$

We shall use this symbol not only for two reflections but also for E and P-reflections. We regard a line perpendicular to a mirror point as a line passing through the point. Thus for p, r $(2 \leq |p|,|r| \leq \infty)$, the group $(p) \overset{2q}{- - -} (r)$ acts on the perpendicular line as a Fuchsian triangle group of type $(|p|, q, |r|)$.

§5. Unitary reflection groups (c.f. [2])

A unitary reflection group generated by two reflections, of order p and r, has a presentation

$$
\begin{aligned}
&R_1{}^p = R_2{}^r = 1, \\
(5.1) \qquad & \\
&R_1R_2R_1 \cdots = R_2R_1R_2 \cdots
\end{aligned}
$$

with $2q$ R's on each side, where $2q$ is a positive integer generator than 2 and

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

and q is an integer if $p \neq r$. It is symbolized by the graph

$$\textcircled{p} \overset{2q}{\text{———}} \textcircled{r} \; .$$

Let $G \subset U(2)$ be a two dimensional unitary reflection group and $\overline{G} \subset PGL(2)$ its propectified group. The group $\overline{G}$ is a polyhedral group. For a given polyhedral group $\langle p,q,r\rangle$, there is a unique maximal unitary reflection group G such that $\overline{G} = \langle p,q,r\rangle$. This group is denoted by $\langle p,q,r\rangle_s$, where

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1,$$

and has a presentation

$$
(5.2) \qquad
\begin{aligned}
&R_1{}^p = R_2{}^q = R_3{}^r = 1, \\
&R_1R_2R_3 = R_2R_3R_1 = R_3R_1R_2.
\end{aligned}
$$

The center is of order $2s$ and generated by $R_1R_2R_3$;

$$1 \to \mathbb{Z}_{2s} \to \langle p,q,r\rangle_s \to \langle p,q,r\rangle \to 1.$$

One can find a graph of the group $\langle p,q,r\rangle_s$ in some literature (c.f. [1]) but we do not use it. The group $\langle p,q,r\rangle_s$ has a subgroup with graph $p \overset{2q}{\rule[0.5ex]{2em}{0.4pt}} r$ and the group is generated by the subgroup and the center. Thus if we regard the group a subgroup of $\mathrm{Aut}(B)$, the group is symbolized by

$$-2s \qquad p \overset{2q}{\rule[0.5ex]{2em}{0.4pt}} r \ .$$

If one regard the group as a transformation group on $\mathbb{C}^2$ then the regular orbit is $\mathbb{C}^2$ minus three lines passing through a point in common.

The Shephard-Todd symbol ([12]) of these groups are as follows :

$$\langle s,2,2\rangle_s = G(2s,2,2), \quad \langle 3,3,2\rangle_6 = (7),$$

$$\langle 4,3,2\rangle_{12} = (11), \quad \langle 5,3,2\rangle_{30} = (19).$$

§6. Parabolic reflection group in Aut(B) ([4])

We shall study here discrete reflection groups in the isotropy subgroup of Aut(B) at $P \in \partial B$. To do so we choose the Hermitian form H so that B is represented by

$$\{[z,u,1] \in \mathbb{P}_2 \mid \operatorname{Im} z - |u|^2 > 0\},$$

and $P = [1,0,0] \in \partial B$. Let $\mathfrak{p}$ be the parabolic subgroup of Aut(B) corresponding to P and C its center. Then

$$\mathfrak{p} = \left\{ [\mu,a,r] = \begin{pmatrix} 1 & 2\sqrt{-1}\mu\bar{a} & r+\sqrt{-1}|a|^2 \\ 0 & \mu & a \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} \mu, a \in \mathbb{C}, \quad r \in \mathbb{R}, \\ |\mu| = 1 \end{array} \right\},$$

$$C = \{[1,0,r] \mid r \in \mathbb{R}\}.$$

Let π be the homomorphism of $\mathfrak{p}$ onto the one dimensional Euclidean motion group $U(1) \ltimes \mathbb{C}$ given by

$$[\mu,a,r] \to \begin{pmatrix} \mu & a \\ 0 & 1 \end{pmatrix},$$

then we have

$$1 \to C \to \mathfrak{p} \xrightarrow{\pi} U(1) \ltimes \mathbb{C} \to 1.$$

Let $G \subset \mathfrak{p}$ be a discrete reflection group of locally finite volume at P then $\bar{G} = \pi(G)$ is a crystallographic group in $U(1) \ltimes \mathbb{C}$ and G has a non-trivial center :

