HOMOTOPY EQUIVALENCE AND DONALDSON INVARIANTS WHEN $b^+ = 1$. I: COBORDISMS OF MODULI SPACES AND CONTINUITY OF GLUING MAPS [1]

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Abstract. The present article is the first in a series whose ultimate goal is to prove the Kotschick-Morgan conjecture concerning the wall-crossing formula for the Donaldson invariants of a four-manifold with $b^+ = 1$. The conjecture asserts that, in essence, the wall-crossing terms due to changes in the metric depend at most on the homotopy type of the four-manifold and the degree of the invariant. Our principal interest in this conjecture is due to (i) the fact that its proof is expected to resemble that of an important intermediate step towards a proof of Witten's conjecture concerning the relation between Donaldson and Seiberg-Witten invariants [63], using PU(2) monopoles as described in [20], [17], and (ii) it affords us another venue in which to address some of the technical difficulties arising in our work on Witten's conjecture. The additional difficulties in the case of PU(2) monopoles are due to the considerably more complicated gluing theory, the presence of obstructions to deformation and gluing, and the need to consider links of positive-dimensional families of 'reducibles' even in the presence of 'simple type' assumptions. Consequently, it is extremely useful to examine the simpler case of the Kotschick-Morgan conjecture in parallel and separately take into account the difficulties that arise in the case of Witten's conjecture. Witten's conjecture should follow from a final step analogous to Göttsche's computation of the wall-crossing terms, assuming that the Kotschick-Morgan conjecture holds [27].

In this article we present the proof of the first half of a general gluing theorem endowed with several important enhancements not present in previous treatments due to Taubes, Donaldson, and Mrowka. We prove that the Uhlenbeck-compactified moduli spaces of instantons on S^4 for two nearby metrics, which *a priori* need only be cobordant, are actually homeomorphic and diffeomorphic on smooth strata. We also show that the gluing maps, with obstructions to deformation permitted, extend to maps on the closure of the gluing data which are continuous with respect to the Uhlenbeck topology. The second half of the proof of our general gluing theorem is proved in the sequel [14].

We note that Kronheimer and Mrowka have conjectured a relation between the Yang-Mills and Seiberg-Witten Floer homologies for certain three-manifolds Y [37], [39], [48] and they have a recently announced a program to prove this conjecture using moduli spaces of PU(2) monopoles over cylinders $Y \times \mathbb{R}$. One of the difficulties in their program, which concerns the contributions of Seiberg-Witten solutions in the presence of instanton bubbling, to some extent mirrors those in our four-manifold program discussed in [20], [21], [17], [18], [19]. Their work is expected to ultimately lead to a proof that all knots in S^3 have 'property P'.

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Contents

1. Introduction

The underlying problem which motivates the present work and its sequels [14], [15], [16] is to prove the strong form of the Kotschick-Morgan conjecture for the Donaldson invariants of a four-manifold with $b^+ = 1$ and $b^1 = 0$, namely Conjecture 6.2.2 in [36]. From work of Göttsche we know that this (in fact, even the weaker Conjecture 6.2.1 of [36]) allows one to completely determine the wall-crossing formula for such four-manifolds. Though the two problems are not directly related, the computational and analytical problems arising in attempts to prove the wall-crossing formula are analogous, but still significantly simpler, than those appearing in our concurrent program to prove Witten's conjecture [20], [17].

Our program to relate the Donaldson and Seiberg-Witten invariants [21], [20], [17], [18] requires a gluing theory, for PU(2) monopoles, possessing most of the properties of the gluing theory established here and in [14] for the easier case of anti-self-dual connections. Thus, we shall present the results we need in the considerably simpler case of anti-self-dual connections separately, so we can address elsewhere the difficulties peculiar to PU(2) monopoles [18]. The other purpose of the present article and its sequels is, of course, to prove the Kotschick-Morgan conjecture. That too bears on our program to prove the relation between Donaldson and Seiberg-Witten invariants, as it allows us to dispose of the technical difficulties in the general method underlying the expected proof of equivalence in an easier case. Our understanding of the work of Kronheimer and Mrowka is that similar difficulties — namely, possible contributions from lower-level 'reducibles' (Seiberg-Witten monopoles in the case of PU(2) monpoles) — afflict their program to relate the Yang-Mills and Seiberg-Witten Floer homologies (and thus prove property P for knots).

The principal difficulties arising in attempts to prove the Kotschick-Morgan conjecture are due to 'bubbling', a phenomenon which also accounts for the problems arising when one considers either PU(2) monopoles or pseudo-holomorphic curves in symplectic manifolds (of any dimension).

1.1. **Statement of results.** The main purpose of the present article and its companion [14] is to prove a general gluing theorem for anti-self-dual connections. Naturally, many statements of gluing theorems have appeared before in the literature [5], [8], [9], [49], [52], [54], [55], [56], so the reader may reasonably ask what is new and different about the present version. The gluing-theorem statements that are closest to those needed for a proof of the Kotschick-Morgan conjecture are given by Theorems 3.4.10 and 3.4.17 in [25]: both assertions are based on work of Taubes, principally [52], [54], [55], [56], though the assertions of [25] go beyond what has been already proved in or is easily adapted from the existing gluing-theory literature. In any event, the statement of the gluing theorem given in [25] is not sufficient to prove the Kotschick-Morgan conjecture; some of the additional requirements are briefly explained below in §1.3.

The present article is intended to bridge the gap between the gluing-theory literature cited above and two assertions of [25, Theorems 3.4.10 & 3.4.17], one implicit and one explicit, as we describe in §1.1.1 and §1.1.2 below.

1.1.1. *Diffeomorphisms between moduli spaces of instantons over S^4.* As is well known, the bundles $\mathrm{Gl}_\kappa^w(\Sigma,\lambda_0)$ of 'gluing data' [25, Eq. (3.4.9)], [25, pp. 250–253] — defining the domain of the gluing maps and determined by a choice of smooth stratum $\Sigma\subset\mathrm{Sym}^\ell(X)$ and small positive constant λ_0 (the maximum 'scale' size) — contain moduli spaces of 'centered' SU(2) connections on S^4, anti-self-dual with respect to its standard metric. Implicit in the statements of [25] is the assumption that Taubes' perturbation scheme can be used to deform almost g-anti-self-dual connections over X, produced by splicing the gluing data, to nearby g-anti-self-dual connections in such a way that the resulting gluing map is a homeomorphism onto an open subset of $M_\kappa^w(X,g)$. (We reserve the term 'splicing' for the process of cutting and pasting connections and the term 'gluing' for the composition of the splicing and the subsequent deformation to a family of anti-self-dual connections; see §3 for definitions of the bundles of gluing data.) However, in the situation considered in [25], the moduli spaces on S^4 are allowed to have arbitrary instanton number and in this case, the Taubes' gluing theory of [56] requires one to compose the splicing map with a certain homotopy of $\mathcal{B}_\kappa^{w,*}(X)$ in order to produce a glued-up family of anti-self-dual connections. The difficulty — a significant one — is due to the discrepancy between the conformally flat metric on S^4 and the non-flat metric on X. This means that the self-dual curvatures of the spliced connections will not satisfy the estimates required to perturb them to a family of anti-self-dual connections when the families of centered instantons over S^4 being spliced are non-compact, which is always the case for the full, centered instanton moduli space when the instanton number is greater than one.

For example, suppose we attempted to use a Taubes deformation scheme to directly establish a diffeomorphism from $M_\kappa(S^4,g_0)$ to $M_\kappa(S^4,g)$, where g_0 is the standard round metric on S^4 and g is a nearby metric; typically, such metrics arise by pulling back the metric on X near a splicing point x_0 via Gaussian coordinates, conformal dilation, and splicing with g_0 to give a smooth, almost-round metric on $S^4=(TX)_{x_0}\cup\{\infty\}$. One might naively expect that one could construct a diffeomorphism by solving the equation

$$P_+(g)(d_Aa+a\wedge a)=-P_+(g)F_A, \tag{1.1}$$

for a deformation $a\in L^2_k(S^4,\Lambda^1\otimes\mathfrak{g}_E)$, yielding an SU(2) connection $A+a$ over S^4 with $P_+(g)F_{A+a}=0$, for each SU(2) connection A over S^4 with $P_+(g_0)F_A=0$. If the points $[A]$ above are constrained to lie in a subset $\mathcal{U}\subset M_\kappa(S^4,g_0)$, then a solution to the anti-self-dual equation (1.1) is possible by the fixed-point method (as in [8], §7 in the present article, or

[56]), provided we have bounds

$$
\begin{aligned}
&\sup_{[A]\in\mathcal{U}} \|P_+(g)F_A\|_{L^p(S^4)} \le M, \\
&\sup_{[A]\in\mathcal{U}} \|P_+(g)F_A\|_{L^{\sharp,2}(S^4)} \le \varepsilon,
\end{aligned} \tag{1.2}
$$

where $M = M(\kappa, g) < \infty$ is a positive constant and $\varepsilon = \varepsilon(\kappa, g)$ is a sufficiently small positive constant; the 'critical exponent' Sobolev $L^{\sharp,2}$ norms are explained in [12]. The size of ε (and what it depends on) varies with the choice of norms used in the perturbation scheme, as does the precise choice of Sobolev norms: we would use $p = \infty$ in §7 (see our Proposition 7.6) while Taubes would use $p = 4$ in [56] (see his Lemma 5.1 and the surrounding discussion). The important point to note is that if $\mathcal{U}$ is a *compact* subset of $M_\kappa(S^4, g_0)$, then we can easily achieve the bounds (1.2) just by requiring g to be suitably close to g_0: indeed, this is exactly the situation in [5], [8], [52], and [54], where the subsets $\mathcal{U}$ are indeed be assumed to be compact with respect to the usual quotient L^2_k topology (for example, the moduli space of centered 1-instantons on S^4, which is a single point). However, if we take $\mathcal{U}$ to be either the full moduli space of κ-instantons on S^4 when $\kappa \ge 1$ or the moduli space of centered k-instantons on S^4 when $\kappa \ge 2$, the subset $\mathcal{U}$ is non-compact and uniform bounds such as (1.2) are impossible to achieve.

Taubes partially circumvents this problem in [56, §5] by showing that one can still homotope a family of connections in $\mathcal{B}^{w,*}_\kappa(X, g)$ with $\|F^+_A\|_{L^2} < \varepsilon$ to a family which does satisfy the required estimates. He can then perform the desired perturbation to a family of anti-self-dual connections; see [56, Lemma 5.1]. However, one no longer knows whether the resulting composite map is a diffeomorphism, or a homeomorphism from the closure of the gluing data. In [5], [8], [52], [54], the families of centered instantons on S^4 are assumed to be compact (the problem then disappears) either by supposing that the instanton number is one or by picking a fixed connection on S^4, that is, a single point in the moduli space over S^4.

The first alternative to employing homotopies, as in [56], might be to simply allow the metrics on S^4 to deviate slightly from the standard round metric of radius one, in such a way that the metrics near the north poles match the metric g on X where the instanton is to be spliced onto the background g-anti-self-dual connection on X. While this strategy would eliminate the problems with *a priori* estimates, the space of gluing data $\mathrm{Gl}^w_\kappa(\Sigma, \lambda_0)$ would not necessarily possess the fiber-bundle structure that seems to be a key ingredient in any attempt to prove the Kotschick-Morgan conjecture: the moduli spaces of instantons on S^4 would vary with the multi-sets $\{x_1, \dots, x_\ell\} \subset \Sigma$ and it is not clear that the space $\mathrm{Gl}^w_\kappa(\Sigma, \lambda_0)$ would be a locally-trivial fiber bundle over Σ as assumed, say, in [25, Eq. (3.4.9)] and in related applications such as [40], [50], [62], [64]. In our own approach to the intersection-theory problem, [15], [16], the existence of locally-trivial fiber-bundle structures plays a crucial role in the calculation of the wall-crossing terms. Furthermore, it is not clear how one could define differentials of the resulting gluing map without the benefit of local triviality.

The second alternative — the one we shall ultimately use — would be to try to show that moduli spaces of instantons on S^4, defined by the standard metric g_0 and a nearby metric g_0' are indeed diffeomorphic, despite the aforementioned difficulties. In practice, this is a much more delicate problem than it might appear and it is not implied (again, because the *a priori* estimates on the self-dual curvature components are too weak) by standard perturbation schemes, such as those described in [8], [56]. The instanton moduli spaces can be viewed as fibers of the projection map $\pi : \mathfrak{M}_\kappa(S^4) \to \mathrm{Met}(S^4)$, where $\mathfrak{M}_\kappa(S^4) \subset \mathcal{B}_\kappa(S^4) \times \mathrm{Met}(X)$ is the universal moduli space of anti-self-dual SU(2) connections on S^4 with second Chern number κ. Because the projection π is not proper, it is not clear that the fibers $\pi^{-1}(g_0)$ and $\pi^{-1}(g_0')$ are necessarily diffeomorphic, no matter how close the metrics g_0 and g_0': by abstract differential topology alone one can at most say that the fibers are cobordant. The difficulty is discussed further in our introduction to §9.

