Smooth integrability of implicit differential systems

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1 Introduction

If P is a smooth manifold and $V : P \to TP$ is a vector field on P, i.e. a section of the tangent bundle $\pi_1 : TP \to P$, then the local integrability of V is a characteristic property, especially for smooth sections V. Traditionally the local integrability of a vector field V (not necessary continuous) on P is defined as an existence at each point p of the domain of V a C^1-curve $\gamma : (-\varepsilon, \varepsilon) \to P$ (for some $\varepsilon > 0$), such that $\gamma(0) = p$ and $\frac{\partial \gamma}{\partial t}(t) = V(\gamma(t))$ for all $t \in (-\varepsilon, \varepsilon)$. The curve γ is called an integral curve of V with initial value p. We may assume that the existence of such γ for V is not an exceptional property , and that for each $p \in P$ there exists a neighbourhood U of p and $\varepsilon > 0$ such that the mapping $U \times (-\varepsilon, \varepsilon) \ni (\bar{p}, t) \to \gamma_{\bar{p}}(t)$, $\gamma_{\bar{p}}(0) = \bar{p}$, is defined and at least continuous. For smooth, compactly supported V this mapping forms a one parameter group of transformations of P.

When we replace V by a smooth embedded manifold M into TP, then the mentioned integrability property becomes rather the restrictive exceptional property for M, especially in the critical point set of the projection $\pi_1 |_M : M \to P$. In general, such systems are called implicit differential systems (equations) of order 1, and were studied by several authors: In connection to the non continuous integrability property by [18, 6]. The local equivalences of singularities of such systems were studies (among others) by [19, 3, 4, 17]. The Hamiltonian implicit case which is especially addressed to mathematical physics was introduced to mathematics by P.A.M. Dirac [5], investigated also by [13, 2, 9].

The local smooth integrability property is important for understanding the local dynamics of the system and the structure of the induced transformation groups. However the integrability property is not a usual property in the implicit case. Then a natural question which arises, is to characterize those implicit systems M which are integrable, i.e. through each point of M there is a local integrable curve passing through. Another natural question is to understand the reason for non integrability - the local topological structure of the strata of non integrable points.

The purpose of this paper is threefold: First we provide necessary and sufficient conditions of integrability for various classes of implicit systems. Second we formulate and investigate the problem of integrability for implicit Hamiltonian systems as a specialized class of implicit systems, and Third we search for the generic peculiarities of non integrable points.

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In Sections 2,3 and 4 we formulate the integrability problem, consider the two dimensional fold and cusp cases and prove the first part of the main results of the paper: the smooth integrability necessary and sufficient conditions for the generic (in the sense of J. Mather) differential systems (Theorem 3.1 and Theorem 3.2). In Section 5 the special case of Morin corank 1 stable singularities is considered in details and in Section 6 the crucial examples and counter examples of various integrability and non integrability situations are constructed. Passing to real analytic category, in Section 7, we extend J. Mather's results weakening his transversality condition to the irreducibility of determinants. This result is formulated in Theorem 7.1 and Theorem 7.2. Along the road to this result we prove several applicable properties of prime ideals having the property of zeroes. Specialization of our differential systems to the Hamiltonian ones, which was a part of the motivation for this paper, was done in Section 8. In this case we applied the main results of the previous sections and obtained the necessary and sufficient conditions for integrability of implicit Hamiltonian systems. Here we develop the computable formula for the integrability check in case of P.A.M. Dirac generalized Hamiltonian systems. Finally in Section 9 the integrability on submanifolds is investigated (Theorem 9.1, Theorem 9.2) and Section 10 is devoted to the non-integrable points of differential systems. We prove here (Theorem 10.1, Theorem 10.2) that the non-integrable points are isolated generically in the sense of J.C. Tougeron.

2 The integrability problem

Let M be a smooth submanifold of $T\mathbf{R}^n$, $M \subset T\mathbf{R}^n \equiv \mathbf{R}^n \times \mathbf{R}^n$, $dimM = m$. M is considered as a first order differential system.

DEFINITION 2.1 *A point $(p,q) \in M \subset T\mathbf{R}^n \equiv \mathbf{R}^n \times \mathbf{R}^n$ is called an integrable point of M if there exists a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbf{R}^n$ such that*

$$\gamma(0) = p, \quad \gamma'(0) = q \quad and \quad (\gamma(t), \gamma'(t)) \in M \quad for\ all \quad t \in (-\varepsilon, \varepsilon).$$

The manifold M is called an integrable manifold if M consists only of integrable points.

Now we have a natural question.

PROBLEM 2.1 *Find a necessary and sufficient condition for a manifold $M \subset T\mathbf{R}^n$ to be integrable.*

We immediately find a necessary condition as follows. Let $\pi_1 : T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be the projection to the first factor:

$$\pi_1 : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n, \qquad \pi_1(p,q) = p, \quad \text{tangent bundle projection.}$$

Let $(p,q) \in M$ be an integrable point of M. Then from the definition there exists a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbf{R}^n$ such that

$$\gamma(0) = p, \quad \gamma'(0) = q \quad \text{and} \quad (\gamma(t), \gamma'(t)) \in M.$$

Then we have

$$q = \gamma'(0) = \frac{d(\pi_1(\gamma, \gamma'))}{dt}(0) = d(\pi_1 \mid M)_{(p,q)}(\gamma'(0), \gamma''(0))$$
$$\in d(\pi_1 \mid M)_{(p,q)}(T_{(p,q)}M).$$

Thus an immediate necessary condition for a point $(p, q) \in M$ to be integrable is that

$$q \in d(\pi_1 \mid M)_{(p,q)}(T_{(p,q)}M).$$

In what follows we will call this condition *N-integrability condition.*

We can ask whether this condition is also a sufficient condition for a submanifold M to be integrable. Although the answer for this question is negative, there is a wide class of submanifolds of $T\mathbf{R}^n$ for which the N-integrability condition is also sufficient. An example of the submanifold M for which N-integrability condition is fulfilled but M is not integrable is given in Section 6, (see Example 6.1).

To come closer to our problem we explain some sufficient conditions for a submanifold M of $T\mathbf{R}^n$ to be integrable. One of them plays an important role in the proof that manifolds with corank 1 stable singularities are integrable. At first we have the following two immediate Lemmas.

LEMMA 2.1 *If a submanifold $M \subset T\mathbf{R}^n$ is a finite union of integrable submanifolds $M_i \subset T\mathbf{R}^n$;*

$$M = \cup_i M_i, \qquad M_i \quad \textit{integrable},$$

then M is integrable.

LEMMA 2.2 *Let $M \subset T\mathbf{R}^n$ be a submanifold satisfying N-integrability condition.*

1) If $\pi_1 |_M : M \to \mathbf{R}^n$ is either an embedding or a submersion, then M is integrable.

2) If $\pi_1(M)$ is a submanifold of $\mathbf{R}^n$ and $\pi_1 |_M : M \to \pi_1(M)$ is a submersion, then M is integrable.

Since our problem is a local one, we may suppose that the manifold M under consideration is the image of an embedding of $\mathbf{R}^m$.

$$M = (f, g)(\mathbf{R}^m), \qquad (f, g) : \mathbf{R}^m \to T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n \quad \text{is an embedding.}$$

Let $Jf(x)$ denote the Jacobian matrix of $f = (f_1, \ldots, f_n)$ at $x = (x_1, \ldots, x_m) \in \mathbf{R}^m$.

$$Jf(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \ldots & \frac{\partial f_1}{\partial x_m}(x) \\ \ldots & \ldots & \ldots \\ \frac{\partial f_n}{\partial x_1}(x) & \ldots & \frac{\partial f_n}{\partial x_m}(x) \end{pmatrix}$$

LEMMA 2.3 *Let $M = (f, g)(\mathbf{R}^m)$, where*

$$(f, g) : \mathbf{R}^m \to T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n \quad \textit{is an embedding.}$$

Let M satisfies the N-integrability condition. If there exist smooth local functions $a_1(x_1,\ldots,x_m),\ldots,a_m(x_1,\ldots,x_m)$ such that

$$\begin{pmatrix} g_1(x) \\ g_2(x) \\ \cdots \\ g_n(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \ldots & \frac{\partial f_1}{\partial x_m}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \ldots & \frac{\partial f_2}{\partial x_m}(x) \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1}(x) & \ldots & \frac{\partial f_n}{\partial x_m}(x) \end{pmatrix} \begin{pmatrix} a_1(x) \\ a_2(x) \\ \cdots \\ a_m(x) \end{pmatrix}, \tag{1}$$

then M is integrable.

Proof. Take any point $(p_0,q_0)=(f,g)(x_0)\in M$. Consider the vector field

$$\xi(x)=a_1(x)\left(\frac{\partial}{\partial x_1}\right)_x + a_2(x)\left(\frac{\partial}{\partial x_2}\right)_x + \ldots + a_m(x)\left(\frac{\partial}{\partial x_m}\right)_x$$

defined on $\mathbf{R}^m$. Consider an integral curve

$$\tilde{\gamma}:(-\varepsilon,\varepsilon)\to\mathbf{R}^m \qquad \text{of} \quad \xi \qquad \text{such that} \quad \tilde{\gamma}(0)=x_0$$

and consider the curve $\gamma:(-\varepsilon,\varepsilon)\to\mathbf{R}^n$, $\qquad \gamma(t)=f(\tilde{\gamma}(t))$.
Then the point $(p_0,q_0)=(f,g)(x_0)\in M$ is an integrable point of M with the curve $\gamma(t)$. This can be seen as follows. Since

$$\frac{d\tilde{\gamma}}{dt}(t)=a(\tilde{\gamma}(t))=(a_1(\tilde{\gamma}(t)),\ldots,a_m(\tilde{\gamma}(t)),$$

we have

$$\frac{d\gamma}{dt}(t)=\frac{d(f\circ\tilde{\gamma})(t)}{dt}=Jf(\tilde{\gamma}(t))\cdot\frac{d\tilde{\gamma}}{dt}(t)$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\tilde{\gamma}(t)) & \ldots & \frac{\partial f_1}{\partial x_m}(\tilde{\gamma}(t)) \\ \frac{\partial f_2}{\partial x_1}(\tilde{\gamma}(t)) & \ldots & \frac{\partial f_2}{\partial x_m}(\tilde{\gamma}(t)) \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1}(\tilde{\gamma}(t)) & \ldots & \frac{\partial f_n}{\partial x_m}(\tilde{\gamma}(t)) \end{pmatrix} \begin{pmatrix} a_1(\tilde{\gamma}(t)) \\ a_2(\tilde{\gamma}(t)) \\ \cdots \\ a_m(\tilde{\gamma}(t)) \end{pmatrix} = \begin{pmatrix} g_1(\tilde{\gamma}(t)) \\ g_2(\tilde{\gamma}(t)) \\ \cdots \\ g_n(\tilde{\gamma}(t)) \end{pmatrix}.$$

Hence we have

$$(\gamma(t),\gamma'(t))=(f(\tilde{\gamma}(t)),g(\tilde{\gamma}(t))\in M=(f,g)(\mathbf{R}^m)$$
$$\text{and} \quad (\gamma(0),\gamma'(0))=(p_0,q_0).$$

Thus (p_0,q_0) is an integrable point of M. *Q.E.D.*

Corollary 2.1 *Let $f:\mathbf{R}^m\to\mathbf{R}^n$ be a smooth mapping with*

$$rank df_x \geq \frac{m}{2}, \qquad \forall x\in\mathbf{R}^m.$$

Then there exist (always) a smooth mapping $g:\mathbf{R}^m\to\mathbf{R}^n$ such that $(f,g):\mathbf{R}^m\to\mathbf{R}^n\times\mathbf{R}^n$ is an embedding and $M=(f,g)(\mathbf{R}^m)$ is an integrable manifold.

Proof. Since we consider the local problem and since

$$r = \quad \mathrm{rank} df_x : T_x(\mathbf{R}^m) \to T_{f(x)}\mathbf{R}^n \geq \frac{m}{2}, \qquad \forall x \in \mathbf{R}^m,$$

we may suppose that f is of the form

$$f(x_1, \dots, x_m) = (x_1, \dots, x_r, f_{r+1}(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)).$$

Let $a_1(x), \dots, a_n(x)$ be the smooth functions chosen in the following form

$$a_1(x) = x_{r+1}, \dots, a_{m-r}(x) = x_m \qquad \text{and} \qquad a_{m-r+1}(x), \dots, a_n(x) \quad \text{being arbitrary.}$$

And consider a mapping $g = (g_1, \dots, g_n) : \mathbf{R}^m \to \mathbf{R}^n$ defined by equation (1). Then $(f, g) : \mathbf{R}^m \to \mathbf{R}^n \times \mathbf{R}^n$ is an embedding and from Lemma 2.3 we see that $M = (f, g)(\mathbf{R}^m)$ is an integrable manifold. *Q.E.D.*

Remark 2.1 Now we came to the natural question: *Does some pathology of the critical point set on M imply the non integrability ?* To some extent the unswer is positive. We may formulate the following supposition: if $f : \mathbf{R}^m \to \mathbf{R}^n$ is stable, then for any $g : \mathbf{R}^m \to \mathbf{R}^n$ such that $(f, g) : \mathbf{R}^m \to \mathbf{R}^n \times \mathbf{R}^n$ is an embedding and $M = (f, g)(\mathbf{R}^m)$ satisfies N-integrability condition, M has to be integrable. Therefore if this conjecture is true, it implies that if M is not integrable, then f is not stable, and therefore $\pi_1 |_M : M \to \mathbf{R}^n$ is not a stable map. Here let us remind that the two maps $f : \mathbf{R}^m \to \mathbf{R}^n$ and $\pi_1 |_M : M \to \mathbf{R}^n$ are right equivalent in J.Mather's sense, since $(f, g) : \mathbf{R}^m \to M = (f, g)(\mathbf{R}^m)$ is a diffeomorphism. Thus the singular point set is not the one of a stable map. On the other hand, the Corollary 2.1 says that if

$$\mathrm{rank}(d\pi_1 |_M) : T_{(p,q)}(M) \to T_p(\mathbf{R}^n) \geq \frac{m}{2},$$

then no matter how much pathological its singularity is, with a good partner g, $M = (f, g)(\mathbf{R}^m)$ becomes integrable.

To complete our sufficient condition formulated in Lemma 2.3 we present Example 6.2, in Section 6 showing that the condition of existence of smooth solution of the linear equation in Lemma 2.3 is not a necessary condition for integrability.

Now continuing the practical search for sufficient conditions we show that the N-integrability condition is also sufficient for fold and cusp singularities of $\pi_1 |_M$. At first we introduce the natural equivalency of equidimensional submanifolds in $T\mathbf{R}^n$

Definition 2.2 *Let M, N be smooth submanifolds of $T\mathbf{R}^n$. We say that M and N are equivalent if there exists a diffeomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$ and so its tangent bundle lifting $H(p, q) = (h(p), Jh(p)q)$, where $Jh(p)$ is the Jacobian matrix of h at $p \in \mathbf{R}^n$, such that $N = H(M)$.*

Lemma 2.4 *Let M and N be smooth submanifolds of $T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$. If M and N are equivalent, then N is integrable if and only if so is M.*

Proof. Since the Jacobian matrix $JH(p,q)$ has the form

$$JH(p,q) = \begin{pmatrix} Jh(p) & O \\ O & Jh(p) \end{pmatrix},$$

H is a diffeomorphism. Suppose that $(p,q) \in M$ is an integrable point of M. Then there exists a smooth curve

$$\gamma : (-\varepsilon, \varepsilon) \to \mathbf{R}^n \quad \text{such that} \quad \gamma(0) = p, \gamma'(0) = q, (\gamma(t), (\gamma'(t)) \in M.$$

Consider a curve $\alpha : (-\varepsilon, \varepsilon) \to \mathbf{R}^n$ given by $\alpha(t) = h(\gamma(t))$. Now we can see that the point $(p', q') = H(p,q) \in N = H(M)$ is an integrable point of N with the curve $\alpha(t)$ as follows.

$$\alpha(0) = h(\gamma(0)) = h(p) = p', \qquad \alpha'(0) = Jh(p)\gamma'(0) = Jh(p)q = q',$$

$$\alpha'(t) = Jh(\gamma(t))\gamma'(t).$$

Therefore

$$(\alpha(t), \alpha'(t)) = (h(\gamma(t)), Jh(\gamma(t))(\gamma'(t))) = H(\gamma(t), \gamma'(t)) \in H(M) = N.$$

The converse is the same. *Q.E.D.*

PROPOSITION 2.1 *Let $M \subset T\mathbf{R}^2 = \mathbf{R}^2 \times \mathbf{R}^2$ be a manifold satisfying N-integrability condition. Let $(p,q) \in M$ be a cusp or fold singularity of $\pi_1 |_M : M \to \mathbf{R}^2$. Then (p,q) is an integrable point of M.*

Proof. It is enough to prove this proposition by lemma 2.4 for the normal form of cusp singularity. The proof for the fold singularity goes in the similar way and it is easier. From Lemma 2.4, we may suppose the manifold M under consideration is the image of a map $(f,g)) : \mathbf{R}^2 \to \mathbf{R}^2 \times \mathbf{R}^2$ such that

1) f has the normal form of cusp singularity

$$f_1(x,y) = x, \quad f_2(x,y) = y^3 + xy,$$

2) $g = (g_1, g_2) : \mathbf{R}^2 \to \mathbf{R}^2$ is any smooth map such that $(f,g) : \mathbf{R}^2 \to \mathbf{R}^2 \times \mathbf{R}^2$ is an embedding and

3) $M = (f,g)(\mathbf{R}^2)$ satisfies N-integrability condition.

We are going to prove that $M = (f,g)(\mathbf{R}^2)$ is integrable.

