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Konstantin A. Makarov, Vladislav V. Melezhik

Dept. of Computational and
Mathematical Physics
St. Petersburg University
198094 St. Petersburg
RUSSIA

Fakultät für Mathematik
Ruhr Universität Bochum
44780 Bochum
GERMANY

Max-Planck Arbeitsgruppe
Potsdam University
14415 Potsdam
GERMANY

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn
GERMANY

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Konstantin A. Makarov[1,2,3], **Vladislav V. Melezhik**[1]

[1] *Department of Computational and Mathematical Physics, St.Petersburg University, 198094 St.Petersburg, Russia;*
[2] *Fakultät für Mathematik, Ruhr Universität Bochum, 44780 Bochum, Germany;*
[3] *Max-Planck Arbeitsgruppe, Potsdam University, 14415 Potsdam, Germany*

Abstract

The three-body systems with point-like interactions with an internal structure are considered. The complete classification of these systems are carried out and the conditions when the corresponding energy operators are semibounded from below are studied.

1 Introduction

The sixties were commemorated by two remarkable events in the spectral theory of the multipartical Schrödinger operators. The first one was the disclosure of the collapse phenomenon (the "fall to the center") in the three-body problem with δ-interactions [1], [2], the second one was the discovery of the Efimov effect which states the presence of infinite discrete spectrum fore some special classes of the three-body systems with short range interactions [3]:

The three-body Hamiltonian with short range interactions has infinitely many bound states if and only if there is at least two two-body subsystems with a zero energy resonance (the virtual level).

The historically first mathematical explanation of the Efimov effect belongs to L.D. Faddeev [4], [5]. Faddeev had noticed that the proof of the Efimov effect can be reduced to the study of solvability conditions for the homogeneous Skornyakov—Ter-Martirosyan equations [6] (in a more precise form proposed by G.S.Danilov [7]) for large negative values of the spectral parameter. It was the task that one had to carry out in order to prove the nonsemiboundedness from below of the three-body Hamiltonian with δ-interactions [1], [2]. Faddeev brought this fact to D.R.Yafaev's attention that resulted in a complete (qualitative) mathematical theory of the Efimov effect [8], slightly different, however, from the initial Faddeev's arguments (see, e.g. [9], Ch. III), where one can find the sketch of the quantative theory of the effect. For further mathematical work on the Efimov effect see [10-12].

A deep relationship between the asymptotics of the negative eigenvalues tending to minus infinity in the three-body problem with δ-interactions and the accumulation law of the eigenvalues to the three-body threshold in the Efimov effect was established in the work [13] by S.Albeverio, R.Hoegh-Krohn and T.T.Wu. The conjecture that (the universality of the Efimov effect) that the leading term of the asymptotics in question is determined by the mass ratio only and does not depend on details of potentials (under the assumption that the two-body energy operators have virtual levels, of cause) was put forward in this work. The proof of a similar but weaker result one can find in [14].

Thus, from the one hand the collapse in the three-body system with δ-interactions and from the other hand the Efimov effect (when at least every two-body Hamiltonians have zero energy resonances) are the two sides of the same phenomenon. This analogy seems to be incomplete, however. Indeed, the class of the three-body systems with short range potentials can be subdivided into four subclasses corresponding to the number of the two-body subsystems possessing virtual level, the Efimov effect taking place in only two of these subclasses. To carry out a similar classification for the three-body systems with δ-interactions by such a way that the energy operator would be unbounded from below for two of such subclasses and semibounded from below for the others subclasses is impossible in principle: any self-adjoint realization of the energy operator for such a system has to be unsemibounded from below [15]. We do not take in mind, of cause, the trivial case when the particles (in one or in two pairs) are simply non-interacting. Nevertheless, the classification desired is possible in a more wide class of the three-body systems, namely in the class of few-body systems with point-like interactions with an internal structure, suggested and partially studied in [16-21].

The first examples of the three-body systems with point-like interactions free of the collapse appeared in the work by Yu.G. Shondin [16] and later in [17] by L.E. Thomas.

After the work of B.S. Pavlov [18] where he had proposed a two-body model of δ-potential with an internal structure (considering self-adjoint extensions of the Laplace operator going out from the basic Hilbert space to an arbitrary Hilbert space of "internal degrees of freedom"), it had become clear that the class of few-body models with point-like interactions can be essentially extended. It was found out that in the two-body sector the scattering theory fore these models is much more "alive" in comparison with that of the case of standard δ-interactions: the corresponding two-body S-matrices are no longer parameterized by a real parameter ω

$$S(k) = \frac{\omega - ik}{\omega + ik}, \tag{1.1}$$

as it took place for δ-interactions (see, e.g. [22]), but by an arbitrary R-function (an analytic function with positive imaginary part in the upper half plane) $\omega(z)$ of the parameter z

$$S(k) = \frac{\omega(k^2) - ik}{\omega(k^2) + ik}, \tag{1.2}$$

the function $\omega(z)$ being now the functional parameter of the model.

To recognize the history one should notice that it was R.Schrader who was the pioneer of " the internal structure" [23], where he had successfully applied the analogous extensions going out to one-dimensional space corresponding to the simplest particular case $\omega(z) = Az + B, A > 0$, when regularizing the Hamiltonian in the Galilean invariant Lee model.

Taking into account the representation theorem

$$\omega(z) = Az + B + \int \frac{sz + 1}{s - z} d\,\mu(s), \tag{1.3}$$

valid for arbitrary R-function $\omega(z)$, where $A \geq 0$, $Im\, B = 0$ and μ is a finite Borel measure, it is natural to subdivide the two-body models suggested in [17] into two classes depending on that whether R-function $\omega(z)$ in representation (1.3) has the "linear term in z" ($A > 0$) or not ($A = 0$).

We distinguish two cases (here and later on we assume that the measure μ in (1.3) has a compact support):

Case a)

S-matrix of the model possesses high energy behavior specific for the case of potential scattering:

$$S(k) \underset{k \to \infty}{\longrightarrow} 1 \tag{1.4}$$

Case b)

S-matrix shows "anomalous" behavior, specific in particular for ordinal δ-potentials:

$$S(k) \underset{k\to\infty}{\longrightarrow} -1. \tag{1.5}$$

Following classification (1.4), (1.5) in the two-body sector we come to a natural subdivision of the class of three-body systems with point-like interactions with an internal structure onto four subclasses. This classification is completely analogous to that of the three particles systems with short range potentials to within the substitution the words "there is a zero energy resonance (a virtual level) in the subsystem α" by "S-matrix of the subsystem α has a normal high energy behavior." The following statement, word-to-word repeating the Efimov effect formulation solves the problem of collapse.

Theorem 1.1

Every self-adjoint realization of the energy operator for the three-body system with point-like interactions with an internal structure is semibounded from below if and only if at least two of three S-matrices corresponding to the two-body subsystems possess a normal high energy behavior, that is

$$S_\alpha(k) \underset{k\to\infty}{\longrightarrow} 1$$

at least for two α from the set $\{\{1,2\},\{2,3\},\{1,3\}\}$.

The part of this statement when all $S_\alpha(k) \to 1$ as $k \to \infty$ for all $\alpha, \alpha = 1,2,3$, has been formulated for the fist time in [19] (see also [20]). The idea to classify the point-like interactions in terms of high energy behavior of the corresponding S-matrices belongs to Pavlov [14].