$$1 \to \mathbb{Z} \to G \xrightarrow{\pi} \bar{G} \to 1.$$

For a given crystallographic group $\bar{G} = \langle p,q,r\rangle (2 \le p,q,r < \infty,$ $1/p + 1/q + 1/r = 1)$ or $\langle 2,2,2,2 \,;\, \tau\rangle (\mathrm{Im}\,\tau > 0)$, there is a unique maximal discrete reflection group G denoted by $\langle p,q,r\rangle_\infty$ or $\langle 2,2,2,2 \,;\, \tau\rangle_\infty$, respectively, such that $\pi(G) = \bar{G}$. The group $\langle p,q,r\rangle_\infty$ has a presentation (5.2) and the group $\langle 2,2,2,2 \,;\, \tau\rangle_\infty$ has a presentation

$$R_1{}^2 = R_2{}^2 = R_3{}^2 = R_4{}^2 = 1,$$

$$[R_1R_2R_3R_4, R_k] = 1, \quad k = 1, \cdots, 4.$$

A discrete group in $\mathfrak{p}$ generated by two reflection of order p and r has a presentation (5.1) and symbolized by the graph

$$\textcircled{p}\overset{2q}{\text{———}}\textcircled{r}$$

where $1/q = 1/p + 1/r - 1$. The group $\langle p,q,r\rangle_\infty$ has a subgroup with graph $\textcircled{p}\overset{2q}{\text{———}}\textcircled{r}$ and is generated by the subgroup and the center. Thus the group has the following diagram :

$$\textcircled{\infty} \qquad \textcircled{p}\overset{2q}{\text{———}}\textcircled{r}$$

The orbit space $B/G \cup \{P\}$ added by a point P is a domain in $\mathbb{C}^2$ and the discriminant consists of three (resp. four) lines passing through a point in common for $G = \langle p,q,r\rangle_\infty$ (resp. $\langle 2,2,2,2\ ;\ \tau\rangle_\infty$).

In [4], these groups were denoted as follows :

$$\langle 2,4,4\rangle_\infty = \Gamma_{IV}(4,0\ ;\ 0),$$

$$\langle 3,3,3\rangle_\infty = \Gamma_{III}(3\ ;\ 0),$$

$$\langle 2,3,6\rangle_\infty = \Gamma_{VI}(6\ ;\ 0),$$

$$\langle 2,2,2,2\ ;\ \tau\rangle_\infty = \Gamma_{II}(\tau\ ;\ 2,0,0\ ;\ 0).$$

§7. PTDM groups (c.f. [11], [13] and [3])

Let x_1, x_2, x_3 be homogeneous coordinates of $\mathbb{P}_2$ and set $x_0 = 0$. Put

$S(ij) = \{x_i = x_j\}$, $\{i,j\} \subset \{0,1,2,3\}$,

$S(ijk) = \{x_i = x_j = x_k\}$, $\{i,j,k\} \subset \{0,1,2,3\}$,

$\mathcal{D} = \mathbb{P}_2 - A$, $A = \bigcup_{i,j} S(ij)$,

$\mathbb{P}_2{}^\sigma$: the manifold obtained from $\mathbb{P}_2$ by blowing up the four points $S(ijk)$,

$\tau : \mathbb{P}^\sigma \to \mathbb{P}_2$ the natural map,

$S(ij)^\sigma \subset \mathbb{P}^\sigma$ proper transform of $S(ij)$,

$S(ijk)^\sigma = \tau^{-1}S(ijk)$,

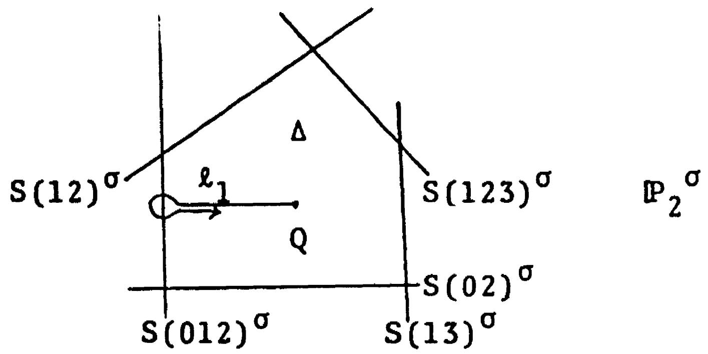
$\Delta = \{[x_1,x_2,1] \in \mathcal{D} \mid x_1, x_2 \in \mathbb{R},\ x_2 > 0,\ x_1 - x_2 > 0,$
$x_1 < 1\}$.

