A FAMILY OF CUBIC RATIONAL MAPS AND MATINGS OF CUBIC POLYNOMIALS

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Abstract. We present an example of mating of two cubic polynomials, which shows that some technique used in quadratic matings fails in higher degree. More precisely, our example has a Thurston's obstruction, does not have a Levy cycle and the quotient of the sphere by the ray-equivalence is homeomorphic to the sphere. We also analyze a certain family of cubic rational maps.

INTRODUCTION

Many mathematicians are interested in the dynamics of complex polynomials and rational maps. In this paper we try to understand the dynamics of a rational map as a mating of two polynomials. This point of view was proposed by Douady and Hubbard [D] in 1982.

There are two kinds of matings.

For the first one, we construct a branched map of the sphere from two polynomials f , g of the same degree, this map is called the *formal mating* of f and g . More precisely, we add to $\mathbb{C}$ a circle at infinity, then f and g can be extended continuously to this circle at infinity in a natural way. We sew up two copies of $\mathbb{C}$ at the circle of infinity to get a topological sphere. We can then define a self branched covering of the sphere which coincides with f on one hemi-sphere and with g on the other hemi-sphere. Thurston defined an isotopy equivalence relation between postcritically finite branched coverings of the sphere (i.e. the orbits orbits of critical points are finite). If f and g are postcritically finite, and their formal mating (which might be modified in some case) is equivalent (in Thurston's sense) to a rational map R , we say that these two polynomials are *matable.*

For the second one, we sew directly the filled-in Julia sets of f and g at their boundary by identifying the point of external angle t for f with the point of external angle $-t$ for g . If the induced map by f and g on this space is conjugate to a rational map R', then we say that f and g are *analytically matable.* This requires in fact three things: the new space should be a topological sphere, the induced map should be a branched covering, and it should be conjugate to R' .

There are a lot of questions on this subject. For example, what are the conditions for two polynomials to be matable? How can we interprete these conditions in the parameter space of polynomials? Can we extend the Thurston's equivalence and hence the definition of matability to non-postcritically finite cases? Is the matability equivalent to the analytical matability, and $R = R'$? Are the correspondences $(f,g) \to R$, $(f,g) \to R'$ injective, continuous?

To analyze these, we need the theory developped by Thurston [Th], Douady, Hubbard [DH2] and others.

In case of mating of quadratic polynomials, some answers to the above questions have been obtained by the work of S. Levy [L], B. Wittner [W], M. Rees [R], and Tan Lei [TL1],[TL2],[TL3]. But the situation in higher degree is no more so clear. In fact besides Tan Lei's work which generelized the degree two results to polynomials of the form $z \to z^d + c$, we had known almost nothing about that until we discovered a mating of two specific cubic polynomials at the Max-Planck-Institut in Bonn in April 1988. This example shows that the main tool in degree two case - Levy's theorem - fails in higher degree, moreover, by sewing together the boundary of the filled-in Julia sets of f and g , one may also get a topological sphere even though the two polynomials are not matable.

In this paper we state our main results on this special mating and we analyze a family of cubic rational maps related this mating. We state also our observations in computer experiments. In Chapter I we review several definitions of matings which were sometimes confused in quadratic case, and we summerize the known results and our main results. The proofs of the main results are given in Chapter II. In Chapter III we state some numerical observations and some related results.

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§I. Review of known results and main results

§I.1 Definitions

In this section, we give definitions of several kinds of matings.

1.1 Definition (formal mating). Let f and g be two monic polynomials of degree d. Let

$$\tilde{C} = C \cup \{\infty \cdot e^{2\pi i s} |\ s \in T = R/Z\}.$$

Then f and g extend continuously to $\tilde{C}$ by setting

$$f(\infty \cdot e^{2\pi i s}) = \infty \cdot e^{2d\pi i s},\ g(\infty \cdot e^{2\pi i s}) = \infty \cdot e^{2d\pi i s}.$$

Let

$$S^2_{f,g} = \tilde{C}_f \sqcup \tilde{C}_g / (\infty \cdot e^{2\pi i s}, f) \approx (\infty \cdot e^{-2\pi i s}, g).$$

The *formal mating* of f and g is defined to be the branched covering $f \perp\!\!\!\perp g : S^2_{f,g} \to S^2_{f,g}$ such that

$$f \perp\!\!\!\perp g = f \text{ on } \tilde{C}_f \text{ and } f \perp\!\!\!\perp g = g \text{ on } \tilde{C}_g.$$

In case there is no ambiguity, we write S^2 instead of $S^2_{f,g}$.

1.2 Definition (postcritical set). Suppose $F : S^2 \to S^2$ is a branched covering. In the following, we always assume that branched coverings are orientation preserving and of degree greater than one. Let

$$\Omega_F = \{\text{critical points of } F\} \quad \text{and} \quad P_F = \bigcup_{n>0} F^n(\Omega_F).$$

A branched covering F is called *postcritically finite*, if P_F is finite.

From now on let us suppose f and g are postcritically finite polynomials of the same degree and let $F = f \perp\!\!\!\perp g$. Then F is also postcritically finite.

See [DH1] for the definitions of the filled-in Julia set K_f for a polynomial f and the external rays in C For $\theta \in T$, let us denote by $R_f(\theta)$ the closure in $\tilde{C}_f$ of the external ray of angle θ (we recall that if f is postcritically finite, then K_f is connected and locally connected, and $R_f(\theta)$ is well defined).

In $S^2_{f,g}$, the external rays $R_f(\theta)$ and $R_g(-\theta)$ are connected at the point $(\infty \cdot e^{2\pi i\theta}, f)$.

1.3 Definition (ray-equivalence). For x and y in $\tilde{C}_f$, we define $x \sim_f y$ if x and y are in the same $R_f(\theta)$ for some θ. The relation $\sim_g$ on $\tilde{C}_g$ is similarly defined. In $S^2_{f,g}$, let $\sim_F$ be the equivalence relation generated by $\sim_f$ on $\tilde{C}_f$ and $\sim_g$ on $\tilde{C}_g$. This relation is called the *ray- equivalence* for F. Denote by $[x]$ the ray-equivalence class of $x \in S^2$.

1.4 DEFINITION (degenerate mating). Let $[x_1]$, $[x_2], \ldots$, $[x_m]$ be the equivalence classes in S^2 such that $\#[x_i] \cap P_F \geq 2$. Let $[y_1]$, $[y_2], \ldots$, $[y_n]$ be the equivalence classes in

$$\bigcup_{n \geq 0} F^{-n}(\bigcup_i [x_i]),$$

such that $[y_i] \cap (P_F \cup \Omega_F) \neq \phi$.

Remark that each $[y_i]$ consists of only a finite number of external rays, since each postcritical point is preperiodic. If none of the $[y_i]$ contain closed curves, then all of $[y_i]$ are topological trees. In this case, collapsing each $[y_i]$ to one point and we get a new space S'^2 which is homeomorphic to S^2 . We modify then F in a neighborhood of each $[y_i]$ and make a new branched covering $F' : S'^2 \to S'^2$. We call F' the *degenerate mating* of f and g .

1.5 REMARK. In the case that there is no such $[x_i]$, set $F' = F$ by convention. In the case that some $[y_i]$ contains a closed curve, the degenerate mating does not exist.

1.6 DEFINITION (Thurston's equivalence). We say two postcritically finite branched coverings F and G are *equivalent*, $F \sim G$, if there exist two orientation preserving homeomorphisms $\theta_1, \theta_2 : S^2 \to S^2$ such that

$$\theta_i(P_F) = P_G \ (i = 1, 2), \ \theta_1 = \theta_2 \text{ on } P_F, \ \theta_1 \text{ and } \theta_2 \text{ are isotopic relative to } P_F$$

(we write rel P_F) , and the following diagram commutes:

$$\begin{array}{ccc} S^2 & \xrightarrow{\theta_1} & S^2 \\ F \downarrow & & \downarrow G \\ S^2 & \xrightarrow{\theta_2} & S^2. \end{array}$$

1.7 DEFINITION (matability). We say that two polynomials f and g are *matable* if the degenerate mating F' exists and is equivalent (in the sense of Thurston's equivalence) to a rational map.

1.8 DEFINITION (topological mating). Let us denote by $\gamma_f(\theta)$ the landing point of $R_f(\theta)$ on ∂K_f . We denote by X the space

$$K_f \sqcup K_g / \gamma_f(\theta) \approx \gamma_g(-\theta)$$

which is equal to

$$S^2 / \sim_F \ .$$

And we can define the induced mapping $F^* = [f \perp\!\!\!\perp g] : X \to X$. If X is homeomorphic to the sphere, we call F^* the *topological mating* of f and g .

1.9 DEFINITION (analytical matability). If F^* is topologically conjugate to a rational map, we say that f and g are *analytically matable* .

§I.2 THURSTON'S AND LEVY'S RESULTS

In this section, we summerize Thurston's and Levy's results. For details and proofs, see [**Th**], [**DH2**].

2.1 DEFINITION. Let $F : S^2 \to S^2$ be a postcritically finite branched covering. A simple closed curve in $S^2 - P_F$ is called *peripheral* if it bounds a disc containing at most one point of P_F. A *multicurve* Γ is a collection of disjoint simple closed curves in $S^2 - P_F$, such that none of them is peripheral and no two curves are homotopic to each other in $S^2 - P_F$. A multicurve Γ is *F-invariant*, if

$$F^{-1}(\Gamma) = \{\text{connected components of } F^{-1}(\gamma) | \gamma \in \Gamma\}$$

consists of peripheral curves and curves which are homotopic to curves in Γ.

2.2 DEFINITION. For a multicurve Γ, the *Thurston's linear transformation* F_Γ is a linear map from $\mathbb{R}^\Gamma = \{\sum_{\gamma \in \Gamma} c_\gamma \gamma | \ c_\gamma \in \mathbb{R}\}$ to itself defined by

$$F_\Gamma(\gamma) = \sum_{\gamma' \subset F^{-1}(\gamma)} \frac{1}{deg(F : \gamma' \to \gamma)} [\gamma']_\Gamma \quad \text{for } \gamma \in \Gamma,$$

where the sum is over all components γ' of $F^{-1}(\gamma)$, and $[\gamma']_\Gamma$ denotes the curve in Γ homotopic to γ' if it exists and $[\gamma']_\Gamma = 0$ otherwise. We denote by λ_Γ the leading eigenvalue of F_Γ.

2.3 THEOREM(THURSTON). *Suppose $F : S^2 \to S^2$ is a postcritically finite branched covering with a hyperbolic orbifold (see* [**DH2**], *for definition). Then F is equivalent to a rational map, if and only if there is no F-invariant multicurve Γ with $\lambda_\Gamma \geq 1$.*

2.4 REMARK. If the orbifold is not hyperbolic, then $F^{-1}(P_F) \subset \Omega_F \cup P_F$ and $\#P_F \leq 4$. Therefore branched coverings with non-hyperbolic orbifolds are considered to be exceptional.

An F-invariant curve Γ with $\lambda_\Gamma \geq 1$ is called *Thurston's obstruction.*

2.5 DEFINITION. A multicurve $\gamma_1, ..., \gamma_n$ is called *Levy cycle*, if each $F^{-1}(\gamma_{i+1})$ contains a component γ_i' homotopic in $S^2 - P_F$ to γ_i and $F : \gamma_i' \to \gamma_{i+1}$ is of degree one $(i = 0, ..., n-1)$, where $\gamma_0 = \gamma_n$.

2.6 THEOREM(LEVY). ([**L**], [**TL2**]) *Let F be a postcritically finite branched covering of degree two. There exists a Thurston's obstruction for F, if and only if there exists a Levy cycle for F.*

See Theorem III.3.3

2.7 REMARK. A branched covering F is called *topological polynomial*, if there is a point $\infty \in S^2$ such that $F^{-1}(\infty) = \infty$). The Levy's theorem also holds for postcritically finite topological polynomials.

§I.3 Degree two case

Douady and Hubbard had a conjecture about the condition for two quadratic polynomials to be matable [D]. By using the Thurston-Levy results (§I.2), Mary Rees and Tan Lei have proved the conjecture almost completely. In this section, we summerize some of their results.

Every quadratic polynomial is affinely conjugate to $f_c : z \to z^2 + c$ for some value c . Let us recall here some notations of Douady and Hubbard:

$K_c = K_{f_c}$ is the filled-in Julia set;

$M_2 = \{ c \in \mathbb{C} \mid K_c \text{ is connected } \}$ is the Mandelbrot set;

$\mathcal{D} = \{ c \in \mathbb{C} \mid f_c \text{ is postcritically finite } \}$;

W_0 = the component of $\mathrm{int}(M_2)$ containing 0 .

3.1 Theorem. ([R] and [TL2]). *For $c, c' \in \mathcal{D}$, f_c and f'_c are matable if and only if for each fixed point α of f_c and f'_c , the equivalence class $[\alpha]$ in $S^2_{c,c'}$ does not contain any closed curve.*

This theorem has a beautiful interpretation in the parameter space M_2 :

3.2 Theorem. ([R] and [TL2]). *Suppose $c, c' \in \mathcal{D}$. Then f_c and f'_c are matable if and only if c and $\bar{c}'$ are not in the same connected component of $M_2 - \overline{W}_0$, where $\bar{c}'$ denotes the complex conjugate of c' .*

Moreover, Mary Rees proved:

3.3 Theorem. ([R]). *Suppose c and c' are periodic for f_c and f'_c . Then the fact that f_c and f'_c are matable implies that the topological mating F^* of f_c and f'_c exists and F^* is topologically conjugate to a rational map. This means in our language that f_c and f'_c are also anatically matable.*

Mary Rees claims that the same result is also true for $c, c' \in \mathcal{D}$ [R, part III].

We give here some more details of the proof of Theorem 3.1 and will show later which part of the proof fails in higher degree case.

3.4 Definition (good and degenerate Levy cycles). Suppose $F : S^2 \to S^2$ is a postcritically finite branched covering, and $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_n\}$ a Levy cycle for F . Then

Γ is *good* if the connected components of $S^2 - \bigcup_i \gamma_i$ are

$$B_1, B_2, ..., B_n, C ,$$

with B_i discs, C not disc, and if $n = 1$, then $C = \phi$ and $F : \gamma' \to \gamma_1$ reverses the orientaiton (where γ' is the component of $F^{-1}(\gamma_1)$ homotopic to γ_1 ; if $n > 1$, then one component C' of $F^{-1}(C)$ is isotopic to C and $F : C' \to C$ is of degree one;

Γ is *degenerate* if the connected components of $S^2 - \bigcup_i \gamma_i$ are

$$B_1, B_2, ..., B_n, C \ ,$$

with B_i discs, C not disc, and each $F^{-1}(B_{i+1})$ has a component B_i' isotopic to B_i (rel P_F) , and $F : B_i' \to B_{i+1}$ is of degree one ($i = 0, 1, .., n-1$), where $B_0 = B_n$.

FIRST REDUCTION. *(Levy's theorem). Suppose F is of degree two. There is a Thurston's obstruction for F if and only if there is a Levy cycle for F .*

SECOND REDUCTION. *Suppose F is of degree two. There is a Levy cycle for F if and only if there is either a degenerate or a good Levy cycle for F .*

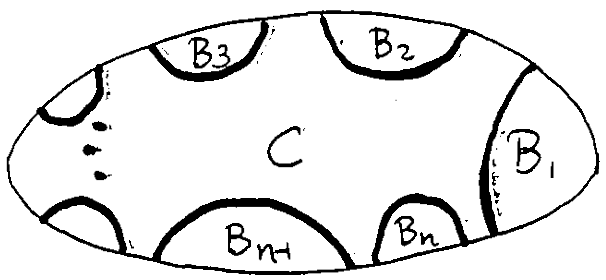
THIRD REDUCTION. *Suppose $F = f_c \perp\!\!\!\perp f_{c'}$, $c, c' \in \mathcal{D}$. Then by using the expansive metric for F near ∂K_c and $\partial K_c'$, we can prove that*

a) each degenerate Levy cycle for F reduces to a cycle of ray-equivalence classes: $[x_0], [x_1], ..., [x_m]$, ($[x_0] = [x_m]$) such that for each i, $F([x_i]) = [x_{i+1}]$, $\#[x_i] \cap P_F \geq 2$, and none of the $[x_i]$ contain closed curves;

b) each non-degenerate Levy cycle for F reduces to a cycle of ray-equivalence classes: $[x_0], [x_1], ..., [x_m]$, ($[x_0] = [x_m]$) such that for each i, $F([x_i]) = [x_{i+1}]$, and $[x_i]$ contains closed curves;

c) each good Levy cycle for F reduces to a ray-equivalence class $[x]$ such that $[x]$ contains at least one closed curve and at least two fixed points of F .

Note that each Levy cycle for the degenerate mating F' lifts to a non-degenerate Lévy cycle for F . By these reductions, Theorem 3.1 is proved.



Good and degenerate Levy Cycle.

§I.4 Main Results

We want to analyze the mating problem of two specific cubic polynomials f_1, f_2 . We will see that some of the results are quite generalizable, and some theorems in degree two case are no more true for cubics.

Let us denote by g_a the cubic polynomial $z \to z^3 + a$.

Let $f_1 = g_c$ for $c = \sqrt[4]{3}e^{7\pi i/12}$. From the point of view of the parameter space, c is the point of the cubic Mandelbrot set

$$M_3 = \{a \mid 0 \in \text{ the filled-in Julia set } K_{g_a} \text{ of } g_a\}$$

of external angle 1/3 . From the point of view of dynamics, the dynamics of the critical point 0 is

$$0 \to c \to \beta \to \beta.$$

Let f_2 be a monic, real cubic polynomial such that it has a periodic cycle:

$$x \to y \to z \to x,$$

where x, y, z are real, x and y are simple critical points of f_2 and $y < x < z$. These conditions uniquely determine f_2 .

Let $F = f_1 \perp\!\!\!\perp f_2$ be the formal mating of f_1 and f_2 . The postcritical set P_F is $\{c, \beta, x, y, z\}$, where c and β denote $c, \beta \in \tilde{C}_{f_1}$ and x, y, z denote $x, y, z \in \tilde{C}_{f_2}$.

4.1 Lemma. *The degenerate mating F' of f_1 and f_2 is equal to F. In other words, there is no ray-equivalence class in $S^2_{f_1,f_2}$ containing more than two points of P_F .*

4.2 Theorem. *The formal mating F has a Thurston's obstruction consisting of two curves, with Thurston's matrix*

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix},$$

whose eigenvalue is 1. Hence $F = F'$ is not equivalent to a rational map, and f_1 and f_2 are not matable.

4.3 Theorem. *The quotient $S^2/\sim_F$ defined in I.1.4 is homeomrphic to a sphere. Hence the topological mating F^* exists.*

4.4 Theorem. *F does not have any Levy cycle.*

These theorems will be proved in §II.

Remark. As we have seen in §I.3, in case of degree two, two polynomials are matable if and only if the quotient $S^2/\sim$ is a sphere and the topological mating F^* exists (M. Rees' theorem). However this is not the case in degree three by Theorems 4.2 and 4.3. Moreover the Levy theorem (or the first reduction in §I.3) fails for this example.

§II. Proofs of main results

In this chapter and the next chapter f_1 and f_2 denote the two specific cubic polynomials defined in §I.4. To simplify the notations, we will use K_i , J_i , R_i , γ_i instead of K_{f_i} , J_{f_i} , R_{f_i} , γ_{f_i} etc. To avoid ambiguity, will denote by ω the critical point of f_1 .