Since regular points of $\pi_1 |_M : M \to \mathbf{R}^2$ are integrable from Lemma 2.2, it suffices to consider singular points of $\pi_1 |_M : M \to \mathbf{R}^2$. Let $(p,q) = (p_1, p_2, q_1, q_2) = (f(x,y), g(x,y))$ be a singular point of $\pi_1 |_M : M \to \mathbf{R}^2$. Then we have

$$Jf(x,y) = \begin{pmatrix} 1 & 0 \\ y & 3y^2 + x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y & 0 \end{pmatrix}.$$

Then N-integrability condition is equivalent to the condition that

$$\begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix} \in Jf(x,y)(\mathbf{R}^2) = \begin{pmatrix} 1 & 0 \\ y & 0 \end{pmatrix} (\mathbf{R}^2) = \mathbf{R} \begin{pmatrix} 1 \\ y \end{pmatrix}.$$

Therefore if $g_1(x,y) = 0$, we have also $g_1(x,y) = 0$ and $q = (0,0)$. We know that any point (p,q) with $q = (0,0)$ is always an integrable point of M.

Suppose now that $g_1(0,0) = c \neq 0$. Then $g_1(x,y)$ has the form

$$g_1(x,y) = c + \alpha(x,y) \qquad \text{with} \quad \alpha(0,0) = 0.$$

This implies that $g_2(x,y)$ coincides with $yg_1(x,y)$ on the singular point set $\Sigma(f)$ of f:

$$g_2(x,y) \mid_{\Sigma(f)} \equiv yg_1(x,y) \mid_{\Sigma(f)} .$$

Let $E_{x,y}$ denote the ring of smooth function-germs of two variables x, y , $m_{x,y}$ denotes its maximal ideal and $E_{x,y} < \beta >$ the ideal in $E_{x,y}$ generated by $\beta \in m_{x,y}$. With these notations we have

$$g_1(x,y) = c + \alpha(x,y) \qquad \text{with} \quad \alpha \in m_{x,y}.$$

Now the condition $g_2(x,y) \mid_{\Sigma(f)} \equiv yg_1(x,y) \mid_{\Sigma(f)}$. implies $(g_2(x,y) - yg_1(x,y)) \mid_{\Sigma(f)} \equiv 0$. But this implies that

$$g_2(x,y) - yg_1(x,y) \in E_{x,y} < 3y^2 + x >,$$

because

$$\Sigma(f) = \{(x,y) \in \mathbf{R}^2 \mid \det Jf(x,y) = 3y^2 + x = 0\}$$

and the generator $3y^2 + x$ of the ideal $E_{x,y} < 3y^2 + x >$ is irreducible. Therefore $g_2(x,y) - yg_1(x,y)$ has the form

$$g_2(x,y) - yg_1(x,y) = (3y^2 + x)\beta(x,y) \qquad \text{for some} \quad \beta \in E_{x,y}.$$

Recalling the form

$$g_1(x,y) = c + \alpha(x,y), \qquad c \neq 0, \quad \alpha \in m_{x,y},$$

the map $(f,g)) : \mathbf{R}^2 \to \mathbf{R}^2 \times \mathbf{R}^2$ turns out to have the form

$$(f,g)(x,y) = (x, y^3 + xy, c + \alpha(x,y), cy + y\alpha(xy) + (3y^2 + x)\beta(x,y)),$$

$$J(f,g)(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \star & \star \\ \star & c \end{pmatrix}, \quad c \neq 0.$$

Thus $(f,g)) : \mathbf{R}^2 \to \mathbf{R}^2 \times \mathbf{R}^2$ is an embedding.

Now let us observe the form of $g(x,y)$;

$$\begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix} = \begin{pmatrix} c + \alpha(x,y) \\ y(c + \alpha(x,y)) + (3y^2 + x)\beta(x,y) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ y & 3y^2 + x \end{pmatrix} \begin{pmatrix} c + \alpha(x,y) \\ \beta(x,y) \end{pmatrix}.$$

Thus the functions

$$a_1(x,y) = c + \alpha(x,y), \quad a_2(x,y) = \beta(x,y)$$

satisfy the condition in Lemma 2.3 and $M = (f,g)(\mathbf{R}^2)$ is integrable. *Q.E.D.*

3 Generic manifolds are integrable

Now we prove that the N-integrability condition is also sufficient for a class of manifolds, which we call the *generic manifolds* and which is much wider than the class of manifolds $M \subset T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$ such that the projections $\pi_1 \mid_M : M \to \mathbf{R}^n$ are C^∞−stable.

Let $J^1(M, \mathbf{R}^n)$ be the 1 jet bundle of 1 jets of maps of M into $\mathbf{R}^n$, and let $\Sigma_r(M, \mathbf{R}^n)$ be the subset of all 1 jets $j^1 f(x) \in J^1(M, \mathbf{R}^n)$ such that the rank of the differential $df_x : T_x(M) \to T_{f(x)}\mathbf{R}^n$ at $x \in M$ is r. Then $\Sigma_r(M, \mathbf{R}^n)$ is a submanifold of $J^1(M, \mathbf{R}^n)$ of codimension $(n-r) \times (m-r)$, where $m = \dim M$.

Definition 3.1 *(following J. Mather [12]) A manifold $M \subset T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$ is called a generic submanifold of $T\mathbf{R}^n$ if the 1 jet section $j^1(\pi_1 \mid_M) : M \to J^1(M, \mathbf{R}^n)$ is transversal to all $\Sigma_r(M, \mathbf{R}^n)$, $r = 0, 1, \dots, min(m, n)$.*

Now we can formulate our first main theorem.

Theorem 3.1 *A generic submanifold M of $T\mathbf{R}^n$ is integrable if and only if it satisfies N-integrability condition. Thus N-integrability condition holds sufficient for generic submanifolds of $T\mathbf{R}^n$.*

Remark 3.1 Remark that generic submanifolds form a very wide class in the set of all submanifolds of $T\mathbf{R}^n$. This can be seen as follows. Since our problem is a local one, we may suppose that the manifold M under consideration is the image of an embedding of $\mathbf{R}^m$ into $T\mathbf{R}^n$:

$$M = (f, g)(\mathbf{R}^m), \qquad (f, g) : \mathbf{R}^m \to T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n \quad \text{an embedding.}$$

Then, the 1 jet section $j^1(\pi_1 \mid_M) : M \to J^1(M, \mathbf{R}^n)$ is transversal to $\Sigma_r(M, \mathbf{R}^n)$ if and only if the 1 jet section $j^1 f : \mathbf{R}^m \to J^1(\mathbf{R}^m, \mathbf{R}^n)$ is transversal to $\Sigma_r(\mathbf{R}^m, \mathbf{R}^n)$. By Thom's transversality theorem, the set of smooth maps $f : \mathbf{R}^m \to \mathbf{R}^n$ such that $j^1 f : \mathbf{R}^m \to J^1(\mathbf{R}^m, \mathbf{R}^n)$ is transversal to all $\Sigma_r(\mathbf{R}^m, \mathbf{R}^n)$, $r = 0, 1, \dots, \min(m, n)$ form an open dense set in the set of smooth maps of $\mathbf{R}^m$ into $\mathbf{R}^n$. Thus generic submanifolds form a very wide class in the set of all submanifolds of $T\mathbf{R}^n$.

Since our problem is a local one, let us consider the set germ

$$(M, (p, q)) = ((f, g)(\mathbf{R}^m), (f(0), g(0)))$$

at $(p, q) \in M$ of a manifold $M \subset T\mathbf{R}^n$. Now we are ready to formulate the second main theorem concerning the local characterization of integrable manifolds.

Theorem 3.2 *Let $M = (f, g)(\mathbf{R}^m) \subset T\mathbf{R}^n$ be a submanifold of $T\mathbf{R}^n$.*

1) The germ $(M, (p, q)) = ((f, g)(\mathbf{R}^m), (f(0), g(0)))$ is generic if and only if $j^1 f : \mathbf{R}^m \to J^1(\mathbf{R}^m, \mathbf{R}^n)$ is transversal to $\Sigma_r(\mathbf{R}^m, \mathbf{R}^n)$ at $0 \in \mathbf{R}^m$, where r is the rank of the Jacobian matrix $Jf(0)$ at $0 \in \mathbf{R}^m$.

2) Let $(M, (p, q))$ be the germ at (p, q) of a generic submanifold M. Then, $(M, (p, q))$ is integrable if and only if there exists a smooth map germ $a : (\mathbf{R}^m, 0) \to \mathbf{R}^m$ such that the equation (1) is fulfilled.

Now we reformulate our problem to have it in more accessible terms.

DEFINITION 3.2 *A smooth map $g : \mathbf{R}^m \to \mathbf{R}^n$ is said to be tangent to a smooth map $f : \mathbf{R}^m \to \mathbf{R}^n$ if the pair $(f, g) : \mathbf{R}^m \to T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$ forms an embedding and the linear equation (1) has a solution $(a_1(x), a_2(x), \dots, a_n(x))$ for every point $x \in \mathbf{R}^m$.*

In this situation, N-integrability condition becomes exactly the tangency of the smooth map $g : \mathbf{R}^m \to \mathbf{R}^n$ to the smooth map $f : \mathbf{R}^m \to \mathbf{R}^n$, defining the embedding $(f, g) : \mathbf{R}^m \to T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$ of M.

Now we came to the following problem.

PROBLEM 3.1 *Find conditions to be posed on a smooth map $f : \mathbf{R}^m \to \mathbf{R}^n$ so that the image $M = (f, g)(\mathbf{R}^m)$ be an integrable manifold for every smooth map $g : \mathbf{R}^m \to \mathbf{R}^n$ tangent to f.*

Also we can reformulate this Problem in other words as follows:
Find conditions to be posed on a smooth map $f : \mathbf{R}^m \to \mathbf{R}^n$ so that the linear equation (1) has a smooth solution $(a_1(x), a_2(x), \dots, a_m(x))$ for every smooth map $g : \mathbf{R}^m \to \mathbf{R}^n$ tangent to f.

The result of [12] concerning solutions of linear equation helps to give a good answer to this problem. In the rest of this section, we will follow the notation from [12]. Let D^k denote the unit disk in $\mathbf{R}^k$: $D^k = \{x \in \mathbf{R}^k : \| x \| \leq 1\}$. Consider an $n \times m$ matrix $M(x)$ of smooth functions on D^k:

$$M(x) = \begin{pmatrix} m_{11}(x) & m_{12}(x) & \dots & m_{1m}(x) \\ m_{21}(x) & m_{22}(x) & \dots & m_{2m}(x) \\ \dots & \dots & \dots & \dots \\ m_{n1}(x) & m_{n2}(x) & \dots & m_{nm}(x) \end{pmatrix},$$

and a column vector

$$g(x) = \begin{pmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix},$$

where $m_{ij}(x)$ and g_i are smooth functions defined on D^k. Consider the linear equation

$$M(x)a = g(x), \tag{2}$$

where a is a column vector of length m.

Let $E(n, m)$ denote the space of $n \times m$ matrices of real numbers. Let Σ_r denote the subset of $E(n, m)$ consisting of matrices of rank r. Then Σ_r is a submanifold of codimension $(n - r) \times (m - r)$ in $E(n, m)$.

DEFINITION 3.3 *An $n \times m$ matrix $M(x)$ of smooth functions on D^k is said to be generic if $M : D^k \to E(n, m)$ and $M : \partial D^k \to E(n, m)$ are both transversal to all $\Sigma_r, r = 0, 1, \dots, min(n, m)$.*

J. Mather's Theorem 1:([12]) If $M(x)$ is generic and the equation $M(x)a = g(x)$ has a solution for every $x \in D^k$, then Equation (2) has a smooth solution $a(x)$.

Let $\mathcal{E}(n, m; k)$ denote the set of $n \times m$ matrices $M(x)$ of smooth functions on D^k. Then $\mathcal{E}(n, m; k)$ may be identified with the set of smooth maps of D^k into $E(n, m)$. We endow $\mathcal{E}(n, m; k)$ with the C^1 topology. Then we have.

J. Mather's Theorem 2:([12]) Generic matrices form an open dense set in $\mathcal{E}(n, m; k)$.

In the proof of J. Mather's Theorem 1, the following local version of the theorem is also proved.

J. Mather's Theorem 3:([12]) Let $M : (\mathbf{R}^k, 0) \to E(n, m)$ be smooth map-germ and let $g : (\mathbf{R}^k, 0) \to \mathbf{R}^n$ be a smooth map-germ such that the linear equation (2) has a solution $a(x)$ for every $x \in \mathbf{R}^k$ close to the origin $0 \in \mathbf{R}^k$. If the map-germ $M : (\mathbf{R}^k, 0) \to E(n, m)$ is transversal to all Σ_r, $r = 0, 1, \ldots, min(n, m)$ at every point $x \in \mathbf{R}^k$ close to the origin $0 \in \mathbf{R}^k$, then the equation (2) has a local smooth solution $a(x)$ defined in a neighbourhood of the origin $0 \in \mathbf{R}^k$.

4 Proof of Main Theorems

Since our problem is local, we use the notation of germs. Let $f : \mathbf{R}^m \to \mathbf{R}^n$ be a smooth mapping. From Lemma 2.4 we may suppose that $f(0) = 0$ without loss of generality. Denote by $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$ its map-germ at $0 \in \mathbf{R}^m$. Consider the map

$$Jf : \mathbf{R}^m \to E(n, m) \quad \text{given by} \quad Jf(x) = \text{the jacobian matrix at } x \in \mathbf{R}^m$$

and its germ

$$Jf : (\mathbf{R}^m, 0) \to E(n, m) \qquad \text{at } 0 \in \mathbf{R}^m.$$

Now the statement of the Theorem 3.1 becomes of the following form.

Theorem 3.1: Let the rank of $Jf(0)$ be equal to r. If $Jf : (\mathbf{R}^m, 0) \to E(n, m)$ is transversal to Σ_r at $0 \in \mathbf{R}^m$, then, for any smooth map-germ $g : (\mathbf{R}^m, 0) \to \mathbf{R}^n$ tangent to $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$, the germ of the manifold

$$M = (f, g)(\mathbf{R}^m) \subset T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$$

at $(0, g(0))$ is integrable.

Now we have immediately.

COROLLARY 4.1 *Let $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$ be a C^∞ stable map-germ. Then for any smooth map-germ $g : (\mathbf{R}^m, 0) \to \mathbf{R}^n$ tangent to $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$, the manifold germ $(M = (f, g)(\mathbf{R}^m), (f(0) = 0, g(0)))$ is integrable.*

Proof. Since stable maps are transversal to every $\mathcal{A}-$orbit in the jet spaces (cf. [11]), so they are also transversal to Σ_r. Thus from the Theorem 3.1, the germ of the manifold $M = (f, g)(\mathbf{R}^m)$ at $(0, g(0))$ is integrable. *Q.E.D.*

The main theorem 3.1 follows from J. Mather's Theorem 3 and the previous Lemma 2.3.

Proof of Theorem 3.1

Let the rank of $Jf(0)$ be equal to r. Suppose that $Jf : (\mathbf{R}^m, 0) \to E(n, m)$ is transversal to Σ_r at $0 \in \mathbf{R}^m$. Since the stratification

$$\{\Sigma_i\}_{i=0}^{min\{m,n\}}$$

of $E(n, m)$ satisfies Whitney's condition a, this implies that $Jf : (\mathbf{R}^m, 0) \to E(n, m)$ is transversal all $\Sigma_i(\mathbf{R}^m, \mathbf{R}^n)$ at every point near the origin $0 \in \mathbf{R}^m$. Thus from J. Mather's Theorem 3, for any smooth map-germ $g : (\mathbf{R}^m, 0) \to \mathbf{R}^n$ tangent to $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$, the linear equation $Jf(x)a = g(x)$ has a smooth solution $a(x)$ defined in a neighbourhood of the origin $0 \in \mathbf{R}^m$. Therefore, from Lemma 2.3, the germ of the manifold $(M = (f, g)(\mathbf{R}^m), (0, g(0)))$ is integrable. *Q.E.D.*

Proof of Theorem 3.2

Proof of 1) The "only if" part is trivial. Now suppose that $j^1 f : \mathbf{R}^m \to J^1(\mathbf{R}^m, \mathbf{R}^n)$ is transversal to $\Sigma_r(\mathbf{R}^m, \mathbf{R}^n)$ at $0 \in \mathbf{R}^m$. Since the stratification

$$\{\Sigma_i(\mathbf{R}^m, \mathbf{R}^n)\}_{i=0}^{\min(m,n)}$$

of $J^1(\mathbf{R}^m, \mathbf{R}^n)$ satisfies Whitney's condition (a) (cf. [8]), this implies that $j^1 f : \mathbf{R}^m \to J^1(\mathbf{R}^m, \mathbf{R}^n)$ is transversal to all $\Sigma_i(\mathbf{R}^m, \mathbf{R}^n)$ at every point near the origin $0 \in \mathbf{R}^m$. Thus the germ $(M, (p, q))$ is generic.

Proof of 2) This follows from the proof of the Theorem 3.1. *Q.E.D.*

5 Integrability for Morin corank 1 stable singularities

The class of generic manifolds which we considered in the previous section includes also those with stable singularities. However J. Mather's theorem used in the proof of the Theorems 3.1 and 3.2 does not explain directly what is the real reason of the integrability. In this section we prove that the N-integrability condition is also sufficient for manifolds M with corank 1 stable singularities of the projection $\pi_1 |_M$ and show the main reason for the integrability being the fact that the Jacobian ideal of the singularity is generated by irreducible functions.