There is a Fuchsian differential equation $E(\lambda)$ defined on $\mathbb{P}_2$, with regular singularity along A, called Appell's hypergeometric differential equations, where $\lambda = (\lambda_0,\lambda_1,\lambda_2,\lambda_3)$ are four complex parameters. The equation defines the projective monodromy representation

$$\rho : \pi_1(\mathcal{E},Q) \to PGL(3,\mathbb{C}). \qquad (Q \in \mathcal{D})$$

For certain values of λ's, the image of ρ (called the projective monodromy group of $E(\lambda)$) is a discrete subgroup of $PU(H)$. There are just 27 such groups, call these as PTDM groups. They are reflection groups in $Aut(B)$ ([15]).

Fix a point Q in Δ and regard $\Delta \subset \mathcal{D} \subset \mathbb{CP}_2{}^{\sigma}$. Let ℓ_1 (resp. $\ell_2, \ell_3, \ell_4, \ell_5$) be a loop with the base point Q going on Δ near to the line $L = S(012)^{\sigma}$ (resp. $S(02)^{\sigma}$, $S(13)^{\sigma}$, $S(123)^{\sigma}$, $S(12)^{\sigma}$), turning around L in $\mathcal{D}$ and coming back on Δ. One may note that $\ell_1, \cdots, \ell_5$ generate $\pi_1(\mathcal{D},Q)$.



Set

$$I(1) = -(\lambda_0 + \lambda_1 + \lambda_2 - 2),$$

$$I(2) = \lambda_0 + \lambda_2 - 1,$$

$$I(3) = \lambda_1 + \lambda_3 - 1,$$

$$I(4) = -(\lambda_1 + \lambda_2 + \lambda_3 - 2),$$

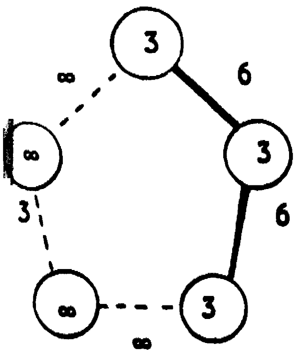
$$I(5) = \lambda_1 + \lambda_2 - 1$$

then, in view of [13], we see that $\rho(\ell_i)$ $(i=1,\cdots,5)$ is a reflection (resp. P-reflection, E-reflection) if $I(i) > 0$ (resp. 0, < 0). Let $F(i)$ be the mirror of $\rho(\ell_i)$. Since Δ is simply connected, we can conclude that $F(1),\cdots,F(5)$ form a (may be degenerated) pentagon (3.3). The intersection of this pentagon and a real geodesic surface H_2 is a pentagon in the (may be degenerated) figure in §1.

For the explicit values of $\lambda_1,\cdots,\lambda_5$, we refer to [13]. We are now ready to give graphs of PTDM groups. Unlike the case of real reflection groups and like the case of unitary reflection groups, one group may have different diagrams.

§8. Coxeter graphs of PTDM groups

We shall give Coxeter graphs for 27 PTDM groups. All conventions are explained in §4, 5 and 6. The number under each pentagon is the group number (Terada number) used in [13].