In [19] Pavlov has investigated the three-body system with point-like interactions with an internal structure such that in every two- particles subsystems case a) is realized. For such systems he had proved the semiboundedness from below of the three-body energy operator. Unfortunately the interactions suggested in [19] turned to be non-pairwise in the strict sense of the word. The corresponding modification to the case of pairwise interactions is contained in the work by one of the authors [20], where the straightforward proof of the semiboundedness from below of the corresponding Hamiltonian based on a direct estimation of its quadratic form has been done. The scattering theory for these systems is developed in [21].

The case of the three-body systems with point-like interactions with an internal structure such that at least one of three the two-body subsystems has a anomalous S-matrix has never been investigated in details.

In the present paper in each case or the suggested classification we shall describe pre-Hamiltonians for the three-body systems with point-like interactions as symmetric operators in an appropriate extended Hilbert space. We shall prove here a part of the theorem (see theorem 4.3) related to the semi-bounded case:

if S-matrix possesses normal high energy behavior (1.4) at least in two of three two-body subsystems then the three-body energy operator is semibounded from below.

The proof of the second part of implication of the theorem which describes the system where the "fall to the center" takes place (the non-semiboundedness from below of the energy operator), when the number of the two-body subsystems with "anomalous" S-matrix (1.5) exceeds one, follows the original Faddeev-Minlos proof [2]. The necessary technical tricks can be extracted from the results of four works [8], [14-15], [24] and we last for ourselves the possibility to publish the "details" in the forthcoming paper.

2 Three-body Hamiltonian as a symmetric operator

In the present section we briefly describe the energy operator of the three-body system with point-like interactions with an internal structure in momentum space representation. Motivations, detailed treatment and discussions of pair-wise character of the interactions in question one can find in [19-21], [25].

First of all we fix in $\mathbf{R}^6$ three orthogonal coordinate systems (of Jacobi's relative coordinates)

$$P = k_\alpha \oplus p_\alpha,\ \alpha = 1, 2, 3, \tag{2.1}$$

related each other by orthogonal transformation

$$\begin{pmatrix} k_\alpha \\ p_\alpha \end{pmatrix} = U_{\alpha\beta} \begin{pmatrix} k_\beta \\ p_\beta \end{pmatrix}, \qquad U_{\alpha\beta} = \begin{pmatrix} c_{\alpha\beta} & s_{\alpha\beta} \\ s_{\alpha\beta} & -c_{\alpha\beta} \end{pmatrix} \tag{2.2}$$

where index α enumerates all possible two-body subsystems of the three-body system and matrices $U_{\alpha\beta}$ satisfy the condition

$$U_{\alpha\beta} U_{\beta\gamma} U_{\gamma\delta} = 1.$$

The coefficients $c_{\alpha\beta}$, $s_{\alpha\beta}$, $(c^2_{\alpha\beta} + s^2_{\alpha\beta} = 1)$ can be explicitly expressed through the masses of the particles [9].

The free Hamiltonian H

$$H = \frac{1}{2m_1}\mathbf{p}_1^2 + \frac{1}{2m_2}\mathbf{p}_2^2 + \frac{1}{2m_3}\mathbf{p}_3^2, \tag{2.3}$$

where $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ are momenta conjugated to the the coordinates x_1, x_2, x_3 of the particles in the configuration space $\mathbf{R}^9$, after splitting off the center mass motion becomes the multiplication operator by the function $P^2 = p_\alpha^2 + k_\alpha^2, P \in \mathbf{R}^6$ (see (2.1)) acting in the space

$$\mathcal{H}^{ex} = L_2(\mathbf{R}^6). \tag{2.4}$$

Let us bind with each two-body subsystem $\alpha, \alpha = 1, 2, 3$, some Hilbert space $\mathcal{H}^{in}_\alpha$ (of internal degrees of freedom), a self-adjoint operator $\mathcal{A}_\alpha$ acting in the space $\mathcal{H}^{in}_\alpha$ (the internal Hamiltonian) and some element $g_\alpha \in \mathcal{H}^{in}_\alpha$ (the channel vector). Furthermore, let us assume that the automorphism of the upper half plane is given

$$z \longrightarrow \frac{a_\alpha + b_\alpha z}{c_\alpha + d_\alpha z}, \tag{2.5}$$

the complex parameters $a_\alpha, b_\alpha, c_\alpha, d_\alpha$, lying on the straight line $\{z = x + iy : x = \theta_\alpha y\}$ in the complex plane $\mathbf{C}$, $\theta_\alpha \in \mathbf{R}$, θ_α is fixed, and

$$\det \begin{pmatrix} a_\alpha & b_\alpha \\ \bar{c}_\alpha & \bar{d}_\alpha \end{pmatrix} = -1. \tag{2.6}$$

The ingredients pointed out are the parameters of the model of point-like interactions with an internal structure suggested by Pavlov [18]. We follow the notations used in [20] where one can also find an explicit description of the two-body model Hamiltonians h_α as self-adjoint operators in the space $L_2(\mathbf{R}^3) \oplus \mathcal{H}^{in}_\alpha$. Let us list the most typical features of the model, referring for the details to [19-21].

The resolvent of the two-body Hamiltonian h_α has block structure and the block corresponding to the space $L_2(\mathbf{R}^3)$ (so-called generalized Krein's resolvent) has the form:

$$P_{L_2(\mathbf{R}^3)}(h_\alpha - z)^{-1}\big|_{L_2(\mathbf{R}^3)} = (\hat{h} - z)^{-1} - T(z). \tag{2.7}$$

Here by $P_{L_2(\mathbf{R}^3)}$ we denote the orthogonal projector from $L_2(\mathbf{R}^3) \oplus \mathcal{H}_\alpha^{in}$ onto $L_2(\mathbf{R}^3)$, by $\hat{h}$ the multiplication operator by the function p^2 in the space $L_2(\mathbf{R}^3)$ (the free Hamiltonian) and, finally $T(z)$ is a rank-one integral operator with the kernel

$$T(p,k;z) = \frac{1}{p^2 - z} t_\alpha(z) \frac{1}{k^2 - \bar{z}}. \tag{2.8}$$

The function $t_\alpha(z)$ which we freely speaking will call the two-body t-matrix corresponding to the subsystem α, is represented in the form:

$$t_\alpha(z) = \frac{1}{\omega_\alpha(z) + i\frac{\sqrt{z}}{4\pi}}, \tag{2.9}$$

where the function $\omega_\alpha(z)$ is expressed within to a constant by linear-fractional transformation

$$\omega_\alpha(z) = \frac{d_\alpha + b_\alpha r(z)}{c_\alpha + a_\alpha r(z)} + \text{const} \tag{2.10}$$

of the quadratic form of the internal operator $\mathcal{A}_\alpha$ resolvent (calculated on the channel vector g_α):

$$r(z) = \langle (\mathcal{A}_\alpha - z)^{-1} g_\alpha, g_\alpha \rangle. \tag{2.11}$$

Note that by (2.5), (2.6) $\omega_\alpha(z)$ is an analytic function (R-function) with positive (negative) imaginary part in the upper (lower) half plane, since the mapping

$$z \longrightarrow \frac{d_\alpha + b_\alpha}{c_\alpha + d_\alpha z}$$

is also an automorphism of the upper half plane.