With the preceding remarks understood, we can state our two main results. The first bears on the basic technical question concerning the comparison of moduli spaces of anti-self-dual connections defined by nearby metrics. In particular, in order for one to have a locally trivial cone bundle neighborhood structure for $\bar{M}_\kappa^w(X,g)$ (see [25] for definitions) one needs to know that the moduli space of anti-self-dual connections for metrics on S^4 which are close to standard are in fact diffeomorphic to that defined by the standard metric of radius one. More precisely, we prove:

Theorem 1.1. *There is a positive constant ε such that the following holds. Let g_0 be the standard metric of radius one on the four-sphere S^4. Suppose $\gamma : I \to C \subset \mathrm{Met}(S^4)$, $t \mapsto g_t$, is a C^∞ path of C^∞ metrics on S^4 through g_0, $t \in I \subset \mathbb{R}$, such that*

$$\sup_{t\in I} \|g_t - g_0\|_{C^1(S^4,g_0)} < \varepsilon.$$

Let κ be a positive integer and let $M_\kappa(S^4, C) := \pi^{-1}(C)$, where $\pi : \mathfrak{M}_\kappa(S^4) \to \mathrm{Met}(S^4)$ is the projection from the universal moduli space $\mathfrak{M}_\kappa(S^4) = \{[A,g] \in \mathcal{B}_\kappa(S^4)\times\mathrm{Met}(S^4) : F_A^{+,g} = 0\}$ onto $\mathrm{Met}(S^4)$. Then the following hold.

- *There is a C^∞ diffeomorphism*
$$\Psi : M_\kappa(S^4, g_0) \times I \cong M_\kappa(S^4, C), \quad ([A], t) \mapsto \Psi([A], t),$$
such that $(\gamma^{-1} \circ \pi) \circ \Psi = \pi_2$, where $\pi_2 : \pi^{-1}(g_0) \times I \to I$ is projection onto the second factor and γ^{-1} is the diffeomorphism $C \to I$, $\gamma(t) \mapsto t$. In particular, the map Ψ restricts to a diffeomorphism
$$\Psi_t = \Psi(\cdot, t) : M_\kappa(S^4, g_0) \cong M_\kappa(S^4, g_t), \quad \textit{for } t \in I.$$
- *The map $\Psi_t = \Psi_t^\kappa$ uniquely extends to a* smoothly stratified diffeomorphism *$\bar{\Psi}_t : \bar{M}_\kappa(S^4, g_0) \to \bar{M}_\kappa(S^4, g_t)$, that is, a homeomorphism of their Uhlenbeck compactifications, which restricts to a diffeomorphism on all smooth strata. The restriction of $\bar{\Psi}_t$ to a stratum $M_\ell(S^4, g_0) \times \Sigma$, where $\Sigma \subset \mathrm{Sym}^\ell(X)$ and $0 < \ell < \kappa$, is given by the diffeomorphism*
$$\bar{\Psi}_t : M_\ell(S^4, g_0) \times \Sigma \cong M_\ell(S^4, g_t) \times \Sigma, \quad ([A], \mathbf{x}) \mapsto (\Psi_t^\ell([A], g), \mathbf{x}).$$

Remark 1.2. Although we shall make no use of generalizations of Theorem 1.1, the conclusions also hold for any oriented, smooth homotopy four-sphere X equipped with a generic Riemannian metric. More generally still, for any closed, connected, oriented, smooth four-manifold X, the basic constraint on the data (X, g, w, κ) is that the Uhlenbeck compactification $\bar{M}_\kappa^w(X, g)$ contain no points $[A, \mathbf{x}]$ with non-trivial H_A^2 and $M_\ell^w(X, g)$ contain no points with nontrivial H_A^0 when $0 < \ell \le \kappa$ and g is near g_0.

We shall keep the standard notational practices of [8], [38]; for completeness, we summarize the relevant conventions in §2.1.

1.1.2. *Continuity of gluing maps.* We now turn to our second principal result. Taubes' gluing maps [56] are defined on the level of connections, rather than ideal connections, and it is not clear from [52], [54], [55], [56] that they necessarily extend continuously to the closure $\bar{\mathrm{Gl}}_\kappa^{w,+}(X,\bar{\mathcal{U}}_0,\bar{\Sigma},\lambda_0)$ of the gluing data $\mathrm{Gl}_\kappa^{w,+}(X,\mathcal{U}_0,\Sigma,\lambda_0)$. See §3.2 for the definitions of the gluing data and its closure defined by a choice of open neighborhood $\mathcal{U}_0$ in $M_{\kappa_0}^w(X,C)$ of the 'background' connection $[A_0]$, together with a smooth stratum $\Sigma\subset\mathrm{Sym}^\ell(X)$ and small positive constant λ_0. Let

$$\bar{\mathcal{B}}_\kappa^w(X) := \bigcup_{\ell=0}^{\lfloor\kappa\rfloor}\mathcal{B}_{\kappa-\ell}^w(X)\times\mathrm{Sym}^\ell(X),$$

where $\lfloor\kappa\rfloor$ denotes the greatest integer less than or equal to κ. Our second main result is a proof of the continuity assertion in [25]:

Theorem 1.3. *Let X be a closed, oriented C^∞ manifold with generic C^∞ Riemannian metric g. Let $w\in H^2(X;\mathbb{Z})$ be a class such that no* $\mathrm{SO}(3)$ *bundle over X with second Stiefel-Whitney class equal to $w \pmod 2$ admits a flat connection. Let E be a rank-two, Hermitian vector bundle over X with $c_1(E)=w$ and $c_2(E)-c_1(E)^2/4=\kappa\geq 1$. Let $0<\ell<\lfloor\kappa\rfloor$ be an integer, let $\Sigma\subset\mathrm{Sym}^\ell(X)$ be a smooth stratum, let $C\subset\mathrm{GL}(T^*X)$ be a compact, connected, smooth submanifold-with-boundary containing the identity, let $g\in\mathrm{Met}(X)$, and let $[A_0]\in M_{\kappa-\ell}^w(X,g)$. Then there are*

- *A smooth* gluing map $\boldsymbol{\gamma}:\mathrm{Gl}_\kappa^{w,+}(X,\mathcal{U}_0,\Sigma,\lambda_0)\to\mathcal{B}_\kappa^{w,*}(X)$ *which extends to a continuous map $\boldsymbol{\gamma}:\bar{\mathrm{Gl}}_\kappa^{w,+}(X,\bar{\mathcal{U}}_0,\bar{\Sigma},\lambda_0)\to\bar{\mathcal{B}}_\kappa^w(X)$. Here, λ_0 is a sufficiently small positive constant, depending at most on X,g,κ.*
- *A smooth* Kuranishi obstruction section $\boldsymbol{\chi}$ *of a smooth* Kuranishi obstruction vector bundle $\boldsymbol{\gamma}^*\mathfrak{V}$ *over $\mathrm{Gl}_\kappa^{w,+}(X,\mathcal{U}_0,\Sigma,\lambda)$, where $\mathfrak{V}$ is a continuous vector bundle over an open neighborhood $\mathcal{U}\subset\bar{\mathcal{B}}_\kappa^w(X)\times C$ of $[A_0]\times\mathrm{Sym}^\ell(X)$ which restricts to a smooth bundle over $\mathcal{U}\cap(\mathcal{B}_\kappa^{w,*}(X)\times C)$. The bundle $\mathfrak{V}$ has real rank equal to the real dimension of $H_{A_0}^2$. The section $\boldsymbol{\chi}$ is the restriction of a continuous section on $\mathcal{U}$.*

Together, the gluing map and obstruction section have the property that

$$\boldsymbol{\gamma}(\boldsymbol{\chi}^{-1}(0))\subset\bar{M}_\kappa^w(X,C)\cap\mathcal{U},$$

where $\bar{M}_\kappa^w(X,C)$ is the Uhlenbeck compactification of the universal moduli space $M_\kappa^w(X,C)$ of connections on $\mathfrak{su}(E)$, anti-self-dual with respect to the family of metrics $\{f(g):f\in C\}\subset\mathrm{Met}(X)$.

The gluing maps and obstruction sections are defined in §3 and §7.

Remark 1.4. In the statements of the gluing theorem given in [25] (see Theorems 3.4.10, 3.4.17, and 3.4.18), it is assumed that the gluing data bundle $\mathrm{Gl}_\kappa^{w,+}(X,\mathcal{U}_0,\Sigma,\lambda_0)\to\Sigma$ is replaced by its restriction $\mathrm{Gl}_\kappa^w(X,\mathcal{U}_0,K,\lambda_0)\to K$, where $K\Subset\Sigma$ is a precompact open subset. We make no such restriction here, so the points $\mathbf{x}\in\Sigma$ are allowed to approach the diagonals of the symmetric product $\mathrm{Sym}^\ell(X)$ containing the non-compact smooth stratum Σ.

There are several important properties which are required of a useful gluing theory — in any context, but particularly for applications to the Kotschick-Morgan conjecture — which are *not* asserted by Theorem 1.3. These extensions are discussed in §1.3 and are derived in the companion article [14].

We shall need the case where C is the image of a path γ of metrics on T^*X and the connection A_0 is a non-flat reducible, with stabilizer S^1. In this instance, an open neighborhood $\mathcal{U}\cap M^w_\kappa(X,\gamma)$ of $[A_0]\times\mathrm{Sym}^\ell(X)$ will be covered by a collection of images of gluing maps, one gluing map per smooth stratum Σ of $\mathrm{Sym}^\ell(X)$.

To satisfy the constraint on the second Stiefel-Whitney class, we use the now standard trick of Morgan and Mrowka of requiring the existence of a spherical class $\beta\in H_2(X;\mathbb{Z})$ such that $\langle c_1(E),\beta\rangle\neq 0 \pmod 2$. This condition can always be satisfied by blowing up if necessary and so, as far as the computation of Donaldson invariants is concerned, there is no loss in generality, but the structure of the moduli space is considerably simpler as flat connections are excluded from the Uhlenbeck compactification $\bar M^w_\kappa(X,g)$. For generic paths of metrics, γ, most, but not all, non-flat reducibles occurring in $\bar M^w_\kappa(X,\gamma)$ can be assumed to have vanishing obstruction spaces, so this explains the appearance of the obstruction map in the statement of Theorem 1.3. After work on the continuity assertions of Theorem 1.3 had been written up, we learned that Graham Taylor [57] had considered the issue of continuity, in a more restricted form, of the gluing maps defined by Taubes in [54].

1.2. **Outline.** We outline the contents of the remainder of the article. The Kotschick-Morgan conjecture and how its solution bears on the expected solution to Witten's conjecture are explained in §2.

In §3 we define the 'splicing map' for connections via the usual cut and paste techniques. Of course, this part is entirely standard: the main new issue is that for later purposes it is necessary to keep track of the constants defining the splicing map itself. The splicing maps yield continuous maps (smooth away from singularities) from bundles of gluing data, one bundle per smooth stratum $\Sigma\subset\mathrm{Sym}^\ell(X)$, into the quotient space $\mathcal{B}^{w,*}_\kappa(X)$ whose image lies suitably close to the stratum $M^w_{\kappa-\ell}(g)\times\Sigma$ of $\bar M^w_\kappa(g)$. The question of exactly how 'close' is addressed in §4 and §5.

In §4 we record the regularity theory for the anti-self-dual equation. The exposition is modeled on the corresponding theory for the PU(2) monopole equations and, indeed, applies to any quasi-linear first-order elliptic equation with a quadratic non-linearity on a four-manifold. The results are used repeatedly throughout the present article and its sequels, so precise statements are given. The general pattern is that one typically has uniform, global L^2_1 estimates for solutions to the anti-self-dual equation (obtained, for example, by deforming an approximate, spliced solution), but when we need estimates with respect to stronger norms — uniform with respect to a non-compact parameter — we must get by with only local estimates.

We shall almost always want estimates — for spliced connections or their curvatures — whose constants depend at most on the L^2 norms of the curvatures, or the L^∞ norms of the self-dual curvatures, of the component connections restricted to suitable open sets. There are different ways these estimates can be achieved, depending on the geometry of the region in X where estimates are required. One option is to use decay estimates, due in varying degrees of generality to Donaldson, Groisser-Parker, Morgan-Mrowka-Ruberman, or Råde. However, since our real interest is in the case of PU(2) monopoles, we provide in §5 a slightly simpler approach here which avoids the use of such decay estimates or their generalizations.