§II.1 Hubbard trees and external angles of f_1 and f_2

The Hubbard tree H_1 of f_1 is shown in Figure 1.1. (See [**DH1**, Exposé IV] for the definition of the Hubbard trees for polunomials.) Figure 1.2 is an extended Hubbard tree $H_1' = f_1^{-1}(H_1)$.

In Figure 1.2, α is a fixed point in $[\omega, c]$; α_1 and α_2 are pre-images of α such that $\alpha_1 \in [\omega, \beta]$ and $\alpha_2 \notin H_1$; and α_{10}, α_{11}, α_{12} are pre-images of α_1 such that $\alpha_{10} \in [\alpha, c]$, $\alpha_{11} \in [\omega, \beta]$, $\alpha_{12} \notin H_1$. Then, by a simple computation, one can show that the external angles of these points are:

$$Ang(\omega) = \{1/9, 4/9, 7/9\}, \ Ang(c) = \{1/3\}, \ Ang(\beta) = \{0\},$$

$$Ang(\alpha) = \{1/8, 3/8\}, \ Ang(\alpha_1) = \{1/24, 19/24\}, \ Ang(\alpha_2) = \{11/24, 17/24\},$$

$$Ang(\alpha_{10}) = \{19/72, 25/72\}, \ Ang(\alpha_{11}) = \{1/72, 67/72\}, \ Ang(\alpha_{12}) = \{43/72, 49/72\}.$$

Figure 1.3 shows the H_1' together with external rays of these angles.

The Hubbard tree H_2 of f_2 and an extended Hubbard tree H_2' are shown in Figure 1.4 and 5.

In Figure 1.5, β', β'' and α' are fixed points such that $[z, \beta'] \cap H_2 = \{z\}$, $[\beta'', y] \cap H_2 = \{y\}$ (or β', β'' real and $\beta'' < y$, $z < \beta'$) and $\alpha' \in [y, x]$; α_1' and α_2' are pre-images of α' such that $\alpha_1' \in [x, z]$ and $\alpha_2' \in [\beta'', y]$; and α_{10}', α_{11}', α_{12}' are pre-images of α_1' such that $\alpha_{10}' \in [y, x]$, $\alpha_{11}' \in [z, \beta']$, $\alpha_{12}' \in [\beta'', y]$; x_*, y_*, z_* are the roots of basins of attraction of x, y, z, respectively (i.e. repelling periodic orbit of period 3 on the boundary of the basins). Then the external angles of these points are:

$$Ang(x_*) = \{3/13, 10/13\}, \ Ang(y_*) = \{4/13, 9/13\}, \ Ang(z_*) = \{1/13, 12/13\},$$

$$Ang(\beta') = \{0\}, \ Ang(\beta'') = \{1/2\},$$

$$Ang(\alpha') = \{1/4, 3/4\}, \ Ang(\alpha_1') = \{1/12, 11/12\}, \ Ang(\alpha_2') = \{5/12, 7/12\},$$

$$Ang(\alpha_{10}') = \{11/36, 25/36\}, \ Ang(\alpha_{11}') = \{1/36, 35/36\}, \ Ang(\alpha_{12}') = \{13/36, 23/36\}.$$

Figure 1.6 shows the H_2' together with external rays of these angles.

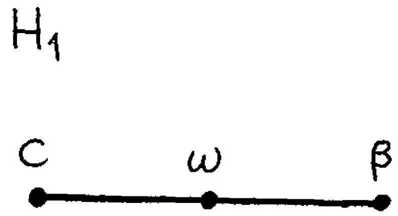


Figure 1.1

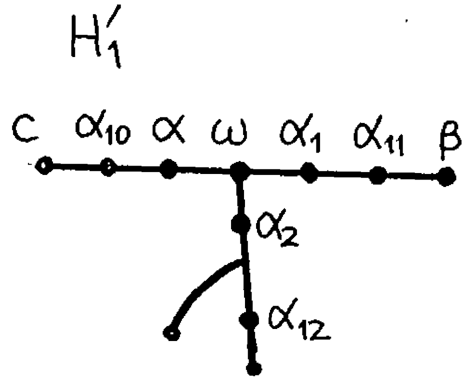


Figure 1.2

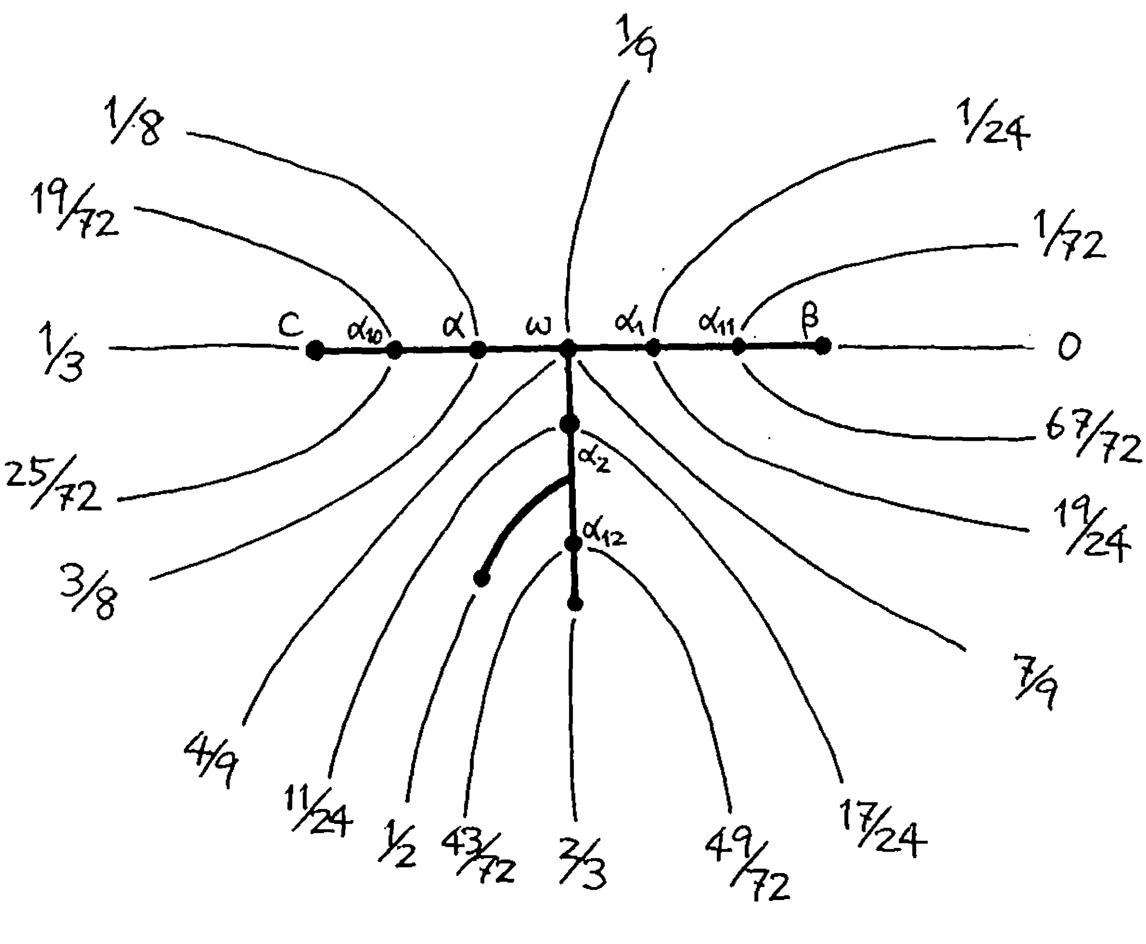


Figure 1.3

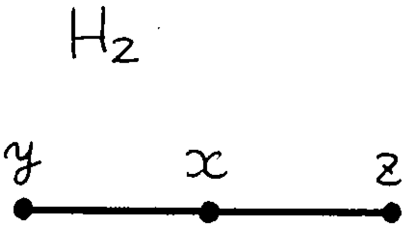


Figure 1.4

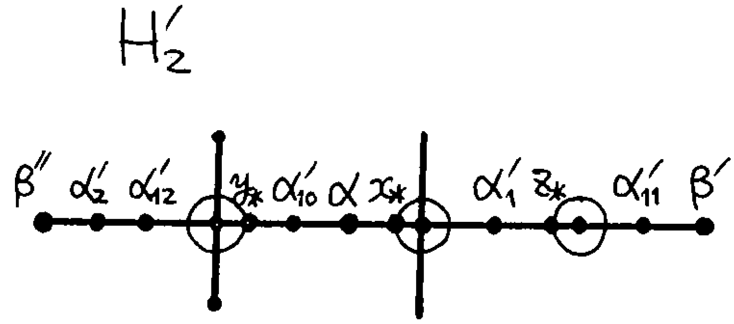


Figure 1.5

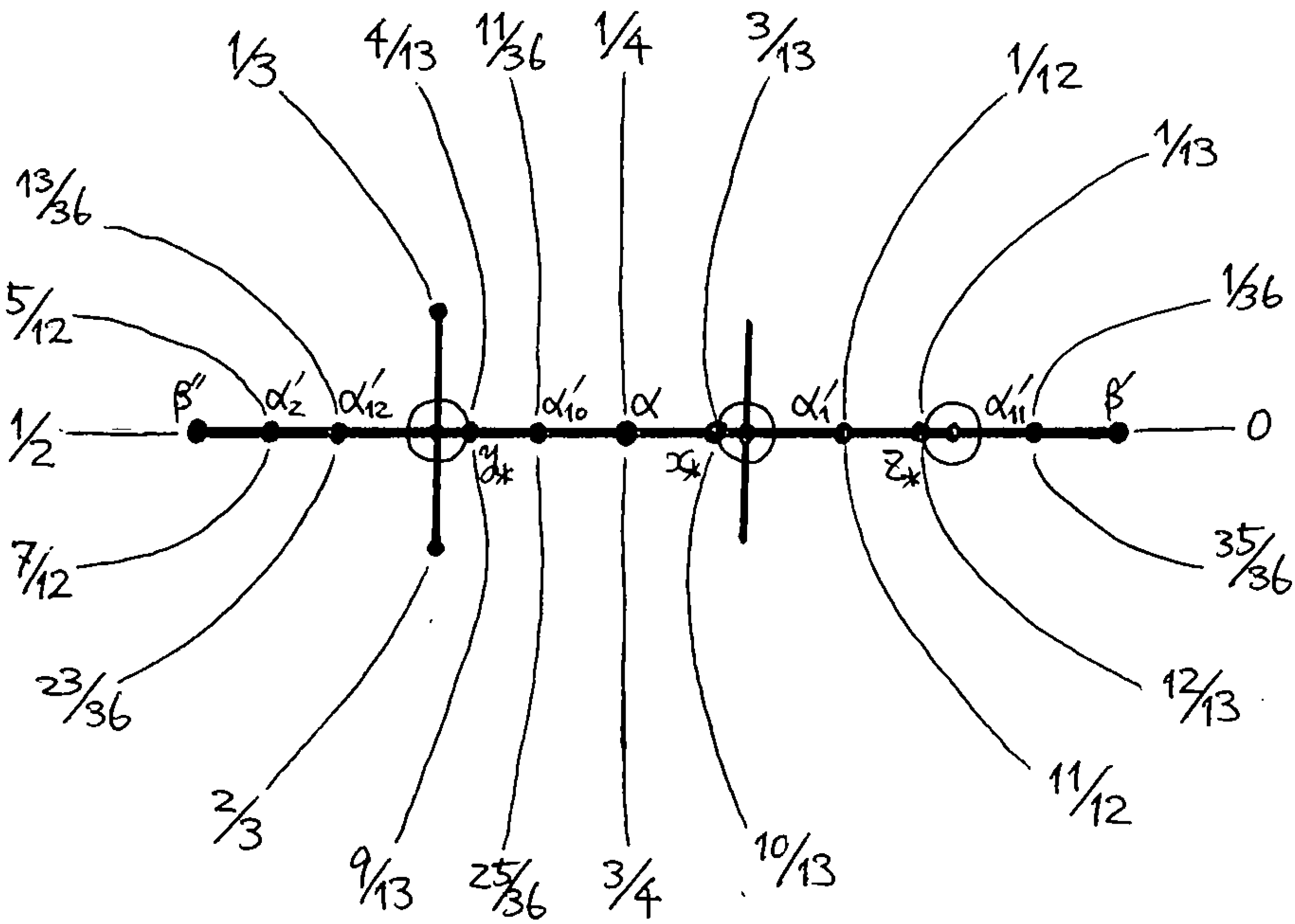


Figure 1.6

§II.2 THE THURSTON'S OBSTRUCTION FOR $F = f_1 \perp\!\!\!\perp f_2$

In this section, we construct the Thurston's obstruction stated in Theorem I.4.2, which is made of external rays and some part of the equator.

2.1 DEFINITION. The *equator* of $S^2_{f_1,f_2}$ is

$$E = \{(\infty \cdot e^{2\pi i s}, f) | s \in \boldsymbol{T}\} = \{(\infty \cdot e^{-2\pi i s}, g) | s \in \boldsymbol{T}\},$$

where $(\infty \cdot e^{2\pi i s}, f) = (\infty \cdot e^{-2\pi i s}, g)$ is a point in $S^2_{f_1,f_2}$.

For $\theta_1, \theta_2 \in \boldsymbol{T}$, we define arcs in $S^2_{f_1,f_2}$ by

$$\theta_1 \overset{f_1}{\underset{z}{\text{---}}} \theta_2 = R_1(\theta_1) \cup R_1(\theta_2) \qquad \text{if } \gamma_1(\theta_1) = \gamma_1(\theta_2) = z,$$
$$\theta_1 \overset{f_2}{\underset{z}{\text{---}}} \theta_2 = R_2(-\theta_1) \cup R_2(-\theta_2) \qquad \text{if } \gamma_2(-\theta_1) = \gamma_2(-\theta_2) = z,$$

$$\theta_1 \overset{E}{\text{---}} \theta_2 = \begin{cases} \{(\infty \cdot e^{2\pi i s}, f) | \theta_1 \le s \le \theta_2\} & \text{if } \theta_1 \le \theta_2 \\ \{(\infty \cdot e^{2\pi i s}, f) | \theta_2 \le s \le \theta_1\} & \text{if } \theta_1 > \theta_2, \end{cases}$$

where θ_1, θ_2 are considered to be in $[0, 1[$, in the last definition.

Let

$$\delta_1 = 1/8 \overset{E}{\text{---}} 1/4 \overset{f_2}{\underset{\alpha'}{\text{---}}} 3/4 \overset{E}{\text{---}} 3/8 \overset{f_1}{\underset{\alpha}{\text{---}}} 1/8,$$
$$\delta_2 = 1/24 \overset{E}{\text{---}} 1/12 \overset{f_2}{\underset{\alpha'_1}{\text{---}}} 11/12 \overset{E}{\text{---}} 19/24 \overset{f_1}{\underset{\alpha_1}{\text{---}}} 1/24,$$
$$\delta_3 = 3/8 \overset{E}{\text{---}} 5/12 \overset{f_2}{\underset{\alpha'_2}{\text{---}}} 7/12 \overset{E}{\text{---}} 11/24 \overset{f_1}{\underset{\alpha_2}{\text{---}}} 17/24 \overset{E}{\text{---}} 3/4 \overset{f_2}{\underset{\alpha'}{\text{---}}} 1/4 \overset{E}{\text{---}} 1/8 \overset{f_1}{\underset{\alpha}{\text{---}}} 3/8,$$
$$\delta_4 = 1/72 \overset{E}{\text{---}} 1/36 \overset{f_2}{\underset{\alpha'_{11}}{\text{---}}} 35/36 \overset{E}{\text{---}} 67/72 \overset{f_1}{\underset{\alpha_{11}}{\text{---}}} 1/72,$$

$$\delta_5 = 25/72 \overset{E}{\text{---}} 13/36 \overset{f_2}{\underset{\alpha'_{12}}{\text{---}}} 23/36 \overset{E}{\text{---}} 43/72 \overset{f_1}{\underset{\alpha_{12}}{\text{---}}} 49/72 \overset{E}{\text{---}} 25/36 \overset{f_2}{\underset{\alpha'_{10}}{\text{---}}} 11/36 \overset{E}{\text{---}} 19/72 \overset{f_1}{\underset{\alpha_{10}}{\text{---}}} 25/72.$$

Then the following lemma can be easily checked. (See Figure 2.1-3.)

2.2 LEMMA. (i) *The above $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ are simple closed curves in $S^2_{f_1,f_2} - P_F$. δ_1 and δ_2 are disjoint.*

(ii) $F^{-1}(\delta_1) = \delta_2 \cup \delta_3$ *and* $\deg(F : \delta_2 \to \delta_1) = 1$, $\deg(F : \delta_3 \to \delta_1) = 2$; $F^{-1}(\delta_2) = \delta_4 \cup \delta_5$ *and* $\deg(F : \delta_4 \to \delta_2) = 1$, $\deg(F : \delta_5 \to \delta_2) = 2$.

(iii) δ_3 *and* δ_5 *are homotopic to* δ_1 *in* $S^2_{f_1,f_2} - P_F$. δ_4 *is peripheral in* $S^2_{f_1,f_2} - P_F$.

