The Hecke algebra on the cohomology of $\Gamma_0(\mathfrak{p}_0)$

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1. Introduction.

Let p_0 be a prime, $p_0 > 3$ and $\Gamma_0(p_0)$, $\Gamma_1(p_0)$, as usual, the congruence subgroups of $\Gamma = PSL_2(\mathbb{Z})$.

$$\Gamma_0(p_0) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, c \equiv 0 \bmod p_0 \right\}, \quad \Gamma_1(p_0) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p_0) \,\middle|\, d \equiv 1 \bmod p_0 \right\}.$$

Denote

$$\Delta = \left\{ r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z}, gcd(a,b,c,d) = 1, det(r) \not\equiv 0 \bmod p_0 \right\}$$

$$\Delta_0 = \left\{ r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \,\middle|\, c \equiv 0 \bmod p_0 \right\}, \quad \Delta_1 = \left\{ r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_0 \,\middle|\, d \equiv 1 \bmod p_0 \right\}.$$

with $\Delta_1 \subset \Delta_0 \subset \Delta$ and $\Delta_0/\Delta_1 \cong (\mathbb{Z}/p_0)^*$. Let $R = \mathbb{Z}[\frac{1}{6}]$. We consider the following R-module $M_n = \left\{ \sum_{v=0}^{n} a_v x^v y^{n-v} \,\middle|\, a_v \in R \right\}$. The semigroup Δ acts on M_n via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^v y^{n-v} = (ax+cy)^v (bx+dy)^{n-v}$$

Let $\eta : \Gamma_0(p_0)/\Gamma_1(p_0) \cong (\mathbb{Z}/p_0)^* \to R^*$ be the Legendre-symbol. We extend η to Δ_0 such that η acts trivially on Δ_1, i.e. η is a character from Δ_0/Δ_1 to R^*. Denote by R_η the R-module of rank 1 with a Δ_0-operation given by $s_0.1 = \eta(s_0) \cdot 1$, $\forall s_0 \in \Delta_0$. Set $M_{n,\eta} = M_n \otimes R_\eta$. This is then a $R[\Delta_0]$-module. The goal of the present paper is to investigate the Hecke algebra on the cohomology group $H^*(\Gamma_0(p_0), M_{n,\eta})$. Let $S_k(\Gamma_0(p_0), \eta)$, as usual, be the cusp forms with the weight k. Then the Eichler-Shimura theorem says that the following sequence

$$0 \to S_{n+2}(\Gamma_0(p_0), \eta) \oplus \overline{S_{n+2}(\Gamma_0(p_0), \eta)} \to H^1(\Gamma_0(p_0), M_{n,\eta} \otimes \mathbb{C}) \to$$

$$\xrightarrow{r} \bigoplus_s H^1(\Gamma_0(p_0)_s, M_{n,\eta} \otimes \mathbb{C}) \to 0$$

is exact, where s runs over cusps of $\Gamma_0(p_0)$ and $\Gamma_0(p_0)_s := \{ r \in \Gamma_0(p_0) \mid r.s = s \} = < T_s >$ is an infinite cyclic group. It is well known that $\Gamma_0(p_0)$ has two cusps $0, \infty$. The dimension of

$$H^1(\Gamma_0(p_0)_s, M_{n,\eta} \otimes \mathbb{C}) \cong M_{n,\eta}/(1-T_s)M_{n,\eta}$$

is 1, which follows in particular that

$$dim(H^1(\Gamma_0(p_0), M_{n,\eta} \otimes \mathbb{C})) = 2dim(S_{n+2}(\Gamma_0(p_0), \eta)) + 2$$

(cf. [Hab] p284). By the above identification, we see that the study of the Hecke algebra on the cusp forms is equivalent to that on the cohomology $H^1(\Gamma_0(p_0), M_{n,\eta})$, see Chap. 1 in [Hab] for more details and backgrounds. Applying the Shapiro lemma to the cohomology group of $\Gamma_0(p_0)$ we get in Section 5 a basis for the cohomology $H^1(\Gamma_0(p_0), M_{n,\eta})$. Using this basis we obtain an algorithm that can be used to compute the Hecke operator T_l on the cohomology $H^1(\Gamma_0(p_0), M_{n,\eta})$. Finally the characteristic polynomials of T_2, T_3, T_5 and T_7 are given in Table 1 for small p_0 and n.

2. The Shapiro-Lemma.

In order to determine the cohomology of $\Gamma_0(p_0)$, we first recall the Shapiro-Lemma. Denote by $W_{n,\eta}$ the induced module of $M_{n,\eta}$ on Γ:

$$W_{n,\eta} = Ind^{\Gamma}_{\Gamma_0(p_0)} M_{n,\eta} = \{ f : \Gamma \to M_{n,\eta} \mid f(r_0 r) = r_0.f(r), \forall r_0 \in \Gamma_0(p_0) \}$$

The operation of Γ on $W_{n,\eta}$ is defined by $(a.f)(r) := f(ra)$, $a, r \in \Gamma$. We extend now this operation to an operation of Δ on $W_{n,\eta}$. For $a \in \Delta$, $r \in \Gamma$, there exist always $a' \in \Delta_0$, $r' \in \Gamma$, such that $ra = a'r'$. We define $(a.f)(r) := a'.f(r')$. It is obvious that this definition coincides with the above definition if $a \in \Gamma$. Now on the cohomology groups

$$H^1(\Gamma_0(p_0), M_{n,\eta}) \quad \text{and} \quad H^1(\Gamma, W_{n,\eta})$$

we can define the Hecke algebra (cf. [Hab] Chap.1). By the Shapiro-Lemma (cf. [Bro] or [AS] $1) there is a canonical isomorphism between

$$H^1(\Gamma_0(p_0), M_{n,\eta}) \cong H^1(\Gamma, W_{n,\eta})$$

as modules under the Hecke algebra.

3. The dimension of the cohomology $H^1(\Gamma, W_{n,\eta})$.

To get startet, we consider the Γ-module $W_{n,\eta}$. Let

$$a_i = \begin{pmatrix} 0 & -1 \\ 1 & i \end{pmatrix}, \quad i = 0, 1, ..., p_0 - 1, \quad a_{p_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\{a_i\}$ is then a set of representatives of Γ with respect to $\Gamma_0(p_0)$:

$$\Gamma = \bigcup_{i=0}^{p_0} \Gamma_0(p_0) a_i.$$

An element $f \in W_{n,\eta}$ is uniquely determined by the values $f(a_0), f(a_1), ..., f(a_{p_0})$ by using the condition $f(r_0 r) = r_0 f(r)$. The dimension of $W_{n,\eta}$ over R is $(p_0 + 1) \cdot dim(M_{n,\eta}) = (p_0 + 1)(n + 1)$. In other words, $W_{n,\eta}$ is generated by the elements $(w_0, w_1, ..., w_{p_0})$ with $w_i \in M_{n,\eta}$.

Now we consider the cohomology $H^1(\Gamma, W_{n,\eta})$. The structur of cohomology $H^1(\Gamma, W_{n,\eta})$ is well known (cf. [Wan] $1):

$$H^1(\Gamma, W_{n,\eta}) \cong W_{n,\eta}/(W^S_{n,\eta} + W^Q_{n,\eta})$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $W^r_{n,\eta} := \{ w \in W_{n,\eta} \mid r.w = w \}$ for $r \in \Gamma$.

We begin with the description of $W^S_{n,\eta}$. It is easy to show that

$$\begin{cases} a_0 S = a_{p_0} \\ a_i S = S_i a_j \quad i \cdot j \equiv -1 \bmod p_0, \quad S_i = \begin{pmatrix} -j & -1 \\ 1 + ij & i \end{pmatrix} \in \Gamma_0(p_0) \\ a_{p_0} S = a_0 \end{cases}$$

and by the definition we obtain

$$\begin{cases} (S.f)(a_0) = f(a_{p_0}) \\ (S.f)(a_i) = S_i.f(a_j),\ i = 1, ..., p_0 - 1 \\ (S.f)(a_{p_0}) = f(a_0) \end{cases}$$

Therefore, $W^S_{n,\eta}$ has the expression:

$$\begin{aligned} W^S_{n,\eta} &= \{ f \in W_{n,\eta} \mid f(a_0) = f(a_{p_0}),\ f(a_i) = S_i.f(a_j) \} \\ &= \{ (w_0, ..., w_{p_0}) \in M_{n,\eta} \times \cdots \times M_{n,\eta} \mid w_0 = w_{p_0},\ w_i = S_i.w_j \} \end{aligned}$$

Let $T = SQ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. One shows immediately that

$$\begin{cases} a_iT = a_{i+1}, \quad i = 0, 1, ..., p_0 - 2 \\ a_{p_0-1}T = \begin{pmatrix} 1 & 0 \\ -p_0 & 1 \end{pmatrix} a_0 \\ a_{p_0}T = Ta_{p_0} \end{cases} \quad \text{and} \quad \begin{cases} a_0Q = Ta_{p_0} \\ a_1Q = T^{-1}a_0 \\ a_iQ = S_ia_{j+1}, \quad i = 2, 3, ..., p_0 - 1 \\ a_{p_0}Q = a_1 \end{cases},$$

from which it follows

$$W^T_{n,\eta} = \left\{ (w_0, ..., w_{p_0}) \in M_{n,\eta} \times \cdots \times M_{n,\eta} \mid w_0 = \cdots = w_{p_0-1} = \begin{pmatrix} 1 & 0 \\ -p_0 & 1 \end{pmatrix} w_0, w_{p_0} = Tw_{p_0} \right\}$$

$$W^Q_{n,\eta} = \{ (w_0, ..., w_{p_0}) \in M_{n,\eta} \times \cdots \times M_{n,\eta} \mid Tw_{p_0} = w_0, w_1 = w_{p_0}, S_iw_{j+1} = w_i \}$$

For the purpose of determing the dimension of $H^1(\Gamma, W_{n,\eta})$ we show now

3.1 *Lemma:* $W^S_{n,\eta} \cap W^Q_{n,\eta} = \{0\}$

Proof: Let $f = (w_0, ..., w_{p_0}) \in W^S_{n,\eta} \cap W^Q_{n,\eta}$. It implies that $f \in W^T_{n,\eta}$, i.e.,

$$w_0 = w_1 = \cdots = w_{p_0-1} = \begin{pmatrix} 1 & 0 \\ -p_0 & 1 \end{pmatrix} w_0, \text{ and } w_{p_0} \in M^T_{n,\eta}$$

Hence it follows that $w_{p_0} = ax^n$, $w_0 = by^n$ for some a, b. For $f \in W^S_{n,\eta}$ we have $w_0 = w_{p_0}$, i.e. $ax^n = by^n$, which implies that $a = b = 0$.∎

Therefore, the dimension of the cohomology $W_{n,\eta}$ is

$$dim(H^1(\Gamma, W_{n,\eta})) = dim(W_{n,\eta}) - dim(W^S_{n,\eta}) - dim(W^Q_{n,\eta})$$

Now we compute the dimension of $W^S_{n,\eta}$ and $W^Q_{n,\eta}$.