Depending on the fact weather the model parameter c_α in (2.5) equals zero or not, the function $\omega_\alpha(z)$ has different asymptotic behavior for $|z| \to \infty$: if $c_\alpha = 0$ then $\omega(z)$ tends to infinity; on the contrary, if $c_\alpha \neq 0$ then the function $\omega_\alpha(z)$ is bounded in the neighbourhood of infinity. In this connection two-body t-matrix (2.9) of the system α displays different behavior also:

case a) $(c_\alpha \neq 0)$

$$t_\alpha(z) = \mathcal{O}(\frac{1}{|z|}), \quad z \to -\infty, \tag{2.12}$$

case b) $((c_\alpha = 0))$

$$t_\alpha(z) = \mathcal{O}(\frac{1}{\sqrt{|z|}}), \quad z \to -\infty. \tag{2.13}$$

Thus the formally defined S-matrix of the model

$$S_\alpha(k) = \frac{\omega_\alpha(k^2) - \frac{ik}{4\pi}}{\omega_\alpha(k^2) + \frac{ik}{4\pi}} \tag{2.14}$$

tends to 1 in the limit (1.4) in case a) and possesses "anomalous" behavior (1.5) in case b).

Let us turn back to the description of the three-body energy operator with pair interactions defined by the two-body Hamiltonians $h_\alpha, \alpha = 1, 2, 3$.

Each coordinate system $\{p_\alpha, k_\alpha\}$ in the space $\mathbf{R}^6$ gives rise to the three-dimensional plane $\mathcal{M}_\alpha$

$$\mathcal{M}_\alpha = \{P : k_\alpha = 0\}, \ \alpha = 1, 2, 3. \tag{2.15}$$

The Hamiltonian H of the three-body system with point-like interactions with an internal structure acts in the extended Hilbert space (of 4-components functions)

$$\mathcal{H} = \mathcal{H}^{ex} \oplus \mathcal{H}^{in}, \tag{2.16}$$

where

$$\mathcal{H}^{in} = \bigoplus_\alpha \{L_2(\mathcal{M}_\alpha) \otimes \mathcal{H}_\alpha^{in}\}. \tag{2.17}$$

We need some additional notations to describe the initial domain of H and its rule of action.

Let $L_2^\delta(\mathbf{R}^n)$ denote the Hilbert space of functions f such that

$$\|f\|_\delta^2 = \int_{\mathbf{R}^n} (1 + P^2)^\delta |f(P)|^2 dP < \infty. \tag{2.18}$$

In the space $\mathcal{H}^{ex} = L_2(\mathbf{R}^6)$ let us consider the domain

$$\mathcal{D} = L_2^2(\mathbf{R}^6) + \mathcal{N}, \tag{2.19}$$

where

$$\mathcal{N} = \{\Phi \in L_2(\mathbf{R}^6) : \Phi(P) = \sum_\alpha \frac{\varphi_\alpha(p_\alpha)}{P^2 + 1}, \ \varphi_\alpha \in L_2^2(\mathcal{M}_\alpha)\}. \tag{2.20}$$

Recall (see 2.1) that $p_\alpha \in \mathbf{R}^3, \alpha = 1,2,3$, denotes the orthogonal projection of the vector $P = p_\alpha \oplus k_\alpha \in \mathbf{R}^6$ on the plane $\mathcal{M}_\alpha$. Note, that any element $u^{ex} \in \mathcal{D}$ can be uniquely represented in the form:

$$u^{ex}(P) = u(P) + \Phi(P), \tag{2.21}$$

where

$$\Phi(P) = \frac{\sum_\alpha \varphi_\alpha(p_\alpha)}{P^2+1}, \tag{2.22}$$

for some "densities" φ_α being elements of the space $L_2^2(\mathcal{M}_\alpha)$ and

$$u \in L_2^2(\mathbf{R}^6). \tag{2.23}$$

In coordinate representation the space $\mathcal{N}$ consists of the functions represented by simple layer potentials with the densities $\hat{\varphi}_\alpha$ from the Sobolev class $W_2^2(\mathbf{R}^3)$ "spread" on the planes $x_i = x_j$ where the particles of the pair $\alpha = \{ij\}$ "interact".

On the domain

$$\mathcal{D}(H_0) = \mathcal{D} \bigoplus_\alpha \{L_2^2(\mathcal{M}_\alpha) \otimes \mathcal{H}_\alpha^{in}\} \tag{2.24}$$

let us consider the operator H_0 acting by the rule

$$H_0 : \begin{pmatrix} u+\Phi \\ \bigoplus_\alpha u_\alpha \end{pmatrix} \longrightarrow \begin{pmatrix} P^2 u - \Phi \\ \bigoplus_\alpha \{(A_\alpha + p_\alpha^2)u_\alpha + l_\alpha g_\alpha\} \end{pmatrix}. \tag{2.25}$$

Here

$$l_\alpha(p_\alpha) = a_\alpha(\mathbf{I}_\alpha u^{ex})(p_\alpha) + b_\alpha(\mathbf{F}_\alpha u^{ex})(p_\alpha), \tag{2.26}$$

$u^{ex} = u + \Phi$ (see (2.21)) and $\mathbf{I}_\alpha$ and $\mathbf{F}_\alpha$ are unbounded embedding operators from the space $L_2(\mathbf{R}^3)$ into the space $L_2(\mathcal{M}_\alpha)$

$$(\mathbf{I}_\alpha u^{ex})(p_\alpha) = \int_{\mathcal{M}_\alpha^\perp} dk_\alpha \left(u^{ex}(P) - \frac{\varphi_\alpha(p_\alpha)}{P^2+1}\right) + 2\pi(\sqrt{1+p_\alpha^2} - 1)\varphi_\alpha(p_\alpha) \tag{2.27}$$

and

$$(\mathbf{F}_\alpha u^{ex})(p_\alpha) = \varphi(p_\alpha). \tag{2.28}$$

We define the three-body pre-Hamiltonian H as the restriction of the operator H_0 on the set $\mathcal{D}(H)$ of those functions from $\mathcal{D}(H_0)$ that satisfy the following "boundary conditions":

$$\langle u_\alpha, g_\alpha \rangle_{\mathcal{H}_\alpha^{in}}(p_\alpha) = c_\alpha \left(\mathbf{I}_\alpha u^{ex}\right)(p_\alpha) + d_\alpha \left(\mathbf{F}_\alpha u^{ex}\right)(p_\alpha), \quad \alpha = 1, 2, 3. \tag{2.29}$$

Using (2.29) and (2.6) one can verify that the operator H on the domain $\mathcal{D}(H)$ is densely defined and symmetric.

If we formally put $g_\alpha = 0$ in the described construction, then the subspace $\mathcal{H}^{ex} = L_2^2(\mathbf{R}^6)$ appears to be invariant with respect to H and boundary conditions (2.29) for $g_\alpha = 0$, $\alpha = 1, 2, 3$, pass in the so-called Skornyakov–Ter-Martirosyan boundary conditions which bind the functions of $\mathbf{I}_\alpha(u^{ex})$ and $\mathbf{F}_\alpha(u^{ex})$ considered as the elements of $L_2(\mathcal{M}_\alpha)$. The Hamiltonian obtained in such a formal passage is nothing else than Faddeev-Minlos [1], [2], Melnikov-Minlos [15] model pre-Hamiltonian which describes the three-body systems with δ-interactions.