PROOF OF THEOREM I.4.2: Let $\Gamma = \{\delta_1, \delta_2\}$. Then by the above lemma, the matrix for the Thurston's linear transformation $F_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ (under basis δ_1, δ_2) is

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}.$$

This matrix has the leading eigenvalue $\lambda_\Gamma = 1$ with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence Γ is a Thurston's obstruction, then by Thurston's theorem, F is not equivalent to a rational map. ∎

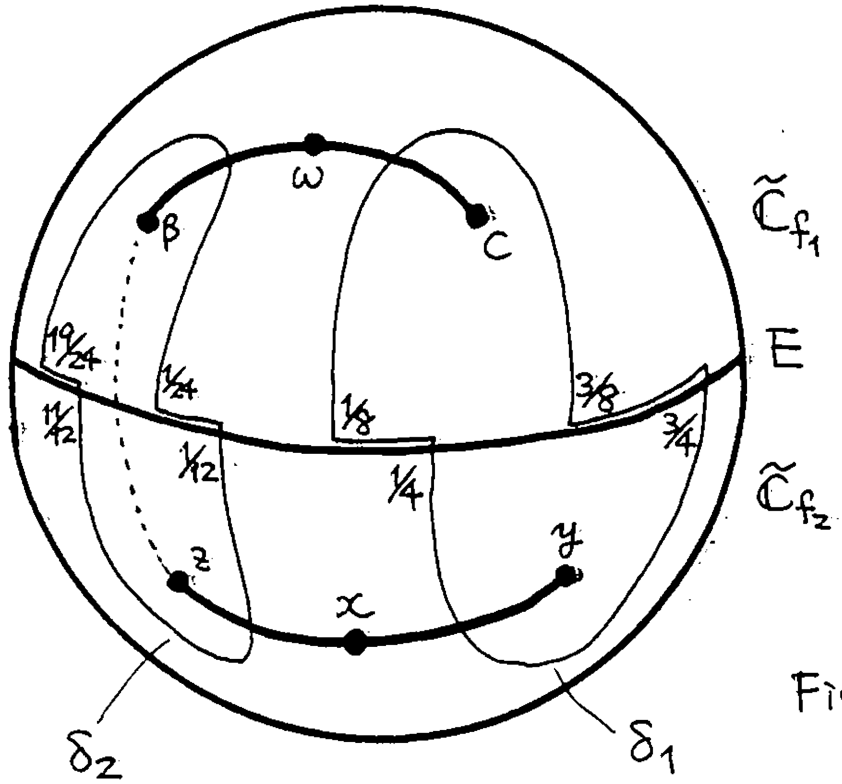


Figure 2.1

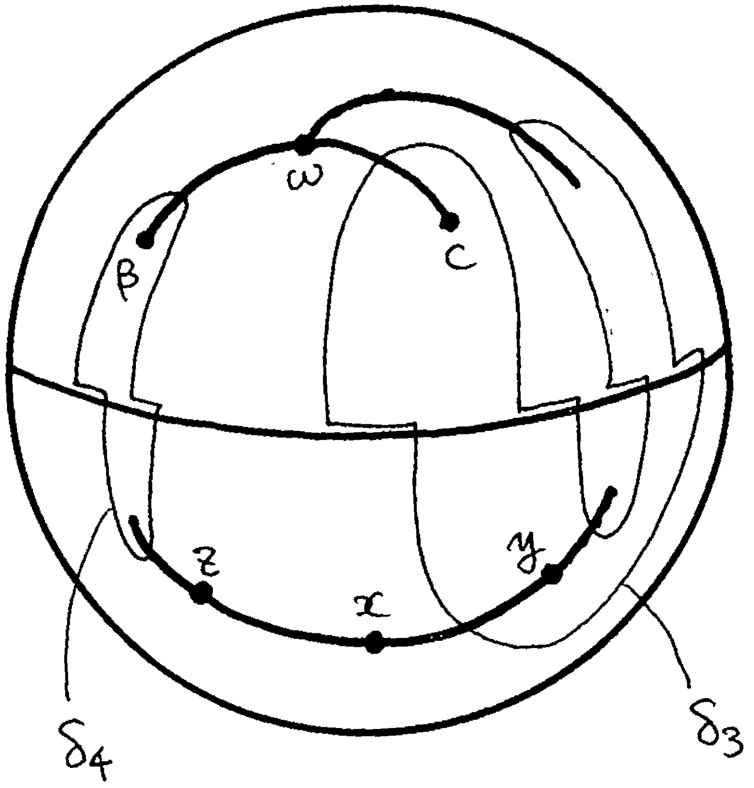


Figure 2.2

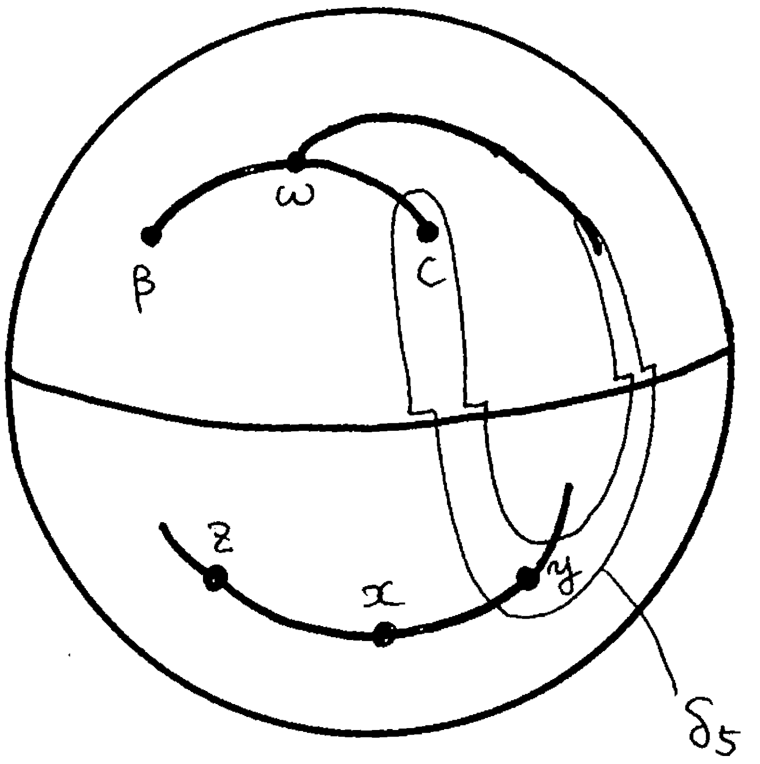


Figure 2.3

§II.3 RAY-EQIVALENCE FOR F

In this section we analyze the ray-eqivalence of the formal mating F of f_1 and f_2 , and we prove Lemma I.4.1 and Theorem I.4.4.

3.1 DEFINITION AND REMARK (EXTENDED HUBBARD TREES). Let $\hat{H}_1$ be the convexe envelope in J_1 of β, β_1, β_2 , where $\beta = \gamma_1(0)$, $\beta_1 = \gamma_1(1/3) = c$ and $\beta_2 = \gamma_1(2/3)$.

Then

$$\hat{H}_1 = H_1 \cup [\omega, \beta_2] .$$

(recall that ω denotes the critical point of f_1 .

Let $\hat{H}_2$ be the convexe envelope in J_2 of $\beta', \beta'_1, \beta'_2$, where $\beta' = \gamma_2(0)$, $\beta'_1 = \gamma_2(1/3)$ and $\beta'_2 = \gamma_2(2/3)$. Then

$$\hat{H}_2 = H_2 \cup [y, \beta'_1] \cup [y, \beta'_2] \cup [\beta', z] .$$

By Douady and Hubbard, we have $F(\hat{H}_i) \subset \hat{H}_i$, and for any point $u \in \hat{H}_i \cap J_i$, the number of external angles $\#Ang(u)$ is equal to the number of access of u relative to $\hat{H}_i$.

In $S^2_{f_1,f_2}$, let

$$R(\theta) = R_1(\theta) \cup R_2(-\theta)$$

to be the *external ray of F* of angle θ .

3.2 DEFINITION. A point u in $J_1 \sqcup J_2 \subset S^2_{f_1,f_2}$ is a *simple point* if u has only one external angle; and u is a *multiple point* otherwise, i.e. u has at least two external angles.

3.3 LEMMA. *If u is multiple and $u \neq \omega$, then $F(u)$ is also multiple.*

PROOF: Since u is not a critical point of F , the mapping F is a local homeomorphism, and F sends external rays to external rays. So $F(u)$ has the same number of external angles as u has, i.e. $\#Ang(u) = \#Ang(F(u))$. ∎

3.4 LEMMA. *Suppose $u \in J_i (i = 1, 2)$, and u is multiple, then there is $k \geq 0$ such that for all $n \geq k$, $F^n(u) \in H_i$.*

PROOF: Since $F(H_i) \subset H_i$, we need only to find a k such that $F^k(u) \in H_i$.

Suppose at first $u \in J_1$. If there is k such that $F^k(u) = \omega$, then we are done. If not, suppose θ, η are two different external angles of u . In the expression of θ, η in base 3, there is i minimal such that the i-th digits of θ, η are different. This gives that $3^i\theta$ and $3^i\eta$ are in different intervals of $\mathbf{T} - \{0, 1/3, 2/3\}$, so $F^i(u) \in \hat{H}_1$. Since $F([\omega, \beta_2]) = [c, \beta]$, the number $k = i + 1$ verifies our condition.

Suppose $u \in J_2$. We can prove as above that any multiple point u has a forward image in $\hat{H}_2$. Since $F([y, \beta'_1]) = F([y, \beta'_2]) = [z, \beta'] \in [x, \beta']$, and in $[x, \beta']$ there is a sequence $\{b_n\}$ such that $b_0 = x$, $b_n \in [b_{n+1}, b_{n-1}]$, $b_n \to \beta'$, and $F(b_{n+1}) = b_n$. So for any point $u \in \hat{H}_2 - \{\beta\}$, there is k such that $F^k(u) \in H_2$. ∎

3.5 LEMMA. *Every multiple point in J_1 has a forward image in $[\omega, c]$.*

PROOF: As in the proof of the above lemma, we can find a sequence $\{a_n\} \in [c, \beta]$ such that $a_0 = c$, $a_n \in [a_{n+1}, a_{n-1}]$, $a_n \to \beta$, and $F(a_{n+1}) = a_n$. We have automatically $a_1 = 0$. So for any multiple point $u \in H_1$ there is $k \geq 0$ such that $F^k(u) \in [\omega, c]$. ∎

3.6 LEMMA. *Every multiple point in J_1 has at most three external angles. Every multiple point in J_2 has at most two external angles.*

PROOF: For a multiple point $u \in J_2$, since all critical points of f_2 are in $\mathrm{int}(K_2)$, the orbit of u contains no critical points. There is k such that $F^k(u) \in \hat{H}_2$ and F^k is a homeomorphism in a neighborhood of u . So $\#Ang(u) = \#Ang(F^k(u))$. Since $\hat{H}_2$ contains only one branched point y which is not in J_2 , any point in $\hat{H}_2$ has at most two external angles.

For a multiple point $u \in J_1$, either there is a unique k such that $F^k(u) = \omega$ (and $F^{k+2}(u) = \beta$) or the orbit of u does not contain ω and there is k such that $F^k(u) \in \hat{H}_1$. In both case $\#Ang(u) = \#Ang(F^k(u))$. But ω is the only branched point in $\hat{H}_1$ and $Ang(\omega) = \{1/9, 4/9, 7/9\}$ (§II.1), we have $\#Ang(F^k(u)) \leq 3$. ∎

3.7 LEMMA. *Suppose u is a multiple point of J_1 , and θ, θ' two angles of u. Let $v = \gamma_2(-\theta)$ and $v' = \gamma_2(-\theta')$ be the landing points of $R(\theta), R(\theta')$ in J_2 . Then at least one of them is a simple point.*

PROOF: Suppose both v and v' are multiple. Then take k large enough such that for all $n \geq k$, $F^n(u) \in H_1$ and $F^n(v), F^n(v') \in H_2$. Moreover, by Lemma 3.5, we can suppose $F^k(u) \in]\omega, c]$. Now we claim $F^k(u) \neq \omega$ otherwise $F^{k+1}(v) = F^{k+1}(v') = \gamma_2(-1/3)$ is a simple point. So one of $3^k\theta, 3^k\theta'$ is in the interval $]1/3, 4/9[$ (§II.1), suppose it is $3^k\theta'$. So $F^k(v')$ has angles in $]1/3, 2/3[$, but no point in H_2 has angles in this interval (§II.1), contradiction. ∎

3.8 COROLLARY. *Each ray-equivalence class of F intersects the equator E at at most 6 points. No ray-equivalence class of F contain closed curve.*

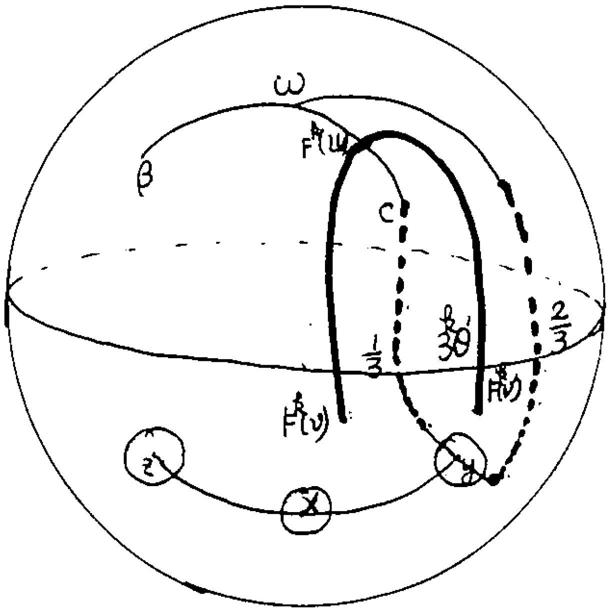
PROOF: Let $v \in J_2$ be a multiple point and $Ang(v) = \{-\theta, -\theta'\}$. Then for $u = \gamma_1(\theta)$, $u' = \gamma_1(\theta')$, we have $\#Ang(u) + \#Ang(u') \leq 6$, and by the above lemma $[v] \cap E = Ang(u) \cup Ang(u')$, and $[v]$ does not contain any closed curve. ∎

3.9 COROLLARY = LEMMA I.4.1. *No ray-equivalence class of F contain more then one postcritical point.*

PROOF: Since each of $[x]$, $[y]$, $[z]$ contains only one point (because $x, y, z, \notin J_2$) , we need only to check $[\omega]$, $[c]$, $[\beta]$. Since $[\beta] = R_1(0) \cup R_2(0)$, we have $\omega, c \notin [\beta]$. Since $F([\omega]) = [c]$ and $F([c]) = [\beta]$, we have $[\omega] \neq [c]$, $[\omega] \neq [\beta]$, $[\beta] \neq [c]$. ∎

3.10 PROPOSITION = THEOREM I.4.4. *F has no Levy cycles.*

PROOF: In fact the third reduction of §I.3 holds also for mating of polynomials of higher degree, because we have always an expansive metric near J_f for any postcritically finite polynomial f . So every Levy cycle of F reduces to a cycle of ray-equivalence classes with some specified conditions. Hence F can not have any degenerate Levy cycle by the above corollary. If F has a non-degenerate Levy cycle, then it will reduces to a cycle of ray-equivalence classes containing closed curves, which is impossible by Corollary 3.8. ∎



Picture for the Lemma 3.7.

§II.4 Quotient is a sphere

In this section we prove Theorem I.4.3 by using Moore's theorem.

4.1 MOORE'S THEOREM. [M] *If G is a partition of S^2 into compact, connected, non-separating sets such that the projection $\pi : S^2 \to S^2/G$ is closed, then the quotient S^2/G is homeomorphic to S^2 .*

4.2 DEFINITION (closed equivalence relation). An equivalence relation $\sim$ in a metric space X is *closed* if the graph of $\sim$ in $X \times X$ is a closed set. This is equivalent to say that for any sequences $x_n \to x$, $y_n \to y$ such that for each n we have $x_n \sim y_n$, then $x \sim y$.

4.3 LEMMA. *If an eqivalence relation $\sim$ in S^2 is closed, then $S^2/\sim$ is Hausdorff and the projection $\pi : S^2 \to S^2/\sim$ is a closed map.*

By Moore's theorem, the above lemma, Corollary 3.8 and Corollary 3.9 , we reduce the proof of Theorem I.4.3 to the following proposition and its corollary.

4.4 PROPOSITION. *Suppose f, g are two postcritically finite monic polynomials of the same degree d . Let $F = f \perp\!\!\!\perp g$ be the formal mating of f and g, and let $\sim\, = \,\sim_F$ be the equivalence relation defined by connected graph of external rays of F in $S^2_{f,g}$.*

If there is $K < \infty$ such that and for every $x \in S^2_{f,g}$

$$\#[x] \cap E \ \leq \ K \ ,$$

then the equivalence relation is closed.

4.5 COROLLARY. *If moreover no equivalence classes of $\sim$ separate $S^2_{f,g}$, then $S^2_{f,g}/\sim$ is homeomorphic to S^2 by Moore's theorem.*

REMARK. Since

$$S^2_{f,g}/\sim\, = \ K_f \sqcup K_g/\gamma_f(\theta) \approx \gamma_g(-\theta) \ ,$$

the above corollary means that we can sew the boundaries of K_f , K_g and get a topological sphere.

PROOF OF THE PROPOSITION: The external ray of F

$$R(\theta) \ = \ R_1(\theta) \cup R_2(-\theta) \ \subset S^2_{f,g}$$

is a connected arc, and by Douady and Hubbard, when $\theta \to \theta_0$, $R(\theta) \to R(\theta_0)$ with respect to the Hausdorff distance on the space of closed subsets of $S^2_{f,g}$.

Suppose $x_n, y_n \in S^2_{f,g}$ such that $\forall\, n$, $x_n \sim y_n$ and $x_n \to x_0$, $y_n \to y_0$. We need to prove $x_0 \sim y_0$.

For each n , there is a collection of angles $\theta_{n,1}$, $\theta_{n,2}$, ..., θ_{n,k_n} such that

$$x_n \in R(\theta_{n,1}) \ , \ y_n \in R(\theta_{n,k_n}) \ , \ R(\theta_{n,i}) \cap R(\theta_{n,i+1}) \neq \phi \ (i = 1, 2, ..., k_n - 1) \ .$$

By the assumption, we have $k_n \le K$ for each n . By taking a good subsequence, we may assume that k_n is a constant k for all n . By taking again subsequences, we may assume that $\theta_{n,i} \to \theta_i$ (as $n \to \infty$) for some $\theta_i \in \boldsymbol{T}$ and for each $i = 1, 2, ..., k$.

By the continuity of external rays $R(\theta)$, we have

$$R(\theta_{n,i}) \to R(\theta_i) \ , \ \text{as } n \to \infty \quad (i = 1, 2, ..., k) \ .$$

It follows that $x_0 \in R(\theta_1)$, $y_0 \in R(\theta_k)$, and $R(\theta_i) \cap R(\theta_{i+1}) \neq \phi$ for $i = 1, 2, ..., k-1$. Hence

$$\bigcup_{i=1}^{k} R(\theta_i)$$

is connected, and therefore $x_0 \sim y_0$. ∎

§III. A FAMILY OF CUBIC RATIONAL MAPS

In this chapter, we analyze a one parameter family of rational maps related to our example in §I.4. In §III.1, we state our numerical experiment, and in the following sections we state some of our results about this family.

§III.1 NUMERICAL EXPERIMENT

As an approach to see the phenomenon of non-matability in the space of cubic polynomials, we have made a computer experiment on a family of cubic rational maps. For the mating of quadratic ploynomials, Ben Wittner [W] made several numerical experiments and gave the interpretation for some phenomena observed in the parameter space. The obtained picture of the parameter space suggests non-matability, shared mating, etc. Applying his methods to cubic rational maps, we found a new phenomenon which does not occur in the quadratic case.

We take one-parameter family

$$\begin{aligned} F_t(z) &= \frac{(3t-2)z^3 - (t^4 - 3t^2 + 5t - 2)(3z-2)}{(3t-2)z^3 - t^3(3z-2)} \\ &= 1 - \frac{(t-1)^3(t+2)(3z-2)}{(3t-2)z^3 - t^3(3z-2)}, \end{aligned}$$

with the parameter $t \in \mathbf{C}$. If $t \neq -2, \frac{2}{3}, 1$, then F_t is a cubic rational map. The critical points of F_t are $0, \infty, 1$ and 0 is a double critical point. Moreover, for any $t \neq -2, \frac{2}{3}, 1$, F_t has a superattracting cycle of period 3 :

$$\infty \to 1 \to t \to \infty.$$

It can be shown that if a cubic rational map has a double critical point and a periodic cycle of period three containing two simple critical points, then it is conjugate by a Möbius transformation to an F_t for some t.

Let us see the result of our numerical experiment. Since 0 is the only *free* critical point, we trace the orbit of 0 to see the dynamics of F_t.

We color the parameter space according to how many iterations it needs for the orbit of 0 to be attracted to certain neighborhoods of $\infty, 1$ and t. We leave the parameter t white if the orbit of 0 is not in these neighborhoods after certain number of iterations.

The Figure 1.1 is the region

$$-5 \le Re\ t \le 5,\ -5 \le Im\ t \le 5\ .$$

in the parameter space.

In this figure we see a bounded set U of white points and an unbounded component C_∞ of the complement of U. The set C_∞ corresponds to the set of t for which 0 is in the immediate basin of ∞ for F_t. There are two other large components C_1, C_* of the complement of U which correspond to parameters for which 0 is in the immediate basin of 1 or t for F_t, respectively.

To interpret the structure of U , we need to consider what is expected in the family F_t .