Let ν_2, ν_3 the number of $\Gamma_0(p_0)$-inequivalent elliptic points of the order 2, 3 respectively.

$$\nu_2 = 0 \text{ or } 2 \equiv p_0 + 1 \bmod 4 \qquad \nu_3 = 0 \text{ or } 2 \equiv p_0 + 1 \bmod 3$$

It is obvious that

$$\nu_2 = 2 \iff p_0 \equiv 1 \bmod 4 \iff \eta(-1) = 1 \iff \text{ there is a } i_0 \text{ with } i_0^2 \equiv -1 \bmod p_0$$

In that case one has $\eta(i_0) = i_0^{\frac{p_0-1}{2}} = (-1)^{\frac{p_0-1}{4}}$ and $a_{i_0}S = S_{i_0}a_{i_0}$. Furthermore it is easy to show that

$$\nu_3 = 2 \Longleftrightarrow p \equiv 1 \bmod 3 \Longleftrightarrow 6|p_0 - 1 \Longleftrightarrow \text{ there is a } i_0 \text{ of order 6 in } (\mathbb{Z}/p_0)^*$$
$$\Longleftrightarrow i_0^3 \equiv -1 \bmod p_0 \Longleftrightarrow i_0(i_0 - 1) \equiv -1 \bmod p_0$$

It follows that $a_{i_0}Q = S_{i_0}a_{i_0}$. Since $(i_0 - 1)^2 \equiv -i_0$ one has $\eta(i_0) = \eta(-1)\eta(i_0 - 1)^2 = \eta(-1) = (-1)^{\frac{p_0-1}{2}}$.

3.2 *Lemma:*

$$dim(W^S_{n,\eta}) = 2[\frac{p_0+1}{4}]\ (n+1) + 2d_S$$
$$dim(W^Q_{n,\eta}) = [\frac{p_0+1}{3}]\ (n+1) + 2d_Q$$

where

$$d_S = \begin{cases} 0 & p_0 \equiv 3 \bmod 4 \\ 2[\frac{n}{4}]\ +1 & p_0 \equiv 1 \bmod 8 \\ 2[\frac{n+2}{4}] & p_0 \equiv 5 \bmod 8 \end{cases}, \qquad d_Q = \begin{cases} 0 & p_0 \equiv 2 \bmod 3 \\ 2[\frac{n}{6}]\ +1 & p_0 \equiv 1 \bmod 12 \\ 2[\frac{n+3}{6}] & p_0 \equiv 7 \bmod 12 \end{cases}.$$

In particular,

$$dim(H^1(\Gamma, W_{n,\eta})) = (p_0 + 1 - 2[\frac{p_0+1}{4}]\ - [\frac{p_0+1}{3}]\ )(n+1) - 2d_S - 2d_Q$$
$$dim(S_{n+2}(\Gamma_0(p_0),\eta)) = \frac{1}{2}(p_0 + 1 - 2[\frac{p_0+1}{4}]\ - [\frac{p_0+1}{3}]\ )(n+1) - d_S - d_Q - 1$$

Proof: For $f = (w_0, ..., w_{p_0}) \in W^S_{n,\eta}$ we have $w_i = S_i w_j$ and $S_j = S_i^{-1}$. If $j \neq i$, then w_j is uniquely determined by w_i. The number of such pair (i,j) is $2[\frac{p_0+1}{4}]$. If $j = i$, that means $p_0 \equiv 1 \bmod 4$, one has $w_i \in Ker(1 - S_i)$. We calculate the dimension of $Ker(1 - S_i)$. Let $m \otimes 1 \in M_{n,\eta}$, then $S_i.(m \otimes 1) = \eta(i)(S_i m \otimes 1)$. For $S_i = \begin{pmatrix} -i & -1 \\ 1+i^2 & i \end{pmatrix}$ there is a regular matrix P with $S_i = PSP^{-1}$. It follows that

$$d_S = dim(M^{S_i}_{n,\eta}) = dim(Ker(1 - S_i)) = dim(Ker(1 - \eta(i)S))$$

For $p_0 \equiv 1 \bmod 8$ one has $\eta(i) = (-1)^{\frac{p_0-1}{4}} = 1$. The dimension of $Ker(1-S)$ can be easily determined, $dimKer(1 - S) = 2[\frac{n}{4}]\ + 1$. Since there are two i with $i^2 \equiv -1$, dimension of $W^S_{n,\eta}$ has the expression:

$$dim(W^S_{n,\eta}) = 2[\frac{p_0+1}{4}]\ (n+1) + 4[\frac{n}{4}]\ + 2$$

The other cases can be proved in the same manner.∎

4. The dimension of $H^1(\Gamma, W_{n,\eta})_\pm$.

Let $\Gamma_\infty = < T >$ be the stabilizer of the cusp ∞ in Γ. We have an exact sequence:

$$0 \to H^0(\Gamma_\infty, W_{n,\eta}) \to H^1_c(\Gamma, W_{n,\eta}) \to H^1(\Gamma, W_{n,\eta}) \to H^1(\Gamma_\infty, W_{n,\eta}) \to \cdots$$

where $H^1_c(\cdot,\cdot)$ is the cohomology with the compact support, referring to [Hab] Chap.1 for details and backgrounds. It has been shown in [Wan] $1 that the cohomology

$$H^1(\Gamma_\infty, W_{n,\eta}) \cong W_{n,\eta}/(1 - T)W_{n,\eta}.$$

4.1 *Lemma:*

$$H^1(\Gamma_\infty, W_{n,\eta} \otimes Q) \cong Q\,\phi_0 + Q\,\phi_\infty$$

where $\phi_0(T) = (x^n, 0, ..., 0) \in W_{n,\eta}$, $\phi_\infty(T) = (0, ..., 0, y^n) \in W_{n,\eta}$.

Proof: For each $w = (w_0, ..., w_{p_0}) \in W_{n,\eta}$, we consider the equation

$$(*) \qquad w = a(x^n, 0, ..., 0) + b(0, ..., 0, y^n) + (T-1)v$$

with $v = (v_0, ..., v_{p_0}) \in W_{n,\eta}$, which means:

$$w_0 = ax^n + v_1 - v_0$$
$$w_i = v_{i+1} - v_i \quad 0 < i < p_0 - 1$$
$$w_{p_0-1} = \begin{pmatrix} 1 & 0 \\ -p_0 & 1 \end{pmatrix} v_0 - v_{p_0-1}$$
$$w_{p_0} = by^n + (T-1)v_{p_0},$$

it follows that

$$\left(1 - \begin{pmatrix} 1 & 0 \\ -p_0 & 1 \end{pmatrix}\right) v_0 = ax^n - \sum_{j=0}^{p_0-1} w_j.$$

We take a as the coefficient of x^n in $\sum_{j=0}^{p_0-1} w_j$ and b as the coefficient of y^n in w_{p_0}. The equations

$$\left(1 - \begin{pmatrix} 1 & 0 \\ -p_0 & 1 \end{pmatrix}\right) v_0 = c_0 y^n + c_1 x y^{n-1} + \cdots + c_{n-1} x^{n-1} y$$
$$(1-T)v_{p_0} = d_1 x y^{n-1} + d_2 x^2 y^{n-2} + \cdots + d_n x^n$$

are always solvable in $M_{n,\eta} \otimes Q$ for any $c_0, ..., c_{n-1}, d_1, ..., d_n \in Q$. Therefore the equation (*) is solvable in $M_{n,\eta} \otimes Q$.∎

Let $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. We define for a cocycle $\phi \in Z^1(\Gamma, W_{n,\eta})$

$$(\epsilon\phi)(r) := \epsilon\phi(\epsilon^{-1} r \epsilon) \qquad \forall r \in \Gamma$$

It induces an automorphism of the order 2 on the cohomologies (cf. [Wan] §1). Hence we obtain two exact sequences:

$$0 \to H^0(\Gamma_\infty, W_{n,\eta})_+ \to H^1_c(\Gamma, W_{n,\eta})_+ \to H^1(\Gamma, W_{n,\eta})_+ \xrightarrow{r^*} H^1(\Gamma_\infty, W_{n,\eta})_+ \to \cdots$$

$$0 \to H^0(\Gamma_\infty, W_{n,\eta})_- \to H^1_c(\Gamma, W_{n,\eta})_- \to H^1(\Gamma, W_{n,\eta})_- \xrightarrow{r^*} H^1(\Gamma_\infty, W_{n,\eta})_- \to \cdots$$

where $H^1(\Gamma, W_{n,\eta})_\pm := \{\, \phi \in H^1(\Gamma, W_{n,\eta}) \mid \epsilon.\phi = \pm\phi \,\}$. Since

$$\begin{cases} a_0\epsilon = \epsilon a_0 \\ a_i\epsilon = E.a_{p_0-i}, \quad i = 1, 2, ..., p_0 - 1 \\ a_{p_0}\epsilon = \epsilon a_{p_0} \end{cases}$$

where $E := \begin{pmatrix} -1 & 0 \\ p_0 & 1 \end{pmatrix}$. The operation of ϵ on $W_{n,\eta}$ is

$$\begin{cases} (\epsilon u)(a_0) = \epsilon.u(a_0) \\ (\epsilon u)(a_i) = E.u(a_{p_0-i}) \\ (\epsilon u)(a_{p_0}) = \epsilon.u(a_{p_0}) \end{cases}$$

In particular, it follows that

$$\epsilon.\phi_\infty(T) = \phi_\infty(T), \quad \epsilon.\phi_0(T) = (-1)^n \phi_0(T)$$