Morin singularities with $m = n$

Let $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^m, 0)$ be a stable smooth map-germ with corank 1. Then it is right-left equivalent to the following canonical form:

$$f(x_1, \ldots, x_{m-1}, y) = (x_1, \ldots, x_{m-1}, f_m(x, y)), \qquad f_m(x, y) = y^{k+1} + \sum_{i=1}^{k-1} x_i y^i.$$

Its Jacobian matrix has the form

$$Jf(x,y) = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & & & O & & \vdots \\ & & \ddots & & & \vdots & \vdots \\ \vdots & & & \ddots & & & \vdots \\ & O & & & \ddots & 0 & \vdots \\ 0 & & \cdots & & 0 & 1 & 0 \\ y & \cdots & y^{k-1} & 0 & \cdots & 0 & \partial f_m/\partial y \end{pmatrix},$$

$$\frac{\partial f_m}{\partial y}(x,y) = x_1 + \sum_{i=2}^{k-1} i x_i y^{i-1} (k+1) y^k.$$

PROPOSITION 5.1 *Let $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^m, 0)$ be a smooth stable corank 1 map-germ. Let $g : (\mathbf{R}^m, 0) \to (\mathbf{R}^m, 0)$ be a smooth map-germ such that*

1) $(f,g) : (\mathbf{R}^m, 0) \to T\mathbf{R}^m = \mathbf{R}^m \times \mathbf{R}^m$ is an embedding,

2) the image manifold-germ $M = (f,g)((\mathbf{R}^m, 0))$ satisfies N-integrability condition, then M is integrable.

Proof.N-integrability condition is equivalent to the condition that the linear equation

$$Jf(x,y) \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} g_1(x,y) \\ \vdots \\ g_m(x,y) \end{pmatrix}$$

has a not necessarily smooth solution $a(x,y)$ for every (x,y). If the above equation has a smooth solution, then from the Lemma 2.3 M is integrable.

At a point (x,y) where $detJf(x,y) \neq 0$, the above equation has a local smooth solution and the point (x,y) is an integrable point. From the form of the equation $Jf(x,y)a(x,y) = g(x,y)$

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & & & O & & \vdots \\ & & \ddots & & & \vdots & \vdots \\ \vdots & & & \ddots & & & \vdots \\ & O & & & \ddots & 0 & \vdots \\ 0 & & \cdots & & 0 & 1 & 0 \\ y & \cdots & y^{k-1} & 0 & \cdots & 0 & \partial f_m/\partial y \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{m-1} \\ a_m \end{pmatrix} = \begin{pmatrix} g_1(x,y) \\ g_2(x,y) \\ \vdots \\ g_{m-1}(x,y) \\ g_m(x,y) \end{pmatrix},$$

$a_1(x,y), \ldots, a_{m-1}(x,y)$ are uniquely determined and smooth:

$$a_1(x,y) = g_1(x,y), \ldots, a_{m-1}(x,y) = g_{m-1}(x,y).$$

At points (x,y) where $detJf(x,y) = 0$, we have

$$g_m(x,y) = \sum_{i=1}^{k-1} y^i a_i(x,y) = \sum_{i=1}^{k-1} y^i g_i(x,y),$$

hence

$$(g_m(x,y) - \sum_{i=1}^{k-1} y^i g_i(x,y)) \mid_{\{Jf(x,y)=0\}} \equiv 0.$$

Since

$$\frac{\partial f_m}{\partial y}(x,y) = x_1 + \sum_{i=2}^{k-1} i x_i y^{i-1} (k+1) y^k$$

is irreducible, $(g_m(x,y) - \sum_{i=1}^{k-1} y^i g_i(x,y))$ is contained in the ideal $\mathcal{E}_{xy} < \partial f_m/\partial y >$, where $\mathcal{E}_{xy}$ is the ring of smooth function-germs at $0 \in \mathbf{R}^m$. Therefore there exists a smooth function $\alpha(x,y) \in \mathcal{E}_{xy}$ such that

$$g_m(x,y) - \sum_{i=1}^{k-1} y^i g_i(x,y) = \alpha(x,y) \frac{\partial f_n}{\partial y}(x,y).$$

Thus we have

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & & & O & & \vdots \\ & & \ddots & & & \vdots & \vdots \\ \vdots & & & \ddots & & & \vdots \\ & O & & & \ddots & 0 & \vdots \\ 0 & & \cdots & & 0 & 1 & 0 \\ y & \cdots & y^{k-1} & 0 & \cdots & 0 & \partial f_m/\partial y \end{pmatrix} \begin{pmatrix} g_1(x,y) \\ \vdots \\ g_{m-1}(x,y) \\ \alpha(x,y) \end{pmatrix}$$

$$= \begin{pmatrix} g_1(x,y) \\ \vdots \\ g_{m-1}(x,y) \\ \sum_{i=1}^{k-1} y^i g_i(x,y) + \alpha(x,y) \partial f_m/\partial y \end{pmatrix} = \begin{pmatrix} g_1(x,y) \\ \vdots \\ g_{m-1}(x,y) \\ g_m(x,y) \end{pmatrix}.$$

Hence the linear equation $Jf(x,y)a = g(x,y)$ has a smooth solution

$$a_1(x,y) = g_1(x,y), \ldots, a_{m-1}(x,y) = g_{m-1}(x,y), a_m(x,y) = \alpha(x,y),$$

and, from the Lemma 2.3, M is integrable. *Q.E.D.*

5.1 Morin Singularities with $m < n$

Let $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$, $m < n$, be a smooth stable map-germ with corank 1. Then it is right-left equivalent to the following canonical form:

$$f(x_1, \ldots, x_{m-1}, y) = (x_1, \ldots, x_{m-1}, f_1(x,y), \ldots, f_{n-m+1}(x,y)),$$

$$\begin{aligned} f_i(x,y) &= \sum_{j=1}^{k} x_{(i-1)k+j} y^j, \quad 1 \le i < n-m+1 \\ f_{n-m+1}(x,y) &= y^{k+1} + \sum_{j=1}^{k-1} x_{(n-m)k+j} y^j. \end{aligned}$$

Its Jacobian matrix $Jf(x,y)$ has the form

$$\left(\begin{array}{ccccccccccccc}
1 & 0 & & & & \cdots & \cdots & \cdots & & & & 0 & 0 \\
0 & \ddots & & & & & & & & & O & & \vdots \\
 & & \ddots & & & & & & & & & \vdots & \vdots \\
 & & \ddots & \ddots & & & & & & & & \vdots & \vdots \\
 & & & \ddots & \ddots & & & & & & & \vdots & \vdots \\
\vdots & & & & \ddots & \ddots & & & & & & \vdots & \vdots \\
\vdots & & & & & \ddots & \ddots & & & & & & \vdots \\
\vdots & & & & & & \ddots & \ddots & & & & & \vdots \\
 & & & & & & & \ddots & \ddots & & & & \vdots \\
 & & & & & & & & \ddots & \ddots & & \vdots & \vdots \\
 & O & & & & & & & & \ddots & \ddots & 0 & \vdots \\
0 & & & & \cdots & & & & & & 0 & 1 & 0 \\
y & \cdots & y^{k-1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \partial f_1/\partial y \\
0 & \cdots & 0 & y & \cdots & y^{k-1} & 0 & \cdots & 0 & 0 & \cdots & 0 & \partial f_2/\partial y \\
 & \cdots & & & & \cdots & \cdots & \cdots & & & \cdots & & \cdots \\
0 & & & & 0 & y & \cdots & y^{k-1} & 0 & \cdots & 0 & \partial f_{n-m+1}/\partial y &
\end{array}\right),$$

$$\begin{aligned}
\frac{\partial f_i}{\partial y}(x,y) &= x_{(i-1)k+1} + \sum_{j=2}^{k} j x_{(i-1)k+j} y^{j-1}, \\
\frac{\partial f_{n-m+1}}{\partial y}(x,y) &= x_{(n-m)k+1} + (k+1)y^k + \sum_{j=2}^{k-1} j x_{(n-m)k+j} y^{j-1}.
\end{aligned}$$

Proposition 5.2 *Let $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$, $m < n$ be a smooth, corank 1 stable map-germ. Let $g : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$ be a smooth map such that*

1) $(f,g) : (\mathbf{R}^m, 0) \to T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$ is an embedding,

2) the image manifold-germ $M = (f,g)((\mathbf{R}^m, 0))$ satisfies N-integrability condition. Then M is integrable.

Proof. N-integrability condition is equivalent to the fact that the linear equation 1 has a solution $a(x,y)$ for every (x,y). By Lemma 2.3, in order to prove that M is integrable, it suffices to prove that the corresponding linear equation has a smooth solution. From the form of $Jf(x,y)$, $a_1(x,y), \ldots, a_{m-1}(x,y)$ are uniquely determined and smooth:

$$a_1(x,y) = g_1(x,y), \ldots, a_{m-1}(x,y) = g_{m-1}(x,y).$$

So the problem is to find a smooth solution $a_m(x,y)$. Note that outside of the singular point set $S(f)$, the solution $a_m(x,y)$ is unique.

We choose a solution $a(x,y)$ and fix it. We are going to find another solution

$$a_1(x,y) = g_1(x,y), \ldots, a_{m-1}(x,y) = g_{m-1}(x,y), \quad \text{and} \quad \alpha(x,y),$$

which is smooth.

Since $a(x,y)$ is a solution of the equation $Jf(x,y)a = g(x,y)$, from the form of Jacobian matrix, we see that at the points (x,y) where $\partial f_1/\partial y(x,y) = 0$, we have

$$g_m(x,y) = \sum_{i=1}^{k-1} a_i(x,y)y^i = \sum_{i=1}^{k-1} g_i(x,y)y^i.$$

Hence we have

$$(g_m(x,y) - \sum_{i=1}^{k-1} g_i(x,y)y^i) \mid_{\{\partial f_1/\partial y(x,y)=0\}} \equiv 0.$$

Since

$$\frac{\partial f_1}{\partial y}(x,y) = x_1 + \sum_{j=2}^{k} i x_j y^{j-1}$$

is irreducible, there exists a smooth function $\alpha_1(x,y) \in \mathcal{E}_{x,y}$ such that

$$g_m(x,y) - \sum_{i=1}^{k-1} g_i(x,y)y^i = \alpha_1(x,y)\frac{\partial f_1}{\partial y}(x,y) = 0.$$

Thus $(g_1(x,y),\ldots,g_m - 1(x,y),\alpha_1(x,y))$ is a smooth solution of the sub equation consisting of the first m lines of the equation $Jf(x,y)a = g(x,y)$:

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & & & O & & \vdots \\ & & \ddots & & & \vdots & \vdots \\ \vdots & & & \ddots & & & \vdots \\ & O & & & \ddots & 0 & \vdots \\ 0 & & \cdots & & 0 & 1 & 0 \\ y & \cdots & y^{k-1} & 0 & \cdots & 0 & \partial f_1/\partial y \end{pmatrix} \begin{pmatrix} g_1(x,y) \\ \vdots \\ g_{m-1}(x,y) \\ \alpha(x,y) \end{pmatrix} = \begin{pmatrix} g_1(x,y) \\ \vdots \\ g_{m-1}(x,y) \\ g_m(x,y) \end{pmatrix},$$

The solution $a(x,y)$ of the equation $Jf(x,y)a(x,y) = g(x,y)$ must be also a solution of the sub equation

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & & & O & & \vdots \\ & & \ddots & & & \vdots & \vdots \\ \vdots & & & \ddots & & & \vdots \\ & O & & & \ddots & 0 & \vdots \\ 0 & & \cdots & & 0 & 1 & 0 \\ y & \cdots & y^{k-1} & 0 & \cdots & 0 & \partial f_1/\partial y \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_{m-1} \\ a_m \end{pmatrix} = \begin{pmatrix} g_1(x,y) \\ \vdots \\ g_{m-1}(x,y) \\ g_m(x,y) \end{pmatrix},$$

which is unique in the region $\{\partial f_1/\partial y(x,y) \neq 0\}$. Thus we have

$$a_m(x,y) = \alpha_1(x,y) \qquad \text{on} \quad \{\partial f_1/\partial y(x,y) \neq 0\}.$$

In the same way, for $i = 2, \ldots, n-m+1$, the sub equation consisting of the first $m-1$ equations and the $(m+i-1)$-th equation of the equation $Jf(x,y)a = g(x,y)$

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & & & O & & \vdots \\ & & \ddots & & & \vdots & \vdots \\ \vdots & & & \ddots & & & \vdots \\ & O & & & \ddots & 0 & \vdots \\ 0 & & \cdots & & 0 & 1 & 0 \\ y & \cdots & y^{k-1} & 0 & \cdots & 0 & \partial f_i/\partial y \end{pmatrix} \begin{pmatrix} g_1(x,y) \\ \vdots \\ g_{m-1}(x,y) \\ \alpha_i(x,y) \end{pmatrix} = \begin{pmatrix} g_1(x,y) \\ \vdots \\ g_{m-1}(x,y) \\ g_{m+i-1}(x,y) \end{pmatrix},$$

has a smooth solution $\alpha_i(x,y)$ which coincides with $a_m(x,y)$ in the region $\{\partial f_i/\partial y(x,y) \neq 0\}$:

$$\alpha_i(x,y) = a_m(x,y) \qquad \forall (x,y) \in \{\partial f_i/\partial y(x,y) \neq 0\}$$

Thus we have

$$\alpha_1(x,y) = \alpha_2(x,y) = \cdots = \alpha_{n-m+1}(x,y) = a_m(x,y)$$

at every point $(x,y) \in \{\partial f_1/\partial y(x,y) \neq 0, \ldots, \partial f_{n-m+1}/\partial y(x,y) \neq 0\}$.
Thus $\alpha_1(x,y), \alpha_2(x,y), \cdots, \alpha_{n-m+1}(x,y)$ coincide with each other on the open and dense set $\{\partial f_1/\partial y(x,y) \neq 0, \ldots, \partial f_{n-m+1}/\partial y(x,y) \neq 0\}$.

Since they are all smooth, they must coincide with each other on the whole region where they are defined. Set

$$\alpha(x,y) = \alpha_1(x,y) = \alpha_2(x,y) = \cdots = \alpha_{n-m+1}(x,y).$$

Thus with this $\alpha(x,y)$, we have found a smooth solution $g_1(x,y), \ldots, g_{m-1}(x,y), \alpha(x,y)$ of the original equation $Jf(x,y)a = g(x,y)$. This completes the proof of the proposition. *Q.E.D.*

5.2 Morin Singularities with $m > n$

Let $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0), \quad m > n$, be a stable smooth map-germ with corank 1. Then it is right-left equivalent to the following canonical form:

$$f(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-n}, z)) = (x_1, \ldots, x_{n-1}, f_n(x,y,z)).$$

$$f_n(x,y,z)) = z^{k-1} \sum_{i=1}^{k-1} x_i z^i + \sum_{i=1}^{m-n} \pm {y_i}^2.$$

Proposition 5.3 *Let $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$ be a smooth corank 1 stable map-germ. Let $g : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$ be a smooth map such that*

1) $(f,g) : (\mathbf{R}^m, 0) \to T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$ is an embedding,

2) the image manifold-germ $M = (f,g)((\mathbf{R}^m, 0))$ satisfies N-integrability condition. Then M is integrable.

Proof. The proof of this Proposition goes the same lines as before. We may leave it as a good exercise. *Q.E.D.*

6 Examples and counter examples

Let us define the following four spaces of surface-germs $M = (f,g)(\mathbf{R}^m) \subset T\mathbf{R}^n$ involved in our integrability conditions.

$$
\begin{array}{rcll}
\mathcal{M}_0 & = & \{M & | \quad M \quad \text{satisfies the N-integrability condition} \quad \} \\
\mathcal{M} & = & \{M & | \quad M \quad \text{integrable} \quad \} \\
\mathcal{M}_1 & = & \{M & | \quad M = (f,g)(\mathbf{R}^m), \quad \text{the linear equation} \\
 & & & \qquad Jf(u)b(u) = g(u) \quad \text{has a smooth solution} \quad \} \\
\mathcal{M}_2 & = & \{M & | \quad M = \quad \text{is generic in the sense of J. Mather} \quad \}
\end{array}
$$

Proposition 6.1 *There is a sequence of proper inclusions.*

$$\mathcal{M}_2 \not\subseteq \mathcal{M}_1 \not\subseteq \mathcal{M} \not\subseteq \mathcal{M}_0.$$

As a prove of this proposition, at each inclusion $\not\subseteq$ we give a simple example which belongs to the latter set but not to the former.

Example 6.1

An example that fulfills the N-integrability condition but is not integrable.

Let

$$f, g : \mathbf{R}^2 \to \mathbf{R}^2, \quad f(u,v) = (u, u^2v + v^3), \quad g(u,v) = (1+v, u)$$
$$M = (f,g)(\mathbf{R}^2).$$

Then M satisfies N-Integrability condition

$$g(u,v) \in Jf(u,v)(\mathbf{R}^2),$$

but M is not integrable at $(1,0) = (f,g)(0,0) \in M$.

First $(f,g) : \mathbf{R}^2 \to T\mathbf{R}^2 = \mathbf{R}^2 \times \mathbf{R}^2$ is an embedding. Since

$$Jf(u,v) = \begin{pmatrix} 1 & 0 \\ 2uv & u^2 + 3v^2 \end{pmatrix},$$

at any point $(u,v) \neq (0,0)$, $Jf(u,v)$ is a regular matrix and $g(u,v) \in Jf(u,v)(\mathbf{R}^2)$. At $(u,v) = (0,0)$,

$$g(0,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R}^2.$$

Thus $M = (f,g)(\mathbf{R}^2)$ satisfies N-Integrability condition.