There are four distinguished t-values for which 0 is fixed under F_t. They are solutions of

$$t^4 - 3t^2 + 5t - 2 = 0.$$

Two of them are real and the other two are non-real and complex conjugate to each other. Let us denote them by t_1, t_2, t_3, t_4, with t_1, t_2 real, $t_1 < t_2$. For each t_i, F_{t_i} is conjugate to a cubic polynomial P_i. We may choose P_i so that P_1, P_2 are real and P_3, P_4 are complex conjugate. It can be shown that P_1 is equal to the f_2 defined in §I.4 (up to affine conjugacy). And any cubic polynomial with a periodic cycle of period three containing two simple critical points is affinely conjugate to one of P_1, P_2, P_3, P_4 .

Let $g_a : z \to z^3 + a$ for $a \in \mathbb{C}$

$$M_3 = \{a \in \mathbb{C} \mid g_a^n(0) \not\to \infty \text{ as } n \to \infty\},$$

$$M' = \{a \in \mathbb{C} \mid g_a \text{ is postcritically finite } \} \subset M_3 \ .$$

The set M_3 is called the cubic Mandelbrot set (see Figure 1.2). If $g_a \in M'$ is matable with some P_i , then the degenerate mating of them is equivalent to $F_{t(a,i)}$ for some $t = t(a,i)$. (In particular, for $a = 0$, $g_0 \perp\!\!\!\perp P_i \sim F_{t_i}$ and $t_i = t(0,i)$.) This $t(a,i)$ is uniquely determined if it exists, since the degenerate mating of g_a and P_i has a hyperbolic orbifold. (See [DH].)

For $i = 1, 2, 3, 4$, let

$$A_i = \{ \ a \in M' \mid g_a \text{ and } P_i \text{ are matable } \} \ ,$$

$$B_i = \{ \ t \in \mathbb{C} \mid F_t \sim \text{ the degenerate mating of } P_i \text{ and some } g_a \ \}$$
$$= \ \{t(a,i) \mid a \in A_i \ \} \ .$$

In §III.5, we prove $A_3 = A_4 = M'$, $A_2 \subset M' - L$ where L is the limb of M_3 of internal angle $-1/4$. (i.e. L is the component of $M_3 - \{a_0\}$ not containing 0 , where a_0 is the point in ∂W_0 of ineternal angle $-1/4$). We conjecture that $A_2 = M' - L$. For A_1 , we know only a few things. For instance, let us return to our example introduced in §I.4: $F = f_1 \perp\!\!\!\perp f_2$ with $f_2 = P_1$ and $f_1 = g_c$, where c is the point of M_3 with external angle 1/3. By our result, f_1 and f_2 are not matable, hence $c \notin A_1$. Moreover, we found some other values of $a \in M'$ such that $a \notin A_1$, i.e. g_a and P_1 are not matable. We conjecture that $A_1 = M' - L_1 \cup \overline{L_1}$, where L_1 is the component of $M_3 - \{\gamma_{M_3}(7/26)\}$ containing c . Here $\gamma_{M_3}(7/26) = \gamma_{M_3}(9/26)$ is the root of period three component on the main vein of internal angle 1/4 .

Let us consider the mapping $(a,i) \to t(a,i)$. We will prove in §III.6 that it is not injective, i.e. one F_t can be equivalent to the matings of several pairs of polynomials. This phenomenon is called *shared matings* by B. Wittner [W].

Our computer observation strongly suggests that this mapping is continuous and can be extended to a considerably large and connected subset of M_3 . This makes it hopeful to define matings for non-postcritically finite polynomials in a reasonable way.

In our computer picture of the parameter space, we see relatively large components D_i of $\mathrm{int}(U)$ containing t_i $(i = 1, 2, 3, 4)$. Attached to these D_i , we also see some other smaller components and filaments.

First, D_3 and D_4 have attached components as many as the main component W_0 of $\mathrm{int}(M_3)$. And they have an attached component in common. However, D_2 is symmetric with respect to the real axis and has three cusp points on the boundary. The unique real one is $t = 2/3$. The period two hyperbolic component D of M_3 attached to W_0 with internal angle $-1/4$ does not seem to have a corresponding component attached to D_2 . We think that $t = 2/3$ corresponds to the cutting point (or the root) of D of M_3.

The degenerate parameter $t = -2$ is much more misterious. The component D_1 looks very much like W_0 . However there is a period 2 component $D_{1,2}$ which is attached to D_1 at two points. This suggests that $D_{1,2}$ can be considered as a self shared mating. The point $t = -2$ is surrounded by $D_{1,2}$ and D_1, and we do not see any white component attached to $t = -2$.

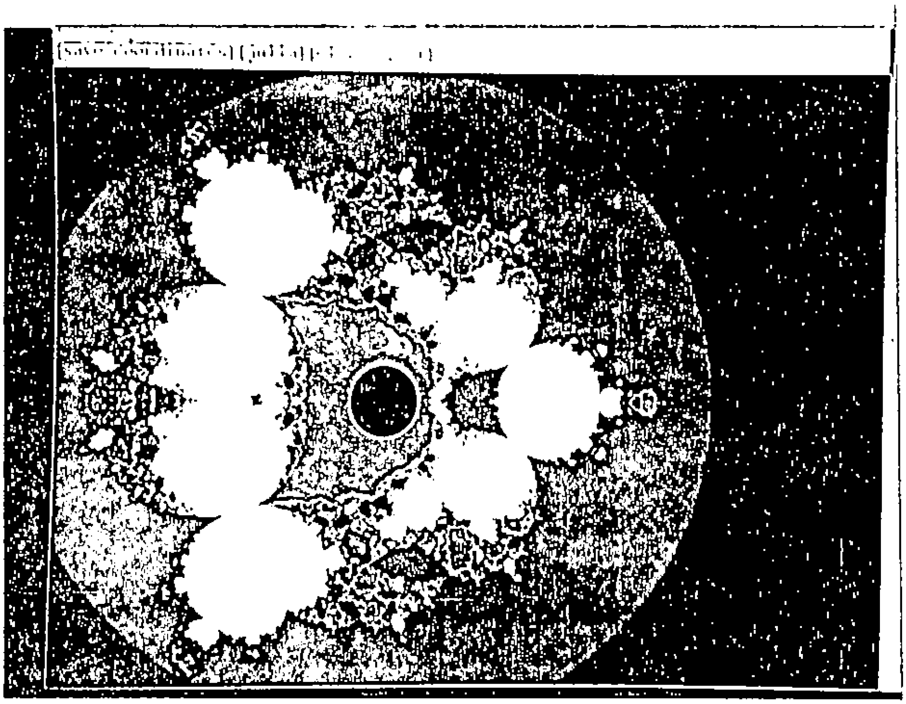
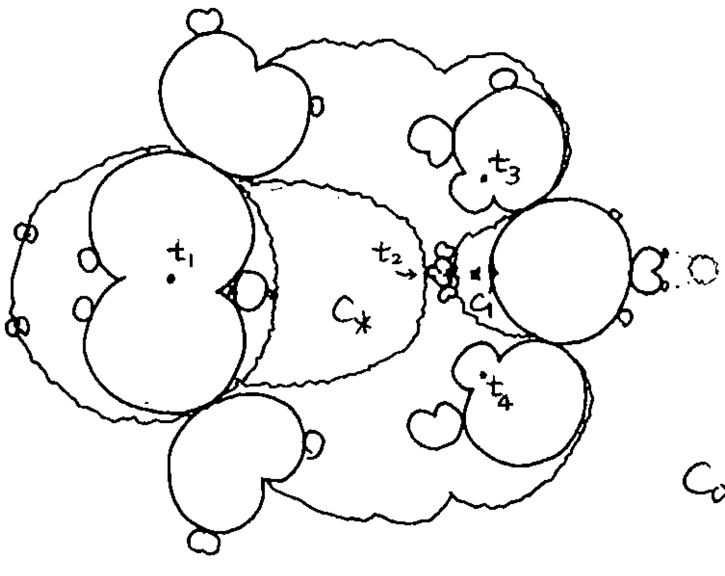


Figure 1.1



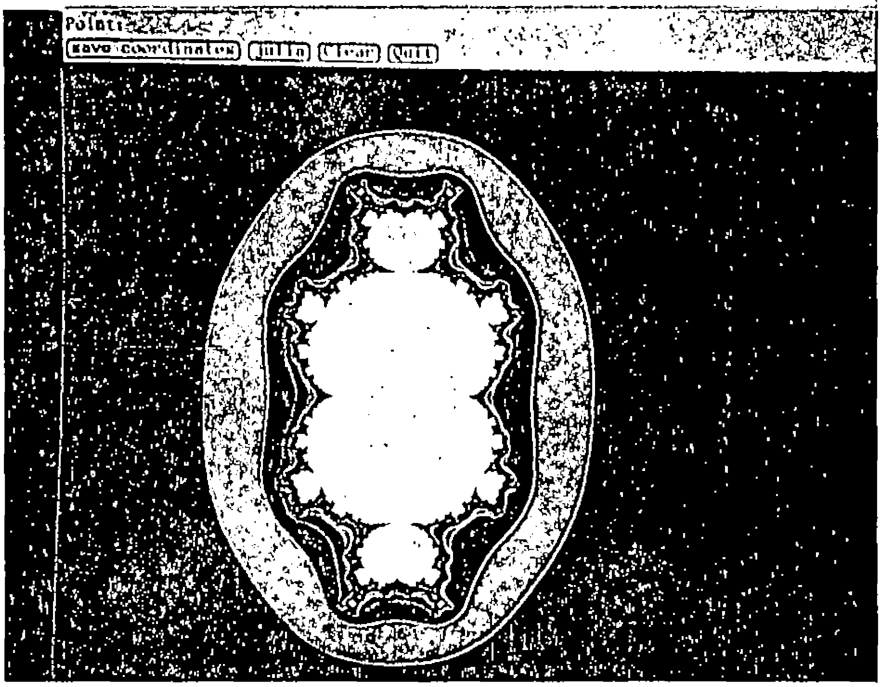
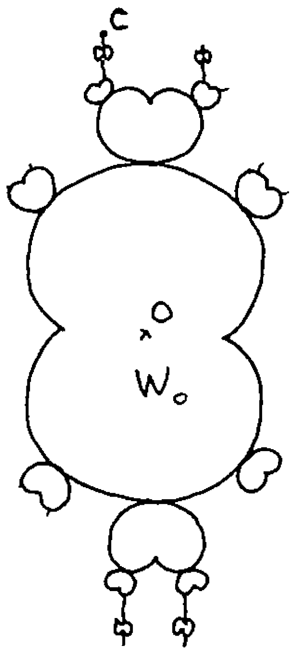


Figure 1.2
The cubic Mandelbrot set



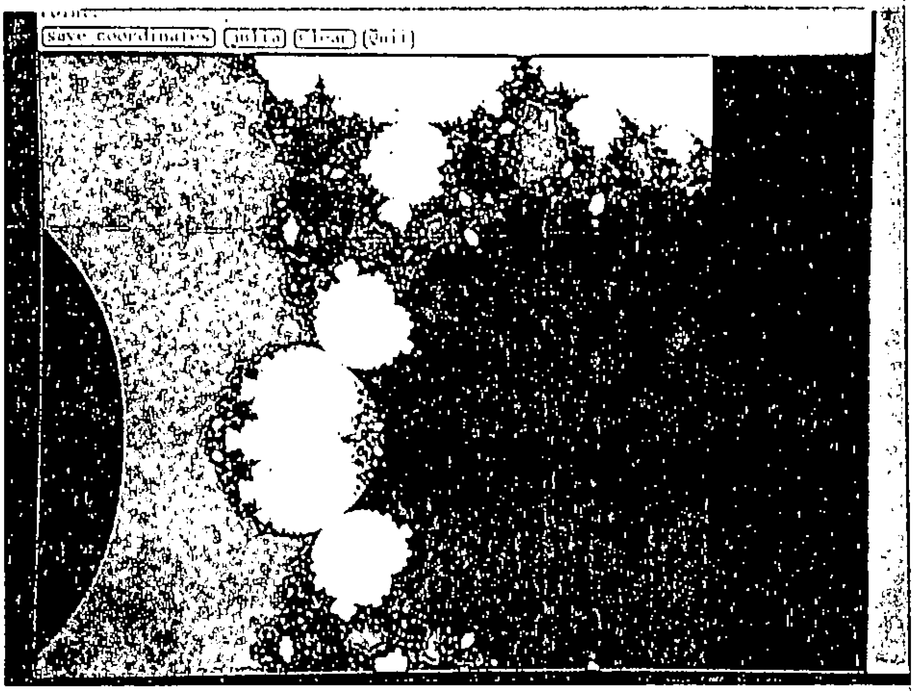


Figure 1.3
A blow-up of D_2

Figure 1.4
A blow-up of $D_{1,2}$

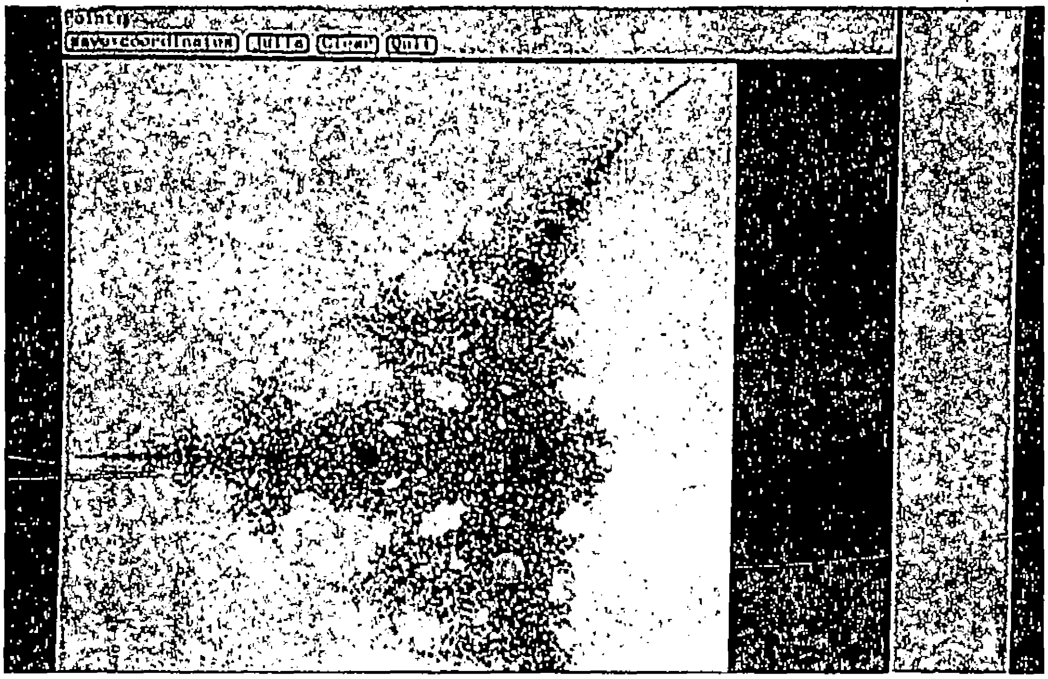


Figure 1.5
A blow-up of the triangle region
in Figure 1.4

Let us prove the "Levy theorem" for the degenerate mating of g_a and P_3 (or P_4).

THEOREM 2.1. *The degenerate mating (or formal mating) of $g_a(z) = z^3 + a$ and P_3 (or P_4) has a Thurston's obstruction if and only if it has a Levy cycle.*

PROOF: Let Γ be a Thurston's obstruction for F . We may assume that Γ is totally invariant, i.e. for any $\gamma \in \Gamma$ there is a curve in $F^{-1}(\Gamma)$ which is homotopic to γ .

There are a fixed point α of P_3 and regularized arcs $l_x = [\alpha, x]$, $l_y = [\alpha, y]$, $l_z = [\alpha, z]$ in $\mathbb{C}_{P_3} \subset S^2_{g_a, P_3}$ such that $F : l_x \to l_y$, $l_y \to l_z$, $l_z \to l_x$ is a homeomorphism. Let $l_1 = l_x \cup l_y$, $l_2 = l_y \cup l_z$, $l_3 = l_z \cup l_x$. Then $F : l_i \to l_{i+1}$ is also a homeomorphism ($i = 1, 2, 3$), where we set $l_4 = l_1$. Note that $l_1 \cup l_2 \cup l_3$ is the Hubbard tree of P_3 .

Let us define the geometric intersection number of l_i and a curve γ in Γ by

$$l_i \cdot \gamma = \inf\{\#(l' \cap \gamma') \mid l' \sim l_i,\ \gamma' \sim \gamma\},$$

where $l' \sim l_i$ means l' is an arc and they are homotopic in $S^2 - P_F$ fixing their end points, and $\gamma' \sim \gamma$ means γ' is a simple closed curve and they are homotopic in $S^2 - P_F$. The geometric intersection is extended bilinearly to $\mathbb{R}^{\{l_1, l_2, l_3\}} \times \mathbb{R}^\Gamma$.

Define a linear transformation $F_{\#,\Gamma} : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ by

$$F_{\#,\Gamma}(\gamma) = \sum_{\gamma' \subset F^{-1}(\gamma)} [\gamma']_\Gamma \quad \text{for } \gamma \in \Gamma,$$

as in Definition I.2.2.

The following inequality holds:

$$l_i \cdot F_{\#,\Gamma}(\gamma) \leq l_{i+1} \cdot \gamma \quad \text{for } \gamma \in \Gamma . \tag{2.2}$$

PROOF: Take l' and γ' such that $l' \sim l_i$, $\gamma' \sim \gamma$ and $\#(l' \cap \gamma') = l_{i+1} \cdot \gamma$. Then there is an arc $l'' \subset F^{-1}(l')$ such that $F : l'' \to l_i$ is a homeomorphism and $l'' \sim l'$. For γ' , there is a one to one correspondance between components of $F^{-1}(\gamma)$ and components of $F^{-1}(\gamma')$ such that corresponding curves are homotopic. Since $F : l'' \to l'$ is one to one, we have

$$l_i \cdot F_{\#,\Gamma}(\gamma) \leq \#(l'' \cap F^{-1}(\gamma')) = \#(l' \cap \gamma') = l_{i+1} \cdot \gamma .$$

∎

Let

$$\tilde{l} = l_1 + l_2 + l_3 \in \mathbb{R}^{\{l_1, l_2, l_3\}}, \quad \tilde{\Gamma} = \sum_{\gamma \in \Gamma} \gamma \ \in \mathbb{R}^\Gamma .$$

By (2.2), we have

$$\tilde{l} \cdot F_{\#,\Gamma}(\tilde{\Gamma}) \le \tilde{l} \cdot \tilde{\Gamma} .$$

On the other hand, since Γ is totally invariant,

$$F_{\#,\Gamma}(\tilde{\Gamma}) \ge \tilde{\Gamma}$$

(i.e. for every $\gamma \in \Gamma$, the coefficient of γ in the left hand side is greater than or equal to that of the right hand side). Hence we conclude that the equality holds in (2.2). So we have

$$\tilde{l} \cdot F_{\#,\Gamma}(\gamma) = \tilde{l} \cdot \gamma \quad \text{for } \gamma \in \Gamma , \tag{2.3}$$

$$\tilde{l} \cdot F_{\#,\Gamma}(\tilde{\Gamma}) = \tilde{l} \cdot \tilde{\Gamma} . \tag{2.4}$$

Let $\Gamma_1 = \{\gamma \in \Gamma \mid \tilde{l} \cdot \gamma \neq 0 \}$ and $\Gamma_2 = \Gamma - \Gamma_1$.