4.2 *Lemma:*

a. $\phi_\infty \in H^1(\Gamma, W_{n,\eta})_-$

b. $\phi_0 \in H^1(\Gamma, W_{n,\eta})_-$ *for n even;* $\phi_0 \in H^1(\Gamma, W_{n,\eta})_+$ *for n odd.*

Proof:

$$\begin{aligned}(\epsilon\phi_\infty)(T) &= \epsilon.\phi_\infty(\epsilon^{-1}T\epsilon) = \epsilon.\phi_\infty(T^{-1}) = -\epsilon T^{-1}\phi_\infty(T) = -T\epsilon.\phi_\infty(T) \\ &= -\epsilon.\phi_\infty(T) + (1-T)\epsilon.\phi_\infty(T) \sim -\epsilon.\phi_\infty(T) = -\phi_\infty(T)\end{aligned}$$

It means that $\epsilon.\phi_\infty = -\phi_\infty$. (b) can be proved in the same way.∎

By applying the Eichler-Shimura isomorphism, together with the observation above, we obtain

4.3 *Corollary:*

a. *For n even we have*

$$dim(H^1(\Gamma, W_{n,\eta})_-) = \frac{1}{2} dim(H^1(\Gamma, W_{n,\eta})) + 1$$

$$dim(H^1(\Gamma, W_{n,\eta})_+) = \frac{1}{2} dim(H^1(\Gamma, W_{n,\eta})) - 1$$

b.. *For n odd we have*

$$dim(H^1(\Gamma, W_{n,\eta})_-) = \frac{1}{2} dim(H^1(\Gamma, W_{n,\eta}))$$

$$dim(H^1(\Gamma, W_{n,\eta})_+) = \frac{1}{2} dim(H^1(\Gamma, W_{n,\eta}))$$

5. The basis of $H^1(\Gamma, W_{n,\eta})$.

It is well known that

$$H^1(\Gamma, W_{n,\eta}) \cong W_{n,\eta}/(W_{n,\eta}^S + W_{n,\eta}^Q)$$

Our goal in this section is to choose a subset V of $W_{n,\eta}$ such that $W_{n,\eta} = W_{n,\eta}^S \oplus W_{n,\eta}^Q \oplus V$. Since the group Γ is generated by S, Q with the relations $S^2 = 1$, $Q^3 = 1$ (cf. [Ser]), the cohomology

$$\begin{aligned} H^1(\Gamma, W_{n,\eta}) &= \frac{\{ (\phi(S), \phi(Q)) \mid \phi(S) \in (1-S)W_{n,\eta}, \phi(Q) \in (1-Q)W_{n,\eta} \}}{\{ ((1-S)u, (1-Q)u) \mid u \in W_{n,\eta} \}} \\ &\cong \frac{\{ \phi(Q) \mid \phi(S) = 0, \phi(Q) \in (1-Q)W_{n,\eta} \}}{\{ (1-Q)u \mid u \in W_{n,\eta}^S \}} \\ &\cong \{ (1-Q)v \mid v \in V \}, \end{aligned}$$

i.e., every class $\phi \in H^1(\Gamma, W_{n,\eta})$ has the form

$$\begin{cases} \phi(S) = 0 \\ \phi(Q) = (1-Q)u, \; u \in V \end{cases}$$

Defining by α_i (resp. β_i) the permutation of $\{0, 1, ..., p_0\}$ induced by the operation of S (resp. Q) on $\{a_0, a_1, ..., a_{p_0}\}$. We have (cf. \$3)

$$\alpha_i \cdot i \equiv -1 \bmod p_0, \quad 0 < i < p_0$$
$$\beta_i = \alpha_i + 1 \quad 1 < i < p_0$$

5.1 *Definition: For $i, j, k \in \{1, 2, ..., p_0 - 1\}$*

a. *The pair (i, j) is called a α-pair if $j = \alpha_i, i = \alpha_j$, or equivalently, $i \cdot j \equiv -1$ mod p_0;*

b. *The triple (i, j, k) is called a β-triple if $j = \beta_i, k = \beta_j, i = \beta_k$, or equivalently, $i \cdot j \cdot k \equiv -1$ mod p_0;*

c. *Let B a subset of $\{1, 2, ..., p_0 - 1\}$. We denote by $< B >$ the subset of $\{1, 2, ..., p_0 - 1\}$ determined by the following conditions:*

 i. *$B \subset < B >$;*

 ii. *if (i, j) is an α-pair and $j \in < B >$ then $i \in < B >$;*

 iii. *if (i, j, k) is a β-triple and $j, k \in < B >$ then $i \in < B >$;*

d. *A subset B of $\{1, 2, ..., p_0 - 1\}$ is called a basis set if it satisfies:*

 i. *$< B > = \{1, 2, ..., p_0 - 1\}$;*

 ii. *$\forall i \in B, < B \setminus \{i\} > \neq \{1, 2, ..., p_0 - 1\}$.*

It follows immediately from the definition that the number of the α-pair is $2[\frac{p_0+1}{4}] - 1$ and the number of the β-triple is $[\frac{p_0+1}{3}] - 1$. Therefore the number of the elements in B is

$$\sharp B = (p_0 - 1) - (2[\frac{p_0+1}{4}] - 1) - ([\frac{p_0+1}{3}] - 1) = p_0 + 1 - 2[\frac{p_0+1}{4}] - [\frac{p_0+1}{3}]$$

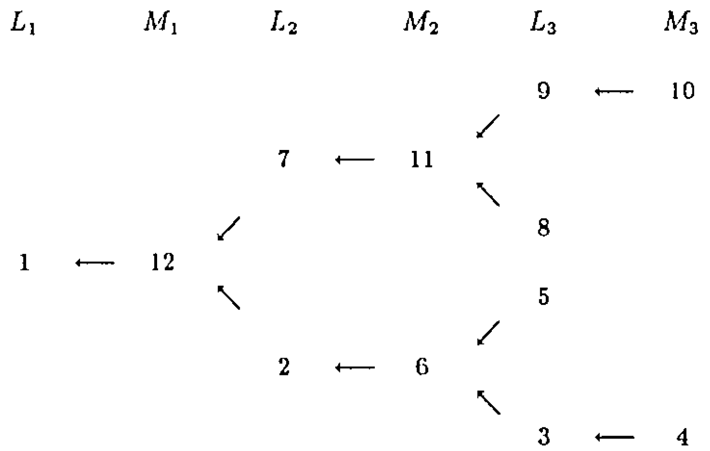
We define inductively two series of subsets of $\{1, 2, ..., p_0 - 1\}$.

$$L_1 = \{1\}$$
$$M_r = \{\alpha_i \mid i \in L_r\} \setminus L_r, \qquad r > 0$$
$$L_{r+1} = \{j = \beta_i, \beta_j \mid i \in M_r\} \setminus M_r$$

5.2 Example: $p_0 = 13$. In that case $\nu_2 = 2$, $\nu_3 = 2$. The permutations of $\{a_0, a_1, ..., a_{p_0}\}$ induced by the operation of S and Q are:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13
S	13	12	6	4	3	5	2	11	8	10	9	7	1	0
Q	13	0	7	5	4	6	3	12	9	11	10	8	2	1

The sets L_r and M_r can be described by the diagram:



5.3 *Lemma:*

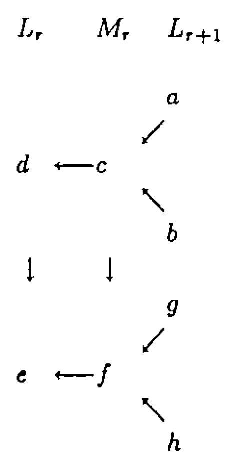
a. $\{1, 2, ..., p_0 - 1\} = \bigcup_{r=1}^{N} (L_r \cup M_r)$ *for some* $N < p_0$

b. *For each* $i \in L_r$ *there exists a* $j \in L_r$ *with* $i \cdot j \equiv 1 \mod p_0$;

c. *For each* $i \in M_r$ *there exists a* $j \in M_r$ *with* $i \cdot j \equiv 1 \mod p_0$.

Proof: a. Assume that a is the smallest element in $\{1, 2, ..., p_0 - 1\}$ with the property $a \notin \bigcup_{r=1}^{\infty} (L_r \cup M_r)$. Let $a = \beta_b$ for some $b \in \{1, 2, ..., p_0 - 1\}$. Then $a = \beta_b = \alpha_b + 1$ and $\alpha_b < a$. By the assumption it implies $\alpha_b \in \bigcup_{r=1}^{\infty} (L_r \cup M_r)$, which follow that $b \in \bigcup_{r=1}^{\infty} (L_r \cup M_r)$ and $a \in \bigcup_{r=1}^{\infty} (L_r \cup M_r)$ by the definition of $< B >$. It contradicts the assumption.

b. We prove the assertion by the induction. The assertion for $r = 1$ is obvious.
Let a be an element in L_{r+1}, then there is an element $c \in M_r$ such that $a = \beta_c$ or $c = \beta_a$. We treat only the case $a = \beta_c$. Let $b = \beta_a \in L_{r+1}$ and $d = \alpha_c \in L_r$. By the induction assumption there is a $e \in L_r$ with $d \cdot e \equiv 1 \mod p_0$. Let $f = \alpha_e$, we see immediately that $f \cdot c \equiv 1 \mod p_0$. Let $g = \beta_f$, $h = \beta_g \in L_{r+1}$, we look at the following diagram:



and assert that $a \cdot h \equiv 1 \mod p_0$. Indeed,

$$\begin{aligned} a = \beta_c = \alpha_c + 1 = d + 1 &\equiv (d+1) \cdot (-ef) \equiv (e+1) \cdot (-f) \\ &\equiv 1 - f = 1 - \beta_h = -\alpha_h \end{aligned}$$

i.e., $a \cdot h \equiv -\alpha_h \cdot h \equiv 1$.

c. It follows immediately from (b).

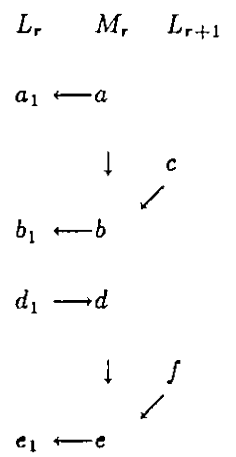
5.4 *Lemma: There is a basis set B with the property: if $a \in B$ then $p_0 - a \in B$.*

The proof of the lemma presents in fact an algorithm to compute the basis set B.

Proof: First note that $< L_r > \subset < M_r > \subset < L_{r+1} >$.