3 Auxiliary operator $K(z)$

The analysis of the essentially self-adjointness conditions for the Hamiltonian H (2.25) on the domain $\mathcal{D}(H)$ (2.24), (2.29) is closely related with the study of an auxiliary operator $K(z)$, the three-body T-matrix is expressed through. Before to define this object let us recall that the two-body t-matrices in the model of point-like interactions with an internal structure have the form:

$$t(z) = \frac{1}{\omega(z) + i\frac{\sqrt{z}}{4\pi}}, \tag{3.1}$$

where $\omega(z)$ is some R-function (see (2.10)). If the internal operator $\mathcal{A}$ in the model (2.25) is bounded, then the measure μ corresponding to the function $\omega(z)$ in representation (1.3) has compact support and hence the following asymptotics are valid for $z \to -\infty$

in case a)

$$t(z) = \frac{1}{Az + B + i\frac{\sqrt{z}}{4\pi}} + \mathcal{O}(|z|^{-2}), \tag{3.2}$$

in case b)

$$t(z) = \frac{1}{B + i\frac{\sqrt{z}}{4\pi}} + \mathcal{O}(|z|^{-\frac{3}{2}}), \tag{3.3}$$

where A and B, $A > 0$, are some real constants. Therefore under sufficiently large negative values of the parameter z the function $t(z)$ is separated from zero and negative. Remark that the leading term of the asymptotics of $t(z)$ in case a) coincides with Shondin's model t-matrix [16] and in case b) corresponds to the case of standard δ-interactions t-matrix [22].

For every α, $\alpha = 1, 2, 3$, consider the self-adjoint strictly positive (unbounded) multiplication operator $W_\alpha(z)$:

$$(W_\alpha(z)f)(p) = (-t_\alpha(z - p^2))^{-1} f(p), \tag{3.4}$$

acting in the space $L_2(\mathbf{R}^3)$ and defined on its natural domain of self-adjointness

$$\mathcal{D}(W_\alpha(z)) = \{f \in L_2(\mathbf{R}^3) : W_\alpha(z)f \in L_2(\mathbf{R}^3)\}. \tag{3.5}$$

For z negative, $|z|$ large enough, the operator $W_\alpha(z)$ is comparable in case a) with the multiplication operator by the function $p^2 + 1$ and in case b) with the multiplication operator by the function $\sqrt{p^2 + 1}$, therefore we have the following description for its domain

in case a)

$$\mathcal{D}(W_\alpha(z)) = L_2^2(\mathbf{R}^3), \tag{3.6}$$

in case b)

$$\mathcal{D}(W_\alpha(z)) = L_2^1(\mathbf{R}^3). \tag{3.7}$$

The auxiliary operator $K(z)$ acts in the space $\mathcal{H}$ of three-components functions

$$\mathcal{H} = L_2(\mathbf{R}^3) \oplus L_2(\mathbf{R}^3) \oplus L_2(\mathbf{R}^3),$$

being a perturbation

$$K(z) = W(z) - R(z) \tag{3.8}$$

of the diagonal operator $W(z)$

$$W(z) = \mathrm{diag}(W_1(z), W_2(z), W_3(z)) \tag{3.9}$$

by a 3×3-operator matrix $R(z)$ with zero diagonal. The matrix elements of $R(z)$ are integral operators having the kernels

$$R_{\alpha\beta}(p, k; z) = \frac{1}{|s_{\alpha\beta}|}(p^2 - 2c_{\alpha\beta}(p, k) + k^2 - zs_{\alpha\beta}^2)^{-1}, \ \alpha \neq \beta, \tag{3.10}$$

where $s_{\alpha\beta}$, $c_{\alpha\beta}$ are the elements of the transition matrix (2.2) from one Jacobi coordinate system to another.

Time to time when the dependence of objects in questiòn on the parameter z seems to be inessential we shall omit the corresponding parametrization in the notations.

First, consider the operator $R(z)$ on the domain

$$\mathcal{D}(R) = \mathcal{H}^{2,2,2},$$

where it is obviously correctly defined and symmetric. Here $\mathcal{H}^{a,b,c}$ denotes the space

$$\mathcal{H}^{a,b,c} = L_2^a(\mathbf{R}^3) \oplus L_2^b(\mathbf{R}^3) \oplus L_2^c(\mathbf{R}^3). \tag{3.11}$$

The operator $K(z)$ is supposed to be defined on the initial domain

$$\mathcal{D}(K(z)) = \mathcal{D}(W(z)) \cap \mathcal{D}(R) = \mathcal{H}^{2,2,2}. \tag{3.12}$$

Recall that there are four types of the three-body systems with point-like interactions with an internal structure:

Case I: case a) realizes in every two-body subsystem

Case II: case b) realizes in the only one of the two-body subsystems

Case III: case a) realizes in the only one of the two-body subsystems

Case IV: case b) realizes in every two-body subsystem

The following two statements play the key role in description of both self-adjoint extensions of the Hamiltonian H (2.25), (2.26)–(2.29) initially defined on $\mathcal{D}(H)$ and the location of the discrete spectrum for such extensions.

Lemma 3.1

The auxiliary operator $K(z)$ is essentially self-adjoint on the domain $\mathcal{H}^{2,2,2}$ for some $z \in \mathbf{R}_-$, $|z|$ large enough, if and only if the three-body pre-Hamiltonian H is essentially self-adjoint on $\mathcal{D}(H)$.

Note, that the fact of essentially self-adjointness of $K(z)$ does not depend on the concrete choice of the value of the parameter z (lying from the left from some fixed z_0), since the difference $K(z) - K(z')$ is a bounded operator, more precisely its closure is a bounded operator.

With every self-adjoint extension of the operator $K(z)$ one can uniquely put in correspondence a self-adjoint extension of the pre-Hamiltonian H. Moreover this correspondence can be established in such a way that the following property holds: let $\hat{K}(z)$ be some self-adjoint extension of the operator $K(z)$ and $\hat{H}$ be the corresponding self-adjoint extension of the operator H and let $\hat{K}(z')$ also be some self-adjoint extension of the operator $K(z')$ corresponding to some another point z' and $\tilde{H}$ be the corresponding self-adjoint extension of the operator H. Then $\hat{H} = \tilde{H}$ if and only if $\mathcal{D}\left(\hat{K}(z)\right) = \mathcal{D}\left(\hat{K}(z')\right)$.

Lemma 3.2 *Let $\tilde{K}(z)$ be some self-adjoint extension of the operator $K(z)$ and let $\tilde{H}$ be the corresponding self-adjoint extension of the pre-Hamiltonian H. Then the real point z belongs to the resolvent set of the operator $\tilde{H}$ if and only if the operator $\tilde{K}(z)$ has a bounded inverse.*

The proofs of these statements are quite standard and in fact differs by nothing from the arguments suggested in [2], [15], [26].

Thus the extensions theory of the pre-Hamiltonian H is reduced to the study of the self-adjoint extensions of $K(z)$ and the proof of semiboundedness from below of the energy operator is reduced to the problem of reversibility of $K(z)$, more precisely of its self-adjoint extensions for all $z < 0$, $|z|$ large enough.

The following lemma is of the key importance in a technical aspect.