2.5 LEMMA. *Γ_2 is F-invariant and $\lambda(\gamma_2) < 1$.*

PROOF: If $\tilde{l} \cdot \gamma = 0$, then $\tilde{l} \cdot F_{\#,\Gamma}(\gamma) = 0$, by (2.3). This implies Γ_2 is F-invariant. Every curve $\gamma \in \Gamma_2$ is homotopic to a curve in the upper hemi-sphere C_{g_a} , since $l_1 \cup l_2 \cup l_3$ is the Hubbard tree of P_3 . Hence Γ_2 can be considered as a multicurve for P_3 . Since P_3 is a rational map, $\lambda(\Gamma_2) < 1$. ∎

It follows from this lemma that $\Gamma_1 \neq \phi$ and $\lambda(\Gamma_1) \ge 1$. By (2.3), for any $\gamma \in \Gamma_1$, at least one component of $F^{-1}(\gamma)$ is in Γ_1 . Moreover by (2.4) and $F_{\#,\Gamma}(\tilde{\Gamma}) \ge \tilde{\Gamma}$, exactly one component of $F^{-1}(\gamma)$ belongs to Γ_1 . Then Γ_1 decomposes into disjoint cycles $\{\gamma_{i,j} | j = 1, ..., k_i\}$ $(i = 1, ..., m)$ such that
there is a component $\gamma'_{i,j}$ of $F^{-1}(\gamma_{i,j+1})$ homotopic to $\gamma_{i,j}$ and no other component is homotopic to a curve in Γ_1 , where we set $\gamma_{i,k_i+1} = \gamma_{i,1}$.
It is easy to see that

$$\lambda(\Gamma_1) = \max_i \left[\prod_{j=1}^{k_i} \frac{1}{\deg(F : \gamma'_{i,j} \to \gamma_{i,j+1})} \right]^{\frac{1}{k_i}} .$$

Since $\lambda(\Gamma_1) \ge 1$, there is an i such that $\deg(F : \gamma'_{i,j} \to \gamma_{i,j+1}) = 1$ for all j . Then $\{\gamma_{i,j} | j = 1, ..., k_i\}$ is a Levy cycle.

Conversely if there is a Levy cycle, then there is an invariant multicurve containing it and this multicurve is obviously a Thurston's obstruction. Therefore the theorem is proved. ∎

§III.3 Degenerate Levy cycle and Levy's theorem

Let F be a postcritically finite branched covering from S^2 to itself. All the isotopies and homotopies in this section are those rel P_F . Let Γ be a multicurve (not necessarily F-invariant) in $S^2 - P_F$. By Γ , we also denote the union $\bigcup_{\gamma \in \Gamma} \gamma$. Let

$$S^2 - \Gamma = D_1 \cup D_2 \cup ... \cup D_k \cup A_1 \cup ... \cup A_l ,$$

where D_i denote the disc components of $S^2 - \Gamma$ and A_i non-disc components.

3.1 Lemma. *Suppose Γ verifies the following conditions:*
(3.1) *each $\gamma \in \Gamma$ is homotopic to a curve in $F^{-1}(\Gamma)$;*
(3.2) *for each $\gamma \in \Gamma$, at least one component of $F^{-1}(\gamma)$ is homotopic to a curve in Γ ;*
Then
a) for B a component of $S^2 - \Gamma$, every component of $F^{-1}(B)$ is isotopically contained in a component of $S^2 - \Gamma$.
b) Let $D = D_i$. Suppose D' is a disc component of $F^{-1}(D)$ and $\partial D'$ is homotopic to some $\delta \in \Gamma$. Then there is j such that $\delta = \partial D_j$ and D' is isotopic to D_j . Moreover, $deg(F : D' \to D) = deg(F : \partial D' \to \partial D)$.
c) Let $D = D_i$. If all components of $F^{-1}(D)$ are discs, then at least one of them is isotopic to some D_j .

Proof:
a) From (3.1), every component of $S^2 - F^{-1}(\Gamma)$ is isotopically (rel P_F) contained in a component of $S^2 - \Gamma$. Since for each component B' of $F^{-1}(B)$, we have $B' \cap F^{-1}(\Gamma) = \phi$ (otherwise $B \cap \Gamma \neq \phi$), so B' is a component of $S^2 - F^{-1}(\Gamma)$.

b) From a) , D' is isotopically contained in a component of $S^2 - \Gamma$. By the assumption, $\partial D'$ is homotopic to $\delta \in \Gamma$, hence D' is isotopic to a component Δ of $S^2 - \delta$, which is a disc. So Δ is a union of components of $S^2 - \Gamma$ and is also contained in a component of $S^2 - \Gamma$. Hence in fact Δ is a component of $S^2 - \Gamma$. Since Δ is a disc, finally $\Delta = D_j$ for some j , and $\delta = \partial D_j$.

c) Since $\partial D \in \Gamma$ and $F^{-1}(\partial D)$ is the union of the boundary of components of $F^{-1}(D)$, from (3.2), at least one disc D' of $F^{-1}(D)$ has the boundary curve γ' homotopic to a curve $\delta \in \Gamma$. From b), we get D_j . ∎

3.2 Proposition. *Suppose Γ verifies the conditions (3.1),(3.2) and*
(3.3) *for each D_i , the set $F^{-1}(D_i)$ consists of only discs.*
Then $\{D_i\}$ is decomposed into several periodic cycles $\{D_{i,j} | i = 1, ..., m,\ j = 1, ..., s_i\}$. More precisely, for $i = 1, ..., m,\ j = 1, ..., s_i$:
there is exactly one component $D'_{i,j}$ of $F^{-1}(D_{i,j+1})$ isotopic to $D_{i,j}$ and none of the other components is isotopic to a component of $S^2 - \Gamma$ (where $D_{i,s_i+1} = D_{i,1}$);
for $\gamma_{i,j} = \partial D_{i,j}$, there is exactly one component $\gamma'_{i,j} = \partial D'_{i,j}$ of $F^{-1}(\gamma_{i,j+1})$ homotopic to $\gamma_{i,j}$ and none of the other components is homotopic to a curve in Γ (where $\gamma_{i,s_i+1} = \gamma_{i,1}$).
Moreover for

$$\Gamma'_i = \{ \gamma_{i,j} = \partial(D_{i,j}) \ ,\ j = 1, ..., s_i \} ,$$

either $\lambda(\Gamma'_i) < 1$ *or* Γ'_i *is a degenerate Levy cycle.*

PROOF: Define

$$\tau : i \to \{\ j \mid D_j \text{ is isotopic (rel } P_F) \text{ to one component of } F^{-1}(D_i)\ \}\ ,\ i = 1, ..., k\ .$$

By the above Lemma, $\tau(i) \neq \phi$. If $i \neq i'$, then $\tau(i) \cap \tau(i') = \phi$, since $F^{-1}(D_i) \cap F^{-1}(D'_i) = \phi$. Hence $\tau(i)$ contains exactly one element, i.e. τ defines a bijection from $\{1, ..., k\}$ to itself. So $\{1, ..., k\}$ decomposes into periodic cycles of τ .

Let $\{i_1, ..., i_s\}$ be a periodic cycle for τ , with $\tau(i_{j+1}) = \{i_j\}, j = 1, ..., s-1,\ \tau(i_1) = \{i_s\}$. Put $D_{i,j} = D_{i_j}$. Then for

$$\{D_{i,j}, j = 1, ..., s\} \text{ and } \Gamma'_i = \{\ \gamma_{i,j} = \partial(D_{i,j})\ ,\ j = 1, ..., s\ \}\ ,$$

exactly one component $D'_{i,j}$ of $F^{-1}(D_{i,j+1})$ is isotopic to $D_{i,j}$ and none of the other components is isotopic to a component of $S^2 - \Gamma$ (where $D_{i,s+1} = D_{i,1}$);
exactly one component $\gamma'_{i,j} = \partial D'_{i,j}$ of $F^{-1}(\gamma_{i,j+1})$ is homotopic to $\gamma_{i,j}$ and none of the other components is homotopic to a curve in Γ (where $\gamma_{i,s+1} = \gamma_{i,1}$) .
And

$$\lambda(\Gamma'_i) = \left[\prod_{j=1}^{s} \frac{1}{\deg(F : \gamma'_{i,j} \to \gamma_{i,j+1})}\right]^{\frac{1}{s}} .$$

So $\lambda(\Gamma'_i) \leq 1$ and $\lambda(\Gamma'_i) = 1$ if and only if for each j , $\deg(F : \gamma'_{i,j} \to \gamma_{i,j+1}) = 1$. But $\deg(F : \gamma'_{i,j} \to \gamma_{i,j+1}) = \deg(F : D'_{i,j} \to D_{i,j+1})$, so if $\lambda(\Gamma'_i) = 1$ then Γ'_i is a degenerate Levy cycle. ∎

DEFINITION 3.3 A Thurston's obstruction Γ for F is called *minimal*, if every $\gamma \in \Gamma$ is homotopic to a curve of $F^{-1}(\Gamma)$, and any invariant proper sub-multicurve of Γ has the leading eigenvalue less than one.

Suppose Γ is a minimal Thurston's obstruction for F . Let us make a decomposition of Γ into $\Gamma_1 \cup \Gamma_2$ ($\Gamma_1 \cap \Gamma_2 = \phi$) , with Γ_2 a maximal invariant proper sub-multicurve of Γ . By the assumption of minimality, we have $\lambda(\Gamma_2) < 1$.

3.4 THEOREM. *Let* Γ_1 *and* Γ_2 *be as above. Suppose moreover that* Γ_1 *verifies the condition (3.3). Then* Γ_1 *is a degenerate Levy cycle.*

PROOF: Since Γ is minimal, any $\gamma \in \Gamma_1$ is homotopic to a curve in $F^{-1}(\Gamma) = F^{-1}(\Gamma_1) \cup F^{-1}(\Gamma_2)$. Since Γ_2 is F-invariant, i.e. $F^{-1}(\Gamma_2)$ is homotopically contained in $'G_2$, so γ cannot be homotopic to a curve in $F^{-1}(\Gamma_2)$. Therefore it is homotopic to a curve in $F^{-1}(\Gamma_1)$. Hence Γ_1 verifies the condition (3.1).

Let $\gamma \in \Gamma_1$. Suppose $F^{-1}(\gamma)$ does not contain any curve homotopic to a curve in Γ_1 . Then $\Gamma' = \{\gamma\} \cup \Gamma_2$ is an invariant sub-multicurve of Γ . Since Γ_2 is a maximal invariant proper sub-multicurve of Γ, we have $\Gamma = \Gamma'$, hence $\lambda(\Gamma) = \lambda(\Gamma') = \lambda(\Gamma_2) < 1$.

This contradicts the fact that Γ is a Thurston obstruction. Thus Γ_1 verifies also the condition (3.2).

So if moreover Γ_1 verifies the condition (3.3) , we can apply the Proposition 3.2 to it. Since for every cycle $\{D_{i,j} | j = 1, ..., s\}$, the set

$$\{\partial D_{i,j} | j = 1, ..., s\} \cup \Gamma_2$$

is F-invariant, and we assumed that Γ_2 is maximal, so there is only one cycle $\{D_j\}$. On the other hand, we should have

$$\lambda(\Gamma_1) = \lambda(\Gamma) \geq 1 \ .$$

Hence by the proposition, Γ_1 is a degenerate Levy cycle. ∎

3.5 COROLLARY. *Suppose F is of degree two and Γ is a Thurston's obstruction for F, then there is a Levy cycle in Γ .*

PROOF: Replacing Γ by a sub-multicurve, we may suppose that Γ is minimal. Decompose $\Gamma = \Gamma_1 \cup \Gamma_2$ as above.

1) If there is $\gamma \in \Gamma_1$ such that the two critical values of F are in different components of $S^2 - \gamma$, then each disc component D of $S^2 - \Gamma_1$ contains at most one critical value. So $F^{-1}(D)$ consists of only discs. By Theorem 3.4, Γ_1 is a degenerate levy cycle.

2) If for each $\gamma \in \Gamma_1$, the two critical values of F are in the same component of $S^2 - \gamma$, then $F^{-1}(\gamma)$ consists of two curves and each of them is sended by F to γ with degree one. This implies every periodic cycle of Γ_1 is a Levy cycle. Since Γ is a Thurston's obstruction, we can find a periodic cycle, hence a Levy cycle. ∎

§III.4 Good Levy cycle

Let $F = g_a \perp\!\!\!\perp P_i$, where $a \in M' \subset M_3$. Then F is a branched covering of S^2 to itself of degree three, $x, y \in \mathcal{C}_{P_i}$ are two simple critical points with $x \to y \to z \to x$ and $\omega \in \mathcal{C}_{g_a}$ is a double critical point with $\omega \to a$.

4.1 THEOREM. *If Γ is a non-degenerate Levy cycle for F , then there is a good Levy cycle Γ' , with $\#\Gamma' \leq 2$.*

PROOF: As a Levy cycle, Γ verifies automatically the conditions (3.1) , (3.2) of §III.3. Let

$$S^2 - \Gamma = D_1 \cup ... \cup D_k \cup A_1 \cup ... \cup A_l ,$$

where $\{D_i, i = 1, ..., k\}$ denotes the set of disc components of $S^2 - \Gamma$. We have $k \geq 2$. Let $X = \{a, x, y, z\}$. Then:

1) For each i , $D_i \cap \{x, y, z\} \neq \phi$. In fact if for some i , $D_i \cap \{x, y, z\} = \phi$, then $F^{-1}(D_i)$ are discs and $F^{-1}(D_i) \cap \{x, y, z\} = \phi$. So for $\gamma = \partial D_i$ and for every n , each component of $F^{-n}(\gamma)$ bounds a disc B in S^2 with $B \cap \{x, y, z\} = \phi$. But $\Gamma \subset \cup_n F^{-n}(\gamma)$, so in fact for each i , $D_i \cap \{x, y, z\} = \phi$, and $F^{-1}(D_i)$ are discs. Hence Γ verifies also the condition (3.3) of §III.3. So by the Proposition 3.2, we conclude that Γ is a degenerate Levy cycle, contradiction.

2) Suppose for some i , $\#D_i \cap X = 1$. Then by 1), $D_i \cap X = \{a\}$ is impossible. If $D_i \cap X = \{y\}$ or $\{z\}$, then $F^{-1}(D_i) = D' \cup D''$ with $D'' \cap \{x, y, z\} = \{x\}$ or $\{y\}$, $D' \cap \{x, y, z\} = \phi$ and $\deg(F : D'' \to D_i) = \deg(F : \partial D'' \to \partial D_i) = 2$. Hence by 1) neither $\partial D'$ nor $\partial D''$ can be in the Levy cycle Γ . This contradicts the condition (3.2).

If $D_i \cap X = \{x\}$, then $F^{-1}(D_i) = D' \cup D'' \cup D'''$, with $z \in D' \cap X$ and $D'' \cap \{x, y, z\} = D''' \cap \{x, y, z\} = \phi$. By Lemma 3.1, D' has to be isotopic to some D_j . By the above, no D_j verifies $D_j \cap X = \{z\}$, so $D' \cap X \neq \{z\}$. But $x, y \notin D'$, we have finally $D' \cap X = \{a, z\}$. Suppose $j = 1$ i.e. D' is isotopic to D_1 , with $D_1 \cap X = D' \cap X = \{a, z\}$. For D_1 , we have $F^{-1}(D_1) = A$, A is an annulus with $y \in A$, $\partial A = \gamma' \cup \gamma''$, $\deg(F : \gamma' \to \gamma_1) = 1$ and $\deg(F : \gamma'' \to \gamma_1) = 2$. Hence $\gamma'' \notin \Gamma$ and γ' has to be homotopic to a curve $\delta \in \Gamma$. By checking the degree, we see that x and y are in different components of $S^2 - \gamma''$. Let B be the annulus bounded by ∂D_1 and δ . Then $B \cap X = \phi$, $F^{-1}(B) = B' \cup B''$, $\deg(F : B' \to B) = 1$, $\deg(F : B'' \to B) = 2$, B' is isotopically contained in B and B'' is isotopically contained in the component of $S^2 - A$ containing x . Set successively $B_1 = B'$, $B_{n+1} =$ the degree one component of $F^{-1}(B_n)$. Then B_{n+1} is isotopically contained in B_n . So for some n , B_{n+1} is isotopic to B_n and $\{\partial B_n\}$ formes a good Levy cycle.

3) Now we can suppose that for each i , $\#D_i \cap X = 2$. In fact if $\#D_1 \cap X \geq 3$ then since $k \geq 2$, D_2 exists and $\#D_2 \cap X \leq 1$. This reduces to the case 1) or 2).

Let $S^2 - \Gamma = D_1 \cup D_2 \cup A_1 \cup ... \cup A_l$ $(l \geq 0)$, with $\#D_1 \cap X = \#D_2 \cap X = 2$. Let $\gamma_1 = \partial D_1$ and $\gamma_2 = \partial D_2$ and suppose $a \in D_1$. There are only three possibilities:

3.1) $D_1 \cap X = \{a, x\}$. Then $F^{-1}(D_1) = D'$ is a disc with $\deg(F : D' \to D_1) = \deg(F : \partial D' \to \gamma_1) = 3$, which contradicts the fact that γ_1 is in the Levy cycle Γ .

3.2) $D_1 \cap X = \{a, y\}$. Then $D_2 \cap X = \{x, z\}$, hence $F^{-1}(D_2) = D' \cup D''$ with $\deg(F : D' \to D_2) = 1$, and $\deg(F : D'' \to D_2) = 2$. So $\partial D''$ cannot be in the Levy cycle, so D'' is isotopic to neither D_1 nor D_2 . Moreover $x \notin F^{-1}(D_2)$, $y, z \in F^{-1}(D_2)$, and $y \in D''$ since $\deg(F : D'' \to D_2) = 2$. Hence $x, y \notin D'$. So D' is isotopic to neither D_1 nor D_2 . This contradicts the Lemma 3.1.c).

3.3) $D_1 \cap X = \{a, z\}$. Then $D_2 \cap X = \{x, y\}$, hence $F^{-1}(D_2) = D' \cup D''$ with $\deg(F : D' \to D_2) = 1$ and $\deg(F : D'' \to D_2) = 2$. We have $y \notin F^{-1}(D_2)$ and $x, z \in F^{-1}(D_2)$, moreover $x \in D''$ because of the degree of F on D'' . So $\partial D'' \notin \Gamma$ and hence $\partial D'$ must be homotopic to a curve in Γ . By Lemma 3.1, D' is isotopic to D_1 or D_2 . Since $x \notin D'$, D' has to be isotopic to D_1 , and $\partial D'$ is homotopic to $\gamma_1 = \partial D_1$. On the other hand, we have $F^{-1}(D_1) = A$, A is an annulus with $y \in A$, $\partial A = \gamma' \cup \gamma''$, $\deg(F : \gamma' \to \gamma_1) = 1$ and $\deg(F : \gamma'' \to \gamma_1) = 2$. Hence $\gamma'' \notin \Gamma$ and γ' has to be homotopic to a curve Γ . By Lemma 3.1, A is isotopically contained in a component of $S^2 - \Gamma$. Since $A \cap D_2 \neq \phi$, we get in fact that A is isotopically contained in D_2 . Hence γ' has to be homotopic to $\gamma_2 = \partial D_2$. Therefore we conclude that $\Gamma = \{\gamma_1, \gamma_2\}$ forms a Levy cycle. If $\gamma_1 = \gamma_2$ then Γ is a good Levy cycle. If $\gamma_1 \neq \gamma_2$ then let B be the annulus bounded by γ_1, γ_2 , $F^{-1}(B) = B' \cup B''$ with B' isotopic to B , B'' is isotopically contained in D_2 and $\deg(F : B' \to B) = 1$. This implies also that Γ is a good Levy cycle. ∎

4.2 PROPOSITION. *For $F = g_a \perp\!\!\!\perp P_i$, if there is a non-degenerate Levy cycle, then there is a ray-equivalence class $[\alpha]$ which is a simple closed curve and which contains one fixed point of g_a and one fixed ponit of P_i . As a consequence, each of these fixed points has exactly two external angles.*

PROOF: To prove this, we use the above proposition and the third reduction of §I.3 (which holds also for matings of higher degree polynomials). ∎

Let g_a and P_i $(i = 1, 2, 3, 4)$ be as in §III.1.