Case 1: If $a, \alpha_a \in L_r$ and $a \notin < B >$, there is an elements $b \in L_r$ with $ab \equiv 1$, which yields $\alpha_a \cdot \alpha_b \equiv 1$. Since $(a + \alpha_b)b = ab + \alpha_b \cdot b \equiv 1 + (-1) = 0$, one has $a + \alpha_b = p_0$ and $\{a, b, \alpha_a, \alpha_b\} \subset < \{a, \alpha_b\} >$. Hence we add a, α_b to B.

Case 2: (a, b, c) is a β-triple, $a, b \in M_r$, $c \in L_{r+1}$ and $a, b \notin B$. For $a, b \in M_r$ there are $d, e \in M_r$ with $ad \equiv 1$, $be \equiv 1$. We consider the following diagram:

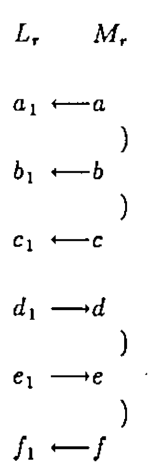


one verifies trivially that $a + d_1 = p_0$ and

$$\{a, b, c, d, e, f, a_1, b_1, d_1, e_1\} \subset < \{a, d_1, c, f\} > .$$

Therefore we add a, d_1 to B.

Case 3: (a, b, c) is a β-triple, $a, b, c \in M_r$ and $a, b, c \notin < B >$. There are $d, e, f \in M_r$ with $ad \equiv 1$, $be \equiv 1$, $cf \equiv 1$. We consider the following diagram:



It is obvious, that $a+d_1=p_0$, $b+e_1=p_0$ and

$$\{a,b,c,d,e,f,a_1,b_1,c_1,d_1,e_1,f_1\} \subset < \{a,b,d_1,e_1\} >$$

We add thus a,b,d_1,e_1 to B.

In such a way we obtain a basis set B.∎

In the example 5.3 we can take the basis set $B=\{5,8,4,9\}$.

We study now the cohomology $H^1(\Gamma,W_{n,\eta})=W_{n,\eta}/(W^S_{n,\eta}+W^Q_{n,\eta})$. Let B be a basis set. Then each element $(0,w_1,...,w_{p_0-1},0)\in W_{n,\eta}$ is congruent mod $W^S_{n,\eta}+W^Q_{n,\eta}$ to an element $g=(v_0,...,v_{p_0})\in W_{n,\eta}$ with $v_i=0$ for $i\notin B$.

If $\nu_2=2$, there is a $i_0\in B$ such that $i_0{}^2\equiv -1$. If $w_{i_0}\in Ker(1-S_{i_0})=M^{S_{i_0}}_{n,\eta}$, then $(0,...,0,w_{i_0},0,...,0)\in W^S_{n,\eta}$. Therefore

$$\{(0,...w_{i_0},...,0)\mid w_{i_0}\in M_{n,\eta}\}/(W^S_{n,\eta}+W^Q_{n,\eta})\cong\{(0,...,v_{i_0},...,0)\mid v_{i_0}\in M_{n,\eta}/M^{S_{i_0}}_{n,\eta}\}$$

Similarly, if $\nu_3=2$ and $i_0\in B$, $i_0^3\equiv -1$, then

$$\{(0,...w_{i_0},...,0)\mid w_{i_0}\in M_{n,\eta}\}/(W^S_{n,\eta}+W^Q_{n,\eta})\cong\{(0,...,v_{i_0},...,0)\mid v_{i_0}\in M_{n,\eta}/M^{S_{i_0}}_{n,\eta}\}$$

Now we consider the index $0,p_0$. Since

$$(0,...,0,w_{p_0})=(-w_{p_0},0,...,0) \bmod W^S_{n,\eta}+W^Q_{n,\eta},$$

we need only to consider only the index 0. Let

$$(w_0,0,...,0)=\underbrace{(a,0,...,0,a)}_{\in W^S_{n,\eta}}+\underbrace{(Tb,b,0...,0,b)}_{\in W^Q_{n,\eta}}+(0,c,0,...,0)$$

for some a,b,c, then $b=-a$, $c=a$, $(1-T)a=w_0$. The equation $(1-T)a=w_0$ can be solved only for $w_0=c_1xy^{n-1}+c_2x^2y^{n-2}+\cdots+c_nx^n$. Therefore the element $(y^n,0,...,0)$ is linear independent to

$$\{(0,...,0,v_i,0,...,0)\mid i\in B, v_i\in M_{n,\eta} \text{ or } M_{n,\eta}/M^{S_i}_{n,\eta} \text{ if } i^2\equiv -1 \text{ or } i^3\equiv -1\} \bmod W^S_{n,\eta}+W^Q_{n,\eta}.$$

On the other hand,

$$(0,x^n,0,...,0)=(-x^n,0,...,0,-x^n)+(Tx^n,x^n,0,...,0,x^n)\in W^S_{n,\eta}+W^Q_{n,\eta}$$

and $(0,x^n,0,...,0)$ can be represented by the elements of

$$\{(0,...,0,v_i,0,...,0)\mid i\in B, v_i\in M_{n,\eta} \text{ or } M_{n,\eta}/M^{S_i}_{n,\eta} \text{ if } i^2\equiv -1 \text{ or } i^3\equiv -1\},$$

which implies that the elements of

$$\{(0,...,0,v_i,0,...,0)\mid i\in B, v_i\in M_{n,\eta} \text{ or } M_{n,\eta}/M^{S_i}_{n,\eta} \text{ if } i^2\equiv -1 \text{ or } i^3\equiv -1\}$$

are linear dependent mod $W^S_{n,\eta}+W^Q_{n,\eta}$. A basis of $H^1(\Gamma,W_{n,\eta})$ is then $(y^n,0,...,0)$ and

$$\{(0,...,0,v_i,0,...,0)\mid i\in B, v_i\in M_{n,\eta} \text{ or } M_{n,\eta}/M^{S_i}_{n,\eta} \text{ if } i^2\equiv -1 \text{ or } i^3\equiv -1\} \quad \bmod \sim,$$

where the relation $\sim$ is given by the equation

$$(0, x^n, 0, ..., 0) \equiv 0 \text{ mod } W^S_{n,\eta} + W^Q_{n,\eta}$$

6. The basis of $H^1(\Gamma, W_{n,\eta})_\pm$.

We shall first deal with the operation of ϵ on $H^1(\Gamma, W_{n,\eta})$. From the definition in \$4 we have for a classe $\phi \in H^1(\Gamma, W_{n,\eta})$, $\phi(S) = 0$, $\phi(Q) = (1-Q)u$,

$$(\epsilon\phi)(S) = \epsilon.\phi(\epsilon S \epsilon) = \epsilon.\phi(S^{-1}) = 0$$
$$(\epsilon\phi)(Q) = \epsilon.\phi(\epsilon Q \epsilon) = \epsilon.\phi(SQ^{-1}S) = -\epsilon SQ^{-1}\phi(Q) = -\epsilon SQ^{-1}(1-Q)u = (1-Q)\epsilon Su = (1-Q)S\epsilon u$$

If $\phi \in H^1(\Gamma, W_{n,\eta})_-$, i.e. $\epsilon\phi + \phi = 0$, it follows that $S\epsilon u + u \in W^S_{n,\eta} + W^Q_{n,\eta}$. Since $S\epsilon u + u = (S+1)\epsilon u + u - \epsilon u$ and $(S+1)\epsilon u \in W^S_{n,\eta}$, we obtain

$$\phi \in H^1(\Gamma, W_{n,\eta})_- \iff u - \epsilon u \in W^S_{n,\eta} + W^Q_{n,\eta}.$$

Similarly,

$$\phi \in H^1(\Gamma, W_{n,\eta})_+ \iff u + \epsilon u \in W^S_{n,\eta} + W^Q_{n,\eta}.$$

In order to determine a basis of $H^1(\Gamma, W_{n,\eta})_-$ we consider the vector space

$$U := \{\, u = (u_0, ..., u_{p_0}) \in W_{n,\eta} \mid u - \epsilon.u = 0 \,\}$$

U has a basis consisting of the elements $(u_0, ..., u_{p_0})$ which satisfy one of the following conditions (cf. \$4):

$$1. \quad \begin{cases} u_0 = x^j y^{n-j}, & j \text{ even} \\ u_i = 0, & i > 0 \end{cases}$$

$$2. \quad \begin{cases} u_{p_0} = x^j y^{n-j}, & j \text{ even} \\ u_i = 0, & i < p_0 \end{cases}$$

$$3. \quad \begin{cases} u_i = x^j y^{n-j} \\ u_{p_0 - i} = E.u_i \\ u_j = 0, & j \neq i, p_0 - i \end{cases}$$

In particular, the classes $\phi \in H^1(\Gamma, W_{n,\eta})$, $\phi(S) = 0$, $\phi(Q) = (1-Q)u$ are classes in $H^1(\Gamma, W_{n,\eta})_-$ for n even, where $u = (u_0, ..., u_{p_0}) \in W_{n,\eta}$ with

$$1. \begin{cases} u_0 = y^n \\ u_j = 0, j > 0 \end{cases}$$

or

$$2. \begin{cases} u_i \in W_{n,\eta} \text{ or } M_{n,\eta}/M^{S_i}_{n,\eta}, \; i \in B, \; i < p_0/2 \\ u_{p_0 - i} = E.u_i \\ u_j = 0, \quad j \neq i, \; p_0 - i \end{cases}$$

The number of the above classes is

$$1 + \frac{\sharp B}{2} dim(M_{n,\eta}) - d_S - d_Q = dim(H^1(\Gamma, W_{n,\eta})_-)$$

By using the fact that the basis set B consists of the pair (i_1, i_2) with $i_1+i_2 = p_0$ we find that the above classes generate the cohomology $H^1(\Gamma, W_{n,\eta})_-$. Therefore this set of classes is a basis of $H^1(\Gamma, W_{n,\eta})_-$ for n even.