This section concludes with an estimate for the self-dual component of the curvature of a spliced connection; as this is a typical application of decay-type estimates for anti-self-dual connection we present the details in this case and then leave subsequent calculations of this type for the reader.

Splicing anti-self-dual connections produces a family, A, of approximately anti-self-dual connections on X and thus a family of Laplacians, $d_A^+ d_A^{+,*}$, on $L^2(X, \Lambda^+ \otimes \mathfrak{su}(E))$. We shall need upper and lower bounds for the small eigenvalues of this Laplacian, along with estimates for its eigenfunctions and variations with respect to A. These and related estimates are derived in §6, the methods and results being modeled on those of Taubes [52], [56].

In §7 we prove the existence and regularity of solutions to the anti-self-dual equation, the essential point being to provide estimates for the solutions using a slightly stronger system of norms than those of [52], [56]: the availability of slightly stronger estimates is important in [14]. Our basic requirement on the Hermitian bundle E is that no $SO(3)$ bundle with second Stiefel-Whitney class equal to $c_1(E)$ (mod 2) admits a flat connection; we do not require that X be simply-connected or even that $b^1(X) = 0$. Taking $w = c_1(E)$, this condition ensures that the Uhlenbeck compactification $\bar{M}_\kappa^w(X, g)$ contains no flat connections and, in particular, no connections A_0 with non-vanishing $H^2_{A_0}$ except reducibles.

In §8 and §9 we are finally in a position to prove the two main results of the present article. The first, which is the subject of §8, concerns the continuity of the gluing map with respect to Uhlenbeck limits: in general, we need to know that if either a proper subset of the scales (used to attach the connections from S^4) tends to zero or if the SU(2) connections on S^4 converge to an Uhlenbeck limit, then the sequence of families of glued-up solutions converge to a family of ideal glued-up solutions. For splicing maps, this property is an immediate consequence of the definition, which is purely local, but for gluing maps it is not obvious because of the global dependence of the solutions on the gluing data via Green's operators. The only continuity statement that is a direct consequence of Taubes' original construction is that if *all* the scales tend to zero then the glued-up solutions converge to the obvious ideal points in the Uhlenbeck compactification.

The proof of our second main result is completed in §9. If one chooses the standard metric on S^4 and splices in the usual way, then one will obtain a family of approximately anti-self-dual connections which only have uniformly bounded C^0 norms of $F_A^{+,g}$ when the metric g on X is (conformally) flat near the gluing sites. It is easy to see that in general, when the metric g on X is not flat near the gluing sites, one will at best have a uniform L^2 bound on $F_A^{+,g}$: this is not sufficient for the fixed-point argument to work, whether one uses the scheme employed here in §7 or the similar one given in [56]. The problem is partly circumvented in [56] by showing that a certain gradient flow (see [53]) yields a homotopy from the image of the splicing map into a neighborhood in $\mathcal{B}_\kappa^w(X, g)$ whose connections satisfy the required *a priori* estimates on $F_A^{+,g}$. While homotopies suffice for Taubes' applications to the stable Atiyah-Jones conjecture in [56], they do not suffice for our construction of links. Instead, we construct a diffeomorphism between the moduli spaces $M_\kappa(S^4, g_0)$ and $M_\kappa(S^4, g_0')$, where g_0 is the standard round metric of radius one and g_0' is a 'nearby' metric which varies with the location of the gluing site $x \in X$. That such a diffeomorphism should exist is less obvious than it might appear, even when the metrics are suitably close, because the spaces are non-compact. Despite this difficulty, we show that a modified version of Taubes' flow does yield a diffeomorphism and this completes the proof of Theorem 1.1.

1.3. **Sequels.** Some remarks are in order concerning the contents of the companion articles [14], [15], [16].

1.3.1. *Surjectivity of gluing maps.* The reader will note that the main result proved here is less than the 'first half' of a desired 'gluing theorem'. To be of use for parameterizing neighborhoods of ideal reducibles, we would also need to know the following:

1. The gluing map $\boldsymbol{\gamma}$ is a smooth embedding of $\mathrm{Gl}_\kappa^{w,+}(X,\mathcal{U}_0,\Sigma,\lambda_0)$ and a topological embedding of the Uhlenbeck closure of the gluing data, $\bar{\mathrm{Gl}}_\kappa^{w,+}(X,\mathcal{U}_0,\Sigma,\lambda_0)$.
2. The image of $\bar{\mathrm{Gl}}_\kappa^{w,+}(X,\mathcal{U}_0,\Sigma,\lambda_0)\cap\boldsymbol{\chi}^{-1}(0)$ under $\boldsymbol{\gamma}$ is an open subset of $\bar{M}_\kappa^w(X,C)$;
3. The space $\bar{M}_\kappa^w(X,C)$ has a finite covering by such open subsets.

These remaining properties (1)–(3) are proved in [14], together with certain C^1 differentiability properties of $\boldsymbol{\gamma}$, and comprise the 'second half' of the gluing theorem. It is important to note that preceding three gluing-map properties are not simple consequences of the gluing method itself and their justification constitutes the more difficult half of the proof of the full gluing theorem. In particular, despite an extensive literature on gluing theory for anti-self-dual connections, the existing accounts (see, for example, [5], [8], [49], [52], [54], [55], [56]) only address very special cases which do not capture all of the difficulties one encounters when attempting to solve the general problem.

1.3.2. *Bubble-tree compactifications and manifolds-with-corners structures.* The proof of injectivity and surjectivity of the gluing maps relies on estimates for the differentials of the gluing maps: these estimates in turn yield further information concerning the C^1 differentiable structure of the bubble-tree compactification of $M_\kappa^w(X,g)$. Indeed, related work of Taubes in [51] suggests that the bubble-tree compactification should be a C^1 manifold with corners (as we would anticipate from known results concerning the moduli space of stable maps on a symplectic manifold — an orbifold). In our case, the C^1 differentiability properties of the gluing maps essentially imply that the bubble-tree compactification of $M_\kappa^w(X,C)$ can be given the structure of a C^1 manifold with corners [15].

1.3.3. *Intersection theory and completion of the proof of the Kotschick-Morgan conjecture.* In work in preparation, [16], we apply the preceding gluing results to compute the wall-crossing contribution of ideal reducibles in a one-parameter family of anti-self-dual moduli spaces, $\bar{M}_\kappa^w(X,g_t)$, $t\in[0,1]$, to the change in the Donaldson invariants of X. Specifically, we show that the changes only depend on the homotopy type of X. The importance of the continuity assertion for gluing maps and that some condition on the differentials of the gluing maps would be needed in order to compare overlaps of different gluing-map images — with applications to a proof of the Kotschick-Morgan conjecture — first arose in discussions with John Morgan and Peter Ozsváth on stratified spaces [43].

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2. The wall-crossing formula for Donaldson invariants

Our first task is to give a precise statement of the conjectured shape of the general wall-crossing formula. There are two slightly different versions given in [36]: the one we aim to prove is a stronger form of the second conjecture. It is only fair to admit straightaway that since the advent of the Seiberg-Witten invariants, the conjecture itself no longer appears to have a direct bearing on smooth four-manifold topology, though it is still an important aspect of Yang-Mills gauge theory. Nevertheless, the conjecture does retain considerable mathematical interest or, more precisely, its method of proof because this is expected to provide a simpler illustration of the proof of the corresponding conjecture for PU(2) monopoles [20].

Our second task in this subsection is to describe the Kuranishi models for essential singularities in parameterized moduli spaces $M_\kappa^w(X,\gamma)$ when $b^+(X) = 1$ and both the class $w \in H^2(X;\mathbb{Z})$ and path of metrics $\gamma : [-1,1] \to \operatorname{Met}(X)$ have been chosen so as to eliminate all singularities except those due to non-flat reducible connections at a critical metric $g_0 = \gamma(0)$. The properties of the model in turn dictate our approach to gluing theory and the construction of models for singularities due to reducibles occurring in all strata of the compactification $\bar M_\kappa^w(X,\gamma)$.

Reducible connections in $M_\kappa^w(X,g_0)$ occur in (possibly obstructed) families of dimension $b^1(X)$. Links of positive-dimensional families of singularities in moduli spaces of PU(2) monopoles are constructed in [17] but, as might be expected, they are considerably more complicated than those of isolated singularities. Neighborhoods of isolated reducibles in lower strata of $\bar M_\kappa^w(X,\gamma)$ may be parameterized via Taubes' approach to gluing, which uses spectral projections to construct an equivariant, finite-rank normal bundle of the singularity in a thickened moduli space which contains the singularity as a smooth submanifold. This is no longer possible in the case of positive-dimensional families, due to spectral flow, and one must use a combination of the link construction in [17] and a gluing method which leans closer to that of Donaldson [5], [8]. Gluing constructions of the latter kind lead to further difficulties: see [18], [19]. However, the formulas for the wall-crossing terms have only been predicted or proved modulo the Kotschick-Morgan conjecture, when $b^1(X) = 0$. Hence, for these reasons, we shall restrict our attention to this case whenever necessary to simplify the discussion at hand.

2.1. **Preliminaries.**

To avoid ambiguity, we gather here the notational conventions we shall employ throughout our work; we follow, whenever possible, the usual practices of [8], [38] and, in a few instances, those of [25].

Let E be a Hermitian, rank-two vector bundle over S^4. Let $\mathcal{A}_\kappa = \mathcal{A}_E = \mathcal{A}_E(S^4)$ be the space of L^2_k connections on the SO(3) bundle $\mathfrak{g}_E = \mathfrak{su}(E)$ over S^4. Let $\mathcal{G}_E$ be the group of L^2_{k+1} unitary gauge transformations of E with determinant one and let $\mathcal{G}_E^s \subset \mathcal{G}_E$ be the subgroup consisting of gauge transformations $u \in \mathcal{G}_E$ such that $u(s) = \mathrm{id}_{E|_s}$.

Let $\mathcal{B}_\kappa^s = \mathcal{B}_E^s = \mathcal{B}_E^s(S^4)$ be the $\mathcal{G}_E^s$-quotient space of L^2_k connections on the bundle $\mathfrak{g}_E$ over S^4. There is a map, defined by the choice $q \in \operatorname{Fr}(\mathfrak{g}_E)|_s$,

$$\mathcal{B}_E^s \simeq \mathcal{A}_E \times_{\mathcal{G}_E} \operatorname{Fr}(\mathfrak{g}_E)|_s, \qquad [A] \mapsto [A,q],$$

which is easily seen to be a homeomorphism. The existence of a slice for the action of the based gauge group, $\mathcal{G}^s_E$, has been established by Groisser and Parker and it then follows that this homeomorphism is in fact a diffeomorphism [30], as there are no non-flat reducible SO(3) connections over S^4. The group $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm\mathrm{id}\}$ acts freely on $\mathcal{B}^s_E$ by the assignment $[A] \mapsto [uAu^{-1}]$ or $[A,q] \mapsto [A,qu]$, if $u \in \mathrm{SU}(2)$.

Let $\mathrm{Fr}(\mathfrak{g}_E)$ denote the principal SO(3) bundle of oriented, orthonormal frames for $\mathfrak{g}_E$. Let $\mathrm{Fr}(TX)$ denote the principal SO(4) bundle of oriented, g-orthonormal frames for TX.

2.2. **The Kotschick-Morgan conjecture.** Let X be a closed, oriented, smooth four-manifold with $b^+(X) = 1$. We briefly recall the definition of Donaldson invariants in this case [6], [26], [27], [28], [35], [36]. Following [26, §1], we call

$$\Omega_X := \{\beta \in H^2(X;\mathbb{R}) : \beta^2 > 0\}/\mathbb{R}^+$$

the *positive cone* of $H^2(X;\mathbb{R})/\mathbb{R}^+$, where we denote $\mathbb{R}^+ = (0,\infty)$. Fix a class $w \in H^2(X;\mathbb{Z})$ and a number $\kappa \in \frac{1}{4}\mathbb{Z}$ such that

$$-4\kappa = w^2 \pmod{\mathbb{Z}} \quad \text{and} \quad 8\kappa - 3(2-b^1) \geq 0. \tag{2.1}$$

A *wall of type* (w,κ) is a nonempty hyperplane in Ω_X,

$$W^\xi := \{\beta \in \Omega_X : \beta\cdot\xi = 0\} = \xi^\perp \cap \Omega_X, \tag{2.2}$$

for some class $\xi \in H^2(X;\mathbb{Z})$ with

$$\xi = w \pmod 2 \text{ and } p \leq \xi^2 < 0, \tag{2.3}$$

where $p = -4\kappa$. Let $\ell = (\xi^2 - p)/4$: observe that $0 \leq \ell < \kappa$ and that ℓ is an integer. [1] According to [6] the set of (w,κ)-walls is locally finite.