Now let us see that M is not integrable at $(1,0) = (f,g)(0,0) \in M$. Suppose that $(1,0) = (f,g)(0,0) \in M$ is an integrable point of M. Then there exists an integral curve $\gamma : (-\epsilon, \epsilon) \to \mathbf{R}^2$ such that

$$(\gamma(0), \gamma'(0)) = (1,0), \quad (\gamma(t), \gamma'(t)) \in M, \quad \forall t \in (-\epsilon, \epsilon).$$

Since $(f,g) : \mathbf{R}^2 \to M \subset T\mathbf{R}^2 = \mathbf{R}^2 \times \mathbf{R}^2$ is a diffeomorphism, we can consider the curve

$$\alpha : (-\epsilon, \epsilon) \to \mathbf{R}^2 \quad defined\ by \quad \alpha(t) = (f,g)^{-1}(\gamma(t), \gamma'(t)).$$

Setting

$$\alpha(t) = (u(t), v(t)), \quad \text{where} \quad \alpha(0) = (u(0), v(0)) = (0,0)$$

we have the equality

$$g(\alpha(t)) = g(u(t), v(t)) = \begin{pmatrix} 1 + v(t) \\ u(t) \end{pmatrix} = Jf(\alpha(t))\alpha'(t)$$

$$= \begin{pmatrix} 1 & 0 \\ 2u(t)v(t) & u(t)^2 + 3v(t)^2 \end{pmatrix} \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}.$$

Therefore we have

$$\begin{aligned} 1 + v(t) &= u'(t) && (3) \\ u(t) &= 2u(t)v(t)u'(t) + (u(t)^2 + 3v(t)^2)v'(t). && (4) \end{aligned}$$

Now we compare the orders of the both sides of the second equation with respect to the variable t. Since $u(0) = v(0) = 0$ and from the first equation, we see that

the order of the left hand side $u(t)$ is equal to 1.

On the other hand, since $u(0) = v(0) = 0$, we see that

the order of the right hand side $2u(t)v(t)u'(t) + (u(t)^2 + 3v(t)^2)v'(t) \geq 2$.

This is a contradiction. Thus we see that there is no integral curve $\gamma : (-\epsilon, \epsilon) \to \mathbf{R}^2$ such that

$$(\gamma(0), \gamma'(0)) = (1,0), \quad (\gamma(t), \gamma'(t)) \in M, \quad \forall t \in (-\epsilon, \epsilon),$$

and $(1,0) = (f,g)(0,0) \in M$ is not an integrable point of M.

Example 6.2

An example that has no smooth solution of the Jacobian linear equation but Integrable

For $n > 2$, set

$$f, g : \mathbf{R}^2 \to \mathbf{R}^2, \quad f(u,v) = (u, v^n), \quad g(u,v) = (1, v),$$

$$M = (f,g)(\mathbf{R}^2).$$

Then M is an integrable manifold, although the Jacobian linear equation $g(u,v) = Jf(u,v)b(u)$ has no smooth solution.

First we easily see that $(f,g) : \mathbf{R}^2 \to T\mathbf{R}^2 = \mathbf{R}^2 \times \mathbf{R}^2$ is an embedding. Next let us check the N-Integrability condition. Since

$$Jf(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & nv^{n-1} \end{pmatrix},$$

at any point (u,v) with $v \neq 0$, $Jf(u,v)$ is a regular matrix and $g(u,v) \in Jf(u,v)(\mathbf{R}^2)$. At $(u,0)$,

$$g(u,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R}^2.$$

Thus $M = (f,g)(\mathbf{R}^2)$ satisfies N-Integrability condition.

Now consider the Jacobian linear equation $g(u,v) = Jf(u,v)b(u)$:

$$g(u,v) = \begin{pmatrix} 1 \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & nv^{n-1} \end{pmatrix} \begin{pmatrix} b_1(u,v) \\ b_2(u,v) \end{pmatrix}.$$

If there exist a smooth solution $b_1(u,v), b_2(u,v)$, it must satisfy

$$v = nv^{n-1} b_2(u,v),$$

which is impossible for a smooth function $b_2(u,v)$ and for $n > 2$. Thus the Jacobian linear equation $g(u,v) = Jf(u,v)b(u,v)$ has no smooth solutions.

Now let us check that M is integrable. The points $(f,g)(u,v) \in M$ where $v \neq 0$ are integrable. For around the points (u,v) where $v \neq 0$, the Jacobian linear equation $g(u,v) = Jf(u,v)b(u,v)$ has smooth solutions.

Now let check the points $(f,g)(u,0) = (u,0,1,0) \in M$. Consider the submanifold

$$S = \{ \quad (u,v) \in \mathbf{R}^2 \quad | \quad v = 0 \quad \} \subset \mathbf{R}^2$$

and restrict the Jacobian linear equation to S:

$$g(u,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_1(u,0) \\ b_2(u,0) \end{pmatrix}$$

Then we have a smooth solution on S:

$$b_1(u,0) = 1, \quad b_2(u,0) = 0.$$

The solution $(b_1(u,0), b_2(u,0)) = (1,0)$ is tangent to the submanifold S at every point $(u,0) \in S$. Consider the integral curves

$$\alpha(t) = (u+t, 0) \quad \text{of the vector field} \quad \frac{\partial}{\partial u} \quad \text{on} \quad S$$

and the curves

$$\gamma : \mathbf{R} \to \mathbf{R}^2, \qquad \gamma(t) = f(\alpha(t)) = f(u+t,0) = (u,0).$$

Then we have

$$(\gamma(t), \gamma'(t)) = (u+t, 1) = (f,g)(u+t,0) \in (f,g)(S) \subset M,$$

$$(\gamma(0), \gamma'(0)) = (u,1) = (f,g)(u,0).$$

Thus the points of the form $(\gamma(0), \gamma'(0)) = (u,1) = (f,g)(u,0)$. are also integrable and hence M is integrable.

EXAMPLE 6.3

Non-generic but there exists a smooth solution of the Jacobian linear equation.

Let

$$f, g : \mathbf{R}^2 \to \mathbf{R}^2, \quad f(u,v) = (u, f_2(u,v)), \quad g(u,v) = (1+v, (1+v)\partial f_2/\partial u(u,v)),$$

$$\text{such that} \quad \frac{\partial^2 f_2}{\partial u \partial v}(0,0) = \frac{\partial^2 f_2}{\partial v^2}(0,0) = 0,$$

$$M = (f,g)(\mathbf{R}^2).$$

Then $Jf : \mathbf{R}^2 \to E(2,2)$ is not transversal to S_1 but the Jacobian linear equation

$$g(u,v) = Jf(u,v)b(u)$$

has a smooth solution, where $E(n,m)$ is the space of $n \times m$ real matrices and S_r is the submanifold of $E(n,m)$ consisting of matrices of rank r.

First we easily see that $(f,g) : \mathbf{R}^2 \to T\mathbf{R}^2 = \mathbf{R}^2 \times \mathbf{R}^2$ is an embedding.

Next let us check that the Jacobian linear equation

$$g(u,v) = Jf(u,v)b(u)$$

has a smooth solution. Since

$$Jf(u,v) = \begin{pmatrix} 1 & 0 \\ \partial f_2/\partial u(u,v) & \partial f_2/\partial v(u,v) \end{pmatrix},$$

we see that

$$g(u,v) = \begin{pmatrix} 1+v \\ (1+v)\partial f_2/\partial u(u,v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \partial f_2/\partial u(u,v) & \partial f_2/\partial v(u,v) \end{pmatrix} \begin{pmatrix} 1+v \\ 0 \end{pmatrix}.$$

Thus the Jacobian linear equation

$$g(u,v) = Jf(u,v)b(u)$$

has the smooth solution $(1+v, 0)$.

On the other hand, since

$$\frac{\partial^2 f_2}{\partial u \partial v}(0,0) = \frac{\partial^2 f_2}{\partial v^2}(0,0) = 0,$$

then $Jf : \mathbf{R}^2 \to E(2,2)$ is not transversal to S_1 at $(0,0)$. Because $Jf : \mathbf{R}^2 \to E(2,2)$ is transversal to S_1 at $(0,0)$ if and only if $\partial \det Jf/\partial u(0,0) \neq 0$ or $\partial \det Jf/\partial v(0,0) \neq 0$.

7 Smooth solutions of linear equations and irreducibility of determinants

In Section 4, by using J. Mather's result on smooth solutions of linear equations, we gave a generic sufficient condition for a submanifold M of $T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$ to be integrable. Now we give an extension of J. Mather's theorem weakening his transversality condition to a kind of irreducibility of the determinant of coefficient matrices of the linear equations.

In order to make our argument simpler, we consider real analytic category rather than C^∞ category.

Since our problem of integrability of submanifolds of $T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$ is a local problem, we need only the local version of J. Mather's theorem, which we recall here. Consider a $n \times m$ matrix $A(x)$ of real analytic function germs defined at the origin of $\mathbf{R}^k$:

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1m}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2m}(x) \\ \dots & \dots & \dots & \dots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nm}(x) \end{pmatrix},$$

where $a_{ij} : (\mathbf{R}^k, 0) \to \mathbf{R}$ are germs at $0 \in \mathbf{R}^k$ of real analytic functions defined on $\mathbf{R}^k$, and a column vector

$$g(x) = \begin{pmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix},$$

where $g_i : (\mathbf{R}^k, 0) \to \mathbf{R}$ are also germs at $0 \in \mathbf{R}^k$ of real analytic functions defined on $\mathbf{R}^k$. Consider the equation $A(x)b = g(x)$, where b is a column vector of length m.

Let $E(n,m)$ denote the space of $n \times m$ matrices of real numbers. Let S_r denote the subset of $E(n,m)$ consisting of matrices of rank r. Then S_r is a submanifold of codimension $(n-r) \times (m-r)$ in $E(n,m)$.

Definition 7.1 *An $n \times m$ matrix $A(x)$ of real analytic function germs, $A : (\mathbf{R}^k, 0) \to E(n,m)$, is said to be generic if $A : (\mathbf{R}^k, 0) \to E(n,m)$ is transversal to S_r, where $r = \mathrm{rank}A(0)$.*

J. Mather's Theorem If $A(x)$ is generic and the equation $A(x)b = g(x)$ has a solution for every $x \in \mathbf{R}^k$ sufficiently close to $0 \in \mathbf{R}^k$, then this equation has a real analytic solution germ $b(x)$.

7.1 Extension of J. Mather's theorem: Case where $m = n$

Let $A(x)$ be $m \times m$ matrix whose entries $a_{ij}(x)$ are germs at $0 \in \mathbf{R}^k$ of real analytic functions defined on $\mathbf{R}^k$. Let $g : (\mathbf{R}^k, 0) \to \mathbf{R}^{m=n}$ be a real analytic map germ such that $g(x) \in A(x)\mathbf{R}^m$, $\forall x$, where we regard $g(x)$ as a column vector of real analytic function germs. Consider the linear equation

$$A(x)b = g(x). \tag{5}$$

Let $\mathcal{O}_k$ denote the ring of germs at $0 \in \mathbf{R}^k$ of real analytic functions of k variables.

PROPOSITION 7.1 *If the ideal* $< \det A(x) >$ *in* $\mathcal{O}_k$ *generated by the determinant of the matrix* $A(x)$ *has property of zeroes, (i.e. if any function h(x) vanish on the variety defined by this ideal then h(x) belongs to the ideal), then the linear equation (5)* $A(x)b = g(x)$ *has a real analytic solution* $b(x)$.

Proof. (cf. J. Mather [12], p.190)
Let $\widetilde{A}(x)$ denote the cofactor matrix of $A(x)$. Then we have

$$\widetilde{A}(x)A(x) = A(x)\widetilde{A}(x) = \det A(x) I_m,$$

where I_m is the identity matrix of size m. Then the equation (5) is equivalent to

$$\det A(x)b = \widetilde{A}(x)g(x).$$

Therefore the component functions $\widetilde{g}_1(x), \ldots, \widetilde{g}_m(x)$ of column vector $\widetilde{A}(x)g(x)$ vanish on the variety $\{\det A(x) = 0\}$. The hypothesis of Proposition 7.1 implies that $\det A(x)$ divides the component functions $\widetilde{g}_1(x), \ldots, \widetilde{g}_m(x)$ and

$$b(x) = \begin{pmatrix} b_1(x) \\ \vdots \\ b_m(x) \end{pmatrix} = \frac{1}{\det A(x)} \widetilde{A}(x)g(x) = \begin{pmatrix} \widetilde{g}_1(x)/\det A(x) \\ \vdots \\ \widetilde{g}_m(x)/\det A(x) \end{pmatrix},$$

is a real analytic solution of the equation (5). Q.E.D.

Now we investigate conditions for the ideal $< \det A(x) >$ in $\mathcal{O}_k$ to have property of zeroes.

Let

$$\mathcal{O}_{k,\mathbf{C}}$$

denote the ring of germs at $0 \in \mathbf{C}^k$ of complex analytic functions of k variables. Any real analytic function germ $f(x) \in \mathcal{O}_k$ can be uniquely extended to a complex analytic function germ $\widetilde{f}(z) \in \mathcal{O}_{k,\mathbf{C}}$, which we call the complexification of $f(x)$. Identifying $f(x) \in \mathcal{O}_k$ with its complexification $\widetilde{f}(z) \in \mathcal{O}_{k,\mathbf{C}}$, we may regard $\mathcal{O}_m$ as a subring of $\mathcal{O}_{k,\mathbf{C}}$.

An element a of a ring R is said to be *irreducible* (in R) if a is not a unit of R and satisfies the condition

$$a = bc \quad \Longrightarrow \quad b \quad \text{or} \quad c \quad \text{is a unit of} \quad R.$$

PROPOSITION 7.2 *If a real analytic function germ* $f(x) \in \mathcal{O}_k$ *is irreducible in* $\mathcal{O}_k$ *as well as so is its complexification* $\widetilde{f}(z)$ *in* $\mathcal{O}_{k,\mathbf{C}}$, *then the ideal* $< f(x) >$ *in* $\mathcal{O}_k$ *generated by* $f(x)$ *has property of zeros.*

Proof. Let $g(x)$ be an element of $\mathcal{O}_k$ vanishing on the germ at $0 \in \mathbf{R}^k$ of the variety

$$V(f) = \{x \in \mathbf{R}^k \mid f(x) = 0\}.$$

Then its complexification $\widetilde{g}(z)$ also vanishes on the germ at $0 \in \mathbf{R}^k$ of complex variety

$$\widetilde{V}(\widetilde{f}) = \{z \in \mathbf{C}^k \mid \widetilde{f}(x) = 0\}.$$

(See, for example, [15], Chapter V, Proposition 1 (p.91)).

Now from Hilbert's Nullstellensatz, for some positive power n, we have

$$\tilde{g}^n(z) \in < \tilde{f}(z) > .$$

Since $\tilde{f}(z)$ is irreducible in $\mathcal{O}_{k,\mathbf{C}}$, $\tilde{f}(z)$ divides $\tilde{g}(z)$ in $\mathcal{O}_{k,\mathbf{C}}$:

$$\tilde{g}(z) = \tilde{b}(z)\tilde{f}(z), \qquad \exists\tilde{b}(z) \in \mathcal{O}_{k,\mathbf{C}}.$$

Consider $\tilde{g}(z), \tilde{b}(z), \tilde{f}(z)$ as power series. Then , since the coefficients of $\tilde{g}(z)$,and $\tilde{f}(z)$ are real, the coefficients of $\tilde{b}(z)$ are also real. Thus $\tilde{b}(z)$ is the complexification of a real analytic function $b(x)$. Thus we have $g(x) = b(x)f(x)$, which implies $f(x)$ divides $g(x)$ in $\mathcal{O}_k$. *Q.E.D.*

PROPOSITION 7.3 *Let $f(x) \in \mathcal{O}_k$ be a real analytic function germ. If $f(x)$ has a decomposition*

$$f(x) = f_1(x)f_2(x)\cdots f_\ell(x), \qquad f_i(x) \in \mathcal{O}_k$$

such that

1) $f_1(x), \ldots, f_\ell(x)$ are all irreducible in $\mathcal{O}_k$ and that any factor $f_i(x)$ is not divisible by other factors $f_j(x)$, $i \neq j$,
and such that

2) the complexifications $\tilde{f}_1(z), \tilde{f}_2(z), \cdots, \tilde{f}_\ell(z)$ of $f_1(x), f_2(x), \cdots, f_\ell(x)$ are all irreducible in $\mathcal{O}_{k,\mathbf{C}}$,
then the ideal $< f(x) >$ in $\mathcal{O}_k$ generated by $f(x)$ has property of zeros.

Proof. Let $g(x)$ be an element of $\mathcal{O}_k$ vanishing on the germ at $0 \in \mathbf{R}^k$ of the variety

$$V(f) = \{x \in \mathbf{R}^k \mid f(x) = 0\}.$$

Then $g(x)$ vanishes on the germ at $0 \in \mathbf{R}^k$ of the variety

$$V(f_i) = \{x \in \mathbf{R}^k \mid f_i(x) = 0\} \quad (\quad \subset V(f) \quad).$$

Now $f_i(x)$ satisfies the condition in Proposition 7.2, $f_i(x)$ divides $g(x)$. Since $f_i(x)$'s are all irreducible and any factor $f_i(x)$ is not divisible by other factors$f_j(x)$, $i \neq j$, then $f(x) = f_1(x)f_2(x)\cdots f_\ell(x)$ divides $g(x)$. *Q.E.D.*

Let $A(x)$ be $m \times m$ matrix whose entries $a_{ij}(x)$ are germs at $0 \in \mathbf{R}^k$ of real analytic functions defined on $\mathbf{R}^k$. Let $g : (\mathbf{R}^k, 0) \to \mathbf{R}^m$ be a real analytic map germ such that $g(x) \in A(x)\mathbf{R}^m$, $\forall x$. Consider here again the linear equation (5)

$$A(x)b = g(x).$$

Combining Proposition 7.1 with Propositions 7.2 and 7.3, we obtain

PROPOSITION 7.4 *If* $\det A(x)$ *satisfies one of the following two conditions I and II, then the linear equation (5) has a real analytic solution $b(x)$.*

Condition I. $\det A(x)$ is irreducible in $\mathcal{O}_k$ as well as so is its complexificationin $\det\widetilde{A}(z)$ in $\mathcal{O}_{k,\mathbf{C}}$,
Condition II $\det A(x)$ has a decomposition

$$\det A(x) = \alpha_1(x)\alpha_2(x)\cdots\alpha_\ell(x), \qquad \alpha_i(x) \in \mathcal{O}_k$$

such that

1) $\alpha_1(x), \ldots, \alpha_\ell(x)$ are all irreducible in $\mathcal{O}_k$ and that any factor $\alpha_i(x)$ is not divisible by other factors $\alpha_j(x)$, $i \neq j$,

and such that

2) the complexifications $\widetilde{\alpha}_1(z), \widetilde{\alpha}_2(z), \cdots, \widetilde{\alpha}_\ell(z)$ of $\alpha_1(x), \alpha_2(x), \cdots, \alpha_\ell(x)$ are all irreducible in $\mathcal{O}_{k,\mathbf{C}}$.