Similarly, we choose a basis of $H^1(\Gamma, W_{n,\eta})_+$ for n odd: $\begin{cases} \phi(S) = 0 \\ \phi(Q) = (1-Q)u \end{cases}$ with

$$\begin{cases} u_i \in M_{n,\eta} \text{ or } M_{n,\eta}/M_{n,\eta}^{S_i},\ i \in B,\ i < p_0/2 \\ u_{p_0-i} = -E.u_i \\ u_j = 0, \quad j \neq i,\ p_0 - i \end{cases}.$$

6.1 Remark: In general it is very difficult to determine the basis of $H^1(\Gamma, W_{n,\eta})_+$ for n even, because the dimension of $H^1(\Gamma, W_{n,\eta})_+$ is $\frac{1}{2}dim(H^1(\Gamma, W_{n,\eta})) - 1$, and the dimension of the vector space generated by the set

$$\begin{cases} u_i \in M_{n,\eta} \text{ or } M_{n,\eta}/M_{n,\eta}^{S_i},\ i \in B,\ i < p_0/2 \\ \\ u_{p_0-i} = E.u_i \end{cases}$$

is $\frac{1}{2}dim(H^1(\Gamma, W_{n,\eta}))$. It implies that there is a relation between the above elements. The case $H^1(\Gamma, W_{n,\eta})_-$ for n odd is similar.

We are now interested in the boundary map r^* on the basis.

6.2 *Lemma: For a class $\phi \in H^1(\Gamma, W_{n,\eta})$ with $\phi(S) = 0$, $\phi(Q) = (1-Q)u$,*

a. *if $u = (y^n, 0, ..., 0)$ then $r^*\phi = \phi_\infty$;*

b. *if $u = (0, .., 0, u_i, 0, ..., 0)$, $0 < i < p_0$ then $r^*\phi = a\phi_0$ for some a.*

Proof: a.

$$(r^*\phi)(T) = \phi(T) = S\phi(Q) = S(1-Q)u = (S-T)u$$
$$= (S-1)u + (1-T)u \sim (S-1)u = (-y^n, 0, ..., 0, y^n).$$

The solution of the equation (*) in \$4.1 is $a = 0$, $b = 1$, i.e., $r^*\phi = \phi_\infty$.

b. $r^*\phi(T) \sim (S-1)u = (0, ..., -u_i, 0, .., S_i^{-1}u_i, 0, ..., 0)$. It is obvious that $b = 0$ (cf. the proof of \$4.1). Hence $r^*\phi = a\phi_0$ for some a. ∎

7. The Hecke operator T_l on $H^1(\Gamma, W_{n,\eta})$.

To get startet, we recall the definition of the Hecke operator T_l auf $H^1(\Gamma, W_{n,\eta})$, where l is a prime, $l \neq p_0$. Let

$$b_i = \begin{pmatrix} 1 & i \\ 0 & l \end{pmatrix},\ i = 0, 1, ..., l-1 \text{ and } b_l = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix},$$

they are a complete set of representatives of $\Gamma \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \Gamma$ with respect to Γ:

$$\Gamma \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \bigcup_{i=0}^{l} \Gamma b_i$$

For each $r \in \Gamma$ there is a $s_i \in \Gamma$ such that $b_i r = s_i b_j$ for some j. Define for a cocycle $f \in Z^1(\Gamma, W_{n,\eta})$

$$(T_l f)(r) := \sum_{i=0}^{l} b_i' f(s_i)$$

where $b_i' := det(b_i)b_i^{-1}$.

All this is discussed in more detail in [AS] $1 or [Wan] $1.2.

7.1 Example: $l = 2$, $p_0 = 5$, $n = 4$

For $l = 2$ the representatives are

$$b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, b_1 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, b_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

A simple calculation shows that

$$\begin{cases} b_0 S = S b_2 \\ b_1 S = SQ^{-1}SQSb_1 \\ b_2 S = S b_0 \end{cases} \qquad \begin{cases} b_0 T = b_1 \\ b_1 T = T b_0 \\ b_2 T = T^2 b_2 \end{cases} \qquad \begin{cases} b_0 Q = QSQb_2 \\ b_1 Q = SQ^{-1}SQ^{-1}b_0 \\ b_2 Q = S b_1 \end{cases}$$

By the definition we get for a class $\phi \in H^1(\Gamma, W_{n,\eta})$

$$\begin{aligned} (T_2\phi)(S) &= b_0'\phi(S) + b_1'\phi(SQ^{-1}SQS) + b_2'\phi(S) = (S-1)b_1'SQ^{-1}\phi(Q) \\ (T_2\phi)(Q) &= b_0'\phi(QSQ) + b_1'\phi(SQ^{-1}SQ^{-1}) + b_2'\phi(S) = (1-Q)(b_0' + b_0'QS)\phi(Q). \end{aligned}$$

Hence the cocycle $T_2\phi$ is cohomolog to

$$T_2\phi \quad \sim \quad \begin{cases} (T_2\phi)(S) = 0 \\ (T_2\phi)(Q) = (1-Q)(b_0' + b_0'QS + b_1'SQ^{-1})\phi(Q) \end{cases}$$

It is easy to see that

$$(1-Q)(b_0' + b_0'QS + b_1'SQ^{-1}) = (1-Q)(b_0' + (Q+Q^2)b_2'Q^{-1}) = (1-Q)(b_0' - b_2'Q^{-1}),$$

we obtain then

$$(T_2\phi)(Q) = (1-Q)(b_0' - b_2'Q^{-1})\phi(Q)$$

For $p_0 = 5$ we choose a basis set $B = \{2, 3\}$. The basis of $H^1(\Gamma, W_{n,\eta})_-$ is then $(y^n, 0, 0, 0, 0, 0)$ and $(0, 0, w_2, Ew_2, 0, 0)$ $\quad w_2 \in M_{n,\eta}/M_{n,\eta}^{S_3}$. For $n = 5$ the numerical computation shows that $M_{n,\eta}/M_{n,\eta}^{S_3} = Rv_1 + Rv_2 + Rv_3$ with

$$\begin{aligned} v_1 &= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4 \\ v_2 &= x^3y - 6x^2y^2 + 12xy^3 - 8y^4 \\ v_3 &= x^2y^2 - 4xy^3 + 4y^4 \end{aligned}$$

Let $v_0 = y^4$, then the basis of $H^1(\Gamma, W_{n,\eta})_-$ is ϕ_i, $i = 0, 1, 2, 3$ with $\phi_i(S) = 0$, $\phi_i(Q) = (1-Q)v_i$. The operation of T_2 is

$$T_2(v_0, v_1, v_2, v_3) = (v_0, v_1, v_2, v_3) \begin{pmatrix} -31 & 0 & 0 & 0 \\ * & 31 & 0 & 0 \\ * & 0 & -10 & 18 \\ * & 0 & -8 & 10 \end{pmatrix}$$

The characteristic polynomial of T_2 on $H^1(\Gamma, W_{n,\eta})_-$ is

$$\chi_2(x) = (x+31)(x-31)(x^2+44)$$

The factors $(x+31)$ and $(x-31)$ come from the operation of T_2 on the boundary cohomology $H^1(\Gamma_\infty, W_{n,\eta} \otimes \mathbb{Q}) \cong \mathbb{Q}\,\phi_0 + \mathbb{Q}\,\phi_\infty$. More precise,

$$T_2\phi_\infty = -31\phi_\infty, \quad T_2\phi_0 = 31\phi_0$$

Therefore the characteristic polynomial of T_2 on $S_6(\Gamma_0(p_0), \eta)$ is x^2+44. The numerical computations of T_2, T_3, T_5 and T_7 for small p_0 and n are given in the table 1.

7.2 Remark: The space $S_{n+2}(\Gamma_0(p_0), \eta)$ carries the Petersson product, a non-degenerate Hermitian product on $S_{n+2}(\Gamma_0(p_0), \eta)$. If ^{t} denotes "transpose"with respect to this product, then $T_l^t = \eta(l)T_l$. Let now λ be an eigenvalue of T_l, we have then $\bar{\lambda} = \eta(l)\lambda$ (cf. [Rib] \$1). Therefore, if $\eta(l) = -1$, then $\lambda = ia$ with $a \in \mathbb{R}$. If $\eta(l) = 1$, $\lambda \in \mathbb{R}$.

(1). $p_0 \equiv 1 \bmod 4$. In that case the dimension of $S_{n+2}(\Gamma_0(p_0), \eta)$ is even.

i. $\eta(l) = -1$. The characteristic polynomial of T_l is

$$\begin{aligned}\chi_l(x) =&(x-ia_1)(x+ia_1)(x-ia_2)(x+ia_2)\cdots(x-ia_r)(x+ia_r)\\ =&(x^2+a_1^2)(x^2+a_2^2)\cdots(x^2+a_r^2)\\ =&x^{2r}+b_1x^{2r-2}+\cdots+b_r\end{aligned}$$

with $b_1, ..., b_r \geq 0$.

ii. $\eta(l) = 1$. The characteristic polynomial of T_l is

$$\chi_l(x) = g(x)^2$$

for some polynomial $g(x)$. The roots of $g(x)$ are all real.

(2). $p_0 \equiv 3 \bmod 4$. In that case the dimension of $S_{n+2}(\Gamma_0(p_0), \eta)$ is odd.

i. $\eta(l) = -1$. There are zero eigenvalues. The characteristic polynomial is

$$\chi_l(x) = x^h(x^{2s}+b_1x^{2s-2}+\cdots+b_s)$$

where h is the class number of the field $\mathbb{Q}(\sqrt{-p_0})$.

ii. $\eta(l) = +1$. The characteristic polynomial is

$$\chi_l(x) = g(x)^2 \cdot f(x)$$

where $f(x)$ is a polynomial generated by the Theta series and $deg(f(x)) = h$.(cf. [Shi])

The results in the table 1 confirm the remark above.

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The characteristical polynomials of the Hecke operators T_2, T_3, T_5, and T_7 on the cups forms $S_k(\Gamma_0(p_0), \eta)$, where η ist the Legendre symbol.