5.1 THEOREM. *For any $a \in M'$, g_a and P_3 (or P_4) are matable, i.e. $A_3 = A_4 = M'$.*

PROOF: Let F be the degenerate mating of g_a and P_3 $(a \in M')$. Suppose F has a Thurston's obstruction. By Theorem III.2.1, there exists a Levy cycle. This Levy cycle lifts to a non-degenerate Levy cycle for $F = g_a \perp\!\!\!\perp P_3$. It follows from Proposition 4.2 that there is a fixed point of P_3 which has exactly two external angles. However, by computation one can show that P_3 has three fixed points α, β, β' with external angles:

$$Ang(\alpha) = \{8/26,\ 20/26,\ 24/26\},\quad Ang(\beta) = \{0\},\quad Ang(\beta') = \{1/2'\}\ .$$

So none of them have exactly two external angles. This gives a contradiction.
Hence g_a and P_3 (or P_4) are matable. ∎

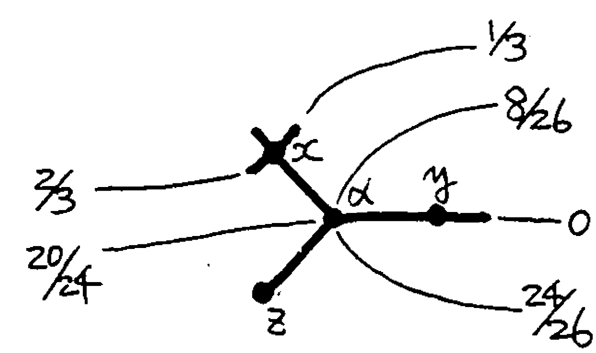
Let $x \to y \to z \to x$ be the periodic cycle of P_2 containing two simple critical points x, y. Suppose $x < y < z$ (otherwise consider $-P_2(-Z)$). We may suppose that the coefficient of Z^3-term is -1. Denote $\tilde{P}_2(Z) = iP_2(-iZ)$ then $\tilde{P}_2$ is monic. Let L be the limb of the cubic Mandelbrot set M_3 of internal angle $-1/4$.

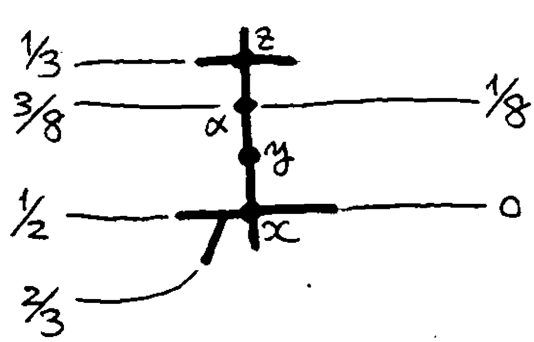
5.2 THEOREM. *For $a \in M' \cap L$, g_a and $\tilde{P}_2$ are not matable. In fact there is a good Levy cycle for $g_a \perp\!\!\!\perp \tilde{P}_2$, which consists of one curve made of two external rays of fixed points of g_a and $\tilde{P}_2$.*

PROOF.: For $a \in M' \cap L$, g_a has a fixed point α with external angle $5/8$ and $7/8$. It is easy to see that $\tilde{P}_2$ has a fixed point with external angles $1/8$ $(\equiv -7/8\ mod 1)$ and $3/8$ $(\equiv -5/8\ mod 1)$. Let

$$\gamma = R(5/8) \cup R(7/8) = \big(R_{g_a}(5/8) \cup R_{\tilde{P}_2}(-5/8)\big) \cup \big(R_{g_a}(7/8) \cup R_{\tilde{P}_2}(-7/8)\big)$$

which is a simple closed curve in $S^2_{g_a, \tilde{P}_2}$. If α is not postcritical, then $\Gamma = \{\gamma\}$ is a good Levy cycle. If α is postcritical, take a thin annulus A along γ (a *tubular neighborhood* of γ) so that $\overline{A} \cap P_F = \{\alpha\}$ and ∂A consists of two simple closed curves γ_+, γ_-. Then $\Gamma = \{\gamma_+, \gamma_-\}$ is a good Levy cycle. ∎





In this section we discuss shared matings.

Recall that M_3 is the cubic Mandelbrot set and M' denotes the set of $a \in M_3$ for which g_a is postcritically finite. Let $A_i \subset M'$, $i = 1,2,3,4$ be as in §III.1.

Let us denote by W_0 the main hyperbolic component of M_3. In M_3 , for each rational angle t there is a hyperbolic component $V(t)$ attached to W_0 , with internal angle t . Let $b(t)$ be the center of $V(t)$. We call the ***limb of*** M_3 with internal angle t , denoted by $L(t)$, the connected component of $M_3 - \overline{W}_0$ containing $b(t)$. There are two period two angles $1/4, 3/4$ and four period three angles $1/6, 1/3, 2/3, 5/6$.

We denote by x, y, z the periodic cycle of P_i of period three containing critical points x, y . To simplify the notation, we make a change of P_1 and P_2: for w the complex variable, replace $P_1(w)$ by $-P_1(-w)$ so that $y < x < z$; replace P_2 by $\tilde{P}_2$ defined in §III.5. The Hubbard trees of P_3, P_4 are stars. To distinguish them, let P_3 be the polynomial so that the Hubbard tree H_3 has $\{x, z, y\}$ as the cyclic order around the unique branched point of the tree, and hence H_4 has $\{x, y, z\}$ as the cyclic order around the branched point.

In $S^2_{g_a, P_i}$, recall that $R(\delta) = R_{g_a}(\delta) \cup R_{P_i}(-\delta)$ is the external ray of $F = g_a \perp\!\!\!\perp P_i$ of angle δ . Let us denote by $B(x), B(y), B(z)$ the immediate attractive basin of x, y, z in $S^2_{g_a, P_i}$ respectively.

1) Let us look at the mating $F = g_{b(1/3)} \perp\!\!\!\perp P_1$. To simplify, put $b = b(1/3)$. We will see that we can find $a \in A_3$ such that F can be also considered as the mating $g_a \perp\!\!\!\perp P_3$.

In $\mathbb{C}_{g_b} \subset S^2_{g_b, P_1}$, there is a unique fixed point α for F with more than one external angles. In fact $Ang(\alpha) = \{6/13, 5/13, 2/13\}$. By our assumption for P_1 , the 0-external ray arrives at β'' in $\mathbb{C}_{P_1}$ (the fixed point such that $[\beta'', y]$ does not intersect x). Then in $\mathbb{C}_{P_1} \subset S^2_{g_b, P_1}$

$R(6/13)$ lands on the point of $\partial B(z)$ with internal angle $1/3$;
$R(5/13)$ lands on the point of $\partial B(x)$ with internal angle $1/3$ and
$R(2/13)$ lands on the point of $\partial B(y)$ with internal angle $2/3$.

Let $\tau(z)$ be the closure of the internal ray of $B(z)$ of angle $1/3$;
$\tau(x)$ be the closure of the internal ray of $B(x)$ of angle $1/3$;
$\tau(y)$ be the closure of the internal ray of $B(y)$ of angle $2/3$.

Then

$$Y = \{\alpha\} \cup R(6/13) \cup \tau(z) \cup R(5/13) \cup \tau(x) \cup R(2/13) \cup \tau(y)$$

is an abstract Hubbard tree ([DH1]) homeomorphic to H_3 , and Y is invariant by F . We can then take a small neighborhood U of Y such that $U \cap P_F = \{x, y, z\}$ (recall that P_F is the postcritical set of F) and $\gamma = \partial U$ is a simple closed curve. It is easy to check that $F^{-1}(\gamma)$ is also a simple closed curve homotopic (rel P_F) to γ and $F : F^{-1}(\gamma) \to \gamma$

is of degree three and preserves the orientation. So we can consider γ as a new equator of the sphere, which separates the sphere into two hemi-spheres, on one of which F is equivalent to P_3 , and on the other one F is equivalent to a cubic polynomial with only one critical value (i.e. is affinely conjugate to a polynomial of our family M_3) (see [W]). So there is a value $a \in M'$ such that F is equivalent to $g_a \perp\!\!\!\perp P_3$, moreover, a is periodic of period 3 for g_a .

In fact instead of $b(1/3)$, for any $b \in M' \cap L(1/3)$, there is a fixed point $\alpha(b)$ of g_b which has the same set of external angles as $\alpha(b(1/3))$. So for any $b \in M' \cap L(1/3)$ such that $\alpha(b)$ is not postcritical, the same argument as above works, i.e. we can find a point $a = a(b) \in M'$ such that $g_b \perp\!\!\!\perp P_1$ is equivalent to $g_a \perp\!\!\!\perp P_3$. Hence we get a lot of shared matings. Moreover, by §III.5, $A_3 = M'$, it means that g_a and P_3 are matable, so g_b and P_1 are also matable.

2) Now let us consider $g_a \perp\!\!\!\perp P_2$. In M_3 , take the point $b = b(5/6)$, and let $F = g_b \perp\!\!\!\perp P_2$.

In $\mathbb{C}_{g_b} \subset S^2_{g_b,P_2}$, there is a unique fixed point α for F with more than one external angles, and $Ang(\alpha) = \{25/26, 23/26, 17/26\}$. By our choice of P_2 , we have: in $\mathbb{C}_{P_2} \subset S^2_{g_b,P_2}$,

$R(25/26)$ lands on the point of $\partial B(x)$ with internal angle 0 ;
$R(23/26)$ lands on the point of $\partial B(y)$ with internal angle 0 and
$R(17/26)$ lands on the point of $\partial B(z)$ with internal angle 0 .

Let $\tau(x)$ be the closure of the internal ray of $B(x)$ of angle 0 ;
$\tau(y)$ be the closure of the internal ray of $B(y)$ of angle 0 ;
$\tau(z)$ be the closure of the internal ray of $B(z)$ of angle 0 .

Then as in 1)

$$Y = \{\alpha\} \cup R(25/26) \cup \tau(x) \cup R(23/26) \cup \tau(y) \cup R(17/26) \cup \tau(z)$$

is an abstract Hubbard tree homeomorphic to H_3 , and Y is invariant by F . Hence we can take a small neighborhood U of Y such that $U \cap P_F = \{x, y, z\}$ and $\gamma = \partial U$ is a simple closed curve. Just as in 1), $F^{-1}(\gamma)$ is also a simple closed curve, is homotopic (rel P_F) to γ and $F : F^{-1}(\gamma) \to \gamma$ is of degree three and preserves the orientation. So we can consider again γ as a new equator of the sphere, on one of the new hemi-sphere F is equivalent to P_3 , and on the other one F is equivalent to g_a for some $a \in M' = A_3$ such that a is periodic of period 3 for g_a . Therefore F is equivalent to $g_a \perp\!\!\!\perp P_3$.

By the same argument we see that instead of $b(5/6)$, for any $b \in M' \cap L(5/6)$ so that $\alpha(b)$ is not postcritical, we can find a point $a = a(b) \in A_3$ such that $F = g_b \perp\!\!\!\perp P_2$ is equivalent to $g_a \perp\!\!\!\perp P_3$. Hence F is a shared mating and g_b and P_2 are matable.

3) Now let us consider matings with P_4 . In M_3 , take the point $b = b(5/6)$, and let $F = g_b \perp\!\!\!\perp P_4$.

As in 2), in $\mathbb{C}_{g_b} \subset S^2_{g_b,P_4}$ we have a fixed point α with $Ang(\alpha) = \{25/26, 23/26, 17/26\}$. For P_4 , let us choose the landing point of 0-external

ray to the fixed point μ such that $\mu \notin H_4$ and $[\mu, y]$ does not intersect x . Then in $\mathbb{C}_{P_4} \subset S^2_{g_b,P_4}$,
$R(25/26)$ lands on the point of $\partial B(y)$ with internal angle $1/3$;
$R(23/26)$ lands on the point of $\partial B(z)$ with internal angle $2/3$ and
$R(17/26)$ lands on the point of $\partial B(x)$ with internal angle $2/3$.

Let $\tau(y)$ be the closure of the internal ray of $B(y)$ of angle $1/3$;
$\tau(z)$ be the closure of the internal ray of $B(z)$ of angle $2/3$;
$\tau(x)$ be the closure of the internal ray of $B(x)$ of angle $2/3$.

Then

$$Y = \{\alpha\} \cup R(25/26) \cup \tau(y) \cup R(23/26) \cup \tau(z) \cup R(17/26) \cup \tau(x)$$

is again an abstract Hubbard tree homeomorphic to H_3 , and Y is invariant by F . Hence as well as in 2) we can take a small neighborhood U of Y such that $U \cap P_F = \{x, y, z\}$, $\gamma = \partial U$ are $F^{-1}(\gamma)$ simple closed curves homotopic (rel P_F) to each other, and $F : F^{-1}(\gamma) \to \gamma$ is of degree three and preserves the orientation. So we can consider γ as a new equator of the sphere, on one of the hemi-sphere F is equivalent to P_3 , on the other one F is equivalent to some g_a with $a \in M' = A_3$ such that a is periodic of period 3 for g_a . So F is equivalent to $g_a \perp\!\!\!\perp P_3$.

We can also generate this result to any $b \in M' \cap L(5/6)$ such that $\alpha(b)$ is not post-critical, and find a point $a = a(b) \in A_3$ such that $g_b \perp\!\!\!\perp P_4$ is equivalent to $g_a \perp\!\!\!\perp P_3$. This gives also that g_b and P_4 are matable.

4) Now let us consider again $g_b \perp\!\!\!\perp P_1$ but with $b = b(1/4)$. This time it has nothing to do with P_3 . The situation is slightly different from the above cases. Let $F = g_b \perp\!\!\!\perp P_1$.

As in 1), the 0-external ray for P_1 lands at the fixed point β'' such that $[\beta'', y]$ does not intersect x . In $\mathbb{C}_{P_1} \subset S^2_{g_b,P_1}$, there is a unique fixed point α' with more than one external angles, and $Ang(\alpha') = \{1/4, 3/4\}$. For g_b , the dynamic of the critical point ω is $\omega \to b \to \omega$. Let us denote by $B(\omega), B(b)$ the immediate attractive basin of ω, b in $S^2_{g_b,P_1}$ respectively. Then in $\mathbb{C}_{g_b} \subset S^2_{g_b,P_1}$
$R(1/4)$ lands on the point of $\partial B(b)$ with internal angle $1/2$;
$R(3/4)$ lands on the point of $\partial B(\omega)$ with internal angle $1/2$.

Let $\tau(b)$ be the closure of the internal ray of $B(b)$ of angle $1/2$,
$\tau(\omega)$ be the closure of the internal ray of $B(\omega)$ of angle $1/2$.

Then

$$Y = \{\alpha'\} \cup R(1/4) \cup \tau(b) \cup R(3/4) \cup \tau(\omega)$$

is an abstract Hubbard tree homeomorphic to $H_{g_{\bar{b}}}$, where $\bar{b} = b(-1/4)$ is the complex conjugate of b . Y is invariant by F . Take a small neighborhood U of Y such that $U \cap P_F = \{\omega, b\}$, and $\gamma = \partial U$ is a simple closed curve, then $F^{-1}(\gamma)$ is also a simple closed curves, and is homotopic (rel P_F) to γ , and $F : F^{-1}(\gamma) \to \gamma$ preserves the orientation. So we can consider γ as a new equator of the sphere on one of the new hemi-sphere F is equivalent to $g_{\bar{b}}$, and on the other one F is equivalent to a polynomial P' , which is one of P_i . We claim it is also P_1 (so F is in fact a *self-shared* mating), however in this case we are not able to draw the abstract Hubbard tree of P' in S^2 explicitly. So we have to determine it in an other way.

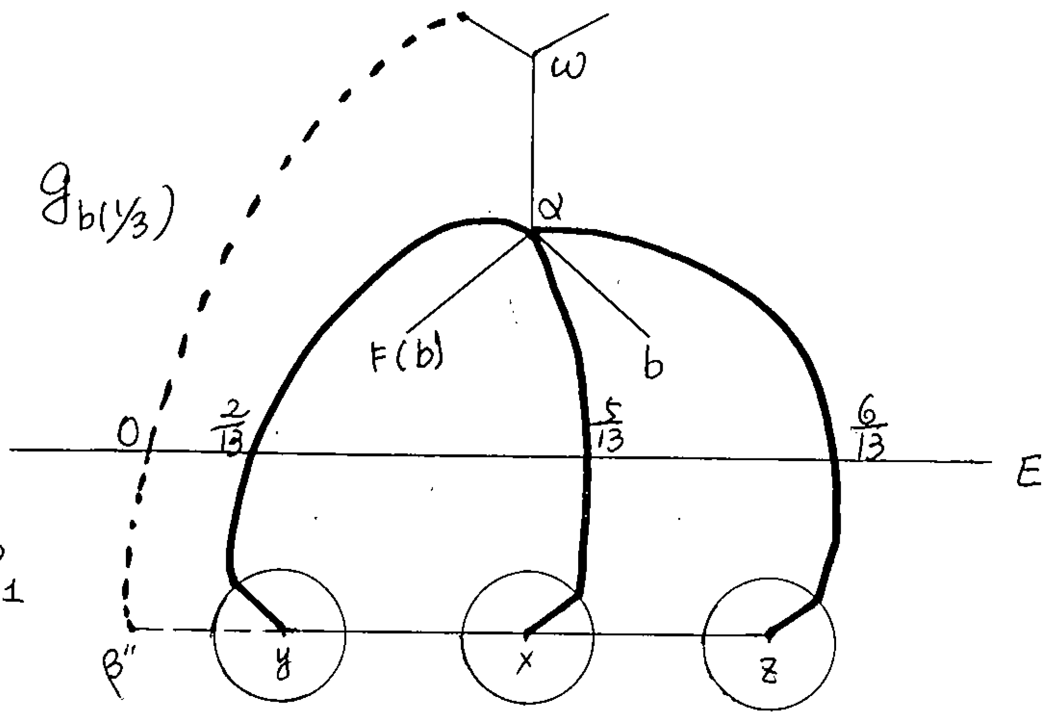
At first let us prove, as in §II.3 and 4, that the topological mating F^* (§I.1.8) of g_b and P_1 exists, i.e. $S^2_{g_b,P_1}/\sim_F$ is homeomorphic to a sphere, where $\sim_F$ denotes the ray-equivalence relation of F . In fact in the Hubbard tree H_{g_b} there is only one point which accepts more than one external rays (i.e. is multiple), it is the fixed point α , and $Ang(\alpha) = \{1/8, 3/8\}$. Since the external rays $R(1/8), R(3/8)$ land in $\mathbb{C}_{P_1} \subset S^2_{g_b,P_1}$ on different simple points, we have $[\alpha] = R(1/8) \cup R(3/8)$, which is an arc. Let E be the equator of $S^2_{g_b,P_1}$, then $\#[\alpha] \cap E = 2$. Since every multiple point in the Julia set of g_b is a preimage of α , and no critical point of F is multiple, we claim that for every $u \in S^2_{g_b,P_1}$, $\#[u] \cap E \leq 2$. By Proposition II.4.4 and Corollary II.4.5, we conclude that $S^2_{g_b,P_1}/\sim_F$ is homeomorphic to a sphere.