The connected components of the complement in Ω_X of the walls of type (w,κ) are called the *chambers of type* (w,κ). Require that $w \pmod 2$ be such that no SO(3) bundle V over X with $w_2(V) = w \pmod 2$ admits a flat connection; see [35], [46] for sufficient criteria and how they may be achieved with no loss of generality [46] for the purposes of computing Donaldson invariants. Then the Donaldson invariants of X associated to (w,κ) are defined with respect to chambers of type (w,κ). We shall write

$$D^w_{X,\mathcal{C}}(z), \qquad z \in \mathbb{A}(X),$$

for the Donaldson invariants defined by the class $w \in H^2(X;\mathbb{Z})$ and the chamber $\mathcal{C}$. Indeed, for a Riemannian metric g on X, let $\omega \in \Omega^{+,g}(X)$ be a g-self-dual, harmonic two-form (so $d^{+,g}d^{*,g}\omega = 0$) such that $\int_X \omega\wedge\omega = 1$. The form is unique up to sign, so fix an ω and let it define an orientation for $H^+(X;\mathbb{R})$. Let $(-1)\wedge\omega \in \det(H^0(X;\mathbb{R})\oplus H^+(X;\mathbb{R}))$ define the choice of homology orientation. For generic g, the point $[\omega] \in H^2(X;\mathbb{R})$ does not lie on a wall W. If $\mathcal{C} \subset H^2(X;\mathbb{R})/\mathbb{R}^+$ is the chamber containing $[\omega]$ then, according to [35, Theorem 3.2], [36, Theorem 3.0.1], one has $D^w_{X,g}(z) = D^w_{X,\mathcal{C}}(z)$, that is, the Donaldson invariants only depend on the chamber $\mathcal{C}$ containing the period point $[\omega(g)]$ and not the metric g.

[1] The case $\xi^2 = 0$ and $\ell = \kappa$ could occur if the background reducible connection were flat. However, this possibility is excluded by choosing w as suggested in [46]; see Lemma 2.4.

Recall that $M^w_\kappa(X,g)$ has the expected dimension

$$\begin{aligned}\dim M^w_\kappa(X,g) &= -2w^2 - 3(b^+ - b_1 + 1)\\ &= -2w^2 - 3(2 - b_1).\end{aligned}$$

If $\deg z \neq -2w^2 - 3(2-b_1) \pmod 8$ then $D^w_{X,\mathcal{C}}(z) = 0$. Otherwise, find $\kappa \in \frac{1}{4}\mathbb{Z}$ such that

$$\deg z = 8\kappa - 3(2 - b_1)$$

and a Hermitian, rank-two vector bundle E over X such that

$$c_1(E) = w \text{ and } -\tfrac{1}{4}p_1(\mathfrak{g}_E) = \langle c_2(E) - c_1(E)^2/4, [X]\rangle = \kappa. \tag{2.4}$$

where, throughout this paper, we denote $\mathfrak{g}_E := \mathfrak{su}(E)$. We now define $D^w_{X,\mathcal{C}}(z)$ following the recipe of [38, §2].

Suppose that $\mathcal{C}_+$ and $\mathcal{C}_-$ are separated by a single wall W^ξ; there may be more than one class ξ of type (w,κ) defining W^ξ. The difference

$$\delta^w_{X,\mathcal{C}_+,\mathcal{C}_-}(z) := D^w_{X,\mathcal{C}_+}(z) - D^w_{X,\mathcal{C}_-}(z), \qquad z \in \mathbb{A}(X),$$

defines a *transition formula*. Based on known examples and their own work, Kotschick and Morgan proposed the following two conjectures:

Conjecture 2.1. The transition formula $\delta^w_{X,\mathcal{C}_+,\mathcal{C}_-}$ is a homotopy invariant of the pair (X,ξ); more precisely, if ϕ is an oriented homotopy equivalence from X' to X, then

$$\delta^{\phi^* w}_{X,\phi^*(\mathcal{C}_+),\phi^*(\mathcal{C}_-)} = \phi^*\delta^w_{X,\mathcal{C}_+,\mathcal{C}_-}.$$

This is expected to be equivalent to the following apparently stronger claim:

Conjecture 2.2. The transition formula $\delta^w_{X,\mathcal{C}_+,\mathcal{C}_-}$ is a polynomial in ξ and the quadratic form Q_X, with coefficients depending only on ξ^2, homotopy invariants of X, and universal constants.

The property of the transition formula which we aim to establish is, in fact, stronger than that predicted by Conjecture 2.2. It is a remarkable result of Göttsche that if one assumes Conjecture 2.1 then the difference terms $\delta^w_{X,\mathcal{C}_+,\mathcal{C}_-}$ can be computed explicitly for any simply-connected four-manifold. It seems very likely that the condition $\pi_1(X) = 1$ can be weakened to the case $b^1(X) = 0$, as Göttsche points out in [27] that he uses this constraint merely to remain consistent with the stated version of the Kotschick-Morgan conjecture in [36] and the Fintushel-Stern blow-up formula [23]: see [27, Corollary 5.1]. Göttsche's argument only requires Conjecture 2.1 in order to replace X by $\mathbb{CP}^1 \times \mathbb{CP}^1$ in [27, Lemma 4.10].

The first result in this direction is due to Donaldson, who gave a formula in the case $w = 0$ and $\kappa = 1$; these were extended by Kotschick in [35]. Using a gauge-theoretic approach, Yang [64] completely determined the transition formulas for $w = 0$ and $\ell = 1$. This was further extended by Leness to the case $\ell = 2$ and arbitrary w [40].

For a rational ruled complex surface X, the transition formulas for $w = 0$ and $2 \leq \kappa \leq 4$ were determined by [41], [42], [58] using methods of algebraic geometry. Using Thaddeus' method, Friedman and Qin completely determined the transition formulas when X is a rational surface with $-K_X$ effective [26, p. 13]. Similar results were independently obtained by Ellingsrud and Göttsche [11], [10].

In a quite different direction, Göttsche [27] assumed the validity of the Kotschick-Morgan conjecture and used this assumption to explicitly compute the transition formulas, for arbitrary closed, oriented, smooth four-manifolds X with $b^1(X)=0$ and $b^+(X)=1$ in terms of modular forms.

2.3. **Uhlenbeck compactness for parameterized moduli spaces.** We recall the Uhlenbeck compactness results [8], [24] for the moduli space of anti-self-dual SO(3) connections and their natural extension to the case of families of metrics. Recall that the process of gluing up connections is essentially the inverse of Uhlenbeck degeneration of connections.

Definition 2.3. We say that a sequence of points $[A_\alpha, g_\alpha]$ in $\mathcal{B}^w_\kappa(X)\times\mathrm{Met}(X)$ *converges* to a point $[A,g,\mathbf{x}]$ in $\mathcal{B}^w_{\kappa-\ell}(X)\times\mathrm{Met}(X)\times\mathrm{Sym}^\ell(X)$, where $0\le\ell\le\kappa$ is an integer, if the following hold:

- The sequence of metrics g_α converges in C^r to a metric g on T^*X.
- Let E, E_0 be rank-two Hermitian bundles over X, with determinant line bundle w and induced SO(3) bundles, $\mathfrak{g}_E$ and $\mathfrak{g}_{E_0}$, with first Pontrjagin numbers -4κ and $-4(\kappa-\ell)$, respectively. There is a sequence of $L^2_{k+1,\mathrm{loc}}(X\setminus\mathbf{x})$, determinant-one, unitary bundle isomorphisms $u_\alpha: E|_{X\setminus\mathbf{x}}\to E_0|_{X\setminus\mathbf{x}}$ such that the sequence of connections $u_\alpha(A_\alpha)$ on $\mathfrak{g}_{E_0}|_{X\setminus\mathbf{x}}$ converges to A on $\mathfrak{g}_{E_0}|_{X\setminus\mathbf{x}}$ in $L^2_{k,\mathrm{loc}}$ over $X\setminus\mathbf{x}$, and
- The sequence of measures $|F_{A_\alpha}|^2$ converges in the weak-* topology on measures to $|F_A|^2+8\pi^2\sum_{x\in\mathbf{x}}\delta(x)$, where $\delta(x)$ is the unit Dirac measure centered at the point $x\in X$.

Let $C\subset\mathrm{Met}(X)$ denote a compact subset of the Banach space of C^r Riemannian metrics on T^*X $(r\ge 3)$ and define the corresponding parameterized moduli space by

$$M^w_\kappa(X,C):=\bigcup_{g\in C}M^w_\kappa(X,g)\times\{g\}\subset\mathcal{B}^w_\kappa(X)\times\mathrm{Met}(X).$$

Let $\bar M^w_\kappa(X,C)$ be the closure of $M^w_\kappa(X,C)$ in the space $\cup_{\ell=0}^{\lfloor\kappa\rfloor}M_{\kappa-\ell}\times\mathrm{Sym}^\ell(X)$ with respect to the induced Uhlenbeck topology, where $\lfloor\kappa\rfloor$ is the greatest integer contained in $\kappa(E):=-\frac{1}{4}p_1(\mathfrak{g}_E)$, where $-\frac{1}{4}p_1(\mathfrak{g}_E)=\langle c_2(E)-\frac{1}{4}c_1(E)^2,[X]\rangle$. We call the intersection of $\bar M^w_\kappa(X,C)$ with $M^w_{\kappa-\ell}\times\mathrm{Sym}^\ell(X)$ a *lower-level* of the compactification $\bar M^w_\kappa(X,C)$ if $\ell>0$ and call $M^w_\kappa(X,C)$ the *top* or *highest level*.

According to [8, Theorem 4.4.3 & Proposition 9.1.2], $\bar M^w_\kappa(X,C)$ is a compact, Haussdorf, second-countable space when endowed with the induced Uhlenbeck topology. We call the intersection of $\bar M^w_\kappa(X,C)$ with $M^w_{\kappa-\ell}(X,C)\times\mathrm{Sym}^\ell(X)$ a *lower-level* of the compactification $\bar M^w_\kappa(X,C)$ if $\ell>0$ and call $M^w_\kappa(X,C)$ the *top* or *highest level*: in general, the spaces $M^w_{\kappa-\ell}(X,C)$ and $\mathrm{Sym}^\ell(X)$ are at best smoothly stratified.

The following lemma gives a simple condition on w which guarantees that there will be no flat SO(3) connections in $M^w_\kappa(X,g)$, for any g. Because it relies only on $w \pmod 2$, the lemma applies simultaneously to all levels of the Uhlenbeck compactification $\bar M^w_\kappa(X,g)$. [2]

Lemma 2.4. [46, p. 226] *Let X be a closed, oriented four-manifold. Then*

[2]According to [22, p. 536], [35, Proposition 4.1], if V is an SO(3) bundle over a closed four-manifold X such that $w_2(V)$ is not pulled back from $H^2(K(\pi_1(X),1);\mathbb{Z}/2\mathbb{Z})$, then there are no flat connections on V. However, one sees more directly that the Morgan-Mrowka criterion is satisfied via blow-ups.

- *If E is a* $\mathrm{U}(2)$ *bundle over X and $\beta \in H_2(X;\mathbb{Z})$ is a spherical class such that $\langle c_1(E), \beta\rangle \neq 0 \pmod 2$, then no* $\mathrm{SO}(3)$ *bundle V over X with $w_2(V) = c_1(E) \pmod 2$ admits a flat connection.*
- *In particular, if E is any* $\mathrm{U}(2)$ *bundle over X and $\hat E$ is the* $\mathrm{U}(2)$ *bundle over $\hat X = X\#\overline{\mathbb{CP}}^2$ with $c_2(\hat E) = c_2(E)$ and $c_1(\hat E) = c_1(E) + e^*$, where $e^* = \mathrm{PD}[e]$ and $e = [\overline{\mathbb{CP}}^1] \in H_2(X;\mathbb{Z})$, then no* $\mathrm{SO}(3)$ *bundle V over $\hat X$ with $w_2(V) = w_2(\mathfrak{su}(\hat E))$ admits a flat connection.* [3]

Proof. Let $S \subset X$ be an embedded two-sphere representing β. If A were a flat $\mathrm{SO}(3)$ connection on V, then its restriction $A|_S$ would be flat and thus a trivial connection on a trivial $\mathrm{SO}(3)$ bundle $V|_S$, as S is simply connected. But $\langle w_2(V|_S), [S]\rangle = \langle c_1(E), \beta\rangle \neq 0 \pmod 2$, yielding a contradiction. This proves the first assertion.