Lemma 3.3

Let us consider in the space $L_2(\mathbf{R}^3)$ the integral operator Q with the kernel

$$Q(p,k) = \frac{1}{(p^2+1)^{\frac{1}{4}}(p^2+k^2+1)(k^2+1)^{\frac{1}{4}}}. \tag{3.13}$$

Then the operator Q is bounded in the space $L_2(\mathbf{R}^3)$. Moreover, Q continuously maps the space $L_2^{\delta}(\mathbf{R}^3)$, $0 \le \delta < 1$, into itself and hence the operator Q continuously maps the space $L_2^{\delta}(\mathbf{R}^3)$, $\delta \ge 1$, into the space $L_2^{\kappa}(\mathbf{R}^3)$ for any $\kappa < 1$.

Proof. The proof of the boundedness of the operator Q in the space $L_2(\mathbf{R}^3)$ one can find in [8]. In order to check its boundedness in the spaces $L_2^{\delta}(\mathbf{R}^3)$, $0 < \delta < 1$, we need a minimal modification of the estimates suggested in [8].

Let $\delta < 1$. Choose $\epsilon > 0$ such that $\delta + \epsilon < 1$. Let $f \in L_2^\delta(\mathbf{R}^3)$ and $q = Qf$. By Cauchy inequality we have

(3.14)
$$|q(p)|^2 \leq \frac{1}{\sqrt{p^2+1}} \int_{\mathbf{R}^3} \frac{dk}{(p^2+k^2+1)(k^2+1)^{\frac{1}{2}+\delta+\epsilon}} \int_{\mathbf{R}^3} \frac{dk|f(k)|^2(k^2+1)^{\delta+\epsilon}}{(p^2+k^2+1)}.$$

For the integral

$$I(p) = \int_{\mathbf{R}^3} \frac{dk}{(p^2+k^2+1)(k^2+1)^{\frac{1}{2}+\delta+\epsilon}} \tag{3.15}$$

the estimate is valid

$$I(p) \leq \int_{\mathbf{R}^3} \frac{dk}{(p^2+k^2+1)|k|^{1+2\delta+2\epsilon}} \leq \mathrm{const}\,(p^2+1)^{-(\delta+\epsilon)}, \tag{3.16}$$

and hence

$$\|q\|^2_{L_2^\delta} \leq \int_{\mathbf{R}^3} \frac{dp}{(p^2+1)^{\epsilon+\frac{1}{2}}} \int_{\mathbf{R}^3} \frac{dk|f(k)|^2(k^2+1)^{\delta+\epsilon}}{(p^2+k^2+1)} =$$

(3.17)
$$= \int_{\mathbf{R}^3} dk|f(k)|^2(k^2+1)^{\delta+\epsilon} \int_{\mathbf{R}^3} \frac{dp}{(p^2+1)^{\epsilon+\frac{1}{2}}(p^2+k^2+1)} \leq \|f\|^2_{L_2^\delta}$$

Note, that "involving in the play" the parameter ϵ provides the convergence of the last integral in estimate (3.17). This inequality proves the first statement of the lemma.

The proof of the remaining statement is based on the observation that any space of "fast decreasing" functions $L_2^\delta(\mathbf{R}^3)$ for $\delta \geq 1$ is a subspace of the space $L_2^\kappa(\mathbf{R}^3)$ for any $\kappa < 1$. The latter is continuously mapped by Q into itself.

Based on the results of lemma 3.3 one can make two important observations.

The first one is that the matrix operator R defined on $\mathcal{D}(R) = \mathcal{H}^{2,2,2}$ admits a symmetric extension $\hat{R}$, still considered as an integral operator with the kernel (3.10), on a more wide domain

$$\mathcal{D}(\hat{R}) = \mathcal{H}^{1,1,1}.$$

To prove this, let us note, that the integral kernels of the operators $R_{\alpha\beta}(z)$ can be estimated as follows

$$|R_{\alpha\beta}(k,p;z)| \le \frac{\text{const}}{p^2+k^2+1}, \tag{3.18}$$

and therefore $R_{\alpha\beta}(z)f \in L_2(\mathbf{R}^3)$ if $f \in L_2^1(\mathbf{R}^3)$, since the operator Q from lemma 3.3 is bounded in the space $L_2^{\frac{1}{2}}(\mathbf{R}^3)$. Hence the operator $\hat{R}$ is correctly defined in the space $\mathcal{H}$ on the domain $\mathcal{D}(\hat{R}) = \mathcal{D}(\mathcal{H}^{1,1,1})$. The symmetriciety of $\hat{R}$ on this domain can be verified directly.

The second one is that as it follows from (3.6), (3.7) the domain of the operator W coincides with $\mathcal{H}^{2,2,2}$ only in case I and it is somewhat wider in the other cases. For example, in case IV we see that

$$\mathcal{D}(W) = \mathcal{H}^{1,1,1,},$$

and in cases II–IV we have the proper inclusion only

$$\mathcal{D}(W) \subset \mathcal{D}(\hat{R}) = \mathcal{H}^{1,1,1}.$$

It allows us to consider the operator K as a perturbation of the operator W not only on the domain $\mathcal{H}^{2,2,2}$ but also to extended it (we denote this extension by $\hat{K}$) on a wider domain

$$\mathcal{D}(\hat{K}) = \mathcal{D}(W) \cap \mathcal{D}(\hat{R}) = \mathcal{D}(W). \tag{3.19}$$

by the formula

$$\hat{K} = W - \hat{R} \tag{3.20}$$

We record the operator $\hat{K}(z)$ in a somewhat different symmetrized form. To this end we introduce the "sandwiched" operators

$$B_{\alpha\beta}(z) = W_\alpha^{-\frac{1}{2}}(z) R_{\alpha\beta}(z) W_\beta^{-\frac{1}{2}}(z), \ \alpha \ne \beta, \tag{3.21}$$

initially defined on the class of the fast decreasing functions. The operators $B_{\alpha\beta}$ are integral operators with the kernels

$$B_{\alpha\beta}(p,k;z) = (-t_\alpha(z-p^2))^{\frac{1}{2}} R_{\alpha\beta}(p,k;z)(-t_\beta(z-k^2))^{\frac{1}{2}} \tag{3.22}$$

which admit the estimate

$$|B_{\alpha\beta}(p,k;z)| \le \frac{\text{const}}{(p^2+1)^{\delta_\alpha}(p^2+k^2+1)(k^2+1)^{\delta_\beta}}, \tag{3.23}$$

where $\delta_\alpha = \frac{1}{2}$ if the two-body t-matrix $t_\alpha(z)$ of the subsystem α satisfies the condition a) and $\delta_\alpha = \frac{1}{4}$ otherwise.