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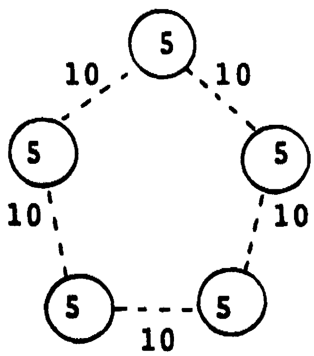

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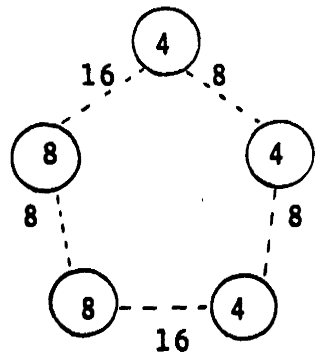

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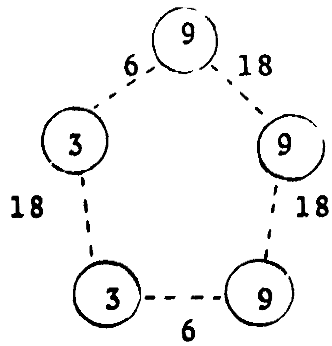

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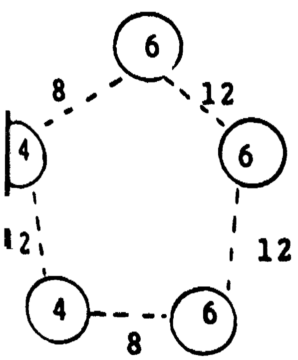

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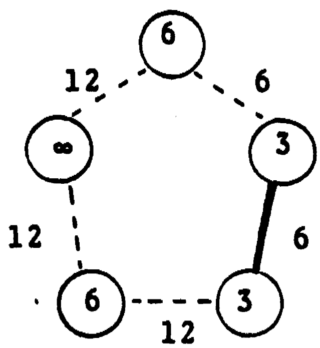

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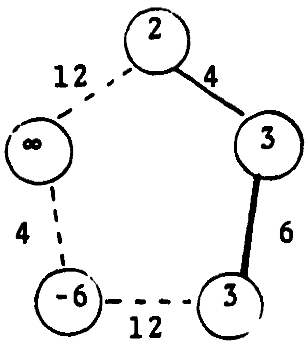

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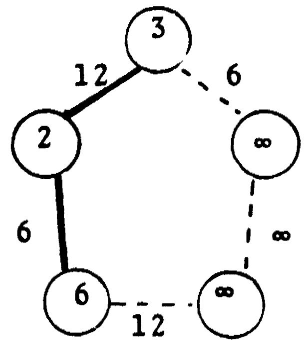

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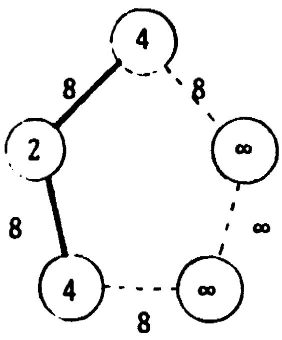

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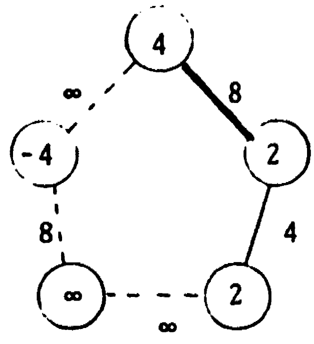

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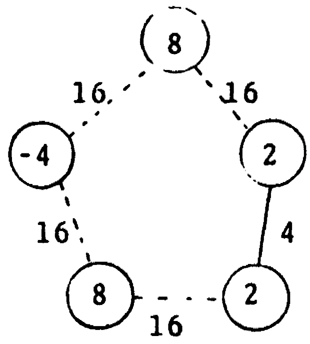

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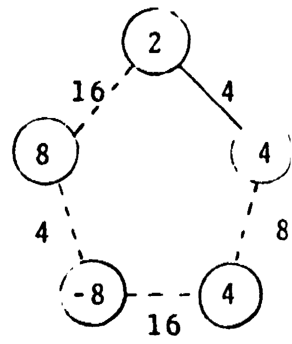

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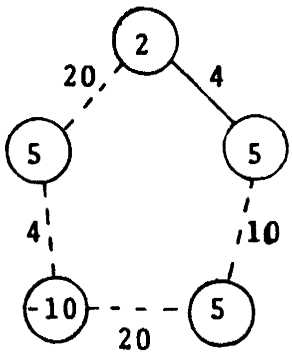