Replacing X by $\hat X = X\#\overline{\mathbb{CP}}^2$ and E by $\hat E$ with $c_2(\hat E) = c_2(E)$ and $c_1(\hat E) = c_1(E) + e^*$ ensures that $\langle c_1(\hat E), e\rangle \neq 0 \pmod 2$, as $c_1(E)\cdot e^* = 0$ and $e^* \cdot e^* = -1$. The second assertion follows. □

2.4. **Transversality for one-parameter families of moduli spaces.** Rather than consider a moduli space of anti-self-dual connections defined by a single Riemannian metric, g_0, we shall need to consider C^∞ paths of metrics $\gamma : [-1,1] \to \mathrm{Met}(X)$, where $\mathrm{Met}(X)$ is the Banach manifold of C^r metrics on T^*X (with $r \in \mathbb{Z}$ large). For this purpose, we shall need the following basic transversality result [8], [24], [38].

Theorem 2.5. *Let X be a closed, oriented, smooth four-manifold and let $r \geq 3$ be an integer. Let $v \in H^2(X;\mathbb{Z}/2\mathbb{Z})$ be a class such that no* $\mathrm{SO}(3)$ *bundle with second Stiefel-Whitney class v admits a flat connection. Then the following hold:*

- *There is a first-category subset of the space of C^r metrics such that for all metrics g in the complement of this subset, the following hold: For any $\kappa \in \frac{1}{4}\mathbb{Z}$ and $w \in H^2(X;\mathbb{Z})$ with $-4\kappa = w^2 \pmod{\mathbb{Z}}$, the moduli space $M^{w,*}_\kappa(X,g)$ of irreducible g-anti-self-dual* $\mathrm{SO}(3)$ *connections is regular (as the zero-locus of the section $F^{+,g}(\,\cdot\,)$ over $\mathcal{B}^{w,*}_\kappa(X)$) and thus a C^∞ manifold of the expected dimension $8\kappa - 3(1 - b^1 + b^+)$.*
- *When $b^+(X) > 0$, there is a first-category subset of the space of C^r metrics such that for all metrics g in the complement of this subset, the following hold: For any $\kappa \in \frac{1}{4}\mathbb{Z}$ and $w \in H^2(X;\mathbb{Z})$ with $-4\kappa = w^2 \pmod{\mathbb{Z}}$, the moduli space $M^w_\kappa(X,g)$ of g-anti-self-dual* $\mathrm{SO}(3)$ *connections contains no reducible connections.*
- *Let g_{-1}, g_1 be metrics satisfying the conditions of the preceding assertion and suppose $b^+(X) > 0$. Then there is a first-category subset of the space of C^r paths $\gamma : [-1,1] \to \mathrm{Met}(X)$ with $\gamma(\pm 1) = g_{\pm 1}$ such that for all paths γ in the complement of this subset, the following hold. For any $\kappa \in \frac{1}{4}\mathbb{Z}$ and $w \in H^2(X;\mathbb{Z})$ with $-4\kappa = w^2 \pmod{\mathbb{Z}}$, the parameterized moduli space of γ-anti-self-dual* $\mathrm{SO}(3)$ *connections,*
$$M^{w,*}_\kappa(X,\gamma) := \{([A],t) \in \mathcal{B}^{w,*}_\kappa(X) \times [-1,1] : F^{\gamma(t)}(A) = 0\},$$
is regular (as the zero-locus of the section $F^{+,\gamma(\cdot)}(\,\cdot\,)$ over $\mathcal{B}^{w,*}_\kappa(X)) \times [-1,1]$ and thus a C^∞ manifold of the expected dimension. In particular, $M^{w,*}_\kappa(X,\gamma)$ gives a C^∞ cobordism between the manifolds $M^{w,*}_\kappa(X,g_{-1})$ and $M^{w,*}_\kappa(X,g_1)$.

[3] Because the Donaldson invariants of $X\#\overline{\mathbb{CP}}^2$ determine and are determined by those of X, (see [23], [46]), no information about these invariants is lost by blowing up.

Proof. All of the above assertions are standard, so we merely list their provenance. The first assertion is proved as Corollary 4.3.18 in [8] and Theorem 3.17 in [24]: the manifold X is not required to be simply connected, as assumed in [8, §4.3], [24, §3], because flat connections are excluded by the condition on w and twisted reducible connections are smooth points by [38, Corollary 2.5]. Similarly, the third assertion follows from [8, Corollary 4.3.18], [38, Corollary 2.5], and the fact that flat connections are excluded. The second assertion is proved as Corollary 4.3.15 in [8]: the condition that X be simply connected is not required. □

2.5. **Kuranishi models and reducible anti-self-dual connections.** It remains to consider the non-flat reducible connections in $\bar{M}^w_\kappa(X,\gamma)$, the only possible singular points of this compact cobordism which cannot be excluded by perturbations or a suitable choice of w (mod 2) or by dimension-counting when considering the intersection of geometric representatives with $\bar{M}^w_\kappa(X,\gamma)$. In this subsection we examine the elliptic deformation complex for a reducible $[A_0]$ in $M^w_\kappa(X,g_0)$, while in the following subsection we describe the Kuranishi model for a neighborhood of a reducible $[A_0,g_0]$ in $M^w_\kappa(X,\gamma)$.

In this section we classify the projectively anti-self-dual connections on E with non-trivial stabilizers in $\mathcal{G}_E$. If A is a connection on E, we write A^{ad} for the connection induced by A on the associated SO(3) bundle $\mathfrak{g}_E$ and denote its stabilizer by $\mathrm{Stab}_A \subset \mathcal{G}_E$. Except when discussing bundle splittings and reducible connections, our usual convention is to consider connections on $\mathfrak{g}_E$ only, with their lifts to E via the fixed connection on $\det E$ being implicit when needed. We call A *irreducible* if $\mathrm{Stab}_A = \{\pm 1\}$, the center of SU(2), and *reducible* otherwise. The following classification is standard; see, for example, Proposition 3.1 in [17].

Lemma 2.6. *Let X be a closed, oriented, smooth four-manifold with Riemannian metric g. Let E be a Hermitian two-plane bundle over X and suppose a unitary connection A on E represents a point $[A] \in M_E$. Then one of the following holds:*

1. *The connection A is irreducible, with stabilizer $\mathrm{Stab}_A = \{\pm 1\}$.*
2. *The connection A is reducible, but not projectively flat: the bundle E splits as an ordered sum of line bundles, $E = L_1 \oplus L_2$, the connection A has stabilizer $\mathrm{Stab}_A \cong S^1$ and $A = A_1 \oplus A_2$, where A_1 is a unitary connection on L_1 and $A_2 = A_e \otimes A_1^*$ is the corresponding connection on $L_2 = (\det E) \otimes L_1^*$, with A_e being the fixed connection on* $\det E$.
3. *The connection A is projectively flat and reducible with respect to a splitting $E = L_1 \oplus L_2$, where $L_2 = (\det E) \otimes L_1^*$ and $c_1(E) - 2c_1(L_1) \in H^2(X;\mathbb{Z})$ is a torsion class.*

For the second case in Lemma 2.6. we remark that a projectively g-anti-self-dual connection A on E is reducible with respect to the splitting $E \cong L_1 \oplus L_2$ if and only if the induced g-anti-self-dual connection A^{ad} on $\mathfrak{g}_E$ is reducible with respect to the splitting $\mathfrak{g}_E \cong i\mathbb{R} \oplus \xi$, where $\xi \cong \det E \otimes L_1^{-2}$. Indeed, we have

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_e \otimes A_1^* \end{pmatrix} \quad \text{and} \quad A^{\mathrm{ad}} = 0 \oplus A_e \otimes 2A_1^*,$$

with $F^{+,g}(A_e \otimes 2A_1^*) = 0$.

From Theorem 2.5 we see that it remains to consider the case of reducible connections in $\bar{M}^w_\kappa(X,\gamma)$. Suppose that a class $\xi \in H^2(X;\mathbb{Z})$ defines a (w,κ)-wall as in definition (2.2) so $\xi = w$ (mod 2) and $-4\kappa \le \xi^2 < 0$ as in (2.3). Let $0 \le \ell < \kappa$ be the integer defined by $-4(\kappa-\ell)=\xi^2$. Then there is a U(2) bundle E and a splitting $E=L\oplus w\otimes L^{-1}$ with $\xi=w\otimes L^{-2}$, where $c_1(E)=w$ and $p_1(\mathfrak{g}_E)=-4(\kappa-\ell)=\xi^2$. The corresponding SO(3) bundle $\mathfrak{g}_E\cong\underline{\mathbb{R}}\oplus\xi$ admits a connection, reducible with respect to this splitting, given by the trivial connection on $\underline{\mathbb{R}}$ and a g-anti-self-dual connection on the line bundle ξ if and only if $c_1(\xi)\cdot[\omega_g]=0$, where ω_g is a g-self-dual, harmonic two-form.

Recall that any (real) linear representation space V of S^1 can be decomposed as $V=V_{\mathbb{C}}\oplus V_{\mathbb{R}}$, where S^1 acts trivially on the real space $V_{\mathbb{R}}$ and by a direct sum of one-dimensional representations on the complex space $V_{\mathbb{C}}$. Furthermore, the only linear irreducible representation which is free away from the origin is the standard representation on $\mathbb{C}$. To determine the possible dimensions for the cohomology of the elliptic deformation complex at a reducible anti-self-dual SO(3) connection, we shall make use of the following general observation of Freed and Uhlenbeck:

Proposition 2.7. [24, Proposition 4.9] *Let X be a closed, oriented four-manifold with Riemannian metric g and $b^1(X)=0$. Let A be a reducible, unitary, projectively anti-self-dual connection on a rank-two, Hermitian bundle E over X. If $H^\bullet_A$ is the cohomology of the deformation complex for the induced reducible, anti-self-dual* SO(3) *connection on $\mathfrak{g}_E$, then there are real isomorphisms*

$$H^0_A\cong H^0(X;\mathbb{R})\cong\mathbb{R},\quad H^1_A\cong\mathbb{C}^q,\quad \textit{and}\quad H^2_A\cong\mathbb{C}^p\oplus H^+(X;\mathbb{R}),$$

where $\mathrm{Stab}_A\cong S^1$ *acts by the standard action on $\mathbb{C}^q$, $\mathbb{C}^p$, and by the trivial action on $H^0(X;\mathbb{R})$, $H^+(X;\mathbb{R})$. Here,* $\mathrm{Stab}_A\cong\mathrm{SO}(2)$ *denotes the stabilizer in $\mathcal{G}_E$ of the anti-self-dual* SO(3) *connection A^{ad} on $\mathfrak{g}_E$ and, as usual, $\mathcal{G}_E$ is the group of unitary automorphisms of E with determinant one.*

The decomposition given by Freed-Uhlenbeck can described more explicitly by borrowing from [24, pp. 69–70] and the corresponding discussion for reducible PU(2) monopoles in [17]. Recall from [17, §3.4] that the elliptic deformation complex for the g_0-anti-self-dual equation at a reducible connection $A=A_1\oplus A_2$ on $E=L_1\oplus L_2$, where $A_2:=A_e\otimes A_1^*$ and $L_2:=(\det E)\otimes L_1^{-1}$, splits into a *tangential* and a *normal* deformation complexes; the tangential complex corresponds to the trivial $\mathrm{Stab}_A\cong S^1$ action. We can thus apply Lemma 3.17 in [17] to partially compute the cohomology groups $H^\bullet_A$ of the deformation complex for A on $\mathfrak{g}_E$,

$$\Omega^0(\mathfrak{g}_E)\xrightarrow{d_A}\Omega^1(\mathfrak{g}_E)\xrightarrow{d_A^+}\Omega^+(\mathfrak{g}_E).$$

The rolled-up full elliptic deformation complex for A has index $-2p_1(\mathfrak{g}_E)-3(1-b^1+b^+)$, that is $-2\xi^2-6$ for $b^1(X)=0$ and $b^+(X)=1$, with

$$\xi=\det E\otimes L_1^{-2}\quad\text{and}\quad\mathfrak{g}_E\cong i\mathbb{R}\oplus\xi. \tag{2.5}$$

When A is reducible, as above, the tangential and normal deformation complexes are

$$\Omega^0(i\mathbb{R})\xrightarrow{d}\Omega^1(i\mathbb{R})\xrightarrow{d^+}\Omega^+(i\mathbb{R}), \tag{2.6}$$

$$\Omega^0(\xi)\xrightarrow{d_{A_\xi}}\Omega^1(\xi)\xrightarrow{d^+_{A_\xi}}\Omega^+(\xi), \tag{2.7}$$

where $A_\xi=A_e\otimes 2A_1^*$. Of course, the cohomology of the tangential deformation complex (2.6) is isomorphic to $H^\bullet(X;\mathbb{R})$, so $H^0(X;\mathbb{R})\cong\mathbb{R}$ while $H^1(X;\mathbb{R})=0$ and $H^+(X;\mathbb{R})\cong\mathbb{R}$

by hypothesis; thus the index of its rolled-up complex is -2. The condition that $H^1(X;\mathbb{R}) = 0$ ensures that reducibles do not occur in obstructed positive-dimensional families and so are isolated points of their moduli spaces. The fact that the reducibles are isolated if and only if $H^1(X;\mathbb{R}) = 0$ can also be seen from [8, Proposition 2.2.6]; see [8, Proposition 4.2.15] in addition.