By lemma 3.3 the operators $B_{\alpha\beta}(z)$ can be extended to bounded operators in the space $L_2(\mathbf{R}^3)$ and, hence, they define a bounded matrix operator $B(z)$ in the space $\mathcal{H}$ with matrix elements $B_{\alpha\beta}(z)$. Moreover, we have the following representation on the domain $\mathcal{D}(\hat{K}(z)) = \mathcal{D}(W)$:

$$\hat{K} = W - W^{\frac{1}{2}} B W^{\frac{1}{2}}. \tag{3.24}$$

The correctness of this representation is provided by the following

Lemma 3.4

The domain of the operator $W^{\frac{1}{2}}$ is invariant with respect to the action of the operator B:

$$B\mathcal{D}(W^{\frac{1}{2}}) \subset \mathcal{D}(W^{\frac{1}{2}}). \tag{3.25}$$

Proof. As we have noticed, the set of the two-body subsystems with point-like interactions brakes up on two classes, which we denote now by $\mathcal{A}$ and $\mathcal{B}$: we say that the subsystem $\alpha \in \mathcal{A}$ if the corresponding two-body t-matrix $t_\alpha(z)$ satisfies condition a) and $\beta \in \mathcal{B}$ if $t_\beta(z)$ satisfies condition b) respectively. Let us consider the integral operators $Q_{\alpha\beta}$ defined by the kernels

$$Q_{\alpha\beta}(p,k;z) = (-t_\alpha(z-p^2))^{\theta_\alpha} R_{\alpha\beta}(p,k;z)(-t_\beta(z-k^2))^{\theta_\beta}, \tag{3.26}$$

where $\theta_\alpha = \frac{1}{4}$, if $\alpha \in \mathcal{A}$ and $\theta_\beta = 1$, if $\beta \in \mathcal{B}$. The kernels of the operators $Q_{\alpha\beta}$ obviously admit the estimates

$$|Q_{\alpha\beta}(p,k;z)| \leq \frac{\text{const}}{(p^2+1)^{\frac{1}{4}}(k^2+p^2+1)(k^2+1)^{\frac{1}{4}}} \tag{3.27}$$

and, hence, all the statements of lemma 3.3 are valid for the operators $Q_{\alpha\beta}$.

Let us note that the operators $B_{\alpha\beta}$ are obviously connected with the operators $Q_{\alpha\beta}$ by the following relations: $B_{\alpha\beta} = W_\alpha^{-\frac{1}{4}} Q_{\alpha\beta} W_\beta^{-\frac{1}{4}}$, if $\alpha,\beta \in \mathcal{A}$, $B_{\alpha\beta} = W_\alpha^{-\frac{1}{4}} Q_{\alpha\beta}$, if $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$, $B_{\alpha\beta} = Q_{\alpha\beta} W_\beta^{-\frac{1}{4}}$ if $\alpha \in \mathcal{B}, \beta \in \mathcal{A}$, and, finally, $B_{\alpha\beta} = Q_{\alpha\beta}$ for $\alpha, \beta \in \mathcal{B}$.

Taking into account (3.6), (3.7) we immediately conclude that in every case I–IV the domain of the operator $W^{\frac{1}{2}}$ can be represented in the form

$$\mathcal{D}(W^{\frac{1}{2}}) = L_2^{\delta_\alpha}(\mathbf{R}^3) \oplus L_2^{\delta_\beta}(\mathbf{R}^3) \oplus L_2^{\delta_\gamma}(\mathbf{R}^3), \tag{3.28}$$

where $\delta_\alpha = 1$ if $\alpha \in \mathcal{A}$ and $\delta_\beta = \frac{1}{2}$ if $\beta \in \mathcal{B}$, and thus, the verification of the statement of the lemma means the proof of the series of inclusions

$$B_{\alpha\beta} L_2^{\delta_\beta}(\mathbf{R}^3) \subset L_2^{\delta_\alpha}(\mathbf{R}^3). \tag{3.29}$$

In each of four cases I–IV such a verification is based on the systematic use of lemma 3.3, the formulae connecting the operators $B_{\alpha\beta}$ and $Q_{\alpha\beta}$ being taken into account.

Here we give such a proof, e.g. in case I.

In this case the statement of the lemma is reduced then to the proof of the fact that the space $L_2^{\frac{1}{2}}(\mathbf{R}^3)$ is invariant with respect to the operators $B_{\alpha\beta}$.

Let $f \in L_2^{\frac{1}{2}}(\mathbf{R}^3)$, then

$$W_\beta^{-\frac{1}{4}} f \in L_2^{\frac{3}{2}}(\mathbf{R}^3)$$

and, hence, by lemma 3.3

$$Q_{\alpha\beta} W_\beta^{-\frac{1}{4}} f \in L_2^{1-\epsilon}(\mathbf{R}^3)$$

for any $\epsilon > 0$. Therefore

$$B_{\alpha\beta} f \in L_2^{\frac{3}{2}-\epsilon} \subset L_2^{\frac{1}{2}}(\mathbf{R}^3)$$

if we take $\epsilon \leq \frac{1}{2}$.

Lemma 3.4. shows that the right hand side of (3.24) makes sense as correctly defined operator. The fact that this operator coincides with $\hat{K}$ is now a simple exercise.

Here we present a result close in spirit to the previous one without proof. The proof can be obtained by using the "$\alpha\beta$-combinatorics" combined with systematic use of lemma 3.3.

Lemma 3.5

In case I if $f \in \mathcal{H}$, then

$$Bf \in \mathcal{D}(W^{\frac{1}{2}}). \tag{3.30}$$

In case II the square of the operator B has the same property: if $f \in \mathcal{H}$ then

$$B^2 f \in \mathcal{D}(W^{\frac{1}{2}}). \tag{3.31}$$

Corollary 3.1
In cases I and II any eigenfunction φ of the operator B

$$B\varphi = \lambda\varphi,$$

corresponding to non-zero eigenvalue $\lambda, \lambda \neq 0$, belongs to the domain of definition of the quadratic form of the operator W:

$$\varphi \in \mathcal{D}(W^{\frac{1}{2}}). \tag{3.32}$$

4 The essentially self-adjointness of $K(z)$ in cases I–II. The KLMN-theorem

In the previous section we have described the symmetric extension $\hat{K}$ of the operator K (see (3.24)) on the domain $\mathcal{D}(\hat{K}) = \mathcal{D}(W)$ more wider (in cases II–IV) than the initial domain $\mathcal{D}(K) = \mathcal{H}^{2,2,2}$, defined by the formula

$$\hat{K} = W - W^{\frac{1}{2}} B W^{\frac{1}{2}}. \tag{4.1}$$

The following lemma gives the qualitative impression on the strength of the perturbation $R = W^{\frac{1}{2}} B W^{\frac{1}{2}}$.

Theorem 4.1
In cases I and II the operator B is compact and belongs to the Hilbert-Schmidt ideal.

In cases III and IV the essential spectrum of the operator B contains point 1

$$1 \in \sigma_{ess}(B) \tag{4.2}$$

together with some its neighborhood.

One can say, slightly abusing the terminology, that the result of theorem 4.1 means that in cases I and II the perturbation $R = W^{\frac{1}{2}} B W^{\frac{1}{2}}$ is relatively form-compact with respect to W and in cases III and IV it is only form-bounded (in fact this form-bound is not less then 1).

Note that the last statement of the theorem concerning the location of the essential spectrum of B in cases III and IV is the crucial point in the proof of the Efimov effect [8], [14] as well as in the one [15] of the unboundedness from below of the three-body energy operator with δ-potentials. We give here a sketch of the proof only.

"Proof". The kernel of the integral operator $B_{\alpha\beta}(z)$ admits estimate (3.23). In cases I and II at least one of the exponents δ_α equals $\frac{1}{2}$, and hence the kernel $B_{\alpha\beta}(p,k;z)$ is a square integrable function in $P = p \oplus k$. Therefore, B is a Hilbert-Schmidt operator.

Let us check the second statement of the theorem.