Suppose P' were P_3 or P_4 . Then there would be a periodic point u of F of period less than or equal to three such that the ray-equivalence class $[u]$ is fixed by F and connects the three attractive basins $B(x), B(y), B(z)$, so that in $S^2_{g_b,P_1}/\sim_F$ the three basins are attached at one point. At first $[u] \neq [\alpha]$, since the landing point of $R(1/8)$ in $\mathbb{C}_{P_1}$ is a periodic point of period two, it cannot be on the boundaries of $B(x), B(y), B(z)$. Hence $[u] \cap H_{g_b} = \phi$. But every non-extremal point of $[u]$ in the Julia set of g_b should be in H_{g_b} (Lemma II.3.4), so in fact $[u]$ does not exist. Hence P' is not P_3 nor P_4 .

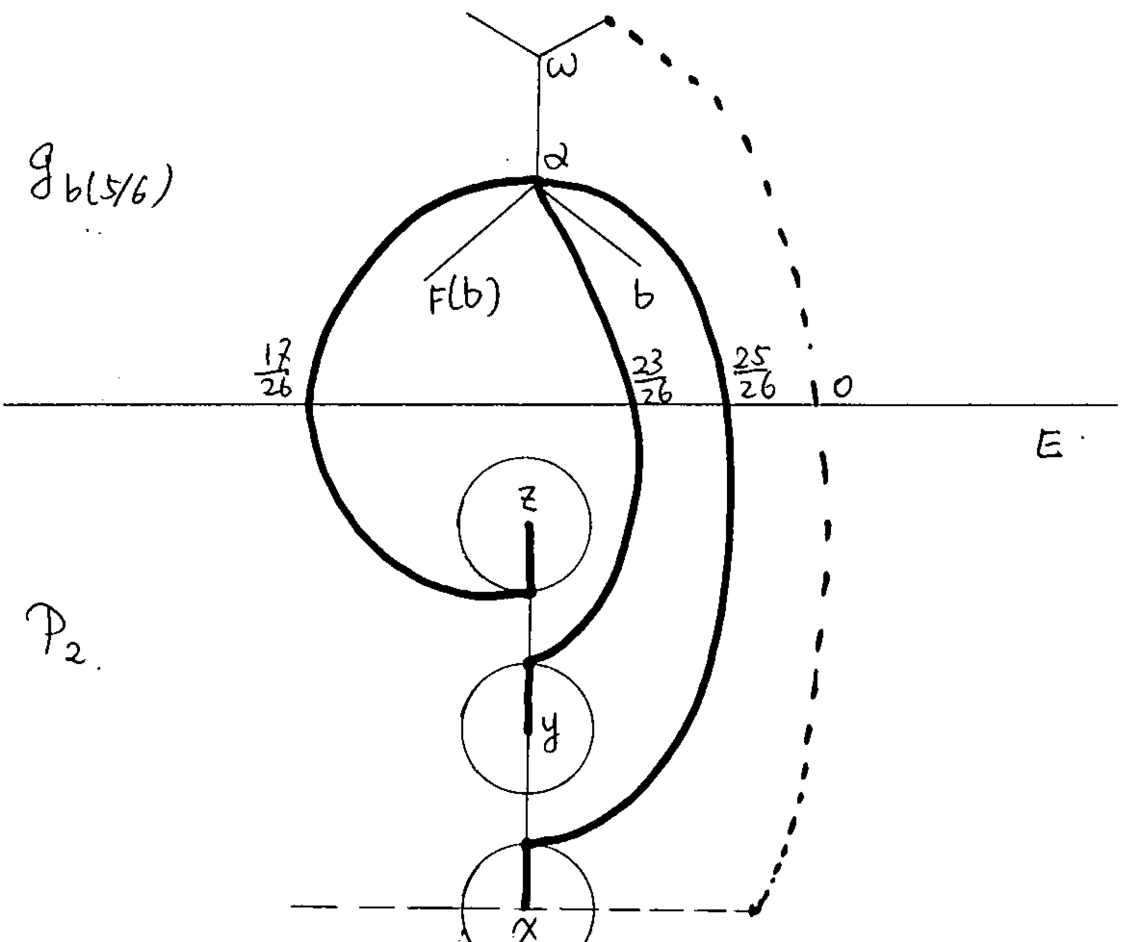
Finally, we will see that P' cannot be P_2 . By the above calculation, $[\alpha']$ connects the basins $B(\omega), B(b)$. So in $S^2_{g_b,P_1}/\sim_F$ the two basins $B(\omega), B(b)$ are attached at two points $[\alpha]$ and $[\alpha']$. In M' there are only two values of b such that the critical point ω is periodic of period two for g_b , they are $b(1/4)$ and $b(-1/4)$. For $b = b(3/4)$, the topological mating of g_b and P_2 does not exist (§III.5) . And for the topological mating of g_b and P_2 , where $b = b(1/4)$, the two basins $B(\omega), B(b)$ are attached only at one point. In fact $R(1/4)$ (resp. $R(3/4)$) lands on the periodic point of period two on $\partial B(\omega)$ (resp. $\partial B(b)$) in $\mathbb{C}_{g_b}$ and lands on a simple point in the Julia set of P_2 . So P' can not be P_2 .

We conclude finally that $P' = P_1$.

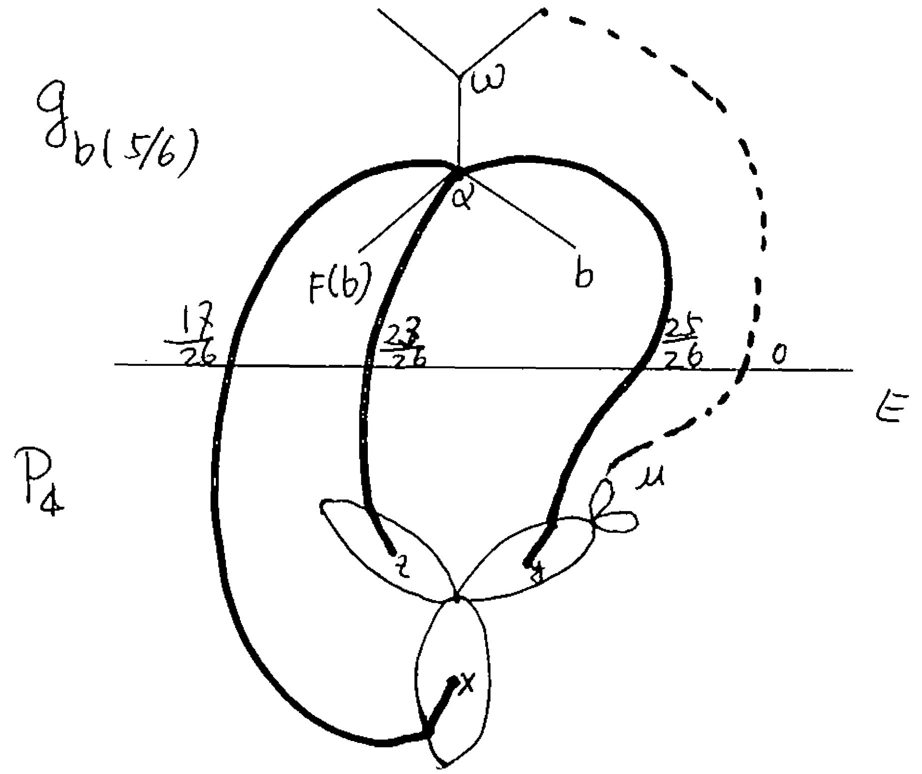
1)


ω
α
$g_{b(1/3)}$
F(b)
b
0
2/13
5/13
6/13
E
P_1
β''
y
x
z

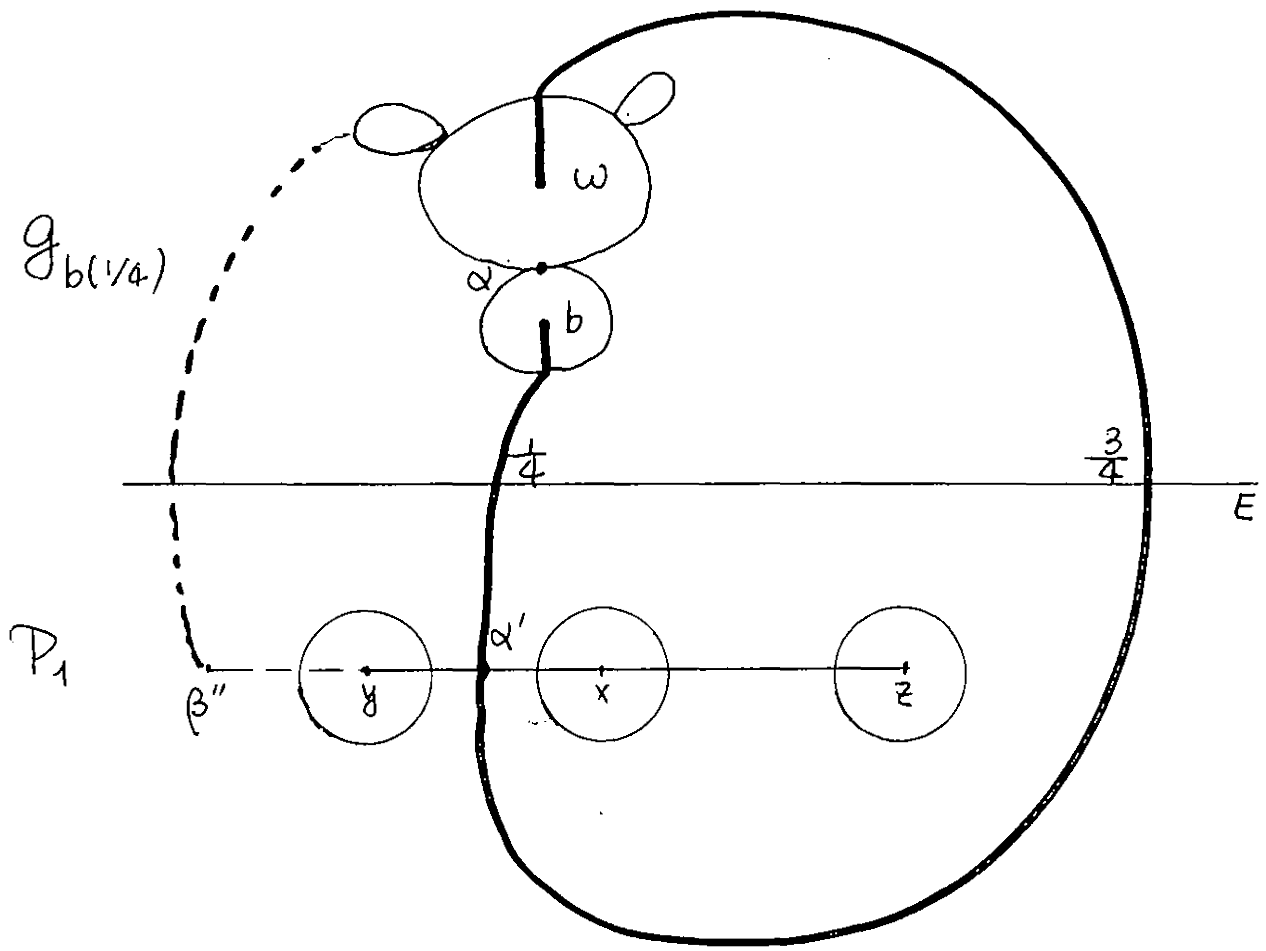
2)


ω
α
$g_{b(5/6)}$
F(b)
b
17/26
23/26
25/26
0
E
z
P_2
y
x

3)



4)



Let F be a branched covering of S^2 to itself of degree three, with x, y two simple critical points and $x \to y \to z \to x$, with 0 a double critical point and $0 \to a \to a' \to a''$. Let $X = \{a, x, y, z\}$. Let Γ be a minimal Thurston's obstruction for F (Definition III.3.3). Let

$$\Gamma_2' = \{\gamma \in \Gamma | \text{ one component of } S^2 - \gamma \text{ does not intersect } \{x, y, z\} \}$$

$$\Gamma_2 = \{\gamma \in \Gamma | \text{ for some } n \text{ every curve in } F^{-n}(\gamma) \text{ is either peripheral or homotopic to a curve in } \Gamma_2'\}$$

and $\Gamma_1 = \Gamma - \Gamma_2$. Let

$$S^2 - \Gamma_1 = D_1 \cup ... \cup D_k \cup A_1 \cup ... \cup A_l ,$$

where $\{D_i, i = 1, ..., k\}$ denotes the set of disc components of $S^2 - \Gamma_1$

7.1 THEOREM. *Let Γ be a minimal Thurston's obstruction and $\Gamma = \Gamma_1 \cup \Gamma_2$ be the decomposition as above. Then Γ_2 is F-invariant, Γ_1 verifies the conditions (3.1),(3.2) in §III.3, and exactly one of the following holds:*

a) $\Gamma_1 = \phi$. In this case Γ_2 contains a degenerate Levy cycle.

b) Γ_1 reduces to a good Levy cycle Γ' with $\#\Gamma' \leq 2$;

or

c) $k = 2$ with $a, y \in D_1$, $a', z \in D_2$, $x \in A_1$ and a'' is not in the component of $S^2 - A_1$ containing a .

PROOF:

Let $\gamma \in \Gamma_2'$ and let D be the component of $S^2 - \gamma$ without intersecting $\{x, y, z\}$. Then the set $F^{-1}(D)$ consists of discs and none of them intersect $\{x, y, z\}$. Hence Γ_2' is F-invariant. It is easy to see that Γ_2 is F-invariant, and $\lambda(\Gamma_2) = \lambda(\Gamma_2')$. If $\Gamma_1 = \phi$, then there is a minimal Thurston's obstruction Γ_3 contained in Γ_2' . And Γ_3 verifies the condition (3.3) in §III.3. Applying the Theorem III.3.4 to Γ_3 , we get a degenerate Levy cycle in $\Gamma_3 \subset \Gamma_2'$.

1) Suppose now $\Gamma_1 \neq \phi$. By the minimality of G , we get $\lambda(\Gamma_2) < 1$ and $\lambda(\Gamma_1) = \lambda(\Gamma) \geq 1$.

2) From the definition, for each i , $D_i \cap \{x, y, z\} \neq \phi$. Moreover, since every curve γ in Γ_1 is homotopic to a curve in $F^{-1}(\Gamma_1 \cup \Gamma_2)$, and Γ_2 is invariant, so γ is in fact homotopic to a curve in $F^{-1}(\Gamma_1)$, i.e. Γ_1 verifies the condition (3.1) . From the definition, for each γ in Γ_1 , at least one curve in $F^{-1}(\gamma)$ is homotopic to a curve in Γ_1 , so Γ_1 verifies also the condition (3.2).

3) For each i , $\#D_i \cap X \neq 3$. Suppose $\#D_1 \cap X = 3$. Since $k \geq 2$, there is D_2 such that $\#D_2 \cap X = 1$. Then $F^{-1}(D_2)$ are discs and a component of it either does

not intersect $\{x, y, z\}$ or is isotopically contained in D_1 but is not isotopic to D_1 . This contradicts the Lemma 3.1.

4) Suppose for each i , either D_i contains at most one critical value, or $D_i \cap X = \{y, z\}$. Then $F^{-1}(D_i)$ are discs and hence Γ_1 verifies also the condition (3.3). From the Proposition 3.2, either $\lambda(\Gamma_1) < 1$ or Γ_1 contains a degenerate Levy cycle, made by the boundary of some disc components of $S^2 - \Gamma_1$. But $\lambda(\Gamma_1) < 1$ is impossible by 1). And since for any Levy cycle Γ_1' in Γ_1 there are at least two disc components $S^2 - \Gamma_1'$, one of them D should verify $D \cap X = \{y\}, \{z\}$ or $\{y, z\}$. Then only one component D' of $F^{-1}(D)$ verifies $D' \cap X \neq \phi$. We get that D' is a disc and $\deg(F : \partial D' \to \partial D) > 1$. Hence ∂D is not in the Levy cycle. Contradiction.

5) From 3) , 4), we can suppose that $D_1 \cap X = \{a, y\}$ or $D_1 \cap X = \{a, z\}$.

6) Suppose $k = 2$ and $\#D_1 \cap X = \#D_2 \cap X = 2$. Let $\gamma_1 = \partial D_1$ and $\gamma_2 = \partial D_2$. then

6.1) $D_1 \cap X = \{a, z\}$ gives us a good Levy cycle as in the Proposition 4.1.

6.2) $D_1 \cap X = \{a, y\}$. We will prove that in this case $\lambda(\Gamma_1) < 1$ and hence get a contradiction.
Since $D_2 \cap X = \{x, z\}$, $F^{-1}(D_2) = D' \cup D''$ with $\deg(F : D' \to D_2) = 1$ and $\deg(F : D'' \to D_2) = 2$. We have $x \notin F^{-1}(D_2)$, and $y \in D''$ because of the degree. So $x, y \notin D'$, hence D' is isotopic neither to D_1 nor to D_2 . By the Lemma 3.1, D'' has to be isotopic to D_1 and $\deg(F : \partial D'' \to \gamma_2) = 2$. We have also $F^{-1}(D_1) = A, x \in A, z \notin A$, A is an annulus with $\partial A = \gamma' \cup \gamma''$, $\deg(F : \gamma' \to \gamma_1) = 1$, and $\deg(F : \gamma'' \to \gamma_1) = 2$. Moreover x and z are in different components of $S^2 - \gamma'$. So by the Lemma 3.1, A is isotopically contained in D_2 and $\gamma' \notin \Gamma_1$. Hence γ'' is homotopic to γ_2 . Set $\Gamma'' = \{\gamma_1, \gamma_2\}$, then $\lambda(\Gamma'') < 1$ and $\Gamma'' \cup \Gamma_2$ is F-invariant. Let $\Gamma_1'' = \Gamma_1 - \Gamma''$, then every curve in Γ_1'' separates $\{a, y\}$ and $\{x, z\}$. Repeating the same argument, we would get finally $\lambda(\Gamma_1) < 1$. Contradiction.

7) $k = 2$, $\#D_1 \cap X = 2$, $c \in D_1$, $\#D_2 \cap X = 1$, and $\#A_1 \cap \{x, y, z\} = 1$. Since $F^{-1}(D_2)$ consists of discs and has only one component D' intersecting $\{x, y, z\}$, if D' intersects A_1 , then D' is isotopic neither to D_1 nor to D_2 , this contradicts the Lemma 3.1. So D' should be isotopic to D_1 . The only possibilities are

7.1) $D_1 \cap X = \{a, z\}$ and $D_2 \cap X = \{x\}$. We will get a good Levy cycle with at most two curves in this case. $F^{-1}(D_1) = A$ is isotopic to A_1 . Let $S^2 - A = B_1 \cup B_2$ with $a \in B_1$. Then $\deg(F : B_1 \to S^2 - D_1) = 1$. Using the same method as in the thesis of Tan Lei [TL2] , we will find a good Levy cycle in B_1 .

7.2) $D_1 \cap X = \{a, y\}$ and $D_2 \cap X = \{z\}$. For this case, see the Proposition 7.2 below.