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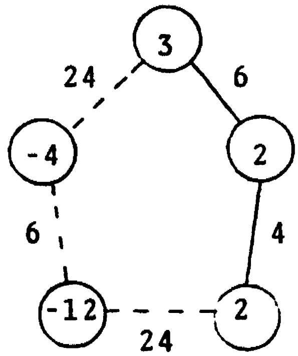

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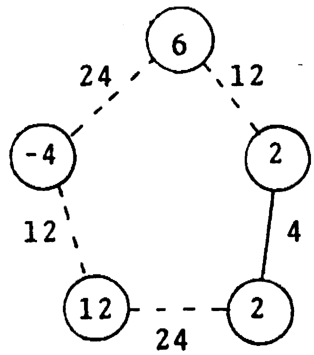

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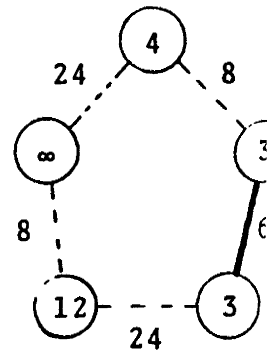

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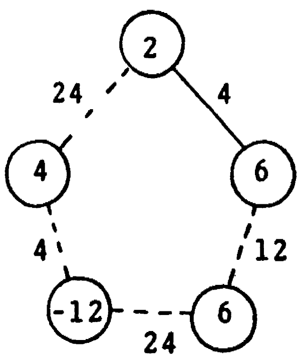

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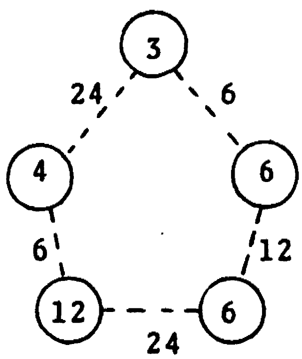

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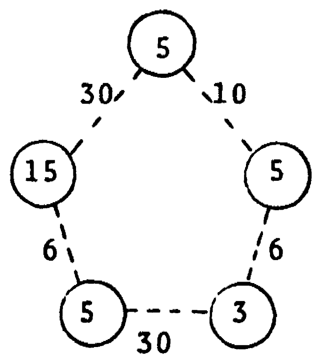

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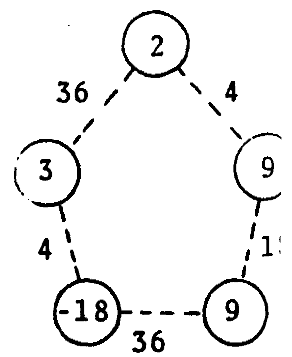

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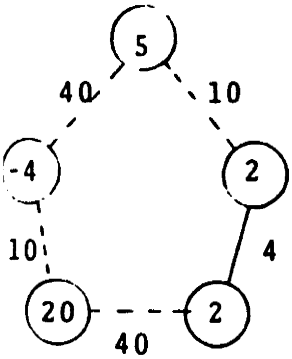

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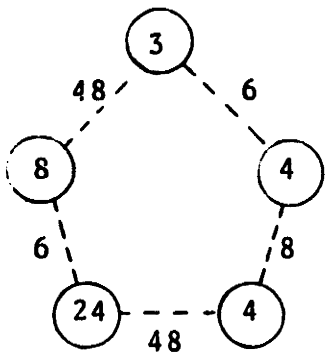

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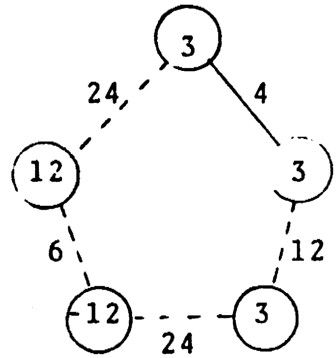

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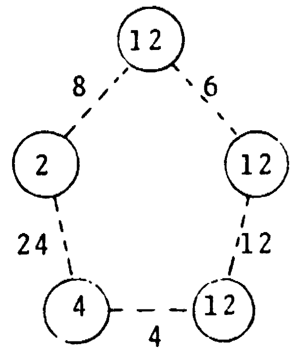

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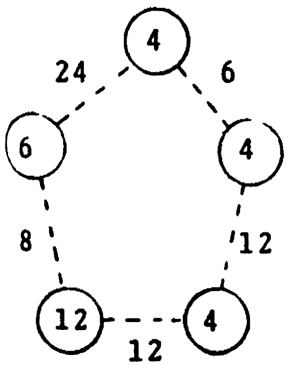