Turning to the normal deformation complex (2.7), we recall from [17, Equation (3.45)] that the (real) index of its rolled-up complex is given by

$$2n := 2n_\xi = -2p_1(\mathfrak{g}_E) - (e(X) + \sigma(X)) = -2\xi^2 - 2(1 - b^1 + b^+). \tag{2.8}$$

For $b^1(X) = 0$ and $b^+(X) = 1$, this becomes $2n = -2\xi^2 - 4$. Observe that the rolled-up normal deformation complex has positive (real) index when $\xi^2 \leq -3$, zero index when $\xi^2 = -2$, and index -2 when $\xi^2 = -1$. From condition (2.3) we need only ever consider $\xi^2 \leq -1$ and thus $n_\xi \geq 0$ (when $b^1(X) = 0$ and $b^+(X) = 1$).

Proposition 2.7 implies that for the normal deformation complex (2.7) we must have

$$H^0_{A_\xi} = 0, \quad H^1_{A_\xi} \cong \mathbb{C}^{p+n}, \quad \text{and} \quad H^2_{A_\xi} \cong \mathbb{C}^p,$$

where $n = -\xi^2 - 2$ is the complex index of the rolled-up normal complex. Hence,

$$H^0_A \cong \mathbb{R}, \quad H^1_A \cong \mathbb{C}^{p+n}, \quad \text{and} \quad H^2_A \cong \mathbb{C}^p \oplus \mathbb{R}. \tag{2.9}$$

The preceding consequence of Proposition 2.7 can be sharpened if the metric g on X is suitably generic, though this refinement is not essential for our purposes.

Theorem 2.8. [24, Theorem 4.19] *Continue the hypotheses and notation of Proposition 2.7, with ξ a complex line bundle over X as in (2.5) and $n := -\xi^2 - 2 \geq -1$, the index of the rolled-up normal deformation complex. If, in addition, the metric g on X is ξ-generic, that is it satisfies $[\omega(g)] \in W_\xi$ but is otherwise generic, then the following hold.*

- *If $n \geq 0$, then $H^1_{A_\xi} \cong \mathbb{C}^n$ and $H^2_{A_\xi} \cong 0$. Thus,*
$$H^0_A \cong \mathbb{R}, \quad H^1_A \cong \mathbb{C}^n, \quad \textit{and} \quad H^2_A \cong \mathbb{R}.$$
- *If $n = -1$, then $H^1_{A_\xi} \cong 0$ and $H^2_{A_\xi} \cong \mathbb{C}$. Thus,*
$$H^0_A \cong \mathbb{R}, \quad H^1_A \cong 0, \quad \textit{and} \quad H^2_A \cong \mathbb{C} \oplus \mathbb{R}.$$

Proof. The argument of Freed-Uhlenbeck in [24, pp. 69–73] leading to the proof of [24, Theorem 4.19] makes no use of the facts that $b^+(X) = 0$, that X is simply-connected, or that E is an SU(2) bundle with $c_2(E) = 1$. Observe that the variations of the metric preserve the curvature, F_{A_ξ}. The period points of these metrics thus remain in the wall W_ξ. Hence, the conclusions follow directly from [24]. □

2.6. **Kuranishi obstructions in the parameterized moduli space.** It remains to extend the remarks of the preceding section concerning singularities in the moduli space $M^w_\kappa(X, g_0)$ to the case of the one-parameter family of moduli spaces $M^w_\kappa(X, \gamma)$.

The usual Kuranishi model for a neighborhood of $[A_0] \in M^w_\kappa(X, g_0)$ results from considering the section $A \mapsto P_+(g_0)F_A$ of the Hilbert bundle $\mathcal{A}_E(X) \times_{\mathcal{G}_E} L^2_\ell(\Lambda^{+,g_0} \otimes \mathfrak{g}_E)$. If g_t, $t \in [-1, 1]$, is a smooth path of metrics on T^*X, we may choose a smooth path $f_t \in \mathrm{GL}(T^*X)$

such that $g_t = f_t(g_0)$, $t \in [-1,1]$; the transformations f_t are uniquely determined by g_t and g_0 by requiring the f_t to be symmetric. Therefore, we have

$$P_+(g_t) = f_t^{-1} \circ P_+(g_0) \circ f_t : \Lambda^2 \to \Lambda^{+,g_t}$$

and so instead of considering the zero-locus of the section $(A,t) \mapsto P_+(g_t)F_A$ of $(\mathcal{A}_E(X) \times [-1,1]) \times_{\mathcal{G}_E} L^2_\ell(\Lambda^{+,g_t} \otimes \mathfrak{g}_E)$, a Hilbert bundle depending on t, we may equivalently consider the zero-locus of the section $(A,t) \mapsto P_+(g_0)f_t(F_A)$ of the *fixed* Hilbert bundle $(\mathcal{A}_E(X) \times [-1,1]) \times_{\mathcal{G}_E} L^2_\ell(\Lambda^{+,g_0} \otimes \mathfrak{g}_E)$. The construction of the local Kuranishi model for a neighborhood of $[A_0, g_0] \in M^w_\kappa(X,\gamma)$ then follows in the usual way [8] to give:

Lemma 2.9. *Let E be a Hermitian two-plane bundle over a closed, oriented, smooth four-manifold X with $b^+(X) = 1$ and $\gamma : [-1,1] \to \mathrm{Met}(X)$ a generic path of Riemannian metrics on T^*X. Let $[A_0, g_0]$ be a point in $M_E(\gamma)$ such that $\gamma(0) = g_0$ and A_0 is a reducible unitary connection on E. Then A_0 has stabilizer $\mathrm{Stab}_{A_0} \cong S^1$ in $\mathcal{G}_E$ and there are*

- *An open, S^1-invariant neighborhood $\mathcal{O} = \mathcal{O}_{A_0}$ of the origin in $H^1_{A_0}$ and a positive constant $\varepsilon = \varepsilon_{A_0}$ together with a smooth, S^1-equivariant embedding $\boldsymbol{\gamma} = \boldsymbol{\gamma}_{A_0,g_0}$,*

  $$\boldsymbol{\gamma} : \mathcal{O} \times (-\varepsilon, \varepsilon) \subset H^1_{A_0} \oplus \mathbb{R} \hookrightarrow \mathcal{A}_E,$$

  with $\boldsymbol{\gamma}(0,0) = (A_0, 0)$ and $(M_E \cap \boldsymbol{\gamma}(\mathcal{O} \times (-\varepsilon,\varepsilon)))/S^1$ an open neighborhood of $[A_0, g_0]$ in $M_E(\gamma)$; and
- *A smooth, S^1-equivariant map $\boldsymbol{\chi} = \boldsymbol{\chi}_{A_0,g_0}$,*

  $$\boldsymbol{\chi} : \mathcal{O} \times (-\varepsilon, \varepsilon) \subset H^1_{A_0} \oplus \mathbb{R} \to H^2_{A_0}$$

  such that $\boldsymbol{\gamma}$ restricts to a smoothly-stratified diffeomorphism from $(\boldsymbol{\chi}^{-1}(0) \cap (\mathcal{O} \times (-\varepsilon,\varepsilon)))/S^1$ onto $(M_E \cap \boldsymbol{\gamma}(\mathcal{O} \times (-\varepsilon,\varepsilon)))/S^1$.

The obstruction map can be defined by

$$\boldsymbol{\chi}(a,t) := \Pi_{A_0,g_0} P_+(g_0) f_t(F(A_0 + a)),$$

while the existence of the embedding $\boldsymbol{\gamma}$ follows from the Banach-space implicit function theorem. Here, Π_{A_0,g_0} is the L^2-orthogonal projection from $L^2(\Lambda^{+,g_0} \otimes \mathfrak{g}_E)$ onto the span of the eigenvectors of $d^{+,g_0}_{A_0} d^{*,g_0}_{A_0}$ on $L^2(\Lambda^{+,g_0} \otimes \mathfrak{g}_E)$ with eigenvalue less than $\frac{1}{2}\mu$, where μ is the first positive eigenvalue of this Laplacian.

The model described in Lemma 2.9 can be simplified: we may remove the factor $H^+(X;\mathbb{R})$ from the obstruction space at the cost of restricting the obstruction map $\boldsymbol{\chi}$ to the preimage $\boldsymbol{\chi}_{\mathbb{R}}^{-1}(0) \subset \mathbb{R} \oplus \mathbb{C}^n$, where $\boldsymbol{\chi}_{\mathbb{R}}$ is defined by (2.10).

Lemma 2.10. *Continue the notation and hypotheses of Lemma 2.9. Assume that $b^1(X) = 0$ and that A_0 is a reducible $\mathrm{SO}(3)$ connection with respect to the splitting $\mathfrak{g}_E \cong i\mathbb{R} \oplus \xi$. Then the following hold.*

- *If $n_\xi \geq 1$, then there are an open, S^1-invariant neighborhood $\mathcal{O}$ of the origin in $\mathbb{C}^n$ and a positive constant ε together with a smooth, S^1-equivariant embedding,*

  $$\boldsymbol{\gamma} : \mathcal{O} \times (-\varepsilon, \varepsilon) \subset \mathbb{C}^{n_\xi} \oplus \mathbb{R} \hookrightarrow \mathcal{A}_E,$$

  with $\boldsymbol{\gamma}(0,0) = (A_0, 0)$ and $M_E(\gamma) \cap \boldsymbol{\gamma}(\mathcal{O}/S^1 \times (-\varepsilon,\varepsilon))$, an open neighborhood of $[A_0, g_0]$ in $M_E(\gamma)$.

The map $\boldsymbol{\gamma}$ *restricts to a smoothly-stratified diffeomorphism from* $\boldsymbol{\chi}_{\mathbb{R}}^{-1}(0) \cap (\mathcal{O}/S^1 \times (-\varepsilon, \varepsilon))$ *onto* $M_E(\gamma) \cap \boldsymbol{\gamma}(\mathcal{O}/S^1 \times (-\varepsilon,\varepsilon))$*. Moreover, there is an* S^1*-equivariant diffeomorphism* $\boldsymbol{\chi}_{\mathbb{R}}^{-1}(0) \cap (\mathcal{O} \times (-\varepsilon,\varepsilon)) \cong \mathbb{C}^n$*.*

- *If* $n_\xi \leq 0$*, then* $[A_0, g_0]$ *is an isolated point of the parameterized moduli space* $M_E(\gamma)$*, with vanishing obstruction when* $n_\xi = 0$ *and obstruction space* $\mathbb{C}$ *when* $n_\xi = -1$*.*

Proof. The proof can be extracted from a similar argument used to prove the wall-crossing formula in [6, Proposition 2.12]. By (2.9), Theorem 2.8, and the fact that the metric g_0 is ξ-generic, $H^1_{A_0} = \mathbb{C}^{n_\xi}$ and $H^2_{A_0} = \mathbb{R}$ when $n_\xi \geq 0$, while $H^1_{A_0} = 0$ and $H^2_{A_0} = \mathbb{R} \oplus \mathbb{C}$ when $n_\xi = -1$. With respect to the splitting $\Lambda^+ \otimes \mathfrak{g}_E \cong i\Lambda^+ \oplus (\Lambda^+ \otimes \xi)$, we may write $\boldsymbol{\chi} = (\boldsymbol{\chi}_{\mathbb{R}}, \boldsymbol{\chi}_{\mathbb{C}})$, where

$$\begin{aligned} \boldsymbol{\chi}_{\mathbb{R}}(z,t) &= \langle \boldsymbol{\chi}(z,t), f_t(\omega_0)\rangle f_t(\omega_0), \\ &= \langle f_t(F(A_0 + \boldsymbol{\gamma}(z,t))), f_t(\omega_0)\rangle f_t(\omega_0), \quad (z,t) \in H^1_{A_0} \oplus (-\varepsilon,\varepsilon). \end{aligned} \tag{2.10}$$

According to Lemma 2.13 we may assume without loss that the path $\omega_t \equiv f_t(\omega_0)$ crosses the wall W_ξ transversely, so $(\partial \boldsymbol{\chi}_{\mathbb{R}}/\partial t)(0,0) \neq 0$. Hence, the map $\pi_{\mathbb{C}} : \boldsymbol{\chi}_{\mathbb{R}}^{-1}(0) \cong \mathbb{C}^{n_\xi}$ is an S^1-equivariant diffeomorphism. □

Remark 2.11. Suppose $\xi \in H^2(X;\mathbb{Z})$ defines a reduction $\mathfrak{g}_{E_0} \cong i\mathbb{R} \oplus \xi$. Because of Lemma 2.10 we see that, provided $b^1(X) = 0$, we need only consider the following cases where the family of reducibles in $M^w_{\kappa_0}(X, C)$ consists of either

- A single point $[A_0, g_0]$ with vanishing obstruction, when $n_\xi \geq 0$, or
- A single, isolated point $[A_0, g_0]$ with obstruction $H^2_{A_0,g_0} \cong \mathbb{C}$, when $n_\xi = -1$.