8) $k = 3$, $\#D_1 \cap X = 2$, $c \in D_1$, $\#D_2 \cap X = \#D_3 \cap X = 1$. Then in fact $\#D_i \cap \{x, y, z\} = 1$, $i = 1, 2, 3$. In any case for $\Gamma'' = \{\partial D_1, \partial D_2, \partial D_3\}$, we have $\lambda(\Gamma'') < 1$ and $\Gamma'' \cup \Gamma_2$ is F-invariant. Let $\Gamma_1'' = \Gamma_1 - \Gamma''$, then Γ_1'' is in case 6), 7) or 8) .

7.2 PROPOSITION. *Suppose* Γ *is a minimal Thurston obstruction for* F . *For the decomposition* $\Gamma = \Gamma_1 \cup \Gamma_2$ *as at the beginning of this section, suppose*

$$S^2 - \Gamma_1 = D_1 \cup D_2 \cup A_1 \cup ... \cup A_l \ ,$$

with only two disc components D_1, D_2 , $D_1 \cap X = \{a, y\}$, $D_2 \cap X = \{z\}$ *and* $x \in A_1$. *Then* $a' = F(a) \in D_2$. *And* $a'' = F(a')$ *is not in the component of* $S^2 - A_1$ *containing* a .

PROOF:

Since $D_1 \cap X = \{a, y\}$, $D_2 \cap X = \{z\}$, the set $F^{-1}(D_2) = D' \cup D''$ is a union of two discs, with $\deg(F : D' \to D_2) = 1$, and $\deg(F : D'' \to D_2) = 2$. Because of the degree, $D'' \cap \{x, y, z\} = \{y\}$ and $D' \cap \{x, y, z\} = \phi$, hence $\partial D' \notin \Gamma_1$. By Lemma 3.1 c), D'' is isotopic to D_1 since $y \in D_1 \cap D''$. Therefore $a' = F(a) \in D_2$.

Let us look at D_1 now. The set $F^{-1}(D_1) = A$ is an annulus with $\partial A = \gamma' \cup \gamma''$ such that $\deg(F : \gamma' \to \partial D_1) = 1$ and $\deg(F : \gamma'' \to \partial D_1) = 2$. By the Lemma 3.1, A is isotopically contained in A_1 . Let B_1, B_2 be the discs of $S^2 - \overline{A}$ bounded by γ', γ'' respectively. We claim that $z \in B_1$. At first $y \in B_2$ because of the degree. If $z \notin B_1$, then $\gamma' \notin \Gamma_1$ hence $\gamma'' \in \Gamma_1$ by the condition (3.2). Since $y, z \notin A$, the disc $A \cup \overline{B}_1$ contains a disc component of $S^2 - \Gamma_1$ which does not contain y, z . This contradicts the assumption. So $z \in B_1$.

We say that a curve $\gamma \in S^2$ *separates* two sets U and V if U is contained in one component of $S^2 - \gamma$ and V is contained in the other component of $S^2 - \gamma$.

Let us define

$$S_1 = \{\gamma \in \Gamma_1 | \gamma \text{ separates } \{a, y\} \text{ and } \{x, z\} \} \ ,$$
$$S_2 = \{\gamma \in \Gamma_1 | \gamma \text{ separates } \{a, y, x\} \text{ and } \{x, z\} \} \ .$$

Since there are only two disc components of $S^2 - \Gamma_1$, all curves in Γ_1 are nested, hence every $\gamma \in \Gamma_1$ separates $\{a, y\}$ and $\{z\}$. So in fact $\Gamma_1 = S_1 \cup S_2$.

Take $\gamma \in \Gamma_1$. Let us denote by Δ the component of $S^2 - \gamma$ containing z , and by Δ' , Δ'' the two components of $F^{-1}(\Delta)$, with $\deg(F : \Delta' \to \Delta) = 1$ and $\deg(F : \Delta'' \to \Delta) = 2$. Then Δ', Δ'' are discs. Moreover, let us denote $\gamma^* = \partial \Delta'$ and $\gamma^{**} = \partial \Delta''$. We have also $\deg(F : \gamma^* \to \gamma) = 1$ and $\deg(F : \gamma^{**} \to \gamma) = 2$.

For each $\gamma \in \Gamma_1$, we have $D_2 \subset \Delta \subset S^2 - D_1$, hence $D' \subset \Delta' \subset B_1$ and $D'' \subset \Delta'' \subset B_2$. Note that $x \in A = S^2 - \overline{B}_1 \cup \overline{B}_2$, $a, y \in D''$ and $z \in B_1$. It follows that for $\gamma \in \Gamma_1$, if γ^* is in Γ_1 then it is in S_2 , and if γ^{**} is in Γ_1 then it is in S_1 . Moreover if $\gamma \in S_2$, then γ^* is not in Γ_1, since $z \notin \Delta'$.

Let

$$v = \sum_{\gamma \in \Gamma_1} c_\gamma \gamma$$

be a positive eigenvector (i.e. every c_γ is positive) of the Thurston's linear transformation F_{Γ_1} with the eigenvalue $\lambda = \lambda(\Gamma_1)$. Such a positive eigenvector exists, since by the assumption Γ is a minimal Thurston's obstruction. Let us denote

$$v_1 = \sum_{\gamma \in S_1} c_\gamma \gamma \ , \quad v_2 = \sum_{\gamma \in S_2} c_\gamma \gamma \ ,$$

$$\text{for } v' = \sum_{\gamma \in \Gamma_1} c'_\gamma \gamma \quad \text{set } |v'| = \sum_{\gamma \in \Gamma_1} c'_\gamma .$$

Recall that $[\eta]_{\Gamma_1}$ denotes the curve in Γ_1 homotopic to η , and by convention $[\eta]_{\Gamma_1} = 0$ if no such curve exists. Using the above notation, we have

$$F_{\Gamma_1}(\sum_{\gamma \in S_1} c_\gamma \gamma) = \sum_{\gamma \in S_1} c_\gamma [\gamma^*]_{\Gamma_1} + \sum_{\gamma \in S_1} \frac{1}{2} c_\gamma [\gamma^{**}]_{\Gamma_1} ,$$

$$F_{\Gamma_1}(\sum_{\gamma \in S_2} c_\gamma \gamma) = \sum_{\gamma \in S_2} \frac{1}{2} c_\gamma [\gamma^{**}]_{\Gamma_1}$$

and

$$\sum_{\gamma \in S_1} \frac{1}{2} c_\gamma [\gamma^{**}]_{\Gamma_1} + \sum_{\gamma \in S_2} \frac{1}{2} c_\gamma [\gamma^{**}]_{\Gamma_1} = \lambda \sum_{\gamma \in S_1} c_\gamma \gamma \; = \; \lambda v_1 ,$$

$$\sum_{\gamma \in S_1} c_\gamma [\gamma^*]_{\Gamma_1} = \lambda \sum_{\gamma \in S_2} c_\gamma \gamma \; = \; \lambda v_2 .$$

Hence we have

$$(1) \qquad \frac{1}{2}(|v_1| + |v_2|) = \sum_{\gamma \in S_1} \frac{1}{2} c_\gamma + \sum_{\gamma \in S_2} \frac{1}{2} c_\gamma \geq \lambda \sum_{\gamma \in S_1} c_\gamma = \lambda |v_1|$$

$$(2) \qquad |v_1| = \sum_{\gamma \in S_1} c_\gamma \geq \lambda \sum_{\gamma \in S_2} c_\gamma = \lambda |v_2| .$$

The equalites in (1) and (2) hold if and only if

(*) for each $\gamma \in S_1$, both γ^* and γ^{**} are homotopic to curves in Γ_1 ; for each $\gamma \in S_2$ γ^{**} is homotopic to a curve in Γ_1 .

Summing twice of (1) to (2), we obtain

$$2|v_1| + |v_2| \geq \lambda(2|v_1| + |v_2|) .$$

Hence $\lambda \leq 1$. But we have $\lambda \geq 1$ by the assumption that Γ is a Thurston's obstruction. It follows that $\lambda = 1$, and the equalities in (1),(2) hold. Hence (*) holds. In particular, for $\gamma = \partial D_1 \in S_1$ and $\gamma^{**} = \gamma''$ the boundary curve of A separating $\{a\}$ and $\{x\}$, we have that γ'' is homotopic to a curve in S_1 . Applying (*) to γ'' , we obtain that $(\gamma'')^*$ is homotopic to a curve in Γ_1 . Let Δ be the disc bounded by γ'' containing z . Then Δ' defined as above contains D_2 , so $a' \in \Delta'$ therefore $a'' = F(a') \in \Delta$. Moreover $x \in \Delta$. This implies that γ'' does not separate x and a'' . ∎

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Let g_a and P_i $(i = 1, 2, 3, 4)$ be as in §III.1.

5.1 THEOREM. *For any $a \in M'$, g_a and P_3 (or P_4) are matable, i.e. $A_3 = A_4 = M'$.*

PROOF: Let F be the degenerate mating of g_a and P_3 $(a \in M')$. Suppose F has a Thurston's obstruction. By Theorem III.2.1, there exists a Levy cycle. This Levy cycle lifts to a non-degenerate Levy cycle for $F = g_a \perp\!\!\!\perp P_3$. It follows from Proposition 4.2 that there is a fixed point of P_3 which has exactly two external angles. However, by computation one can show that P_3 has three fixed points α, β, β' with external angles:

$$Ang(\alpha) = \{8/26,\ 20/26,\ 24/26\}, \quad Ang(\beta) = \{0\}, \quad Ang(\beta') = \{1/2'\} .$$

So none of them have exactly two external angles. This gives a contradiction.

Hence g_a and P_3 (or P_4) are matable. ▮

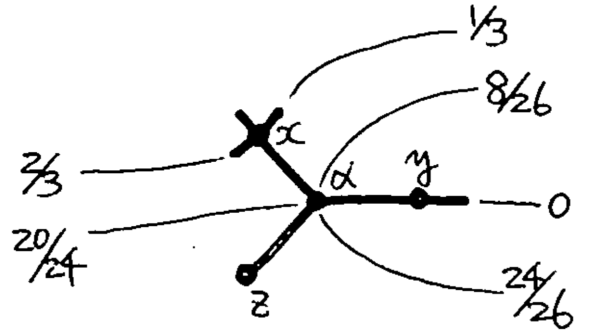
Let $x \to y \to z \to x$ be the periodic cycle of P_2 containing two simple critical points x, y . Suppose $x < y < z$ (otherwise consider $-P_2(-Z)$). We may suppose that the coefficient of Z^3-term is -1. Denote $\tilde{P}_2(Z) = iP_2(-iZ)$ then $\tilde{P}_2$ is monic. Let L be the limb of the cubic Mandelbrot set M_3 of internal angle $-1/4$.

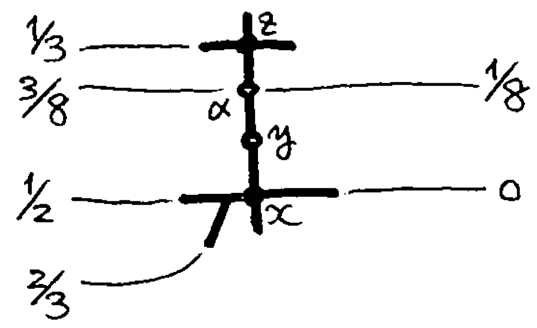
5.2 THEOREM. *For $a \in M' \cap L$, g_a and $\tilde{P}_2$ are not matable. In fact there is a good Levy cycle for $g_a \perp\!\!\!\perp \tilde{P}_2$, which consists of one curve made of two external rays of fixed points of g_a and $\tilde{P}_2$.*

PROOF.: For $a \in M' \cap L$, g_a has a fixed point α with external angle $5/8$ and $7/8$. It is easy to see that $\tilde{P}_2$ has a fixed point with external angles $1/8$ $(\equiv -7/8\ mod 1)$ and $3/8$ $(\equiv -5/8\ mod 1)$. Let

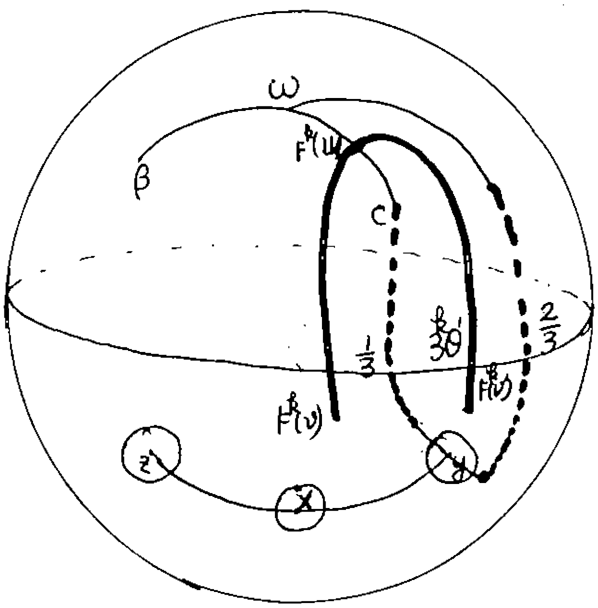
$$\gamma = R(5/8) \cup R(7/8) = \big(R_{g_a}(5/8) \cup R_{\tilde{P}_2}(-5/8)\big) \cup \big(R_{g_a}(7/8) \cup R_{\tilde{P}_2}(-7/8)\big)$$

which is a simple closed curve in $S^2_{g_a, \tilde{P}_2}$. If α is not postcritical, then $\Gamma = \{\gamma\}$ is a good Levy cycle. If α is postcritical, take a thin annulus A along γ (a *tubular neighborhood* of γ) so that $\overline{A} \cap P_F = \{\alpha\}$ and ∂A consists of two simple closed curves γ_+, γ_- . Then $\Gamma = \{\gamma_+, \gamma_-\}$ is a good Levy cycle. ▮





polynomial f . So every Levy cycle of F reduces to a cycle of ray-equivalence classes with some specified conditions. Hence F can not have any degenerate Levy cycle by the above corollary. If F has a non-degenerate Levy cycle, then it will reduces to a cycle of ray-equivalence classes containing closed curves, which is impossible by Corollary 3.8. ∎



Picture for the Lemma 3.7.

(iii) *δ_3 and δ_5 are homotopic to δ_1 in $S^2_{f_1,f_2} - P_F$. δ_4 is peripheral in $S^2_{f_1,f_2} - P_F$.*

PROOF OF THEOREM I.4.2: Let $\Gamma = \{\delta_1, \delta_2\}$. Then by the above lemma, the matrix for the Thurston's linear transformation $F_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ (under basis δ_1, δ_2) is

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}.$$

This matrix has the leading eigenvalue $\lambda_\Gamma = 1$ with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence Γ is a Thurston's obstruction, then by Thurston's theorem, F is not equivalent to a rational map. ∎

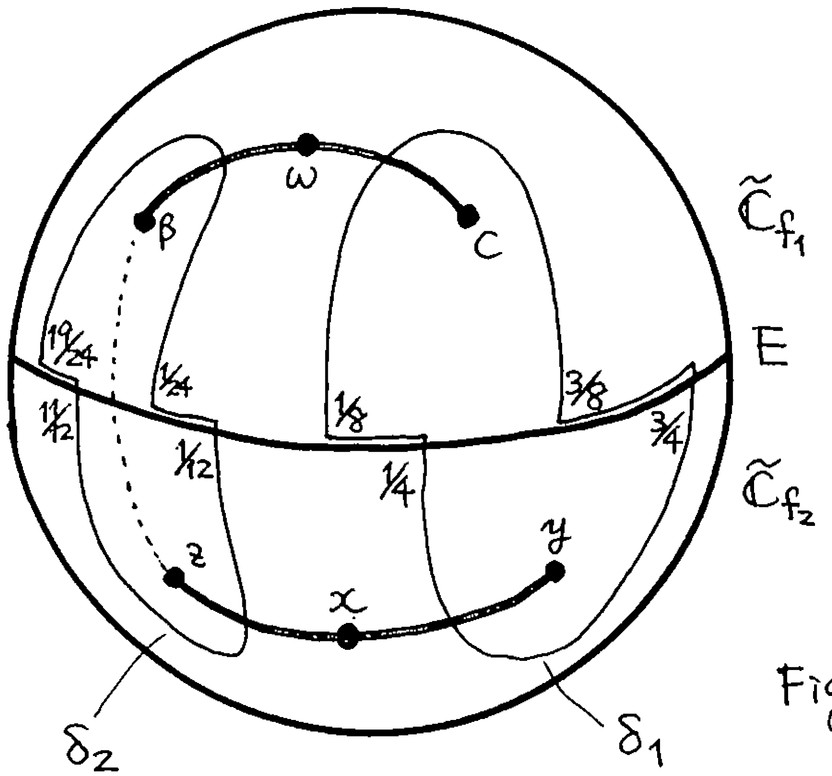


Figure 2.1

Γ is *degenerate* if the connected components of $S^2 - \bigcup_i \gamma_i$ are

$$B_1, B_2, ..., B_n, C \ ,$$

with B_i discs, C not disc, and each $F^{-1}(B_{i+1})$ has a component B'_i isotopic to B_i (rel P_F) , and $F : B'_i \to B_{i+1}$ is of degree one ($i = 0, 1, .., n-1$), where $B_0 = B_n$.

FIRST REDUCTION. *(Levy's theorem). Suppose F is of degree two. There is a Thurston's obstruction for F if and only if there is a Levy cycle for F .*

SECOND REDUCTION. *Suppose F is of degree two. There is a Levy cycle for F if and only if there is either a degenerate or a good Levy cycle for F .*

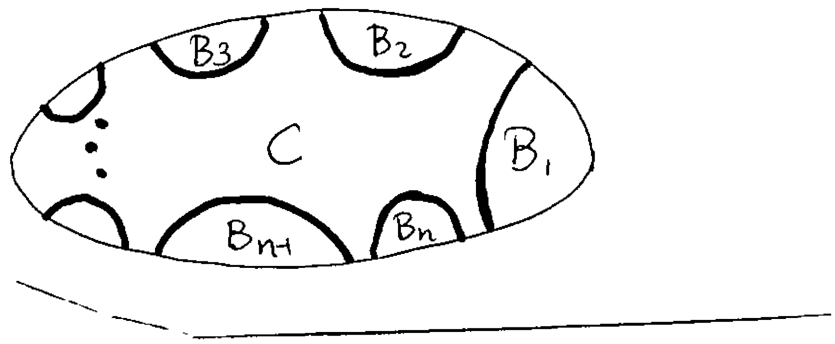
THIRD REDUCTION. *Suppose $F = f_c \perp\!\!\!\perp f_{c'}$, $c, c' \in \mathcal{D}$. Then by using the expansive metric for F near ∂K_c and $\partial K'_c$, we can prove that*

a) each degenerate Levy cycle for F reduces to a cycle of ray-equivalence classes: $[x_0], [x_1], ..., [x_m]$, ($[x_0] = [x_m]$) such that for each i, $F([x_i]) = [x_{i+1}]$, $\#[x_i] \cap P_F \geq 2$, and none of the $[x_i]$ contain closed curves;

b) each non-degenerate Levy cycle for F reduces to a cycle of ray-equivalence classes: $[x_0], [x_1], ..., [x_m]$, ($[x_0] = [x_m]$) such that for each i, $F([x_i]) = [x_{i+1}]$, and $[x_i]$ contains closed curves;

c) each good Levy cycle for F reduces to a ray-equivalence class $[x]$ such that $[x]$ contains at least one closed curve and at least two fixed points of F .

Note that each Levy cycle for the degenerate mating F' lifts to a non-degenerate Levy cycle for F . By these reductions, Theorem 3.1 is proved.



Good and degenerate Levy Cycle.