Now let's go back further to the original problem of integrability. The following is an immediate consequence of Proposition 7.4.

Theorem 7.1 *Let $f : \mathbf{R}^m \to \mathbf{R}^m$ be a real analytic map and $g : \mathbf{R}^m \to \mathbf{R}^n$ be a real analytic map tangent to f. If $\det Jf(x)$ satisfies one of the conditions I and II in Proposition 7.4, then the germ at $(f(0), g(0))$ of $M = (f, g)(\mathbf{R}^m) \subset T\mathbf{R}^m$ is integrable.*

7.2 Extension of J. Mather's theorem: Case where $m < n$

In this section we observe the case where $m < n$. Suppose $m < n$. We consider an $n \times m$ matrix $A(x)$ of real analytic function germs defined at the origin of $\mathbf{R}^k$:

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \ldots & a_{1m}(x) \\ a_{21}(x) & a_{22}(x) & \ldots & a_{2m}(x) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(x) & a_{n2}(x) & \ldots & a_{nm}(x) \end{pmatrix},$$

where $a_{ij} : (\mathbf{R}^k, 0) \to \mathbf{R}$ are germs at $0 \in \mathbf{R}^k$ of real analytic functions defined on $\mathbf{R}^k$, and a column vector

$$g(x) = \begin{pmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix},$$

where $g_i : (\mathbf{R}^k, 0) \to \mathbf{R}$ are also germs at $0 \in \mathbf{R}^k$ of real analytic functions defined on $\mathbf{R}^k$. Consider the equation

$$A(x)b = g(x), \tag{6}$$

where b is a column vector of length m.

Suppose that the rank of the matrix $A(0)$ at the origin $0 \in \mathbf{R}^k$ is r. Then we may suppose that the matrix $A(x)$ be of the form

$$A(x) = \begin{pmatrix} I_r & O \\ O & A' \end{pmatrix}, \qquad \text{where } I_r \text{ is an identity matrix of size } r$$

$$\text{and} \qquad A' = \begin{pmatrix} a_{r+1,r+1}(x) & a_{r+1,r+2}(x) & \cdots & a_{r+1,m}(x) \\ a_{r+2,r+1}(x) & a_{r+2,r+2}(x) & \cdots & a_{r+2,m}(x) \\ \multicolumn{4}{c}{\dotfill} \\ a_{n,r+1}(x) & a_{n,r+2}(x) & \cdots & a_{n,m}(x) \end{pmatrix}$$

Consider the set

$$\mathcal{I} = \{I = (i_1, i_2, \ldots, i_{m-r}) \quad | \quad i_j \in \mathbf{N}, \quad r < i_1 < i_2 < \ldots < i_{m-r} \le n\},$$

and for each element $I = (i_1, i_2, \ldots, i_{m-r}) \in \mathcal{I}$ consider the $(m-r)\times(m-r)$ minor of $A'(x)$:

$$\det{}_I A(x) = \det \begin{pmatrix} a_{i_1,r+1}(x) & a_{i_1,r+2}(x) & \cdots & a_{i_1,m}(x) \\ a_{i_2,r+1}(x) & a_{i_2,r+2}(x) & \cdots & a_{i_2,m}(x) \\ \multicolumn{4}{c}{\dotfill} \\ a_{i_{m-r},r+1}(x) & a_{i_{m-r},r+2}(x) & \cdots & a_{i_{m-r},m}(x) \end{pmatrix}.$$

PROPOSITION 7.5 *If the equation $A(x)b = g(x)$ has a solution $b(x)$for every $x \in \mathbf{R}^k$ and if for every element $I \in \mathcal{I}$ the ideal $< \det_I A(x) >$ in $\mathcal{O}_k$ generated by the $(m-r)\times(m-r)$ minor $\det_I A(x)$ of the matrix $A(x)$ has property of zeroes, then the linear equation $A(x)b = g(x)$ has a real analytic solution $b(x)$.*

Proof. Although the proof of this proposition is almost the same as the argument in §7(pp.191-192) of J. Mather [12], we recall it here.

First we investigate the equation (6) at points $x \in \mathbf{R}^k$ where $\operatorname{rank} A(x) = m = \min(m,n)$. Set

$$\begin{aligned} \Omega &= \{x \in \mathbf{R}^k \quad | \quad \operatorname{rank} A(x) = m\}, \\ \Sigma &= \mathbf{R}^k - \Omega = \{x \in \mathbf{R}^k \quad | \quad \operatorname{rank} A(x) < m\}. \end{aligned}$$

LEMMA 7.1 *1) Ω is an open dense set of $\mathbf{R}^k$.*

2) For every point $x \in \Omega$, the equation (6) has a unique solution, which we denote by $b_\Omega(x)$.

3) The unique solution $b_\Omega(x)$, $x \in \Omega$, is analytic in Ω.

Proof. 1) Is trivial.

2) Since $\operatorname{rank} A(x) = m = \min(m,n)$, if there exists a solution $b(x)$, which does from the assumption of the Lemma, then $b(x)$ is a unique solution of (6).

3) First note that equation (1) is of the form

$$\begin{pmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix} = \begin{pmatrix} I_r & O \\ O & A' \end{pmatrix} \begin{pmatrix} b_1(x) \\ \vdots \\ b_m(x) \end{pmatrix}.$$

Since $x \in \Omega$, there exists $I = (i_1, i_2, \ldots, i_{m-r}) \in \mathcal{I}$ such that the sub matrix $A'_I(x)$ has rank $m-r$, where

$$A'_I = \begin{pmatrix} a_{i_1,r+1}(x) & a_{i_1,r+2}(x) & \cdots & a_{i_1,m}(x) \\ a_{i_2,r+1}(x) & a_{i_2,r+2}(x) & \cdots & a_{i_2,m}(x) \\ \multicolumn{4}{c}{\dotfill} \\ a_{i_{m-r},r+1}(x) & a_{i_{m-r},r+2}(x) & \cdots & a_{i_{m-r},m}(x) \end{pmatrix}.$$

Set

$$A_I(x) = \begin{pmatrix} I_r & O \\ O & A'_I \end{pmatrix},$$

and consider the sub equation

$$g_I(x) = A_I(x)b,$$

$$\text{where} \quad g_I(x) \quad = \quad {}^t(g_1(x), \ldots, g_r(x), g_{i_1}(x), \ldots, g_{i_{m-r}}(x)).$$

Since $b_\Omega(x)$ is a solution of (6), $b_\Omega(x)$ is also a solution of the sub equation $g_I(x) = A_I(x)b$, which is analytic since $g_I(x)$ is analytic and $\det A_I(x) \neq 0$. *Q.E.D.*

Now consider the solution at points $x \in \Sigma$ where $\mathrm{rank} A(x) < m$. Take any $I = (i_1, i_2, \ldots, i_{m-r}) \in \mathcal{I}$ and consider sub equation

$$g_I(x) = A_I(x)b,$$

where $A_I(x)$ and $g_I(x)$ are defined in the same way as those in the proof of Lemma 7.1. Then, from the assumption of Proposition 7.5, the ideal $< \det A_I(x) >$ satisfies the condition posed in Proposition 7.1 that it has property of zeros. Therefore the sub equation $g_I(x) = A_I(x)b$ has an analytic solution $b_I(x)$.

Now consider the set

$$\Omega_I = \{x \in \mathbf{R}^k \quad | \quad \det\, A_I(x) \neq 0\} \quad \subset \quad \Omega.$$

Note that Ω_I is also open and dense in $\mathbf{R}^k$.

We compare the restriction $(b_I \mid_{\Omega_I})\,(x)$ of $b_I(x)$ to Ω_I with the restriction $(b_\Omega \mid_{\Omega_I})(x)$ of $b_\Omega(x)$ to Ω_I. Since both of $(b_I \mid_{\Omega_I})(x)$ and $(b_\Omega \mid_{\Omega_I})(x)$ are solutions of the sub equation

$$g_I(x) = A_I(x)b, \qquad x \in \Omega_I,$$

and the solution of the sub equation $g_I(x) = A_I(x)b$ is unique on the region $\Omega_I \subset \Omega$, we have

$$b_I(x) = b_\Omega(x) \qquad \forall x \in \Omega_I.$$

Now take any other $J \in \mathcal{I}$, $J \neq I$ and consider the sub equation

$$g_J(x) = A_J(x)b.$$

With the same reason, the sub equation $g_J(x) = A_J(x)b$ has an analytic solution $b_J(x)$ and we have

$$b_J(x) = b_\Omega(x) \qquad \forall x \in \Omega_J.$$

Hence we have

$$b_J(x) = b_I(x) = b_\Omega(x) \qquad \forall x \in \Omega_J \cap \Omega_I.$$

Thus $b_J(x)$ and $b_I(x)$ coincide on an open dense subset $\Omega_J \cap \Omega_I$ of $\mathbf{R}^k$ and they are analytic. Therefore they coincide on the whole $\mathbf{R}^k$. Set

$$b(x) = b_I(x) \qquad \forall I \in \mathcal{I}.$$

Then $b(x)$ is an analytic solution of all the sub equations $g_I(x) = A_I(x)b$, and it is an analytic solution of the whole equation $g(x) = A(x)b$. Thus we proved the Proposition 7.5 *Q.E.D.*

Using Proposition 7.5 and Propositions 7.2 and 7.3, in the same way as in the proof of Proposition 7.4, we can prove the following theorem :

PROPOSITION 7.6 *If the equation $A(x)b = g(x)$ has a solution $b(x)$ for every $x \in \mathbf{R}^k$ and if, for every element $I \in \mathcal{I}$, $\det_I A(x)$ satisfies one of the two conditions I and II in Proposition 7.4, then the linear equation (1) has a real analytic solution $b(x)$.*

Now let's go back to the original problem of integrability. The following is an immediate consequence of Proposition 7.6.

THEOREM 7.2 *Let $f : \mathbf{R}^m \to \mathbf{R}^n$, $m < n$, be a real analytic map and $g : \mathbf{R}^m \to \mathbf{R}^n$ be a real analytic map tangent to f. If for every element $I \in \mathcal{I}$, $\det_I Jf(x)$ satisfies one of the two conditions I and II in Proposition 7.4, then the germ at $(f(0), g(0))$ of $M = (f,g)(\mathbf{R}^m) \subset T\mathbf{R}^m$ is integrable.*

8 Integrability of implicit Hamiltonian systems

Now we consider the special class of implicit differential systems - the generalized Hamiltonian systems, which play an important role in field theory and mechanics.

Let $(\mathbf{R}, \omega)$ be the symplectic Euclidean space, endowed with the symplectic structure in the Darboux form, $\omega = \sum_{i=1}^n dy_i \wedge dx_i$. By the vector bundle morphism $\beta : T\mathbf{R}^{2n} \ni u \to u\rfloor\omega \in T^*\mathbf{R}^{2n}$ we introduce the canonical symplectic structure $\bar{\omega}$ on $T\mathbf{R}^{2n}$, namely the pullback of the Liouville symplectic form $d\theta$ defined on $T^*\mathbf{R}^{2n}$.

$$\bar{\omega} = \beta^* d\theta = \sum_{i=1}^n (d\dot{y}_i \wedge dx_i - d\dot{x}_i \wedge dy_i),$$

where $(p,q) = ((x,y),(\dot{x},\dot{y}))$ are coordinates on $T\mathbf{R}^{2n} \equiv \mathbf{R}^{2n} \times \mathbf{R}^{2n}$

DEFINITION 8.1 *A smooth submanifold $M \subset T\mathbf{R}^{2n} \equiv \mathbf{R}^{2n} \times \mathbf{R}^{2n}$ is called generalized Hamiltonian system if M is isotropic according to $\bar{\omega}$, i.e. $\bar{\omega}\mid_M = 0$. In this case $dimM \leq 2n$. The case of M such that $dimM = 2n$ is called the implicit Hamiltonian system.*

By the Hörmander-Arnold-Weinstein generating family representation of M [1], we have the following Lemma.

LEMMA 8.1 *Let $M \subset T\mathbf{R}^{2n} \equiv \mathbf{R}^{2n} \times \mathbf{R}^{2n}$ be an implicit Hamiltonian system, and let $(p,q) \in M$. Suppose that* corank$d(\pi_1\mid_M)(p,q) = k$. *Then there exists an open neighbourhood $\mathcal{O}$ of (p,q) in $T\mathbf{R}^{2n}$ and a smooth function $F : \mathbf{R}^{2n} \times \mathbf{R}^k \to \mathbf{R}$ defined on some open neighbourhood of $(p,0)$, such that*

$$M \cap \mathcal{O} = \{((x,y),(\dot{x},\dot{y})) \in \mathcal{O}; \quad \exists_{\lambda\in\mathbf{R}^k} \quad \text{such that} \quad \dot{x}_i = \frac{\partial F}{\partial y_i}(x,y,\lambda), \tag{7}$$

$$\dot{y}_j = -\frac{\partial F}{\partial x_j}(x,y,\lambda), 0 = \frac{\partial F}{\partial \lambda_l}(x,y,\lambda)\}, \tag{8}$$

where $1 \leq i,j \leq n, \quad 1 \leq l \leq k,$ *and such that*

$$rank(\frac{\partial^2 F}{\partial x_j \partial \lambda_l}, \frac{\partial^2 F}{\partial y_i \partial \lambda_l}, \frac{\partial^2 F}{\partial \lambda_s \partial \lambda_l})(p,0) = k, \quad and \quad \frac{\partial^2 F}{\partial \lambda_s \partial \lambda_r}(p,0) = 0, \tag{9}$$

where $1 \leq s,r \leq k,$

In what follows we will consider the Hamiltonian case, i.e. $dimM = 2n$, and because we are considering the local properties of M, we assume that M is generated by some generating family-germ F as in the above Lemma 8.1 and denote it also by M_F.

LEMMA 8.2 *Let* $M_F \subset T\mathbf{R}^{2n} \equiv \mathbf{R}^{2n} \times \mathbf{R}^{2n}$ *be an implicit Hamiltonian system generated by the Morse family germ* $F : (\mathbf{R}^{2n} \times \mathbf{R}^k, (0,0)) \to \mathbf{R}$*, then the N-integrability condition of* M_F *is equivalent to the existence of the solution* $\mu = (\mu_1, \ldots, \mu_n) \in \mathbf{R}^k$ *of the following equation*

$$\begin{pmatrix} \frac{\partial^2 F}{\partial \lambda_1 \partial \lambda_1} & \cdots & \frac{\partial^2 F}{\partial \lambda_1 \partial \lambda_k} \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 F}{\partial \lambda_k \partial \lambda_1} & \cdots & \frac{\partial^2 F}{\partial \lambda_k \partial \lambda_k} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} = \begin{pmatrix} \{\frac{\partial F}{\partial \lambda_1}, F\} \\ \vdots \\ \{\frac{\partial F}{\partial \lambda_k}, F\} \end{pmatrix}, \tag{10}$$

for each $(p,q) \in M$*, where* $\{.,.\}$ *denotes the Poisson bracket on* $\mathbf{R}^{2n}$ *induced by* ω*.*

Proof. Assume that $\gamma : (-\varepsilon, \varepsilon) \to \mathbf{R}^{2n}$ and $\delta : (-\varepsilon, \varepsilon) \to \mathbf{R}^k$, $\gamma(0) = (x,y), \quad \delta(0) = \lambda$, is a smooth solution of (7), throug the point (x,y,λ) of M. Then we have

$$\frac{\partial F}{\partial \lambda_l}(\gamma(t), \delta(t)) \equiv 0, \quad \forall_{t \in (-\varepsilon,\varepsilon)}, \quad 1 \leq l \leq k,$$

and also the derivative.

$$\sum_{i=1}^{k} \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_l}(\gamma(t), \delta(t))\delta_i'(t) + \sum_{i=1}^{k} \frac{\partial^2 F}{\partial x_i \partial \lambda_l}(\gamma(t), \delta(t))\dot{x}_i + \sum_{i=1}^{k} \frac{\partial^2 F}{\partial y_i \partial \lambda_l}(\gamma(t), \delta(t))\dot{y}_i =$$
$$\sum_{i=1}^{k} \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_l}(\gamma(t), \delta(t))\delta_i'(t) + \sum_{i=1}^{k} \frac{\partial^2 F}{\partial x_i \partial \lambda_l}(\gamma(t), \delta(t))\frac{\partial F}{\partial y_i}(\gamma(t), \delta(t))$$
$$-\sum_{i=1}^{k} \frac{\partial^2 F}{\partial y_i \partial \lambda_l}(\gamma(t), \delta(t))\frac{\partial F}{\partial x_i}(\gamma(t), \delta(t)), \quad 1 \leq l \leq k.$$

Taking $\gamma(0) = (x,y), \quad \delta(0) = \lambda$, we get the solvability of the equation

$$\sum_{i=1}^{k} \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_l}(x,y,\lambda)\mu_i - \{\frac{\partial F}{\partial \lambda_l}, F\} = 0, \quad 1 \leq l \leq k$$

on the smooth manifold given by equations, $\frac{\partial F}{\partial \lambda_l}(x,y,\lambda) = 0.$ Q.E.D.

N-integrability condition is the necessary condition for integrability of M. Now we give an example which fulfills the N-integrability condition but it is not integrable.