First, consider case IV, which is simpler in technical respect. We recall that in this case for all α, $\alpha = 1,2,3$, the two-body t-matrices $t_\alpha(z)$ satisfy condition b) and hence we have the asymptotic representations

$$t_\alpha(z) = \frac{1}{B_\alpha + i\frac{\sqrt{z}}{4\pi}} + \mathcal{O}(|z|^{-\frac{3}{2}}), \ \alpha = 1,2,3. \tag{4.3}$$

Let $\tilde{t}(z)$ be the two-body t-matrix corresponding the the special case of usual δ-interaction with a zero-energy resonance (a virtual level)

$$\tilde{t}(z) = \frac{1}{i\frac{\sqrt{z}}{4\pi}}. \tag{4.4}$$

Then, by (4.3) we have

$$t_\alpha(z) = \tilde{t}(z) + \mathcal{O}(\frac{1}{|z|}). \tag{4.5}$$

Let us introduce as before the multiplication operators $\tilde{W}_\alpha(z)$ (which are now the same for all α)

$$\tilde{W}_\alpha(z) = (-\tilde{t}(z-p^2))^{-1}, \ \alpha = 1,2,3, \tag{4.6}$$

and the operators $\tilde{B}_{\alpha\beta}(z)$ defined by the formula

$$\tilde{B}_{\alpha\beta}(z) = \tilde{W}_\alpha^{\frac{1}{2}}(z) R_{\alpha\beta}(z) \tilde{W}_\beta^{\frac{1}{2}}(z). \tag{4.7}$$

By inspection, using (4.5) we conclude that the kernel of the difference of integral operators $B_{\alpha\beta}(z) - \tilde{B}_{\alpha\beta}(z)$ is a Hilbert-Schmidt kernel and therefore the operator $\tilde{B}_{\alpha\beta}(z)$ is a compact perturbation of the operator $B_{\alpha\beta}(z)$. Let $\mathbf{B}$ be the unit ball in the space $\mathbf{R}^3$ and let $\mathcal{P}$ be the orthogonal projector from the space $L_2(\mathbf{R}^3) = L_2(\mathbf{B}) \oplus L_2(\mathbf{R}^3 \setminus \mathbf{B})$ onto the subspace $L_2(\mathbf{R}^3 \setminus \mathbf{B})$. The analogous estimate of the kernel

$$(4.8) \qquad \frac{1}{(-z+p^2)^{\frac{1}{4}}} R_{\alpha\beta}(p,k;z) \frac{1}{(-z+k^2)^{\frac{1}{4}}} - \frac{1-\chi(p)}{\sqrt{p}} R_{\alpha\beta}(p,k;z) \frac{1-\chi(k)}{\sqrt{k}},$$

where $\chi(p)$ is the characteristic function of the unit ball $\mathbf{B}$ in $\mathbf{R}^3$, shows that the difference $\tilde{B}(z) - \mathcal{P}\tilde{B}(0)\mathcal{P}$ is also a Hilbert-Schmidt operator. Hence, by the Weyl theorem the essential spectra of $B(z)$ and $\mathcal{P}\tilde{B}(0)\mathcal{P}$ coincide.

The spectral analysis of $\tilde{B}(0)$ as well as that of $\mathcal{P}\tilde{B}(0)\mathcal{P}$ can be performed explicitly in a certain sense. The first observation is that the subspaces $L^2(\mathbf{R}^3)$ corresponding to a fixed angular momentum value are reducing subspaces for the operators $\tilde{B}_{\alpha\beta}(0)$. The second observation is that the kernels of integral operators $\tilde{B}_{\alpha\beta}$ are homogeneous (of degree $-3/2$) functions of their arguments. In particular, the part of the operator $\tilde{B}_{\alpha\beta}(0)$ corresponding to the invariant subspace of spherically symmetric functions is unitary equivalent to a multiplication operator by the function (admitting the meromorphic continuation on the whole complex plane in our particular case)

$$(4.9) \qquad M_{\alpha\beta}(s) = \frac{\pi}{2|s_{\alpha\beta}|} \frac{\mathrm{sh}(\arcsin|s_{\alpha\beta}|s)}{s\ \mathrm{ch}\frac{\pi s}{2}}$$

acting in the space $L_2(\mathbf{R})$. Here $s_{\alpha\beta}$ are the elements of the transition matrix (2.2) connecting different coordinate systems (k_α, p_α) and (k_β, p_β). The most close on spirit for us derivation of the formula (4.9) based on the Mellin transform one can find in [15] (see also [8]).

Thus the operator $\tilde{B}(0)$ has the part which is unitary equivalent to a multiplication operator in the space $L_2(\mathbf{R}) \oplus L_2(\mathbf{R}) \oplus L_2(\mathbf{R})$ by a 3×3-matrix function $M(s)$ having the matrix elements $M_{\alpha\beta}(s)$, $\alpha \neq \beta$, given by (4.9), with zero diagonal.

In turn, the operator $\mathcal{P}\tilde{B}(0)\mathcal{P}$ has the part which is unitary equivalent to a matrix operator of Wiener-Hopf type having the same spectrum [27] as the one of the multiplication operator by the matrix-function $M(s)$ "on semiaxis", more precisely in the space $L_2(\mathbf{R}_+) \oplus L_2(\mathbf{R}_+) \oplus L_2(\mathbf{R}_+)$. Hence, the essential spectrum of the operator $\mathcal{P}\tilde{B}(0)\mathcal{P}$, and by the Weyl theorem that of the operator $B(z)$ as well includes the set of values attained by the eigenvalues $\lambda_\alpha(s), \alpha = 1, 2, 3,$ of the matrix $M(s)$ for real s. The fact that this set includes point 1 is based on the following arguments (see [15]).

Consider the (continuous) function

$$f(s) = \det(I - M(s)). \tag{4.10}$$

Using (4.9) we infer that

$$\lim_{s\to\infty} f(s) = 1. \tag{4.11}$$

Computing of the determinant $f(s)$ at point zero results in the formula

$$f(0) = 1 - \prod_{(\alpha\to\beta)\in g} M_{\alpha\beta}(0) - \sum_{(\alpha\to\beta)\in g} M^2_{\alpha\beta}(0), \tag{4.12}$$

where the sum is carried on over the arrows of the cyclic graph g: $1 \to 2 \to 3 \to 1$. Furthermore, making use of the elementary inequality

$$\frac{2}{\pi} < \frac{\sin\varphi}{\varphi}$$

valid for $0 \le \varphi < \frac{\pi}{2}$, by (4.9) we immediately conclude that

$$M_{\alpha\beta}(0) > 1, \;\; \alpha \neq \beta. \tag{4.13}$$

Hence $f(0) < 0$, and then using the continuity of $f(s)$ we infer that there is s_0 such that $f(s_0) = 1$.

As far as the inequality $f(0) < 0$ is strict that means that certain neighborhood of apoint 1 is a subset of the essential spectrum of the operator B.