PO=5, K=N+2: KRO(2,PO)=-1,KRO(3,PO)=-1,KRO(7,PO)=-1

N=4

$T2 := X^{2} + 44$

$T3 := X^{2} + 396$

$T7 := X^{2} + 3564$

N=6

$T2 := X^{2} + 116$

$T3 := X^{2} + 1044$

$T7 := X^{2} + 176436$

N=8

$T2 := X^{4} + 1708 \cdot X^{2} + 1216$

$T3 := X^{4} + 33552 \cdot X^{2} + 45529776$

$T7 := X^{4} + 104167728 \cdot X^{2} + 2144749073480496$

N=10

$T2 := X^{4} + 4132 \cdot X^{2} + 2496256$

$T3 := X^{4} + 341568 \cdot X^{2} + 18385718256$

$T7 := X^{4} + 4904976672 \cdot X^{2} + 2087691277621558896$

N=12

$T2 := X^{6} + 41052 \cdot X^{4} + 440779968 \cdot X^{2} + 617678127104$

$T3 := X^{6} + 8329788 \cdot X^{4} + 17708569483248 \cdot X^{2} + 1517182687182390336$

$T7 := X^{6} + 213997084092 \cdot X^{4} + 10526623838205776341488 \cdot X^{2} + 46528027403146207719038230676544$

N=14

$T2 := X^{6} + 117588 \cdot X^{4} + 2455515648 \cdot X^{2} + 4160982695936$

$T3 := X^{6} + 48755052 \cdot X^{4} + 160831293357168 \cdot X^{2} + 7991454328138726 7904$

$T7 := X^{6} + 8435989101708 \cdot X^{4} + 21799671533824901950559088 \cdot X^{2} + 1756039103173248326616347118672836 0256$

N=16

$T2 := X^{8} + 813836 \cdot X^{6} + 197805587136 \cdot X^{4} + 15212877148553216 \cdot X^{2} + 33802260467179 6903936$

$T3 := X^{8} + 634018824 \cdot X^{6} + 123866741829162816 \cdot X^{4} +$

$805235918890685234435 3664 \cdot X^{2} + 625567943601835645403415787752 96$

$T7 := X^{8} + 1358809234759656 \cdot X^{6} + 571583885437582806176526269376 \cdot X^{4} +$

$71743845253248677409589367384237677906875776 \cdot X^{2} + 122564938710388624712653679087106802412105511455875917 0816$

N=18

$T2 := X^{8} + 2907524 \cdot X^{6} + 2568216374016 \cdot X^{4} + 678867689422782464 \cdot X^{2} + 830184914753253 1204096$

$T3 := X^{8} + 4476366576 \cdot X^{6} + 6998614044948851616 \cdot X^{4} +$

$4394102925151257527276257536*X^2 + 859178610673769519506507390330864896$

$T7 := X^8 + 51160209747400944*X^6 + 649955449858844816462059274614176*X^4$

$1364277688497122259242343356898905016125205537024*X^2 + 669240116784884405332807029722912360484202369263457687836124416$

N=20

$T2 := X^{10} + 15122620*X^8 + 74461069946560*X^6 + 143355636201404579840*X^4$

$920507960428929611517132 80*X^2 + 14584363461253989437721829965824$

$T3 := X^{10} + 75700218780*X^8 + 1690293073124929870560*X^6 + 10589033423492535098094901061760*X^4 +$

$10613905392864453389568881849143800910080*X^2 + 2839805815981800681177617222924350898397211646976$

$T7 := X^{10} + 2623942726584980220*X^8 + 2393834166138243432310096381198875360*X^6 +$

$897532555190091115792311471245276662831526251090803840*X^4 +$

$121300745152481981309943878223945412910348541980942424246523288877748480 *x^2 +$

$4612704276869570987318804156552408528294314218510153268776173059937738127467213792820224$

N=22

$T2 := X^{10} + 62579380*X^8 + 1269587477762560*X^6 + 9620767823712245596160*X^4 +$

$19648398991934117012339425280*X^2 + 3574276364739503586982992256434176$

$T3 := X^{10} + 565341209820*X^8 + 118033406092349714504160*X^6 + 10931210697192722327499640220787840*X^4 +$

$406914738284133623534685754882233338060775680*X^2 + 2922982673270172565978306559380807929420812626129030624$

$T7 := X^{10} + 1111965553847879947 80*X^8 + 3435262712787547437076787484432246075360*X^6 +$

$3641203833345308738917856736733886777356038503872276919616 0*X^4$

$+ 855866842508375850528107157082441937915049683030625853470656823315813705484*X^2$

$+ 348799178342000751433475154597246860220286451662647328870444081898307411589887020773579776$

PO=7, K=N+2: KRO(2,PO)=1,KRO(3,PO)=-1,KRO(5,PO)=-1

N=1

$T2 := X + 3$

$T3 := X$

$T5 := X$

N=3

$T2 := X - 1$

$T3 := X$

$T5 := X$

N=5

$T2 := (X + 8)^2 *(X - 9)$

$T3 := X*(X^2 + 2040)$

$T5 := X*(X^2 + 2040)$

N=7

$T2 := (X^2 - 16*X - 120)^2 *(X + 31)$

```
              4         2
     T3 := X*(X  + 17184*X  + 40430880)

              4           2
     T5 := X*(X  + 1809120*X  + 736852788000)

N=9
             2             2
     T2 := (X  + 24*X - 592) *(X -57)

              4           2
     T3 := X*(X  + 132480*X  + 4381776000)

              4            2
     T5 := X*(X  + 11349120*X  + 25531635024000)

N=11
             3                     2
     T2 := (X  -10216*X + 172800) *(X + 47)

              6           4                 2
     T3 := X*(X  + 2434704*X  + 1858882957920*X  + 429665499302054400)

              6              4                      2
     T5 := X*(X  + 1290415440*X  + 544550093091324000*X  + 7525211490074395101600000)

N=13

             4       3         2                              2
    T2 := (X  - 88*X  -49600*X  +3161344*X +199833600) *(X +87)

              8            6                    4                             2
    T3 := X*(X  + 32897856*X  + 307339393288320*X  + 678298556041314989600*X  + 3197232909629570972160000)

              8               6                         4
     T5 := X*(X  + 30327873600*X  + 30349045935845547840*X  +

                                                     2
            120328279654140363917091456000000*X  + 164399490757004988415036879412640000000)

N=15
             4        3         2                                2
    T2 := (X  + 272*X  -98776*X  -15713792*X + 773514240) *(X - 449)

              8             6                    4
    T3 := X*(X  + 193153824*X  + 13542540815792160*X  +

                                          2
           40791453850842013992960*X  + 4459777119693624095941077504000)

              8                6                          4
     T5 := X*(X  + 579368436960*X  + 8173036213126267035600*X  +

                                                           2
                2948421249394654853254317120000000*X  + 193895861327178772224183734880266400000000)

N=17

             5        4          3              2                                             2
    T2 := (X  - 456*X  - 716336*X  +195823104*X  +124785737728*X - 13438656184320) *(X+999)

              10              8                         6
     T3 := X*(X   + 2541979176*X  + 198119467658051424 0*X  + 470805560399816932850265600*X  +

                                                   2
               3391496795541799179551606827622 4000*X  + 46359858718913402222477382760183848960000)

              10                    8
     T5 := X*(X   + 26205473373480*X  + 22951348729014501081133920 0*X  +

                                                      4                                                       2
            7711461430642655377888630974647932800000*X  + 727302726371763893278482096922796468827756800000000*X

            25839613550107353408022273387686110935250026161600000000000000)

-----------------------------------------------------------------
P0=11, K=N+2:    KRO(2,P0)=-1,KRO(3,P0)=1,KRO(5,P0)=1,KRO(7,P0)=-1
-----------------------------------------------------------------
N=1

     T2 := X

     T3 := X + 5

     T5 := X + 1

     T7 := X

N=3
```

```
               2
     T2 :- X*(X  + 30)

                  2
     T3 :- (X + 3) *(X - 7)

                    2
     T5 :- (X - 31) * (X + 49)

               2
     T7 :- X*(X  + 3000)

N=5

               4         2
     T2 :- X*(X  + 270*X  + 16680)

             2                 2
     T3 :- (X  - 12*X - 1509) *(X - 10)

                   4
     T5 :- (X + 65) * (X - 74)

               4            2
     T7 :- X*(X  + 393000*X  + 38537472000)

N=7

               6         4           2
     T2 :- X*(X  + 1374*X  + 436560*X  + 40320000)

             3       2                       2
     T3 :- (X  + 18*X  - 6285*X - 201150) *(X + 113)

             3        2                          2
     T5 :- (X  + 224*X  - 525475*X - 31988350) * (X - 1151)

               6              4                    2
     T7 :- X*(X  + 22327704*X  + 102738589578240*X  + 1345440482426880000000)

N=9

               8         6              4                 2
     T2 :- X*(X  + 6030*X  + 11712120*X  + 7669330560*X  + 564269690880)

             4       3          2                              2
     T3 :- (X  + 201*X  -98919*X  -1150929*X+ 1149750126) *(X - 475)

             4         3            2                                    2
     T5 :- (X  - 1215*X - 21311915*X - 2265218325*X + 17429871112150) * (X + 3001)

               8               6                     4
     T7 :- X*(X  + 767889840*X  + 1025822677676496000*X  +

                                     2
           156624989439810976358400000*X  +  633032585807963484596679475200)

N=11

               10          8              6                 4                    2
     T2 :- X*(X   + 30654*X  + 318945120*X  + 1305642637440*X  + 2049564619929600*X  + 957721368231936000)

             5        4          3              2                                          2
     T3 :- (X  - 1218*X  -775914*X  +838214892*X  +189020241225*X + 120422340866250) *(X + 1358)

             5        4             3                  2                                                      2
     T5 :- (X - 13246*X - 413004050*X + 7878939523400*X - 3229823088024375*X + 12308222362848968750) * (X + 25774)

               10               8                            6                                 4
     T7 :- X*(X   + 72369291504*X  + 1579588871009845139520*X + 12964051646785030759215833088000*X  +

                                                  2
          37709819138673185762480264655566929920000*X  + 2318744185066423214284238927254874788724736000000)

-----------------------------------------------------------------
P0=13, K=N+2:    KRO(2,PO)=-1,KRO(3,PO)=1,KRO(5,PO)=-1,KRO(7,PO)=-1
-----------------------------------------------------------------
N=2

             2
     T2 :- X  + 9

                  2
     T3 :- (X + 1)

             2
     T5 :- X  + 81

             2
     T7 :- X  + 225
```