Example 8.1 *As in the case of Example 6.1 in Section 6, we show that the N-integrability condition is also only a necessary condition for implicit Hamiltonian systems.*

Let us consider the implicit Hamiltonian system surface-germ $M \subset T\mathbf{R}^2 \equiv \mathbf{R}^2 \times \mathbf{R}^2$ *around* $(p,q) = (0,0,1,0)$. M *is generated by the generating family-germ* $F : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$, *at* $(x, y, \lambda_1, \lambda_2) = (0,0,1,0)$ *of the form*

$$F(x,y,\lambda) = \frac{1}{2}\lambda_2^2(1-(\lambda_1-1)^2) - \frac{1}{4}(\lambda_1-1)^4 + \lambda_1 y - \lambda_2 x.$$

So the corresponding equations for M *one can write in the form.*

$$\begin{aligned}
\dot{x} &= \frac{\partial F}{\partial y}(x,y,\lambda) = \lambda_1, \\
\dot{y} &= -\frac{\partial F}{\partial x}(x,y,\lambda) = \lambda_2, \\
0 &= \frac{\partial F}{\partial \lambda_1}(x,y,\lambda) = y - \lambda_2^2(\lambda_1-1) - (\lambda_1-1)^3, \\
0 &= \frac{\partial F}{\partial \lambda_2}(x,y,\lambda) = -x + \lambda_2(1-(\lambda_1-1)^2).
\end{aligned}$$

Certainly N-integrability condition is fulfilled around $(0,0,1,0)$. *The determinant*

$$\begin{pmatrix} \frac{\partial^2 F}{\partial\lambda_1\partial\lambda_1} & \frac{\partial^2 F}{\partial\lambda_1\partial\lambda_2} \\ \frac{\partial^2 F}{\partial\lambda_2\partial\lambda_1} & \frac{\partial^2 F}{\partial\lambda_2\partial\lambda_2} \end{pmatrix}(x,y,\lambda)$$

has isolated zero at $(0,0,1,0)$ *and so there exists a solution of the linear equation*

$$\begin{pmatrix} \frac{\partial^2 F}{\partial\lambda_1\partial\lambda_1} & \frac{\partial^2 F}{\partial\lambda_1\partial\lambda_2} \\ \frac{\partial^2 F}{\partial\lambda_2\partial\lambda_1} & \frac{\partial^2 F}{\partial\lambda_2\partial\lambda_2} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \{\frac{\partial F}{\partial \lambda_1}, F\} \\ \{\frac{\partial F}{\partial \lambda_2}, F\} \end{pmatrix},$$

in some neighbourhood outside of $(0,0,1,0)$. *At that point we have*

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and it is also solvable. So N-integrability condition is fulfilled.

If M *is integrable, say at* $(p,q) = (0,0,1,0)$, *then there exists an integral curve* $\gamma : (-\varepsilon, \varepsilon) \to \mathbf{R}^2$ *such that*

$$(\gamma(0), \gamma'(0)) = (0,0,1,0) \quad \textit{end} \quad (\gamma(t), \gamma'(t)) \in M \forall_{t \in (-\varepsilon,\varepsilon)}.$$

So there exists a smooth curve $\lambda(t) = (\lambda_1(t), \lambda_2(t))$, $\lambda(0) = (1,0)$ *such that*

$$\begin{pmatrix} \frac{\partial^2 F}{\partial\lambda_1\partial\lambda_1} & \frac{\partial^2 F}{\partial\lambda_1\partial\lambda_2} \\ \frac{\partial^2 F}{\partial\lambda_2\partial\lambda_1} & \frac{\partial^2 F}{\partial\lambda_2\partial\lambda_2} \end{pmatrix}(\gamma(t), \lambda(t)) \begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} = \begin{pmatrix} -\lambda_2(t) \\ \lambda_1(t)\} \end{pmatrix},$$

i.e. the following equations are fulfilled.

$$\lambda_2(t) = 3(\lambda_1(t)-1)^2\lambda_1'(t) + \lambda_2^2(t)\lambda_1'(t) + 2\lambda_2(t)(\lambda_1(t)-1)\lambda_2'(t) \tag{11}$$

$$\lambda_1(t) = (1-(\lambda_1(t)-1)^2)\lambda_2'(t) - 2\lambda_2(t)(\lambda_1(t)-1)\lambda_1'(t). \tag{12}$$

From the equation (12) we get

$$\lambda_2'(0) = 1,$$

so from the first equation (11) we have a contradiction:
order of $\lambda_2(t)$ *at* 0 *is equal* 1, *but*
order of

$$3(\lambda_1(t)-1)^2\lambda_1'(t) + \lambda_2^2(t)\lambda_1'(t) + 2\lambda_2(t)(\lambda_1(t)-1)\lambda_2'(t)$$

at 0 *is greater than equal to* 2.

Let $\tilde{M}_F \subset T\mathbf{R}^{2n}\times\mathbf{R}^k \equiv \mathbf{R}^{2n}\times\mathbf{R}^{2n}\times\mathbf{R}^k$ denotes a manifold-germ defined by equations (7) in Lemma 8.1. Consider the projection

$$\pi : \mathbf{R}^{2n}\times\mathbf{R}^{2n}\times\mathbf{R}^k \to \mathbf{R}^{2n}\times\mathbf{R}^k, \quad (p,q,\lambda)\to(p,\lambda)$$

and the image $C_F = \pi(\tilde{M}_F)$, which is a smooth manifold-germ by transversality condition (9) in Lemma 8.1.

Let

$$H = \begin{pmatrix} \frac{\partial^2 F}{\partial\lambda_1\partial\lambda_1}(x,y,\lambda) & \dots & \frac{\partial^2 F}{\partial\lambda_1\partial\lambda_k}(x,y,\lambda) \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 F}{\partial\lambda_k\partial\lambda_1}(x,y,\lambda) & \dots & \frac{\partial^2 F}{\partial\lambda_k\partial\lambda_k}(x,y,\lambda) \end{pmatrix}.$$

By

$$\tilde{H} = \begin{pmatrix} [\frac{\partial^2 F}{\partial\lambda_1\partial\lambda_1}(x,y,\lambda)] & \dots & [\frac{\partial^2 F}{\partial\lambda_1\partial\lambda_k}(x,y,\lambda)] \\ \vdots & \vdots & \vdots \\ [\frac{\partial^2 F}{\partial\lambda_k\partial\lambda_1}(x,y,\lambda)] & \dots & [\frac{\partial^2 F}{\partial\lambda_k\partial\lambda_k}(x,y,\lambda)] \end{pmatrix}$$

we denote the $k\times k$ matrix of corresponding minors in $(p,q)\in M_F$. Then, by Cramer's rule the Equation (10) can be rewritten in the form $(detH)\mu = \tilde{H}\{\frac{\partial F}{\partial\lambda}, F\}$, i.e.

$$(detH)(x,y,\lambda)\mu_i = \sum_{j=1}^{k}[\frac{\partial^2 F}{\partial\lambda_i\partial\lambda_j}]\{\frac{\partial F}{\partial\lambda_j}, F\}(x,y,\lambda).$$

So we have an equivalent formulation of the N-integrability condition.

Corollary 8.1 *Let* $M_F \subset T\mathbf{R}^{2n} \equiv \mathbf{R}^{2n}\times\mathbf{R}^{2n}$ *be an implicit Hamiltonian system generated by the Morse family germ* $F : (\mathbf{R}^{2n}\times\mathbf{R}^k,(0,0))\to\mathbf{R}$, *then the N-integrability condition of* M_F *is equivalent to the following condition.*

$$\sum_{j=1}^{k}[\frac{\partial^2 F}{\partial\lambda_i\partial\lambda_j}]\{\frac{\partial F}{\partial\lambda_j}, F\}\mid_{\{detH=0\}\cap C_F}\equiv 0$$

Now we have the result corresponding to Lemma 2.3.

LEMMA 8.3 *Let $F : \mathbf{R}^{2n} \times \mathbf{R}^k \to \mathbf{R}$ be smooth generating family for M_F. If the linear equation*

$$\begin{pmatrix} \frac{\partial^2 F}{\partial\lambda_1\partial\lambda_1}(x,y,\lambda) & \dots & \frac{\partial^2 F}{\partial\lambda_1\partial\lambda_k}(x,y,\lambda) \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 F}{\partial\lambda_k\partial\lambda_1}(x,y,\lambda) & \dots & \frac{\partial^2 F}{\partial\lambda_k\partial\lambda_k}(x,y,\lambda) \end{pmatrix} \begin{pmatrix} \mu_1(x,y,\lambda) \\ \vdots \\ \mu_k(x,y,\lambda) \end{pmatrix} = \begin{pmatrix} \{\frac{\partial F}{\partial\lambda_1}, F\}(x,y,\lambda) \\ \vdots \\ \{\frac{\partial F}{\partial\lambda_k}, F\}(x,y,\lambda) \end{pmatrix},$$

has a smooth solution $(\mu_1(x,y,\lambda),\dots,\mu_k(x,y,\lambda))$ on the critical manifold

$$C_F = \{(x,y,\lambda); \frac{\partial F}{\partial\lambda_i}(x,y,\lambda) = 0, \quad 0 \le i \le k\}.$$

Then the manifold M_F is integrable.

Proof. We can easily see, on the basis of Lemma 8.1 that if we find the smooth solution of the linear equation (10) with respect to μ, then the system of equations

$$\begin{aligned} \dot{x}_i &= \frac{\partial F}{\partial y_i}(x,y,\lambda), \quad 1 \le i \le n \\ \dot{y}_j &= -\frac{\partial F}{\partial x_j}(x,y,\lambda), \quad 1 \le j \le n \\ \dot{\lambda}_l &= \mu_l(x,y,\lambda), \quad 1 \le l \le k \end{aligned}$$

is locally integrable over the surface $\{(x,y,\lambda); \frac{\partial F}{\partial\lambda_l}(x,y,\lambda) = 0, \quad 1 \le l \le k\}$. Q.E.D.

PROBLEM 8.1 *Find conditions to be posed on a smooth generating family $F : \mathbf{R}^{2n} \times \mathbf{R}^k \to \mathbf{R}$, so that the linear equation (10) has a smooth solution on C_F.*

Let $E_s(k,k)$ denote the space of $k \times k$ symmetric matrices of real numbers. For each integer $r \ge 0$ let Σ_r denote the subset of $E_s(k,k)$ consisting of all symmetric matrices of rank r. Then Σ_r is a submanifold of $E_s(k,k)$ of codimension $(k-r)(k-r+1)/2$. Now we have a well defined mapping of M_F into symmetric matrices $E_s(k,k)$. Because the projections $\pi_{T\mathbf{R}^{2n}} \mid_{\tilde{M}_F}: \tilde{M}_F \to M_F$ and $\pi \mid_{\tilde{M}_F}: \tilde{M}_F \to C_F$ are embeddings, then we can uniquely represent this mapping by $H : C_F \to E_s(k,k)$,

$$H(x,y,\lambda) \mid_{\{(x,y,\lambda)\in C_F\}} = \begin{pmatrix} \frac{\partial^2 F}{\partial\lambda_1\partial\lambda_1}(x,y,\lambda) & \dots & \frac{\partial^2 F}{\partial\lambda_1\partial\lambda_k}(x,y,\lambda) \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 F}{\partial\lambda_k\partial\lambda_1}(x,y,\lambda) & \dots & \frac{\partial^2 F}{\partial\lambda_k\partial\lambda_k}(x,y,\lambda) \end{pmatrix} \tag{13}$$

DEFINITION 8.2 *An implicit Hamiltonian system $M_F \subset T\mathbf{R}^{2n} \equiv \mathbf{R}^{2n} \times \mathbf{R}^{2n}$, generated by generating family $F : \mathbf{R}^{2n} \times \mathbf{R}^k \to \mathbf{R}$ is called generic if the map $H : C_F \to E_s(k,k)$ is transversal to all Σ_r, $\quad r = 0,\dots,k-1$.*

Now we can formulate the main result of this section.

THEOREM 8.1 *The generic implicit Hamiltonian system $M \subset T\mathbf{R}^{2n}$ is integrable if and only if it satisfies the N-integrability condition.*

Proof. Proof of this theorem follows from the J. Mather's theorem (cf. [12], p. 185) concerning the smooth solvability of linear equations with symmetric matrix. Q.E.D.

Using Section 7 we can formulate the corresponding results concerning integrability of implicit Hamiltonian systems. Now our function-matrix $H\mid_{\{(x,y,\lambda)\in C_F\}}: C_F \to E_s(k,k)$ corresponds to matrix $A(x)$ in Proposition 7.4. So on the basis of Propositions 7.1 - 7.4 we get the result corresponding to Theorem 7.1.

THEOREM 8.2 *Let $F : (\mathbf{R}^{2n} \times \mathbf{R}^k, 0) \to \mathbf{R}$ be a real analytic function-germ. Let M_F generated by F fulfills the N-integrability condition. If $det(H\mid_{C_F})(x,y,\lambda)$ satisfies one of the conditions* **I** *or* **II** *in Proposition 7.4, then the germ at $(0,0)$ of M_F is integrable.*

The class of implicit Hamiltonian systems for which the N-integrability condition is also the sufficient condition is provided by the generating family germs of the form

$$F(x,y,\lambda) = f(x,y) + \sum_{i=1}^{k} \lambda_i f_i(x,y),$$

where $\tilde{f}(x,y) = (f_1(x,y),\ldots,f_k(x,y))$, $1 \le k \le$ is a nonsingular smooth map-germ $\mathbf{R}^{2n},0 \to \mathbf{R}^k,0$. Thus M_F, called the constrained Hamiltonian system, is a smooth Lagrangian submanifold germ over the submanifold $K = \tilde{f}^{-1}(0)$.

Now the N-integrability condition from the Lemma 8.2 becomes the system of equations.

$$\{\frac{\partial F}{\partial \lambda_i}, F\}\mid_{\tilde{f}^{-1}(0)} = 0, \quad 1 \le i \le k \tag{14}$$

This system is equivalent to the following one.

$$\begin{aligned} &\{f_i(x,y), f_j(x,y)\}\mid_{\tilde{f}^{-1}(0)} = 0, \quad 1 \le i,j \le k, \\ &\{f(x,y), f_l(x,y)\}\mid_{\tilde{f}^{-1}(0)} = 0, \quad 1 \le l \le k, \end{aligned}$$

which in fact defines K to be a coisotropic submanifold of $(\mathbf{R}^{2n},\omega)$, i.e. geometrically $T_qK \supset \{u \in T_q\mathbf{R}^{2n} \mid \omega(u,v) = 0, \forall_{v\in T_qK}\}$, and f restricts to those who are constant on the leaves of the characteristic filiation of K. Now we have the integrability result proved also in geometric way in ([13], Theorem 8.1).

COROLLARY 8.2 *Let M_F be a constrained Hamiltonian system over the submanifold $K \subset \mathbf{R}^{2n}$ and F fulfills the N-integrability condition. Then K is a coisotropic submanifold of $(\mathbf{R}^{2n},\omega)$ and M_F is integrable.*

9 Integrability on submanifolds

In the preceding sections we studied the sufficient conditions on the existence of smooth solutions of Linear Equations which imply the integrability of submanifolds M of the tangent bundle $T\mathbf{R}^n$, where the resultant family of integral curves $(\gamma(t), \gamma'(t))$ were smooth on the whole manifold M. However according to the definition of integrability, the family of integral curves need not to be smooth on the whole manifold M. In this section we observe the cases where there exist no smooth family of integral curves on the whole manifold M but there exists a submanifold N of M on which there exists a smooth family of integral curves. Now we have the basic Lemma.

LEMMA 9.1 *Let* $g : \mathbf{R}^m \to \mathbf{R}^n$ *be a smooth map tangent to a smooth map* $f : \mathbf{R}^m \to \mathbf{R}^n$. *If there exists a submanifold* S *of* $\mathbf{R}^m$ *such that the linear equation*

$$\begin{pmatrix} \frac{\partial f_1}{\partial u_1}(u) & \dots & \frac{\partial f_1}{\partial u_m}(u) \\ \frac{\partial f_2}{\partial u_1}(u) & \dots & \frac{\partial f_2}{\partial u_m}(u) \\ \dots & \dots & \dots \\ \frac{\partial f_n}{\partial u_1}(u) & \dots & \frac{\partial f_n}{\partial u_m}(u) \end{pmatrix} \begin{pmatrix} a_1(u) \\ a_2(u) \\ \dots \\ a_m(u) \end{pmatrix} = \begin{pmatrix} g_1(u) \\ g_2(u) \\ \dots \\ g_n(u) \end{pmatrix}, \qquad u \in S$$

has a smooth solution tangent to S, *i.e*

$$(a_1(u), a_2(u), \dots, a_n(u)) \in T_u(S), \quad u \in S,$$

then the submanifold $N = (f, g)(S)$ *is an integrable submanifold of* $T\mathbf{R}^n$. *In particular, points of* $N = (f, g)(S)$ *are integrable points of* $M = (f, g)(\mathbf{R}^m)$.

Proof Since the smooth solution

$$(a_1(u), a_2(u), \dots, a_n(u)) \in T_u(S), \quad u \in S,$$

is a tangent vector field of S, for any point

$$(x, x') = (f, g)(u) \in N = (f, g)(S),$$

there exists an integral curve

$$\alpha : (-\epsilon, \epsilon) \to S \quad \text{such that} \quad \alpha'(t) = (a_1(\alpha(t)), \dots, a_n(\alpha(t))) \quad \text{and} \quad \alpha(0) = u.$$

Then the curve

$$\gamma : (-\epsilon, \epsilon) \to \mathbf{R}^n, \qquad \gamma(t) = f(\alpha(t))$$

satisfies

$$(\gamma(t), \gamma'(t)) \in N = (f, g)(S), \qquad \text{and} \quad (\gamma(0), \gamma'(0)) = (x, x') = (f, g)(u).$$

Thus, the point $(x, x') = (f, g)(u) \in N = (f, g)(S)$ is an integrable point of $N = (f, g)(S)$ and hence $N = (f, g)(S)$ is an integrable submanifold of $T\mathbf{R}^n$. *Q.E.D.*

Corollary 9.1 *Let* $g : \mathbf{R}^m \to \mathbf{R}^n$ *be a smooth map tangent to a smooth map* $f : \mathbf{R}^m \to \mathbf{R}^n$. *If there exists an embedding* $\phi : \mathbf{R}^k \to \mathbf{R}^m$ *such that the linear equation*

$$J(f \circ \phi)(v)b(v) = g(\phi(v)), \qquad v \in \mathbf{R}^k$$

has a smooth solution $b(v)$, *then* $N = ((f,g) \circ \phi)(\mathbf{R}^k)$ *is an integrable submanifold of* $T\mathbf{R}^n$. *In particular, points of* $N = ((f,g)\circ\phi)(\mathbf{R}^k)$ *are integrable points of* $M = (f,g)(\mathbf{R}^m)$.