In case III there is the only one two-body subsystem, say with index γ, such that $\gamma \in \mathcal{A}$, i.e. the corresponding scattering matrix possesses "regular" high energy behavior. Therefore, by the statement of the first part of the theorem the matrix elements $B_{\alpha\gamma}(z)$ and $B_{\gamma\beta}(z)$, $\alpha, \beta \neq \gamma$, are Hilbert-Schmidt operators. This means , just as in case IV, that B is a compact perturbation of a certain operator the part of which has the same spectrum as the one of the multiplication operator (acting in the space $L_2(\mathbf{R}_+) \oplus L_2(\mathbf{R}_+) \oplus L_2(\mathbf{R}_+)$) by the matrix-function $\tilde{N}(s) = \{N_{\alpha\beta}\}$ with only two non-zero matrix elements:

$$N_{\alpha^\star\beta^\star}(s) = M_{\alpha^\star\beta^\star}(s). \tag{4.14}$$

Here $\alpha^\star, \beta^\star$ are the indicies of two-particles subsystems ($\alpha^\star \neq \beta^\star$) different from the subsystem γ.

Just as in case IV we infer that

$$\lim_{s\to\infty} \det(I - N(s)) = 1$$

and

$$\det(I - N(0)) = 1 - N^2_{\alpha^*\beta^*}(0) < 0, \tag{4.15}$$

that proves that $1 \in \sigma_{ess}(B(z))$ together with its certain neighborhood.

Making use the relative form-compactness (with respect to W) of the perturbation $R = W^{\frac{1}{2}} B W^{\frac{1}{2}}$ and corollary 3.1 we shall prove that in cases I and II the relative W-form-bound of the perturbation R equals zero. Then applying the KLMN-theorem ([28], Theorem X.17) we can prove that the operator $\hat{K}(z)$ is self-adjoint on the domain $\mathcal{D}(\hat{K}(z))$.

The fact that $1 \in \sigma_{ess}(B)$ in cases III and VI prevents to apply the KLMN-theorem and one can show that in these cases the operator $\hat{K}(z)$ defined on $\mathcal{D}(\hat{K}(z))$ is only symmetric but is not essentially self-adjoint.

Theorem 4.2
In cases I and II the operator $\hat{K}$ on the domain $\mathcal{D}(\hat{K})$ is self-adjoint.

Proof. On the domain $\mathcal{D}[\mathbf{k}] = \mathcal{D}(W^{\frac{1}{2}})$ we consider the quadratic form

$$\mathbf{k}[\varphi] = \mathbf{w}[\varphi] - \mathbf{r}[\varphi], \tag{4.16}$$

where $\mathbf{w}$ is the quadratic form of the operator W

$$\mathbf{w}[\varphi] = \langle W^{\frac{1}{2}}\varphi, W^{\frac{1}{2}}\varphi \rangle, \qquad \varphi \in \mathcal{D}(W^{\frac{1}{2}}), \tag{4.17}$$

and the form $\mathbf{r}$ defined on $\mathcal{D}[\mathbf{r}] = \mathcal{D}(W^{\frac{1}{2}})$ "corresponds" to the perturbation R

$$\mathbf{r}[\varphi] = \langle B W^{\frac{1}{2}}\varphi, W^{\frac{1}{2}}\varphi \rangle. \tag{4.18}$$

We shall show that the form $\mathbf{r}$ is $\mathbf{w}$-bounded form with zero $\mathbf{w}$-bound, i.e. for all $\epsilon > 0$ there is $b > 0$ such that the inequality holds

$$|\mathbf{r}[\varphi]\,| \leq \epsilon \mathbf{w}[\varphi] + b||\varphi||^2, \tag{4.19}$$

for all $\varphi \in \mathcal{D}(W^{\frac{1}{2}})$.

The first step of the proof of (4.19) makes use the compactness of the operator B in cases I and II.

Let P be the spectral projector of the operator B corresponding to the component of its spectrum lying in the interval $[-\epsilon, \epsilon]$. Then

$$B = PBP + Q, \tag{4.20}$$

where Q is the finite rank operator

$$Q\,\cdot = \sum_{|\lambda_i|>\epsilon} \lambda_i \langle\,\cdot\,,\varphi_i\rangle\varphi_i. \tag{4.21}$$

Here λ_i and φ_i denote the eigenvalues and the corresponding eigenfunctions of the compact operator B.The norm of the operator PBP does not exceed ϵ and hence the following inequality holds

$$|\mathbf{r}[\varphi]| \leq \epsilon\mathbf{w}[\varphi] + \sum_{|\lambda_i|>\epsilon} |\lambda_i|\,|\langle W^{\frac{1}{2}}\varphi,\varphi_i\rangle|^2. \tag{4.22}$$

By corollary 3.1 the second summand in the right hand side of (4.22) is a bounded form that proves the inequality (4.19) with

$$b = \sum_{|\lambda_i|>\epsilon} |\lambda_i|\,\|W^{\frac{1}{2}}\varphi_i\|^2 \tag{4.23}$$

Applying the KLMN-theorem we conclude that in cases I and II the form $\mathbf{k}$ is closed and semibounded from below and hence this form is the quadratic from of some self-adjoint semibounded from below operator. It is not difficult, however, to understand that this operator is nothing else as the operator $\hat{K}$. This completes the proof of the theorem.

Theorem 4.3

In cases I and II the three-body Hamiltonian H on the domain $\mathcal{D}(H)$ is essentially self-adjoint and semibounded from below.

Proof. The KLMN-theorem states as well that the operator $\hat{K}$ is essentially self-adjoint on any core of the non-perturbed operator, the operator W in the present case. As a consequence, the operator K is essentially self-adjoint on the domain $\mathcal{D}(K) = \mathcal{H}^{2,2,2}$ in cases I and II (moreover, K is self-adjoint on $\mathcal{D}(K) = \mathcal{H}^{2,2,2}$ in case I). Using lemma 3.1 we conclude that in cases I and II the three-body Hamiltonian H is essentially self-adjoint on $\mathcal{D}(H)$. Thus, we have proved the first statement of the lemma.

The proof of reversibility of the self-adjoint operator $\hat{K}(z)$, for z negative, $|z|$ large enough, is based on significantly more rough arguments as those in the proof of theorem 4.2: the existence of integrable majorant for the square of the kernel of the operators $B_{\alpha\beta}(z)$ and the fact that

$$|B_{\alpha\beta}(p,k;z)| \underset{z\to-\infty}{\longrightarrow} 0 \tag{4.24}$$

for fixed arguments p and k show, by dominated convergence theorem, that $B(z) \to 0$ as $z \to -\infty$ in the Hilbert-Schmidt norm. Hence, for all z, lying from the left from a certain $z_0 \in \mathbf{R}_0$, the operator $B(z)$ is a contraction and therefore the inequality holds

$$|\mathbf{r}| \leq (1 - \|B(z)\|)\mathbf{w}. \tag{4.25}$$

On the other side, it follows from the representations (3.2)–(3.4) that the lower bound of the form $\mathbf{w}(z)$ tends to plus infinity in case I and as $\sqrt{|z|}$ in case II in the limit $z \to -\infty$, i.e.

$$\inf_{\|\varphi\|=1} \langle W^{\frac{1}{2}}(z)\varphi, W^{\frac{1}{2}}(z)\varphi\rangle = \mathcal{O}(|z|^{\delta}), \tag{4.26}$$

where $\delta = 1$ in case I and $\delta = \frac{1}{2}$ in case II. Hence, for $z < z_0$ the operator $\hat{K}(z)$ has a bounded inverse. By lemma 3.2 it just proves the semiboundedness from below of the three-body energy operator H in cases I and II, that shows how to avoid the "fall to the center" in the problem with point-like interactions.

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