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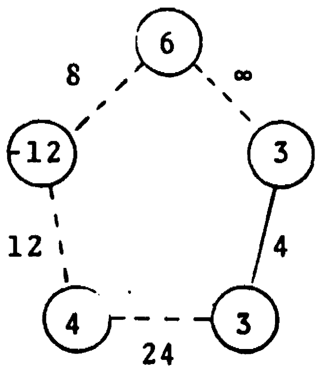

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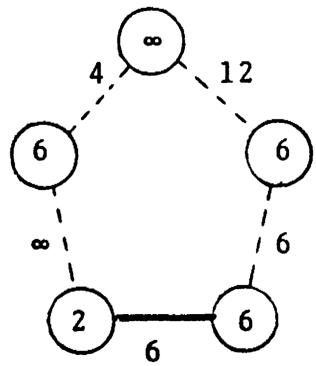

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Note. The groups 3 and 19 are used in [14] and the group 1 is treated in [11].

§9. A two dimensional complex crystallographic group

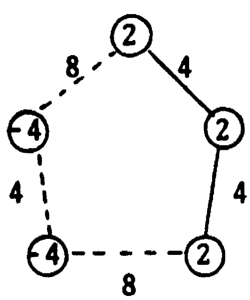
Let $\mathrm{Aut}(\mathbb{C}^2)$ be the complex Euclidean motion group $U(2) \ltimes \mathbb{C}^2$ acting on $\mathbb{C}^2$. We call an element of $\mathrm{Aut}(\mathbb{C}^2)$ a reflection or an E-reflection if it is conjugate to $\begin{bmatrix}1 & \\ & \mu\end{bmatrix}$ or $\begin{bmatrix}\mu & \\ & \mu\end{bmatrix}$ $(\mu = 1^{1/p})$, and symbolize as p or -p, respectively. Note that this definition is essentially the same as we did in §4.

There exists a crystallographic group $\Gamma_0 \subset (\mathrm{Aut}(\mathbb{C}^2)$ such that the orbit space is $\mathbb{P}_2$ and the regular orbit is $\mathcal{D}$. The group Γ_0 admits an exact sequence

$$1 \to \mathbb{Z} \oplus \mathbb{Z} \to \Gamma_0 \to \langle 2,2,2\rangle_2 \to 1.$$

This group is denoted by $(4.2)_1$ in [9], and generating reflections and their relations are known.

By using E-reflections also, by the analogous arguments and conventions as in §4, 5 and 7, we can symbolize the group Γ_0 by the pentagonal graph :

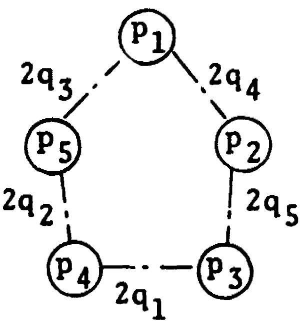


Here the subdiagram

$$(p) \overset{2q}{- - - -} (r) \qquad \frac{1}{|p|} + \frac{1}{q} + \frac{1}{|r|} = 1$$

stands for the subgroup of $\mathrm{Aut}(\mathbb{C}^2)$, which is isomorphic to the one dimensional crystallographic group $\langle |p|, |q|, |r| \rangle$, acting on the perpendicular affine line.

The group Γ_0 appears as a quotient of the PTDM groups Γ of Terada number 3,5,9,10,11,12,15,17,24. Because each of them has a diagram



—·— stands for one of the three kinds.

such that $|p_1|$, $|p_2|$, $|p_3|$, q_2, q_4, q_5 are even and $|p_4|$, $|p_5|$, q_1, q_3 are divisible by 4. The kernel K of the natural homomorphism $\Gamma \to \Gamma_0$ is again a reflection group in $\mathrm{Aut}(B)$, requires infinitely many generating reflections, whose arrangement of mirrors are same as Γ and the orders are half or one fourth of those of Γ. Galois correspondence may be put into the following diagram.

$$\begin{tikzcd}
B - \{\text{mirros of } \Gamma\} \arrow[d] \arrow[r, leftrightarrow] & 1 \arrow[d, dash] & \\
\mathbb{C}^2 - \{\text{mirrors of } \Gamma_0\} \arrow[d] \arrow[r, leftrightarrow] & K \arrow[d, dash] & \Gamma_0 = \Gamma/K \\
\mathcal{D} \arrow[r, leftrightarrow] & \Gamma &
\end{tikzcd}$$

This situation is very much related to the Hirzebruch's construction [6] of surfaces covered by a ball, which are finite ramified coverings of abelian surfaces.