Proof Set

$$S = \phi(\mathbf{R}^k) \qquad \text{and} \qquad a(\phi(v)) = J\phi(b(v))$$

and apply Fundamental Lemma 9.1 to S and $a(u = \phi(v))$. *Q.E.D.*

Now we describe a type of examples which are the generalization of Example 6.2.
Let M be the image of an embedding of $\mathbf{R}^m$:

$$M = (f,g)(\mathbf{R}^m), \qquad (f,g) : \mathbf{R}^m \to T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n \quad \text{an embedding.}$$

Suppose the Jacobian matrix $Jf(0)$ at $0 \in \mathbf{R}^m$ of f has rank r:

$$\mathrm{rank} Jf(0) = r.$$

Then there exists a local coordinate system in a neighborhood of the origin of $\mathbf{R}^m$

$$(v,w) = (v_1, \ldots, v_r, w_{r+1}, \ldots, w_m)$$

such that

$$f(v,w) = (v_1, \ldots, v_r, f_{r+1}(v,w), \ldots, f_n(v,w)).$$

Let us denote

$$\begin{aligned}
\mathcal{E}_{v,w} &= \text{the ring of } C^\infty \text{ function germs at } (u,v) = (0,0) \\
&\qquad\qquad \text{of the variables } v_1, \ldots, v_r, w_{r+1}, \ldots, w_m, \\
\mathcal{E}_w &= \text{the ring of } C^\infty \text{ function germs at } w = 0 \\
&\qquad\qquad \text{of the variables } w_{r+1}, \ldots, w_m, \\
\mathcal{M}_w &= \text{the maximal ideal of } \mathcal{E}_w
\end{aligned}$$

Theorem 9.1 *Let* $M = (f,g)(\mathbf{R}^m)$ *be the image of the germ at* $0 \in \mathbf{R}^m$ *of an embedding* $(f,g) : \mathbf{R}^m \to T\mathbf{R}^n$ *satisfying* N*-integrability condition. Let*

$$\mathrm{rank} Jf(0) = r$$

and let $(v,w) = (v_1, \ldots, v_r, w_{r+1}, \ldots, w_m)$ *be a local coordinate system in a neighborhood of the origin of* $\mathbf{R}^m$ *such that*

$$f(v,w) = (v_1, \ldots, v_r, f_{r+1}(v,w), \ldots, f_n(v,w)).$$

Let

$$S = \{(v, w) \in \mathbf{R}^m \mid w_{r+1} = w_{r+2} = \cdots = w_m = 0\} = \underline{R}^r \times \{0\}.$$

If

$$f_{r+1}(v, w), \ldots, f_n(v, w) \in {\mathcal{M}_w}^2 \mathcal{E}_{v,w},$$

then $N = (f, g)(S)$ *is an integrable submanifold of* $T\mathbf{R}^n$.

Proof Since

$$f_{r+1}(v, w), \ldots, f_n(v, w) \in {\mathcal{M}_w}^2 \mathcal{E}_{v,w},$$

we see that the Jacobian matrix $Jf(v, 0)$ has the form

$$Jf(v, 0) = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix},$$

where I_r is the identity matrix of size r, and the linear equation has the form

$$g(v, 0) = \begin{pmatrix} g_1(v, 0) \\ \vdots \\ g_n(v, 0) \end{pmatrix} = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} \begin{pmatrix} a_1(v, 0) \\ \vdots \\ a_m(v, 0) \end{pmatrix}.$$

Therefore we have

$$g(v, 0) = \begin{pmatrix} g_1(v, 0) \\ \vdots \\ g_r(v, 0) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} \begin{pmatrix} g_1(v, 0) \\ \vdots \\ g_r(v, 0) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = Jf(v, 0) \begin{pmatrix} g_1(v, 0) \\ \vdots \\ g_r(v, 0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus the linear equation

$$g(u) = Jf(u)a(u), \qquad u \in S = \{u = (v, 0)\}$$

has a smooth solution

$$(a_1(v, 0), \ldots, a_n(u, 0)) = (g_1(v, 0), \ldots, g_r(v, 0), 0, \ldots, 0)$$

tangent to S. Therefore, from Fundamental Lemma 9.1, $N = (f, g)(S)$ is an integrable submanifold of $T\mathbf{R}^n$. *Q.E.D.*

Now we consider the relation between integral curves at regular points of $\pi_1 \mid M \to \mathbf{R}^n$ and the integral curves on the submanifold S given by Theorem 9.1 .

Remark 9.1 *Let us define*

$$\Sigma = \{(v,w) \in \mathbf{R}^m \quad | \quad \mathrm{rank} Jf(v,w) < \min\{m,n\} \quad \}.$$

Then we have

$$S \quad \subset \quad \Sigma.$$

Since all the points

$$(x,x') \in M - (f,g)(\Sigma)$$

are regular points of the projection

$$\pi_1 : T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n, \qquad \pi_1(x,x') = x,$$

from Lemma 2.2, $M - (f,g)(\Sigma)$ *is an integrable submanifold of* $T\mathbf{R}^n$*. Concerning the integrability of points of the set* $(f,g)(\Sigma) - N$ *, we need more precise observation, which will be done in the forthcoming paper on Stratification of singular point sets and N-Integrability conditions.*

Remark 9.2 *Let* M *be the image of an embedding* $(f,g) : \mathbf{R}^m \to T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$*. Moreover suppose that* $f : \mathbf{R}^m \to \mathbf{R}^n$ *be of the form*

$$f(v,w) = (v_1, \ldots, v_r, f_{r+1}(v,w), \ldots, f_n(v,w)).$$

In this situation, if

$$g_{r+1}(v,w), \ldots, g_n(v,w) \in \mathcal{M}_w \mathcal{E}_{v,w},$$

then $M = (f,g)(\mathbf{R}^m)$ *satisfies N-integrability condition. However if*

$$f_{r+1}(v,w), \ldots, f_n(v,w) \in {\mathcal{M}_w}^3 \mathcal{E}_{v,w}$$

and if

$$g_{r+1}(v,w), \ldots, g_n(v,w) \in \mathcal{M}_w \mathcal{E}_{v,w} - {\mathcal{M}_w}^2 \mathcal{E}_{v,w},$$

then the linear equation

$$g(v,w) = Jf(v,w)a(v,w)$$

has no smooth solution on the whole $\mathbf{R}^m$*, while it has a smooth solution* $b(v,0)$ *defined on* $S = \{(v,w) \in \mathbf{R}^m \mid w = 0\}$*;*

$$g(v,0) = Jf(v,0)b(v,0), \qquad (v,0) \in S,$$

and also a smooth solution $c(v,w)$ *on* $\mathbf{R}^m - \Sigma$*;*

$$g(v,w) = Jf(v,w)c(v,w), \qquad (v,w) \notin \Sigma,$$

$$\text{where} \quad \Sigma = \{(v,w) \in \mathbf{R}^m \quad | \quad \mathrm{rank} Jf(v,w) < \min\{m,n\} \quad \}.$$

Let us now observe the solution $c(v,w)$ *at points* (v,w)*, which are very close to* S*. The Jacobian matrix* $Jf(v,w)$ *has the form*

$$Jf(v,w) = \begin{pmatrix} I_r & O \\ \frac{\partial f_{r+i}}{\partial v_j} & \frac{\partial f_{r+i}}{\partial w_{r+k}} \end{pmatrix}_{1 \le i \le n-r, \quad 1 \le j \le r, \quad 1 \le k \le m-r},$$

Therefore the first r solutions $c_1(v,w),\ldots,c_r(v,w)$ *has the form*

$$c_1(v,w)=g_1(v,w), \quad \ldots, \quad c_r(v,w)=g_r(v,w).$$

The rest solutions $c_{r+1}(v,w),\ldots,c_m(v,w)$ *must satisfy the equality*

$$\begin{pmatrix} g_{r+1}(v,w) \\ \vdots \\ g_n(v,w) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_{r+1}}{\partial v_1} & \cdots & \frac{\partial f_{r+1}}{\partial v_r} & \frac{\partial f_{r+1}}{\partial w_{r+1}} & \cdots & \frac{\partial f_{r+1}}{\partial w_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial v_1} & \cdots & \frac{\partial f_n}{\partial v_r} & \frac{\partial f_n}{\partial w_{r+1}} & \cdots & \frac{\partial f_n}{\partial w_m} \end{pmatrix} \begin{pmatrix} g_1(v,w) \\ \vdots \\ g_r(v,w) \\ c_{r+1}(v,w) \\ \vdots \\ c_m(v,w) \end{pmatrix}.$$

Therefore they must satisfy the equality

$$\begin{pmatrix} g_{r+1}(v,w) \\ \vdots \\ g_n(v,w) \end{pmatrix} - \begin{pmatrix} \frac{\partial f_{r+1}}{\partial v_1} & \cdots & \frac{\partial f_{r+1}}{\partial v_r} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial v_1} & \cdots & \frac{\partial f_n}{\partial v_r} \end{pmatrix} \begin{pmatrix} g_1(v,w) \\ \vdots \\ g_r(v,w) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f_{r+1}}{\partial w_{r+1}} & \cdots & \frac{\partial f_{r+1}}{\partial w_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial w_{r+1}} & \cdots & \frac{\partial f_n}{\partial w_m} \end{pmatrix} \begin{pmatrix} c_{r+1}(v,w) \\ \vdots \\ c_m(v,w) \end{pmatrix}.$$

Since

$$f_{r+1}(v,w),\ldots,f_n(v,w) \in {\mathcal{M}_w}^3\mathcal{E}_{v,w},$$

and

$$g_{r+1}(v,w),\ldots,g_n(v,w) \in \mathcal{M}_w\mathcal{E}_{v,w} - {\mathcal{M}_w}^2\mathcal{E}_{v,w},$$

comparing the orders with respect to $w_{r+1},\ldots,w_m$ *of the both hand sides of the equation, we see*

$$\lim_{\|w\|\to 0} c_{r+i}(v,w)=\infty, \qquad i=1,\ldots,m-r.$$

On the other hand the solution $b(v,0)$ *has the form*

$$(b_1(v,0),\ldots,b_n(v,0))=(g_1(v,0),\ldots,g_r(v,0),0,\ldots,0).$$

Thus

$$\lim_{\|w\|\to 0} \angle(c(v,w),b(v,0))=\frac{\pi}{2},$$

where $\angle(\alpha,\beta)$ *denotes the angle of the vectors* α *and* β*.*

Thus we can never have a smooth solution of the linear equation

$$g(v,w)=Jf(v,w)a(v,w)$$

by glueing $c(v,w)$ *with* $b(v,0)$*.*

Now we investigate the codimension 1 singularities. For simplicity, we consider the case $m = n$. Let M be the image of an embedding of $\mathbf{R}^m$:

$$M = (f, g)(\mathbf{R}^m), \qquad (f, g) : \mathbf{R}^m \to T\mathbf{R}^m = \mathbf{R}^m \times \mathbf{R}^m \quad \text{an embedding.}$$

Suppose the Jacobian matrix $Jf(0)$ at $0 \in \mathbf{R}^m$ of f has rank $m-1$:

$$\mathrm{rank} Jf(0) = m - 1.$$

Then there exists a local coordinate system in a neighborhood of the origin of $\mathbf{R}^m$

$$(v, w) = (v_1, \dots, v_{m-1}, w_m)$$

such that

$$f(v, w) = (v_1, \dots, v_{m-1}, f_m(v, w)).$$

Theorem 9.2 *Let $M = (f, g)(\mathbf{R}^m)$ be the image of the germ at $0 \in \mathbf{R}^m$ of an embedding $(f, g) : \mathbf{R}^m \to T\mathbf{R}^m$ satisfying N-integrability condition. Let*

$$\mathrm{rank} Jf(0) = m - 1$$

and let $(v, w) = (v_1, \dots, v_{m-1}, w_m)$ be a local coordinate system in a neighborhood of the origin of $\mathbf{R}^m$ such that

$$f(v, w) = (v_1, \dots, v_{m-1}, f_m(v, w)).$$

Suppose that

$$\det Jf(v, w) = \frac{\partial f_m}{\partial w_m}(v, w)$$

has the form

$$\det Jf(v, w) = (h_1(v, w))^k h_2(v, w) \qquad \text{with} \quad \frac{\partial h_1}{\partial w_m}(0, 0) \neq 0.$$

Set

$$S = \{(v, w) \in \mathbf{R}^m \quad | \quad h_1(v, w) = 0\}, \qquad N = (f, g)(S).$$

Then N is an integrable submanifold of $T\mathbf{R}^m$.

Proof Consider the linear equation

$$g(v, w) = Jf(v, w) a(v, w), \qquad (v, w) \in S.$$

We are going to construct a smooth solution $a(v, w)$, $(v, w) \in S$ of the equation tangent to S. Then from Fundamental Lemma 9.1, we see that S is an integrable submanifold of $T\mathbf{R}^m$.

Since $Jf(v, w)$ has the form

$$Jf(v, w) = \begin{pmatrix} I_{m-1} & O \\ \frac{\partial f_m}{\partial v_j}(v, w) & (h_1(v, w))^k h_2(v, w) \end{pmatrix},$$

we see that

$$Jf(v,w)=\begin{pmatrix} I_{m-1} & O \\ \frac{\partial f_m}{\partial v_j}(v,w) & 0 \end{pmatrix}, \qquad \forall (v,w)\in S.$$

Therefore first of all, $g_m(v,w)$ must satisfy the equality

$$g_m(v,w)=\sum_{j=1}^{m-1}\frac{\partial f_m}{\partial v_j}(v,w)g_j(v,w), \qquad \forall (v,w)\in S.$$

And if we choose $a_1(v,w),\ldots,a_{m-1}(v,w)$ as

$$a_1(v,w)=g_1(v,w),\ldots,a_{m-1}(v,w)=g_{m-1}(v,w),$$

then for any function $a_m(v,w)$,

$$a(v,w)=(a_1(v,w),\ldots,a_{m-1}(v,w),a_m(v,w))=(g_1(v,w),\ldots,g_{m-1}(v,w),a_m(v,w))$$

is a solution of the equation

$$g(v,w)=Jf(v,w)a(v,w), \qquad (v,w)\in S.$$

Now we choose $a_m(v,w)$ so that the vector field $a(v,w)$ is tangent to S. Consider the gradient vector field of $h_1(v,w)$:

$$\mathrm{grad}h_1(v,w)=(\frac{\partial h_1}{\partial v_1}(v,w),\ldots,\frac{\partial h_1}{\partial v_{m-1}}(v,w),\frac{\partial h_1}{\partial w_m}(v,w)).$$

Choose a function $a_m(v,w)$, $(u,v)\in S$, so that the vector field

$$(g_1(v,w),\ldots,g_{m-1}(v,w),a_m(v,w))$$

is orthogonal to $\mathrm{grad}h_1(v,w)$:

$$g_1(v,w)\frac{\partial h_1}{\partial v_1}(v,w)+\cdots+g_{m-1}(v,w)\frac{\partial h_1}{\partial v_{m-1}}(v,w)+\frac{\partial h_1}{\partial w_m}(v,w)a_m(v,w)=0.$$

From the assumption that $\frac{\partial h_1}{\partial w_m}(0,0)\neq 0$, we have such $a_m(v,w)$, namely

$$a_m(v,w)=\frac{-1}{\frac{\partial h_1}{\partial w_m}(v,w)}\sum_{j=1}^{m-1}g_j(v,w)\frac{\partial h_1}{\partial v_j}(v,w).$$

Then, with the coefficient function $a_m(v,w)$, the vector field

$$(a_1(v,w),\ldots,a_m(v,w))=(g_1(v,w),\ldots,g_{m-1}(v,w),a_m(v,w)) \qquad (v,w)\in S$$

is tangent to S. In fact $(a_1(v,w),\ldots,a_m(v,w))$ is orthogonal to the gradient vector field $\mathrm{grad}h_1(v,w)$ and $h_1(v,w)$ is the defining function of S. Thus $a(v,w)$, $(v,w)\in S$ is a smooth solution of the linear equation

$$g(v,w)=Jf(v,w)a(v,w), \qquad (v,w)\in S.$$

tangent to S. Now, from Fundamental Lemma 9.1, we see that S is an integrable submanifold of $T\mathbf{R}^m$. *Q.E.D.*

In the same way as Theorem 9.2, we can prove the following

Corollary 9.2 *Let* $M = (f,g)(\mathbf{R}^m)$ *be the image of the germ at* $0 \in \mathbf{R}^m$ *of an embedding* $(f,g) : \mathbf{R}^m \to T\mathbf{R}^m$ *satisfying* N*-integrability condition. Let*

$$\mathrm{rank} Jf(0) = m-1$$

and let $(v,w) = (v_1, \ldots, v_{m-1}, w_m)$ *be a local coordinate system in a neighborhood of the origin of* $\mathbf{R}^m$ *such that*

$$f(v,w) = (v_1, \ldots, v_{m-1}, f_m(v,w)).$$

Suppose that

$$\det Jf(v,w) = \frac{\partial f_m}{\partial w_m}(v,w)$$

has the form

$$\det Jf(v,w) = h_1(v,w)^{k_1} h_2(v,w)^{k_2} \cdots h_s(v,w)^{k_s} h(v,w)$$

$$\textit{with} \quad \frac{\partial h_i}{\partial w_m}(0,0) \neq 0, \quad i = 1, \ldots, s.$$

Set

$$S_i = \{(v,w) \in \mathbf{R}^m \quad | \quad h_i(v,w) = 0\}, \quad \textit{and} \quad N_i = (f,g)(S_i).$$

Then N_i*'s are integrable submanifolds of* $T\mathbf{R}^m$.