In the first case we would glue up an open neighborhood $\mathcal{U}_0 = \tilde{\mathcal{U}}_0/S^1$ $M^w_{\kappa_0}(X,C)$ of the point $[A_0, g_0]$; in the second, $\mathcal{U}_0$ could simply be the isolated point $[A_0, g_0]$ itself when gluing up. Here, in either case, we can assume that $\tilde{\mathcal{U}}_0 \subset \mathcal{A}^w_{\kappa_0}(X) \times \mathrm{Met}(X)$ is contained in the slice $(A_0 + \mathrm{Ker}\, d^{*,g_0}_{A_0}) \times C$, so $\tilde{\mathcal{U}}_0/\mathcal{G}_E = \mathcal{U}_0/S^1$. We shall make use of this simplification — namely, that we need only glue up anti-self-dual connections rather than *extended* anti-self-dual connections even when gluing up non-smooth points — when defining the space of gluing data, $\mathrm{Gl}^w_\kappa(X, \mathcal{U}_0, \Sigma, \lambda_0)$, in §3.

2.7. **Paths of metrics and the period map.** There is a technical point with regard to the definition of the family of background connections over X, in the presence of a path of metrics, which requires some comment. Suppose $g_{\pm 1}$ are two C^∞ metrics on TX and $\omega_{\pm 1}$ are the corresponding self-dual harmonic two-forms on X (we assume $b^+(X) = 1$). The metrics $g_{\pm 1}$ thus define points $[\omega_{\pm 1}]$ in $H^+(X;\mathbb{R}) - \{0\}$, lying in the positive cone Ω_X of $H^+(X;\mathbb{R})$ since if $\omega = \omega_{-1}$ or ω_1 then

$$\langle [\omega] \cup [\omega], [X]\rangle = \int_X \omega \wedge \omega = \int_X \omega \wedge *\omega = \|\omega\|^2_{L^2(X)} > 0.$$

In particular, the points $[\omega_{\pm 1}]$ lie in chambers $\mathcal{C}_\pm$ of Ω_X. If $\mathcal{C}_+ = \mathcal{C}_- = \mathcal{C}$, that is, these points lie in the *same* chamber, there is a question of whether a path $[\omega_t] \in \mathcal{C}$, $t \in [-1,1]$, joining $[\omega_{-1}]$ to $[\omega_1]$ can be lifted to a path of metrics g_t in $\mathrm{Met}(X)$ joining g_{-1} to g_1. For this purpose, one needs to consider the 'period map'

$$\pi : \mathrm{Met}(X) \to \mathbb{P}(H^2(X;\mathbb{R})) := H^2(X;\mathbb{R})/\mathbb{R}^*, \qquad g \mapsto \pi(g) := \mathcal{H}^+(g),$$

where $\mathcal{H}^{\pm}(g)$ are the spaces of self-dual and anti-self-dual harmonic two-forms. As remarked in [36, p. 434] it is not known whether two metrics whose period points lie in the same chamber can be connected by a path of metrics whose period points lie in the same chamber. However, following result of Donaldson will suffice for our needs: [4]

Proposition 2.12. [5, §6], [8, Proposition 4.3.14] *Let X be a closed, oriented four-manifold and let $\mathbb{G}(b^2, b^+)$ be the real Grassmann manifold of $b^+(X)$-dimensional subspaces of $H^2(X;\mathbb{R})$. Let $\mathbb{U} \subset \mathbb{G}(b^2, b^+)$ be the open subset of planes on which the intersection form Q of X is positive definite. Let $\pi : \mathrm{Met}(X) \to \mathbb{U}$ be the period map $g \mapsto \mathcal{H}^+(g)$. Let $\xi \in H^2(X;\mathbb{Z})$ be a class such that $\xi^2 < 0$ and let $W_\xi \subset \mathbb{U} = \{K \in \mathbb{U} : \xi \perp K\}$. Then W_ξ is a $b^+(X)$-codimensional submanifold and the period map π is transverse to W_ξ.*

When $b^+(X) = 1$ then $\mathbb{G}(b^2, b^+) = \mathbb{P}(H^2(X;\mathbb{R}))$, $\mathbb{U} = \Omega_X$, and W_ξ is a wall. For the purposes of proving the wall-crossing formula, it is enough to consider paths of metrics satisfying the conclusions of the following lemma, which comprise the preamble to Theorem 5.2.2 in [36].

Lemma 2.13. *Let X be a closed, oriented, smooth four-manifold with $b^+(X) = 1$ and suppose that $\mathcal{C}_\pm$ are chambers in Ω_X separated by a single geometric wall W. Then there is a generic path of smooth metrics $\gamma : [-1,1] \to \mathrm{Met}(X)$, $t \mapsto g_t$, with the following properties:*

1. *Fix a class $w \in H^2(X;\mathbb{Z})$ such that no $\mathrm{SO}(3)$ bundle V over X with $w_2(V) = w$ $(\mathrm{mod}\ 2)$ admits a flat connection. Then for every $\mathrm{U}(2)$ bundle E with $c_1(E) = w$ and $\kappa \equiv -\frac{1}{4}p_1(\mathfrak{g}_E) > 0$, the parameterized moduli space $M^w_\kappa(X,\gamma)$ contains no flat connections and is a C^∞ manifold away from the reducible connections.*
2. *For all $t \neq 0$, the compactification $\bar{M}^w_\kappa(X,\gamma)$ contains no points whose background connection is reducible or flat.*
3. *The self-dual two-form $\omega(g_0)$ lies in a single geometric (w,κ)-wall, W, while $\omega(g_t) \in \mathcal{C}_-$ for $t < 0$ and $\omega(g_t) \in \mathcal{C}_+$ for $t > 0$.*
4. *The path of self-dual two-forms $\omega(g_t)$ defined by γ crosses the wall W transversely.*

Proof. We may restrict our attention to an open neighborhood $U \subset \mathbb{P}(H^2(X;\mathbb{R}))$ such that $U \subset \mathcal{C}^- \cup W \cup \mathcal{C}^+$ and so we can express U as a disjoint union $U = U \cap \mathcal{C}^- \sqcup U \cap W \sqcup U \cap \mathcal{C}^+$. Furthermore, we choose U so that it only meets a single geometric wall W. The wall W is defined by some, not necessarily unique class $\xi \in H^2(X;\mathbb{Z})$. According to Proposition 2.12, the period map π is transverse to W and so $\pi^{-1}(W)$ is a smooth, codimension-one submanifold of $\mathrm{Met}(X)$. Plainly, we may choose a generic path $\gamma : [-1,1] \to \mathrm{Met}(X)$ whose image is contained in the open subset $\pi^{-1}(U)$ and which is transverse to $\pi^{-1}(W)$. Consequently, the induced path $\omega(\gamma) \equiv \pi(\gamma)$ in $H^2(X;\mathbb{R})/\mathbb{R}^+$ is transverse to W. Finally, we may choose γ short enough that $\omega(g_t) \in \mathcal{C}_-$ for all $t < 0$ and $\omega(g_t) \in \mathcal{C}_+$ for all $t > 0$. □

2.8. **Links of reducibles and the computational problem.** It remains to state the computational problem which concerns the present article and its sequels. Given the situation of Lemma 2.13 one can prove (see [36, Theorem 5.2.2], though we shall give a different

[4] The statement of Proposition 2.12 is taken from [5, p. 336]: the period map defined there is precisely dual to the one considered in [8].

argument elsewhere) the following local wall-crossing result:

$$D^w_{X,g_{+1}}(z) - D^w_{X,g_{-1}}(z) = \sum_\xi (-1)^{(w-\xi)^2/4}\delta^w_\xi(z), \quad z \in \mathbb{A}(X),$$

where the sum is over all $\xi \in H^2(X;\mathbb{Z})$ which define the geometric (w,κ)-wall W containing $\omega(g_0)$ and which satisfy $\xi \cdot \omega(g_{-1}) < 0 < \xi \cdot \omega(g_1)$, with $\deg z = 8\kappa - 3(2-b^1)$. The terms $\delta^w_\xi(z)$ have the following properties:

- δ^w_ξ is independent of where the path γ crosses the wall W,
- δ^w_ξ changes sign when the direction of crossing is reversed,
- $\delta^w_{-\xi} = (-1)^{\xi^2}\delta^w_\xi$.

It follows easily from these properties that $D^w_{X,g}(z) = D^w_{X,g'}(z)$ for any pair of metrics g, g' whose period points lie in the same chamber $\mathcal{C}$ of Ω_X, so $D^w_{X,\mathcal{C}}(z) = D^w_{X,g}(z)$ is well-defined and the preceding wall-crossing formula can be replaced by its global analogue:

$$D^w_{X,\mathcal{C}_+}(z) - D^w_{X,\mathcal{C}_{-1}}(z) = \sum_\xi (-1)^{(w-\xi)^2/4}\delta^w_\xi(z), \quad z \in \mathbb{A}(X),$$

with $\mathcal{C}_\pm$ in the same component of Ω_X and the sum is over all $\xi \in H^2(X;\mathbb{Z})$ which define geometric (w,κ)-walls satisfying $\xi \cdot \mathcal{C}_- < 0 < \xi \cdot \mathcal{C}_+$, with $\deg z = 8\kappa - 3(2-b^1)$.

Technicalities aside, each term $\delta^w_\xi(z)$ is defined by pairings

$$\delta^w_\xi(z) = \langle \mu(z), [\bar{\mathbf{L}}^w_{\kappa,\xi}]\rangle,$$

where $\mu : H_i(X;\mathbb{Q}) \to H^{4-i}(\mathcal{B}^w_\kappa(X)\times\gamma;\mathbb{Q})$ and $\mathbf{L}^w_{\kappa,\xi}$ is the "link" in $M^{w,*}_\kappa(X,\gamma)$ of the level containing the reducible SO(3) connection defined by w, ξ, κ,

$$[A_0] \times \operatorname{Sym}^\ell(X) \subset \bar{M}^w_\kappa(X,\gamma),$$

while $\bar{\mathbf{L}}^w_{\kappa,\xi}$ is its closure in $\bar{M}^w_\kappa(X,\gamma)$. Here, γ is a path satisfying the conclusions of Lemma 2.13 such that A_0 is a reducible g_0-anti-self-dual SO(3) connection on an SO(3) bundle $\mathfrak{g}_{E_0}$ with $w_2(\mathfrak{g}_{E_0}) = w \pmod 2$ and $-\frac{1}{4}p_1(\mathfrak{g}_{E_0}) = \kappa - \ell$. The reduction of the SO(3) bundle $\mathfrak{g}_{E_0}$ is defined by the class ξ: we have $E_0 = L \oplus (w \otimes L^{-1})$ and $\mathfrak{g}_{E_0} \cong \xi \oplus \underline{\mathbb{R}}$, where $\xi = w \otimes L^{-2}$, according to our standard convention, and thus $p_1(\mathfrak{g}_{E_0}) = \xi^2$.

The heart of the problem then is to show, using a more precise version of this "definition", that the δ^w_ξ depend at most on the cohomology classes $w, \xi \in H^2(X;\mathbb{Z})$, the homotopy type of X, and the degree of z. The above computation has been carried out directly when $\ell = 0,1,2$ (see [6], [40], [35], [36], [64]), which verifies the conjecture in these cases, so our task is to extend this work so we may conclude that the $\delta^w_\xi(z)$ have the desired invariance properties even if an explicit formula cannot be deduced by purely topological methods.

3. Splicing connections

We construct a *splicing map* $\boldsymbol{\gamma}'$ which will give a homeomorphism from the total space of a gluing data bundle into an open neighborhood in $\bar{\mathcal{B}}^w_\kappa(X)$ of a level $M^w_{\kappa-\ell}(X) \times \Sigma$, where $\Sigma \subset \operatorname{Sym}^\ell(X)$ is a smooth stratum. In this section we keep the Riemannian metric g on TX fixed, though we shall later allow the metrics g to vary in a precompact open subset of the space $\operatorname{Met}(X)$ of all C^∞ Riemannian metrics on TX. When defining our spaces of connections and gauge transformations, we shall always choose to work L^2_k connections modulo L^2_{k+1} gauge transformations with $k \geq 4$: though only $k \geq 2$ is strictly necessary, we shall occasionally need to consider estimates whose constants depend on the C^0 norm of F^+_A. Thus, *for the remainder of this article and its sequels* [14], [15], [16], we shall assume $k \geq 4$ in order to make use of the Sobolev embedding $L^2_4 \subset C^0$.