§10. Hessian configuration and HH groups

Hessian configuration $A \subset \mathbb{P}_2$ consists of the following 12 lines :

$$A_{oj} : x_j = 0 \qquad i = 1,2,3,$$

$$A_{ij} : \omega^i x_1 + \omega^j x_2 + x_3 = 0 \quad i,j = 1,2,3,$$

where $\omega = \exp 2\pi\sqrt{-1}/3$. Put

$\mathcal{D}' = \mathbb{P}_2 - A,$

$E = \{e_{ij} = A_{ij} \cap A_{oj} \mid i,j=1,2,3\}$ the nine 4-fold points of A,

$\mathbb{P}_2{}^\sigma$: the manifold obtained from $\mathbb{P}_2$ by blowing up E,

$\tau : \mathbb{P}_2{}^\sigma \to \mathbb{P}_2$ the natural map,

$A^\sigma_{k\ell} \subset \mathbb{P}_2{}^\sigma$: proper transformation of $A_{k\ell}$ $(k,\ell=0,1,2,3)$,

$E^\sigma = \tau^{-1}E, \quad e^\sigma_{ij} = \tau^{-1}e_{ij}.$

It is known ([5], [7]) that the universal branched covering of $\mathbb{P}_2{}^\sigma$ with ramification indices n on $A^\sigma_{k\ell}$ $(k,\ell=0,1,2,3)$ and m on E^σ is biholomorphic to the ball B for $(n,m) = (4,2)$, (3.3) and $(2,\infty)$. Let $\Gamma(n,m) \subset \mathrm{Aut}(B)$ be the Deck transformation group. These three are the HH groups corresponding to the Hessian configuration. These are reflection groups generated by eleven reflections. I can not give any Coxeter graphs for them but shall describe the arrangements of mirrors.

Let Δ be a triangle

$$\{[x_1,x_2,1] \in \mathcal{D}' \mid x_1,x_2 \in \mathbb{R},\ \ x_1<0,\ x_2<0,\ x_1+x_2+1>0\}.$$

Regarding $\Delta \subset \mathcal{D} \subset P_2{}^{\sigma}$, the triangle is bounded by five lines A^{σ}_{02}, e^{σ}_{30}, A^{σ}_{33}, e^{σ}_{03} and A^{σ}_{01} in this order. Thus just as we did in §7, we can define loops $\ell_1,\cdots,\ell_5$, with base point at $Q \in \Delta$, turning around the above five lines, the representation

$$\rho : \pi_1(\mathcal{D}',Q) \to \Gamma(n,m)$$

and the reflection or P-reflection $\rho(\ell_i)$. The mirrors $F(i)$ of $\rho(\ell_i)$ $(i=1,\cdots,5)$ again form a pentagon (3.3). For $(n,m) = (2,\infty)$, the pentagon degenerates to a triangle.

Unlike the PTDM groups, $\rho(\ell_i)$ $(i=1,\cdots,5)$ do not generate the group $\Gamma(n,m)$. The stabilizer $\Gamma(n,m)_i$ of $F(i)$ in $\Gamma(n,m)$ is a reflection group containing $\rho(\ell_{i+2})$ and $\rho(\ell_{i+3})$, regarding $6 = 1$. If $F(i)$ is a line in B then $\Gamma(n,m)_i$ acts as a Fuchsian group of the first kind on $F(i)$ and if $F(i)$ is a point on ∂B then $\Gamma(n,m)_i$ is isomorphic to the group $\langle 2,2,2,2;\tau\rangle_{\infty}$

for som τ. One can easily check that the five groups $\Gamma(n,m)_i$ $(i=1,\cdots,5)$ generates the whole group. Thus we can say that the group $\Gamma(n,m)$ is determined by the skelton pentagon and the five Fuchsian groups generated by elliptic transformation ; for $(n,m) = (2,\infty)$, three Fuchsian and two parabolic reflection groups.