N=4

$$T2 := X^{6} + 161*X^{4} + 5856*X^{2} + 18864$$

$$T3 := (X^{3} - 8*X^{2} - 549*X + 4068)^{2}$$

$$T5 := X^{6} + 8018*X^{4} + 13754433*X^{2} + 2485690416$$

$$T7 := X^{6} + 82950*X^{4} + 1662348177*X^{2} + 423560602764$$

N=6

$$T2 := X^{6} + 449*X^{4} + 37224*X^{2} + 205776$$

$$T3 := (X^{3} +28*X^{2} - 2601*X - 71748)^{2}$$

$$T5 := X^{6} + 243506*X^{4} + 1206410825*X^{2} + 93756690000$$

$$T7 := X^{6} + 847206*X^{4} + 231424342425*X^{2} + 20471634652072500$$

N=8

$$T2 := X^{10} + 3841*X^{8} + 5134480*X^{6} + 2823572208*X^{4} + 614223235584*X^{2} + 43308450164736$$

$$T3 :=(X^{5} + X^{4} - 66033*X^{3} + 1260423*X^{2} + 530326440*X + 14266185264)^{2}$$

$$T5 := X^{10} + 14820283*X^{8} + 74785768290163*X^{6} + 146559998245698565881*X^{4}$$
$$+ 873305044665886448091944000*X^{2} + 12065478109519129517186006240000$$

$$T7 := X^{10} + 252125259*X^{8} + 23724928789729587*X^{6} + 1025407325324195954977977*X^{4} +$$
$$19661129805887483504404526084736*X^{2} + 121307703706137674344780717867862132400$$

N=10

$$T2 := X^{12} + 18433*X^{10} + 121088056*X^{8} + 340607607312*X^{6} +$$
$$380893885719552*X^{4} + 134825856231997440*X^{2} + 14974254765897154 56$$

$$T3 := (X^{6} + 244*X^{5} - 665334*X^{4} - 129598956*X^{3} + 109163403621*X^{2} + 14522233287672*X - 255121008509808)^{2}$$

$$T5 := X^{12} + 289917556*X^{10} + 32326953002900950*X^{8} + 1726712418063587931532500*X^{6} +$$
$$44108094881553049831926298640625*X^{4} + 430033290962195234920750132329450000000*X^{2} +$$
$$10886105645673774994569770130605197500000000$$

$$T7 := X^{12} + 13650769356*X^{10} + 6446583670028092126 2*X^{8} + 139418894150631875357617076028*X^{6} +$$
$$143785268511480525150168789070017931401*X^{4} + 63753954827004609548776322006655133952858100000*X^{2} +$$
$$9054507376401194828902343707876292621596570213493750000$$

P0=17, K=N+2: KRO(2,P0)=1,KRO(3,P0)=-1,KRO(5,P0)=-1,KRO(7,P0)=-1

N=2

$$T2 := (X^{2} + X - 8)^{2}$$

$$T3 := X^{4} + 74*X^{2} + 1072$$

$$T5 := X^{4} + 480*X^{2} + 38592$$

$$T7 := X^{4} + 530*X^{2} + 68608$$

N=4

$$T2 := (X^3 + X^2 - 68 \cdot X - 36)^2$$

$$T3 := X^6 + 668 \cdot X^4 + 145216 \cdot X^2 + 10185984$$

$$T5 := X^6 + 9488 \cdot X^4 + 8442048 \cdot X^2 + 40743936$$

$$T7 := X^6 + 71708 \cdot X^4 + 1048887424 \cdot X^2 + 2346850713600$$

N=6

$$T2 := (X^5 + 9 \cdot X^4 - 452 \cdot X^3 - 2988 \cdot X^2 + 27904 \cdot X + 83616)^2$$

$$T3 := X^{10} + 16832 \cdot X^8 + 93191572 \cdot X^6 + 192821327856 \cdot X^4 + 116860780245888 \cdot X^2 + 9421474370420736$$

$$T5 := X^{10} + 351440 \cdot X^8 + 44989957632 \cdot X^6 + 2580556932172800 \cdot X^4$$
$$+ 65876023734658560000 \cdot X^2 + 6029743597069271040000000$$

$$T7 := X^{10} + 4233136 \cdot X^8 + 5824132863636 \cdot X^6 + 2871845375371443376 \cdot X^4 +$$
$$497996015831560956471424 \cdot X^2 + 14855656017369895192889851904$$

N=8

$$T2 := (X^6 - 15 \cdot X^5 - 1892 \cdot X^4 + 20460 \cdot X^3 + 770176 \cdot X^2 - 3195840 \cdot X - 6636441)^2$$

$$T3 := X^{12} + 122690 \cdot X^{10} + 5157152560 \cdot X^8 + 87983684680032 \cdot X^6 +$$
$$612743619071665152 \cdot X^4 + 1335826553351738886144 \cdot X^2 + 203949399568932198678528$$

$$T5 := X^{12} + 13939648 \cdot X^{10} + 67854209805568 \cdot X^8 + 136905662305154384896 \cdot X^6 + 103030638845234136672153600 \cdot$$
$$X^4 + 23873047875692895959460126720000 \cdot X^2 + 2132021607333312666110860984320000000$$

$$T7 := X^{12} + 181444282 \cdot X^{10} + 8551923317087424 \cdot X^8 +$$
$$145015964651608425915232 \cdot X^6 + 922072716536803810905054408448 \cdot X^4$$
$$+ 2318685324256549381944604148484046848 \cdot X^2 + 196767658578850959128594953227006671585 2800$$

PO=19, K=N+2: KRO(2,PO)=-1,KRO(3,PO)=-1,KRO(5,PO)=1,KRO(7,PO)=1

N=1

$$T2 := X \cdot (X^2 + 13)$$

$$T3 := X \cdot (X^2 + 13)$$

$$T5 := (X - 4)^2 \cdot (X + 9)$$

$$T7 := (X + 5)^3$$

N=3

$$T2 := X \cdot (X^4 + 35 \cdot X^2 + 142)$$

$$T3 := X \cdot (X^4 + 301 \cdot X^2 + 5112)$$

$$T5 := (X^2 + 21 \cdot X + 92)^2 \cdot (X - 31)$$

$$T7 := (X^2 - 68 \cdot X + 499)^2 \cdot (X + 73)$$

N=5

$$T2 := X \cdot (X^8 + 483 \cdot X^6 + 75582 \cdot X^4 + 4242376 \cdot X^2 + 71047680)$$

$$T3 := X\cdot(X^{8} + 3442\cdot X^{6} + 4292649\cdot X^{4} + 2281096296\cdot X^{2} + 432254085120)$$

$$T5 := (X^{4} - 54\cdot X^{3} - 49415\cdot X^{2} + 3367200\cdot X + 292006000)^{2} \cdot (X + 54)$$

$$T7 := (X^{4} + 70\cdot X^{3} - 157380\cdot X^{2} - 29481334\cdot X - 1276939885)^{2} \cdot (X-160)$$

610

N=7

$$T2 := X\cdot(X^{12} + 2323\cdot X^{10} + 2010462\cdot X^{8} + 803113072\cdot X^{6} + 150633270400\cdot X^{4} + 12173735396352\cdot X^{2} + 333034797957120)$$

$$T3 := X\cdot(X^{12} + 59719\cdot X^{10} + 1354569075\cdot X^{8} + 14270784462117\cdot X^{6} + 66670855305320376\cdot X^{4} + 99071703704871505152\cdot X^{2} + 33664128506976532561920)$$

$$T5 := (X^{6} - 4\cdot X^{5} - 1446203\cdot X^{4} + 95652050\cdot X^{3} + 409166434600\cdot X^{2} - 102103842940000\cdot X + 6563900254320000)^{2} \cdot (X + 289)$$

$$T7 := X^{6} - 1843\cdot X^{5} - 25578196\cdot X^{4} + 37453164210\cdot X^{3} + 157007096825425\cdot X^{2} - 139069305605381375\cdot X - 21933742012221418750)^{2} \cdot (X - 527)$$

PO=23, K=N+2: KRO(2,PO)=1, KRO(3,PO)=1, KRO(5,PO)=-1, KRO(7,PO)=-1

N=1

$$T2 := X^{3} - 12\cdot X + 7$$

$$T3 := X^{3} - 27\cdot X + 38$$

$$T5 := X^{3}$$

$$T7 := X^{3}$$

N=3

$$T2 := (X^{2} + 4\cdot X - 2)^{2} \cdot (X^{3} - 48\cdot X + 79)$$

$$T3 := (X^{2} + 6\cdot X - 45)^{2} \cdot (X^{3} - 243\cdot X + 14)$$

$$T5 := X^{3} \cdot (X^{4} + 2556\cdot X^{2} + 1270188)$$

$$T7 := X^{3} \cdot (X^{4} + 11988\cdot X^{2} + 31754700)$$

N=5

$$T2 := (X^{4} - 4\cdot X^{3} - 162\cdot X^{2} + 920\cdot X + 832)^{2} \cdot (X + 7)\cdot(X^{2} - 7\cdot X - 143)$$

$$T3 := (X^{4} - 15\cdot X^{3} - 957\cdot X^{2} + 13293\cdot X - 12870)^{2} \cdot (X + 38) \cdot (X^{2} - 38\cdot X - 743)$$

$$T5 := X^{3} \cdot (X^{8} + 95100\cdot X^{6} + 3184494300\cdot X^{4} + 44006549508000\cdot X^{2} + 214214641502400000)$$

$$T7 := X^{3} \cdot (X^{8} + 660492\cdot X^{6} + 152231816700\cdot X^{4} + 13982809796769600\cdot X^{2} + 400834240321008384000)$$

N=7

$$T2 := (X^{6} + 4\cdot X^{5} - 850\cdot X^{4} - 3248\cdot X^{3} + 147872\cdot X^{2} + 268672\cdot X - 5317760)^{2} \cdot (X^{3} - 768\cdot X + 1951)$$

$$T3 := (X^{6} + 38\cdot X^{5} - 20508\cdot X^{4} - 1030644\cdot X^{3} + 86837139\cdot X^{2} + 5371429140\cdot X + 55514443500)^{2} \cdot (X^{3} - 19683\cdot X + 1062686)$$

$$T5 := X^{3} \cdot (X^{12} + 3434556\cdot X^{10} + 4503520431468\cdot X^{8} + 2817283398730424640\cdot X^{6} + 847955819735403719760000\cdot X^{4} + 103782437973914306469472512000\cdot X^{2} + 2208782254549937077079536204800000)$$

$$T7 := X^{3}*(X^{12} + 40494132*X^{10} + 605958970060332*X^{8} + 409682115221540142240 0*X^{6} + 12087187496206708701149510400*X^{4} + 1154031169130311733611855781068 8000*X^{2} + 2526073889115665118986184384442368 00000)$$