We can prove similar results for the cases $m \neq n$. However we leave it to the forthcoming paper.

10 Non-integrable points are generically isolated

In this section we will investigate the non integrable points. We prove that *non-transversal points*, i.e. the points where $Jf : \mathbf{R}^m \to E(n,m)$ is not transversal to S_r, are generically isolated, which implies that *non-integrable points are generically isolated.*

Here, in this section, the *genericity* is in the strongest sense of Tougeron [20]:

Definition 10.1 *We say that the non-transversal points of* $M = (f,g)(\mathbf{R}^m)$ *are generically isolated if there exists an* ∞ *codimensional subset* $\Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n)$ *of the space* $C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ *of* C^∞ *mappings of* $\mathbf{R}^m$ *into* $\mathbf{R}^n$ *such that for any* $f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n) - \Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n)$ *non-transversal points are isolated.*

Let us fix the notation now. Set

$$\begin{aligned}
C^\infty(\mathbf{R}^m, T\mathbf{R}^n) &= \{(f,g) : \mathbf{R}^m \to \mathbf{R}^n \times \mathbf{R}^n = T\mathbf{R}^n \mid C^\infty \text{ mapping}\} \\
&= C^\infty(\mathbf{R}^m, \mathbf{R}^n) \times C^\infty(\mathbf{R}^m, \mathbf{R}^n) \\
Emb(\mathbf{R}^m, T\mathbf{R}^n) &= \{(f,g) : \mathbf{R}^m \to \mathbf{R}^n \times \mathbf{R}^n = T\mathbf{R}^n \mid \quad \text{embedding}\}, \\
\mathcal{N}(\mathbf{R}^m, T\mathbf{R}^n) &= \{(f,g) \in C^\infty(\mathbf{R}^m, T\mathbf{R}^n) \mid (f,g) \quad \text{satisfies N-integrability} \\
&\qquad \text{condition} \quad g(u) \in Jf(u)(T_u(\mathbf{R}^n)) \quad \forall u \in \mathbf{R}^m\} \\
\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) &= Emb(\mathbf{R}^m, T\mathbf{R}^n) \cap \mathcal{N}(\mathbf{R}^m, T\mathbf{R}^n),
\end{aligned}$$

where all these spaces are endowed with Whitney C^∞ topology, and $Jf(u)$ is the Jacobian matrix of f at $u \in \mathbf{R}^m$.

Definition 10.2 *Let $(f,g)\in \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n)$. A point $(x,\dot{x}) = (f(u), g(u)) \in M = (f,g)(\mathbf{R}^m)$ is a transversal point of M if $Jf : \mathbf{R}^m \to E(n,m)$ is transversal to S_r at $u \in \mathbf{R}^m$ for $r = \mathrm{rank} Jf(u)$, equivalently $j^1(\pi_1 \mid_M) : M \to J^1(M,\mathbf{R}^n)$ is transversal to*

$$S_r(M,\mathbf{R}^n) = \{j^1h(v) \in J^1(M,\mathbf{R}^n) \mid \mathrm{rank}Jh(v) = r\},$$

where $\pi_1 : T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is the projection defined by $\pi_1(x,\dot{x}) = x$.

We are interested in the structure of the non-generic (in the sense of the previous sections) points and to investigate how they are located in M. We prove the following main theorems.

Theorem 10.1 *There exists an ∞ codimensional subset $\Sigma_\infty\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n)$ of $\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n)$ such that for any embedding*

$$(f,g) \in \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) - \Sigma_\infty\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n)$$

non-transversal points of $M = (f,g)(\mathbf{R}^m)$ are isolated.

Combining this theorem with J. Mather's theorem, we obtain

Theorem 10.2 *Let $\Sigma_\infty\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n)$ be the ∞ codimensional subset of $\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n)$ given in Theorem 10.1. Then for any*

$$(f,g) \in \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) - \Sigma_\infty\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n)$$

non-integrable points of $M = (f,g)(\mathbf{R}^m)$ are isolated.

Remark 10.1 **Remark** *Let*

$$\Sigma_{tr}\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) = \{(f,g) \in \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) \mid Jf : U \to E(n,m) \text{ is not transversal to } S_r \text{ for some } r\}.$$

Then we see

$$\begin{aligned} \Sigma_\infty\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) &\subset \Sigma_{tr}\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n), \\ \mathrm{codim}\Sigma_{tr}\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) \text{ in } \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) &< \infty, \\ \mathrm{codim}\Sigma_\infty\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) \text{ in } \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) &= \infty. \end{aligned}$$

This means that even if $M = (f,g)(\mathbf{R}^m)$ is not a generic submanifold in the sense of section 3, non-transversal points and non-integrable points of $M = (f,g)(\mathbf{R}^m)$ are isolated points for a very generic (f,g), i.e. for any $(f,g) \in \Sigma_{tr}\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) - \Sigma_\infty\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n)$.

For the strong genericity result we need an appropriate transversality theorem. Let us denote

$$\begin{aligned} J^r(m,n) &= \{j^rf(0) \mid f \in C^\infty(\mathbf{R}^m,\mathbf{R}^n) \text{ and } f(0)=0\} \\ J^r(\mathbf{R}^m,\mathbf{R}^n) &= \{j^rf(u) \mid f \in C^\infty(\mathbf{R}^m,\mathbf{R}^n)), \quad u \in \mathbf{R}^m\}. \end{aligned}$$

For positive integers $s > r > 0$, we define the canonical projections

$$\pi^s_r : J^s(m,n) \to J^r(m,n), \quad \pi^s_r(j^sf(0)) = j^rf(0).$$

Theorem 10.3 *(Transversality) Let W be a semi-algebraic subset of $J^r(m,n)$ and let $X_1, \ldots, X_\ell$ be a finite number of semi-algebraic submanifolds of $J^k(\mathbf{R}^m, \mathbf{R}^n)$. Then there exists a closed semi-algebraic subset*

$$\Sigma_W \quad \text{of} \quad (\pi_r^{r+k+1})^{-1}(W) \quad \text{having codimension} \quad \geq 1$$

such that for any C^∞ mapping $f : \mathbf{R}^m \to \mathbf{R}^n$ with

$$j^{r+k+1}(0) \in (\pi_r^{r+k+1})^{-1}(W) - \Sigma_W,$$

there exists a neighborhood U of the origin 0 of $\mathbf{R}^m$ such that $j^r f : U - \{0\} \to J^k(\mathbf{R}^m, \mathbf{R}^n)$ is transversal to $X_1, \ldots, X_\ell$ at every point $u \in U$ except at the origin $0 \in \mathbf{R}^m$

For the proof we refer to the more general version of Theorem 10.3 which is proved in [7]. (Theorem 1 in [7], p. 229)

10.1 Proof of Theorem 10.1

Let us identify $J^1(m,n)$ with $E(n.,m)$ and set

$$S_r(\mathbf{R}^m, \mathbf{R}^n) = \{j^1 f(u) \in J^1(\mathbf{R}^m, \mathbf{R}^n) \mid \mathrm{rank} Jf(u) = r\}$$

$$= \mathbf{R}^m \times \mathbf{R}^n \times S_r \subset \mathbf{R}^m \times \mathbf{R}^n \times J^1(m,n) = J^1(\mathbf{R}^m, \mathbf{R}^n).$$

For $(u_0, x_0) \in \mathbf{R}^m \times \mathbf{R}^n$, set

$$C^\infty(\mathbf{R}^m, \mathbf{R}^n; u_0, x_0) = \{f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)) \mid f(u_0) = x_0\}.$$

We prove Main Theorem 10.1 using the following result

Theorem 10.4 *There exists an infinite codimensional subset $\Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n; u_0, x_0)$ of $C^\infty(\mathbf{R}^m, \mathbf{R}^n; u_0, x_0)$ such that for any C^∞ mapping $f : \mathbf{R}^m \to \mathbf{R}^n$ with*

$$f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n; u_0, x_0) - \Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n; u_0, x_0)$$

there exists a neighborhood $V(u_0)$ of u_0 in $\mathbf{R}^m$ such that $j^1 f : V(u_0) - \{u_0\} \to J^1(\mathbf{R}^m, \mathbf{R}^n)$ is transversal to $S_r(\mathbf{R}^m, \mathbf{R}^n)$ for $\forall r = 0, \ldots, \min(m,n)$ at every point $u \in V(u_0) - \{u_0\}$ except at u_0.

Proof of Theorem 10.1 Set

$$\begin{aligned}
\Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n) &= \cup_{(u,x) \in \mathbf{R}^m \times \mathbf{R}^n} \Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n; u, x) \subset C^\infty(\mathbf{R}^m, \mathbf{R}^n) \\
\Sigma_\infty(\mathbf{R}^m, T\mathbf{R}^n) &= \{(f,g) \in C^\infty(\mathbf{R}^m, T\mathbf{R}^n) \mid f \in \Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n)\} \\
&= \Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n) \times C^\infty(\mathbf{R}^m, \mathbf{R}^n) \\
\Sigma_\infty \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) &= \Sigma_\infty(\mathbf{R}^m, T\mathbf{R}^n) \cap \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n),
\end{aligned}$$

where we note that $T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$ and hence that

$$C^\infty(\mathbf{R}^m, T\mathbf{R}^n) = C^\infty(\mathbf{R}^m, \mathbf{R}^n) \times C^\infty(\mathbf{R}^m, \mathbf{R}^n).$$

We claim that $\Sigma_\infty \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n)$ is the required one.

It is easy to see that it satisfies the condition:
if $(f,g) \in \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) - \Sigma_\infty \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n)$, then non-transversal points of $M = (f,g)(\mathbf{R}^m)$ are isolated.

Now we investigate the codimension of this set. Since $\Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n; u_0, x_0)$ is ∞ codimensional in $C^\infty(\mathbf{R}^m, \mathbf{R}^n; u_0, x_0)$, so is $\Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n)$ in $C^\infty(\mathbf{R}^m, \mathbf{R}^n)$. Therefore

$$\mathrm{codim}\Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n) \times C^\infty(\mathbf{R}^m, \mathbf{R}^n) \quad \text{in} \quad C^\infty(\mathbf{R}^m, T\mathbf{R}^n) \quad \text{is equal to} \quad \infty.$$

Recall that we defined

$$\mathcal{N}(\mathbf{R}^m, T\mathbf{R}^n) = \{(f,g) \in C^\infty(\mathbf{R}^m, T\mathbf{R}^n) \mid g(u) \in Jf(u)(\mathbf{R}^m), \quad \forall u \in \mathbf{R}^m\}.$$

and

$$\mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) = Emb(\mathbf{R}^m, T\mathbf{R}^n) \cap \mathcal{N}(\mathbf{R}^m, T\mathbf{R}^n).$$

Since

$$\Sigma_\infty(\mathbf{R}^m, T\mathbf{R}^n) = \Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n) \times C^\infty(\mathbf{R}^m, \mathbf{R}^n)$$

and

$$\mathrm{codim}\Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n) \times C^\infty(\mathbf{R}^m, \mathbf{R}^n) \quad \text{in} \quad C^\infty(\mathbf{R}^m, T\mathbf{R}^n) \quad \text{is equal to} \quad \infty,$$

we have

$$\mathrm{codim}\Sigma_\infty(\mathbf{R}^m, T\mathbf{R}^n) \cap \mathcal{N}(\mathbf{R}^m, T\mathbf{R}^n) \quad \text{in} \quad \mathcal{N}(\mathbf{R}^m, T\mathbf{R}^n) \quad \text{is equal to} \quad \infty.$$

Since

$$Emb(\mathbf{R}^m, T\mathbf{R}^n) \quad \text{is open in} \quad C^\infty(\mathbf{R}^m, T\mathbf{R}^n),$$

we see that

$$\begin{aligned} &\mathrm{codim}\Sigma_\infty(\mathbf{R}^m, T\mathbf{R}^n) \cap \mathcal{N}(\mathbf{R}^m, T\mathbf{R}^n) \cap Emb(\mathbf{R}^m, T\mathbf{R}^n) \\ &\text{in} \quad \mathcal{N}(\mathbf{R}^m, T\mathbf{R}^n) \cap Emb(\mathbf{R}^m, T\mathbf{R}^n) \quad \text{is equal to} \quad \infty. \end{aligned}$$

Since

$$\begin{aligned} \Sigma_\infty \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) &= \Sigma_\infty(\mathbf{R}^m, T\mathbf{R}^n) \cap \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) \\ &= \Sigma_\infty(\mathbf{R}^m, T\mathbf{R}^n) \cap \mathcal{N}(\mathbf{R}^m, T\mathbf{R}^n) \cap Emb(\mathbf{R}^m, T\mathbf{R}^n), \\ \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) &= \mathcal{N}(\mathbf{R}^m, T\mathbf{R}^n) \cap Emb(\mathbf{R}^m, T\mathbf{R}^n) \end{aligned}$$

we see that

$$\mathrm{codim}\Sigma_\infty \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) \quad \text{in} \quad \mathcal{N}Int(\mathbf{R}^m, T\mathbf{R}^n) \quad \text{is equal to} \quad \infty.$$

This completes the proof of Theorem 10.1.

Proof of Theorem 10.4

It suffices to prove Theorem 10.4 for $(u_0, x_0) = (0, 0)$, so we reformulate it in an accessible way.

Theorem 10.4 There exists an infinite codimensional subset $\Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n; 0, 0)$ of $C^\infty(\mathbf{R}^m, \mathbf{R}^n; 0, 0)$ such that for any C^∞ mapping $f : \mathbf{R}^m \to \mathbf{R}^n$ with

$$f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n; 0, 0) - \Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n; 0, 0)$$

there exists a neighborhood $V(0)$ of 0 in $\mathbf{R}^m$ such that $j^1 f : V(0) - \{0\} \to J^1(\mathbf{R}^m, \mathbf{R}^n)$ is transversal to $S_r(\mathbf{R}^m, \mathbf{R}^n)$ for $\forall r = 0, \ldots, \min(m, n)$ at every point $u \in V(0) - \{0\}$ except at the origin 0.

Proof. First we apply Theorem 10.3 to

$$W = W_1 = J^1(m, n) \quad \text{and}$$

$$X_r = S_r(\mathbf{R}^m, \mathbf{R}^n) \subset J^1(\mathbf{R}^m, \mathbf{R}^n), \quad r = 0, \ldots, \min(m, n).$$

Then there exists a closed semi-algebraic subset

$$\Sigma_1 \quad \text{of} \quad (\pi_1^{1+1+1})^{-1}(W) = J^3(m, n) \quad \text{having codimension} \quad \geq 1$$

such that for any C^∞ mapping $f : \mathbf{R}^m \to \mathbf{R}^n$ with

$$j^3(0) \in J^3(m, n) - \Sigma_1,$$

there exists a neighborhood U of the origin 0 of $\mathbf{R}^m$ such that

$$j^1 f : U - \{0\} \to J^1(\mathbf{R}^m, \mathbf{R}^n) \quad \text{is transversal to} \quad S_r(\mathbf{R}^m, \mathbf{R}^n), \quad \forall r.$$

.

Next we apply Theorem 10.3 to

$$W = \Sigma_1 \subset J^3(m, n) \quad \text{and}$$

$$X_r = S_r(\mathbf{R}^m, \mathbf{R}^n) \subset J^1(\mathbf{R}^m, \mathbf{R}^n), \quad r = 0, \ldots, \min(m, n).$$

Then, again there exists a closed semi-algebraic subset

$$\Sigma_2 \quad \text{of} \quad (\pi_3^{3+1+1})^{-1}(\Sigma_1) \subset J^5(m, n)$$

such that

$$\text{codimension of} \quad \Sigma_2 \quad \text{in} \quad (\pi_3^5)^{-1}(\Sigma_1) \quad \geq 1$$
$$\text{codimension of} \quad \Sigma_2 \quad \text{in} \quad J^5(m, n) \geq 2$$

and such that for any C^∞ mapping $f : \mathbf{R}^m \to \mathbf{R}^n$ with

$$j^5(0) \in J^5(m, n) - \Sigma_2,$$

there exists a neighborhood U of the origin 0 of $\mathbf{R}^m$ such that

$$j^1 f : U - \{0\} \to J^1(\mathbf{R}^m, \mathbf{R}^n) \quad \text{is transversal to} \quad S_r(\mathbf{R}^m, \mathbf{R}^n), \quad \forall r = 0.$$

In this way, we can prove inductively that there exists a closed semi-algebraic subset

$$\Sigma_k \quad \text{of} \quad J^{2k+1}(m,n) \quad \text{with codimension} \geq k$$

such that for any C^∞ mapping $f : \mathbf{R}^m \to \mathbf{R}^n$ with

$$j^{2k+1}(0) \in J^{2k+1}(m,n) - \Sigma_k,$$

there exists a neighborhood U of the origin 0 of $\mathbf{R}^m$ such that

$$j^1 f : U - \{0\} \to J^1(\mathbf{R}^m, \mathbf{R}^n) \quad \text{is transversal to} \quad S_r(\mathbf{R}^m, \mathbf{R}^n), \quad \forall r.$$

Set

$$\Sigma_k(\mathbf{R}^m, \mathbf{R}^n; 0, 0) = \{f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n; 0, 0) \mid j^{2k+1}(0) \in \Sigma_k\},$$
$$\Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n; 0, 0) = \cap_{k=1}^\infty \Sigma_k(\mathbf{R}^m, \mathbf{R}^n; 0, 0).$$

Then $\Sigma_\infty(\mathbf{R}^m, \mathbf{R}^n; 0, 0)$ satisfies the condition required in Theorem 4.
This completes the proof of Theorem 10.4 and hence also the proof of Main Theorem 10.1. *Q.E.D.*

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