Let us study the group $\Gamma(n,m)_i$. The Hessian configuration A is invariant under the Hessian group $G_{216} \subset PGL(3)$ generated by 3-fold reflections with mirror $A_{k\ell}$ $(k,\ell=0,1,2,3)$ ([12]). This implies the existence of a reflection group $\tilde{\Gamma}(n,m)$ in Aut(B) which admits an exact sequence

$$1 \to \Gamma(n,m) \to \tilde{\Gamma}(n,m) \to G_{216} \to 1.$$

This tells us the following. Let $\langle p,q,r\rangle$ $(1/p+1/q+1/r = 1)$ be a triangle group such that p and q are multiples of 3 then there exists, uniquely, a subgroup of index 3 with sign $(p/3,q,q,q,r/3)$, which we shall denote by $\langle p/3;q,q,q;r/3\rangle$. Then if $\bar{\Gamma}(n,m)_i$ denotes the restriction of $\Gamma(n,m)_i$ on the line $f(i)$, we have

$$\bar{\Gamma}(4,2)_1 \cong \bar{\Gamma}(4,2)_3 \cong \bar{\Gamma}(4,2)_5 \cong \langle 4;2,2,2;4\rangle,$$

$$\bar{\Gamma}(4,2)_2 \cong \bar{\Gamma}(4,2)_4 \cong \langle 4;4,4,4;1\rangle,$$

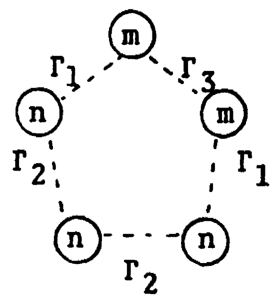
$$\bar{\Gamma}(3,3)_1 \cong \bar{\Gamma}(3,3)_3 \cong \bar{\Gamma}(3,3)_5 \cong \langle 3;3,3,3;3\rangle,$$

$$\bar{\Gamma}(3,3)_2 \cong \bar{\Gamma}(3,3)_4 \cong \langle 3;3,3,3;1\rangle,$$

$$\bar{\Gamma}(2,\infty)_1 \cong \bar{\Gamma}(2,\infty)_3 \cong \bar{\Gamma}(2,\infty)_5 \cong \langle 2;\infty,\infty,\infty;2\rangle,$$

$$\Gamma(2,\infty)_2 \cong \Gamma(2,\infty)_4 \cong \langle 2,2,2,2;\omega\rangle_\infty.$$

If $(a) \overset{\langle\bar{a};\bar{b},b,b;c\rangle}{\text{- - - - - - - -}} (b)$ denotes a reflection group in Aut(B) which is isomorphic to $\langle a;b,b,b;c\rangle$, with two specified , on the common perpendicular, reflection of order a and b, and $(2) \xrightarrow{\langle 2,2,2,2;\omega\rangle_\infty} (2)$ denotes the parabolic group $\langle 2,2,2,2;\omega\rangle_\infty$ with two specified reflection, then the above result can be abriviated by the following diagrams :

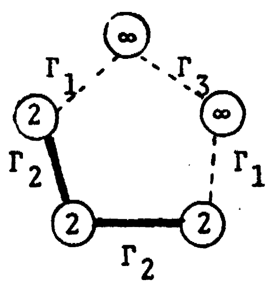


$(m,n) = (4,2),\quad (3,3),$

$\Gamma_1 = \langle\bar{n}\ ;\ \bar{m},\ m,\ m\ ;\ n\rangle,$

$\Gamma_3 = \langle n\ ;\ \bar{m},\ \bar{m},\ m\ ;\ n\rangle,$

$\Gamma_2 = \langle\bar{n}\ ;\ \bar{n},\ n,\ n\ ;\ 1\rangle,$



$\Gamma_1 = \langle\bar{2}\ ;\ \bar{\infty},\ \infty,\infty\ ;\ 2\rangle,$

$\Gamma_3 = \langle 2\ ;\ \bar{\infty},\ \bar{\infty},\ \infty\ :\ 2\rangle,$

$\Gamma_2 = \langle 2,\ 2,\ 2,\ 2\ ;\ \omega\rangle_\infty.$

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