P0=29, K=N+2: KRO(2,P0)=-1,KRO(3,P0)=-1,KRO(5,P0)=1,KRO(7,P0)=1

N=2

$$T2 := X^{6} + 38*X^{4} + 301*X^{2} + 560$$

$$T3 := X^{6} + 61*X^{4} + 791*X^{2} + 875$$

$$T5 := (X^{3} - 11*X^{2} - 133*X + 1071)^{2}$$

$$T7 := (X^{3} + 14*X^{2} - 108*X - 1192)^{2}$$

N=4

$$T2 := X^{12} + 278*X^{10} + 28285*X^{8} + 1260472*X^{6} + 22944832*X^{4} + 140087936*X^{2} + 966400$$

$$T3 := X^{12} + 2245*X^{10} + 1884878*X^{8} + 715200530*X^{6} + 112977325989*X^{4} + 4281127461369*X^{2} + 46577165867100$$

$$T5 := (X^{6} - 23*X^{5} - 12280*X^{4} + 235866*X^{3} + 33953337*X^{2} - 384523443*X - 2627317458)^{2}$$

$$T7 := (X^{6} - 10*X^{5} - 76080*X^{4} + 1925088*X^{3} + 1377655664*X^{2} - 73626194400*X - 519034134784)^{2}$$

N=6

$$T2 := X^{16} + 1382*X^{14} + 744077*X^{12} + 200869632*X^{10} + 28931822432*X^{8} + 2155663113216*X^{6} + 71710495842560*X^{4} + 663330761523200*X^{2} + 590388176896000$$

$$T3 := X^{16} + 22051*X^{14} + 187767701*X^{12} + 793510274339*X^{10} + 1809033803032599*X^{8} + 2281021494195869649*X^{6} + 1527214705246483000335*X^{4} + 458931418705423915202025*X^{2} + 30465487147014831010162500$$

$$T5 := (X^{8} + 99*X^{7} - 276993*X^{6} - 31299849*X^{5} + 17584369885*X^{4} + 1686634037625*X^{3} - 196943514064875*X^{2} - 14966521618921875*X - 200278684287731250)^{2}$$

$$T7 := (X^{8} - 330*X^{7} - 3716228*X^{6} + 233875960*X^{5} + 3438911219312*X^{4} + 1091616310004000*X^{3} - 293827217111058624*X^{2} - 89873092347162858880*X + 8119186526578407384064)^{2}$$

P0=31, K=N+2: KRO(2,P0)=1,KRO(3,P0)=-1,KRO(5,P0)=1,KRO(7,P0)=1

N=1

$$T2 := (X + 1)^{2} * (X^{3} - 12*X + 15)$$

$$T3 := X^{3}*(X^{2} + 26)$$

$$T5 := (X - 2)^{2} * (X^{3} - 75*X + 246)$$

$$T7 := (X - 8)^{2} * (X^{3} - 147*X + 430)$$

N=3

$$T2 := (X^{3} + X^{2} - 30*X + 6) * (X^{2} - 48*X - 97)^{3}$$

$$T3 := X^{3}*(X^{6} + 398*X^{4} + 49236*X^{2} + 1934136)$$

$$T5 := (X^{3} + 4*X^{2} - 291*X + 1014) * (X^{2} - 1875*X - 29256)^{3}$$

$$T7 := (X + 66 \cdot X - 1005 \cdot X - 31688) \cdot (X - 7203 \cdot X + 50398)$$

N=5

$$T2 := (X^{6} + X^{5} - 222 \cdot X^{4} - 370 \cdot X^{3} + 9416 \cdot X^{2} + 13440 \cdot X - 90624) \cdot (X + 15)^{2} \cdot (X^{2} - 15 \cdot X + 33)$$

$$T3 := X^{3} \cdot (X^{12} + 7208 \cdot X^{10} + 19859688 \cdot X^{8} + 26566749360 \cdot X^{6} +$$

$$17884354852944 \cdot X^{4} + 5570285336959680 \cdot X^{2} + 590986232936084000)$$

$$T5 := (X^{6} + 73 \cdot X^{5} - 51615 \cdot X^{4} - 3624325 \cdot X^{3} + 522398750 \cdot X^{2}$$

$$+ 25671172500 \cdot X - 103336)^{2} \cdot (X^{2} - 246 \cdot X + 13641) \cdot (X+246)$$

$$T7 := (X^{6} - 3 \cdot X^{5} - 207897 \cdot X^{4} - 2308819 \cdot X^{3} + 13269144858 \cdot X^{2}$$

$$+ 215614693848 \cdot X - 247)^{2} \cdot (X^{2} - 430 \cdot X - 168047) \cdot (X+430)$$

PO=37, K=N+2: KRO(2,PO)=-1,KRO(3,PO)=1,KRO(5,PO)=-1,KRO(7,PO)=1

N=2

$$T2 := X^{8} + 50 \cdot X^{6} + 709 \cdot X^{4} + 3000 \cdot X^{2} + 1764$$

$$T3 := (X^{4} + 3 \cdot X^{3} - 50 \cdot X^{2} - 57 \cdot X + 427)^{2}$$

$$T5 := X^{8} + 431 \cdot X^{6} + 29521 \cdot X^{4} + 588072 \cdot X^{2} + 2039184$$

$$T7 := (X^{4} - 2 \cdot X^{3} - 587 \cdot X^{2} + 2460 \cdot X + 53892)^{2}$$

N=4

$$T2 := X^{16} + 390 \cdot X^{14} + 60701 \cdot X^{12} + 4799932 \cdot X^{10} + 203487156 \cdot X^{8}$$

$$+ 4519465040 \cdot X^{6} + 48993644736 \cdot X^{4} + 211923220224 \cdot X^{2} + 178006118400$$

$$T3 := (X^{8} + 9 \cdot X^{7} - 1280 \cdot X^{6} - 11016 \cdot X^{5} + 422488 \cdot X^{4} + 2751084 \cdot X^{3} - 25673805 \cdot X^{2} - 30714957 \cdot X + 141986196)^{2}$$

$$T5 := X^{16} + 31026 \cdot X^{14} + 373650779 \cdot X^{12} + 2220056867434 \cdot X^{10} + 6834316986168825 \cdot X^{8} + 10475224449621004436 \cdot X^{6}$$

$$+ 6885539411711705092656 \cdot X^{4} + 1110302609356408225416384 \cdot X^{2} + 19726944242324026399110144$$

$$T7 := (X^{8} - 95 \cdot X^{7} - 54561 \cdot X^{6} + 2410919 \cdot X^{5} + 907038560 \cdot X^{4} + 534484632 \cdot X^{3}$$

$$- 3494999585616 \cdot X^{2} - 2731120576272 \cdot X + 3409346511153792)^{2}$$

PO=41, K=N+2: KRO(2,PO)=1,KRO(3,PO)=-1,KRO(5,PO)=1,KRO(7,PO)=-1

N=2

$$T2 := (X^{5} + 3 \cdot X^{4} - 25 \cdot X^{3} - 51 \cdot X^{2} + 104 \cdot X + 32)^{2}$$

$$T3 := X^{10} + 180 \cdot X^{8} + 10910 \cdot X^{6} + 276172 \cdot X^{4} + 2531856 \cdot X^{2} + 524672$$

$$T5 := (X^{5} + 2 \cdot X^{4} - 282 \cdot X^{3} - 1400 \cdot X^{2} + 9016 \cdot X + 43904)^{2}$$

$$T7 := X^{10} + 1912 \cdot X^{8} + 1274822 \cdot X^{6} + 344662636 \cdot X^{4} + 30875879696 \cdot X^{2} + 87767656832$$

PO=43, K=N+2: KRO(2,PO)=-1,KRO(3,PO)=-1,KRO(5,PO)=-1,KRO(7,PO)=-1

N=1

$$T2 := X \cdot (X^{6} + 20 \cdot X^{4} + 121 \cdot X^{2} + 214)$$

$$T3 := X \cdot (X^{6} + 45 \cdot X^{4} + 431 \cdot X^{2} + 214)$$

$$T5 := X \cdot (X^{6} + 117 \cdot X^{4} + 3863 \cdot X^{2} + 25894)$$

$$T7 := X*(X^{6} + 150*X^{4} + 4896*X^{2} + 3424)$$

P0=47, K=N+2: KRO(2,P0)=1,KRO(3,P0)=1,KRO(5,P0)=-1,KRO(7,P0)=1

N=1

$$T2 := (X + 1)^{2} * (X^{5} - 20*X^{3} + 80*X - 17)$$

$$T3 := (X + 2)^{2} * (X^{5} - 45*X^{3} + 405*X - 298)$$

$$T5 := X^{5} *(X^{2} + 78)$$

$$T7 := (X + 4)^{2} * (X^{5} - 245*X^{3} + 12005*X - 31922)$$

N=3

$$T2 := (X^{5} + X^{4} - 40*X^{3} + 12*X^{2} + 300*X - 316)^{2} * (X^{5} - 80*X^{3} + 1280*X + 1759)$$

$$T3 := (X^{5} - 4*X^{4} - 207*X^{3} + 576*X^{2} + 7803*X + 9558)^{2} * (X^{5} - 405*X^{3} + 32805*X + 29294)$$

$$T5 := X^{5} *(X^{10} + 5490*X^{8} + 10917588*X^{6} + 9407020248*X^{4} + 3230761626000*X^{2} + 270690407718048)$$

$$T7 := (X^{5} - 14*X^{4} - 2905*X^{3} - 45230*X^{2} - 141377*X + 94796)^{2} * (X^{5} - 12005*X^{3} + 28824005*X - 454063586)$$

P0=53, K=N+2: KRO(2,P0)=-1,KRO(3,P0)=1,KRO(5,P0)=-1,KRO(7,P0)=1

N=2

$$T2 := X^{12} + 60*X^{10} + 2356*X^{8} + 30996*X^{6} + 176575*X^{4} + 393232*X^{2} + 285376$$

$$T3 := X^{12} + 215*X^{10} + 16178*X^{8} + 505118*X^{6} + 5738621*X^{4} + 15503831*X^{2} + 673036$$

$$T5 := X^{12} + 789*X^{10} + 196604*X^{8} + 18690640*X^{6} + 682399088*X^{4} + 6573121072*X^{2} + 2960741056$$

$$T7 := (X^{6} - 12*X^{5} - 1052*X^{4} + 10868*X^{3} + 215348*X^{2} - 624840*X - 9386656)^{2}$$