3.1. **Uhlenbeck neighborhoods.** In order to associate sequences of mass centers and scales to a sequence of points $[A_\alpha]$ in $M_E(X,g)$ converging to a point $[A_0,\mathbf{x}]$ in $\bar M_E(X,g)$, we need a suitable finite open cover of $\bar M_E(X,g)$; for simplicity, we take $C = \{g\}$. Let ϱ be the injectivity radius of the Riemannian manifold (X,g) and let $\operatorname{dist}_g(\cdot,\cdot)$ be the distance function on X defined by the metric g.

Definition 3.1. Suppose $[A_0,\mathbf{x}]$ is a point in $\bar M_E(X,g)$, where A_0 is a connection on a Hermitian, rank-two bundle E_0 over X with $c_2(E_0) < c_2(E)$, $c_1(E_0) = c_1(E)$ and ε,δ,M are positive constants with $M \geq 1$, and $\varepsilon < 2\pi$, and $M\delta \leq \frac{1}{4}\{1,\varrho,d_{\mathbf{x}}\}$, where $d_{\mathbf{x}} := \min_{i\neq j}\operatorname{dist}(x_i,x_j)$ is the minimum distance separating the distinct points x_i of the multiset $\mathbf{x} = (x_1,\dots,x_m)$. Assume $M\delta$ is small enough that

$$\int_{B(x_i,M\delta)} |F_{A_0}|^2\,dV < \tfrac{1}{4}\varepsilon^2.$$

Then $[A] \in M_E(X,g)$ lies in an (ε,δ,M) *Uhlenbeck open neighborhood* $\mathcal{U}$ of $[A_0,\mathbf{x}]$ if the following hold:

- There is an $L^2_{k+1,\mathrm{loc}}$ bundle map $u : E|_{X\setminus\{\mathbf{x}\}} \to E_0|_{X\setminus\{\mathbf{x}\}}$ such that the connections $u(A)$ and A_0 are L^2_k close on $X \setminus B_{\mathbf{x}}(\frac{1}{2}\delta)$:
$$\|u(A) - A_0\|_{L^2_k(A_0;X\setminus B(\mathbf{x},\delta/2))} < \varepsilon;$$
- The measure $|F_A|^2$ is close to $|F_{A_0}|^2 + 8\pi^2\kappa_{\mathbf{x}}\delta_{\mathbf{x}}$:
$$\left|\int_{B(x_i,\delta)} \left(|F_A|^2 - |F_{A_0}|^2 - 8\pi^2\kappa_i\delta_{x_i}\right)\,dV\right| < \tfrac{1}{4}\varepsilon^2,$$
$$\left|\int_{\Omega(x_i;\delta,M\delta)} \left(|F_A|^2 - |F_{A_0}|^2\right)\,dV\right| < \tfrac{1}{4}\varepsilon^2,$$
for $i = 1,\dots,m$ and where $(x_1,\dots,x_m)$ is a representative for $\mathbf{x}$ with multiplicities $\kappa_1,\dots,\kappa_m$.

The constraints on the curvature F_A in Definition 3.1 imply that

$$\begin{aligned}
&\left|8\pi^2\kappa_i - \int_{B(x_i,\delta)} |F_A|^2\,dV\right| < \tfrac{1}{2}\varepsilon^2,\\
&\int_{\Omega(x_i;\delta,M\delta)} |F_A|^2\,dV < \tfrac{1}{2}\varepsilon^2,\\
&\left|8\pi^2\kappa_i - \int_{B(x_i,M\delta)} |F_A|^2\,dV\right| < \varepsilon^2.
\end{aligned} \tag{3.1}$$

Observe that no cutoff functions are required to smooth out the implied characteristic functions of the balls with center x_i and radii δ, $M\delta$ above as Definition 3.1 rules out the possibility of the curvature F_A concentrating on their boundaries.

3.2. **Gluing data.** In this section we describe the gluing-data bundles we shall need to parameterize an open neighborhood in $\bar M^w_\kappa(X,C)$ of a level $(M^w_{\kappa-\ell}(X,C)\times\Sigma)\cap\bar M^w_\kappa(X,C)$.

3.2.1. *Connections over the four-sphere.* We begin by defining the required gluing data associated with the four-spheres: this will, essentially, define the fibers of our gluing-data bundles. Let $\kappa\geq 1$ be an integer and let S^4 have its standard round metric of radius one. Let E be a Hermitian, rank-two vector bundle over S^4 with $c_2(E)=\kappa$ and let $\mathfrak{g}_E=\mathfrak{su}(E)$ be the corresponding SO(3) bundle with $-\frac{1}{4}p_1(\mathfrak{g}_E)=\kappa$.

A choice of frame f in the principal SO(4) frame bundle $\mathrm{Fr}(TS^4)$ for TS^4, over the north pole $n\in S^4$, defines a conformal diffeomorphism,

$$\varphi_n:\mathbb{R}^4\to S^4\setminus\{s\}, \tag{3.2}$$

that is inverse to a stereographic projection from the south pole $s\in S^4\subset\mathbb{R}^5$; let $y(\,\cdot\,)$: $S^4\setminus\{s\}\to\mathbb{R}^4$ be the corresponding coordinate chart.

Definition 3.2. [55, pp. 343–344] Continue the notation of the preceding paragraph. The *center* $z=z([A],f)\in\mathbb{R}^4$ and the *scale* $\lambda=\lambda([A],f)$ of a point $[A]\in\mathcal{B}_E(S^4)=\mathcal{B}_\kappa(S^4)$ are defined by

$$z([A],f):=\frac{1}{8\pi^2\kappa}\int_{\mathbb{R}^4}y|F_A|^2\,d^4y, \tag{3.3}$$

$$\lambda([A],f)^2:=\frac{1}{8\pi^2\kappa}\int_{\mathbb{R}^4}|y-z([A],f)|^2|F_A|^2\,d^4y. \tag{3.4}$$

A point $[A]$ is *centered* if $z([A],f)=0$ and $\lambda([A],f)=1$. An ideal point $[A,\mathbf{x}]\in\mathcal{B}_{\kappa-\ell}(S^4)\times\mathrm{Sym}^\ell(S^4)$ is *centered* if $z([A,\mathbf{x}],f)=0$ and $\lambda([A,\mathbf{x}],f)=1$, where the center and scale are defined by replacing $|F_A|^2$ with $|F_A|^2+8\pi^2\sum_{x\in\mathbf{x}}\delta_x$ in equations (3.3) and (3.4).

For any $(\lambda,z)\in\mathbb{R}^+\times\mathbb{R}^4$, we define a conformal diffeomorphism of $\mathbb{R}^4$ by

$$f_{\lambda,z}:\mathbb{R}^4\to\mathbb{R}^4,\qquad y\mapsto(y-z)/\lambda. \tag{3.5}$$

It is easy to see that the point $[(f^{-1}_{\lambda,z})^*A]$ is centered if $z=z([A],f)$ and $\lambda=\lambda([A],f)$. Via the standard action of $\mathrm{SO}(4)\subset\mathrm{GL}(\mathbb{R}^4)$ on $\mathbb{R}^4$ and the conformal diffeomorphism $\varphi_n^{-1}:S^4\setminus\{s\}\to\mathbb{R}^4$, we obtain a homomorphism from SO(4) into the group of conformal diffeomorphisms of S^4 fixing the south pole s. The group SO(4) acts on the quotient space of connections based at the south pole, $\mathcal{B}^s_\kappa(S^4)\cong\mathcal{A}_\kappa(S^4)\times_{\mathcal{G}_E}\mathrm{Fr}(\mathfrak{g}_E)|_s$, by

$$\mathrm{SO}(4)\times\mathcal{B}^s_\kappa(S^4)\to\mathcal{B}^s_\kappa(S^4),\quad(u,[A,q])\mapsto u\cdot[A,q]=[(u^{-1})^*A,q]. \tag{3.6}$$

The assignment $f\mapsto z([A],f)$ is SO(4)-equivariant, that is, $z([A],uf)=uz([A],f)$ for $u\in\mathrm{SO}(4)$, and so $\lambda[A]=\lambda([A],f)$ is independent of the choice of $f\in\mathrm{Fr}(TS^4)|_n$. The center and scale functions define a smooth map $(z,\lambda):\mathcal{B}_\kappa(S^4)\to\mathbb{R}^4\times\mathbb{R}^+$ and the preimage of $(0,1)\in\mathbb{R}^4\times\mathbb{R}^+$, namely the codimension-five submanifold of *centered connections*

$$\mathcal{B}^\diamond_\kappa(S^4):=\{[A]:z[A]=0\text{ and }\lambda[A]=1\}\subset\mathcal{B}_\kappa(S^4), \tag{3.7}$$

serves as a canonical slice for the action of $\mathbb{R}^4 \times \mathbb{R}^+$ on $\mathcal{B}_\kappa(S^4)$; we simply write $z[A]$ when the frame f for $TS^4|_n$ is understood and employ the decoration $\diamond$ to indicate a subspace of centered connections. Since the action of SO(4) preserves the north and south poles of S^4, it acts on $\mathcal{B}_\kappa^{\diamond,s}(S^4)$, and so $\mathcal{B}_\kappa^{\diamond,s}(S^4)$ admits an action of $\mathrm{SO}(3) \times \mathrm{SO}(4)$, with SO(3) varying the frame of $\mathfrak{g}_E|_s$.

A simple *a priori* decay estimate for the curvature of connections is provided by the Chebychev inequality:

Lemma 3.3. *Let $[A] \in \mathcal{B}_\kappa(S^4)$ and suppose $z([A], f) = 0$. If $\lambda = \lambda[A]$ and $R > 0$, then*

$$\frac{1}{8\pi^2\kappa}\int_{|y|\geq R\lambda} |F_A|^2\, d^4y \leq R^{-2}.$$

Proof. From equation (3.4) we see that

$$\lambda^2 \geq \frac{1}{8\pi^2\kappa}\int_{|y|\geq R\lambda} |y|^2 |F_A|^2\, d^4y \geq \frac{R^2\lambda^2}{8\pi^2\kappa}\int_{|y|\geq R\lambda} |F_A|^2\, d^4y$$

and this gives the required bound. □

An immediate consequence of the Chebychev inequality is that if A is a centered anti-self-dual connection over S^4, with second Chern number κ so its energy is equal to $8\pi^2\kappa$, then A cannot bubble outside the ball $B(0, 1/2\pi) \subset \mathbb{R}^4$, as it has most energy $4\pi^2$ in this region by Lemma 3.3.

We let $M_\kappa^\diamond(S^4) \subset \mathcal{B}_\kappa^\diamond(S^4)$ denote the subspace of anti-self-dual connections, centered at the north pole, while $M_\kappa^s(S^4) \subset \mathcal{B}_\kappa^s(S^4)$ is the subspace of anti-self-dual connections, framed at the south pole. Finally, we let $M_\kappa^{\diamond,s}(S^4) \subset \mathcal{B}_\kappa^{\diamond,s}(S^4)$ denote the subspace of anti-self-dual connections, which framed at the south pole and centered at the north pole. The definition of Uhlenbeck convergence (Definition 2.3) for sequences of points $[A_\alpha]$ in $\mathcal{B}_\kappa(S^4)$ generalizes easily to the case of sequences $[A_\alpha, q_\alpha]$ in $\mathcal{B}_\kappa^s(S^4)$, when $\mathrm{Fr}(\mathfrak{g}_E|_s) \cong \mathrm{SO}(3)$ is given its usual topology. Since Lemma 3.3 implies that points in $M\diamond, s_\kappa(S^4)$ cannot bubble near the south pole, the proof of [8, Theorem 4.4.3] adapts with no change to show that $M_\kappa^{\diamond,s}(S^4)$ has compact closure $\bar{M}_\kappa^{\diamond,s}(S^4)$ in the space of framed, ideal anti-self-dual connections,

$$\bigcup_{\ell=0}^{\kappa} \left(M_{\kappa-\ell}^s(S^4) \times \mathrm{Sym}^\ell(S^4) \right).$$

Note that it is the ideal points $[A_0, \mathbf{x}]$ in $\bar{M}_{\diamond,s}^s\kappa(S^4)$ which are centered in the sense of Definition 3.2 and not the background connections A_0. Note also that the points in $S^4 = \mathbb{R}^4 \cup \{\infty\}$ representing the multisets $\mathbf{x}$ will be constrained to lie in the ball $\bar{B}(0, 1/2\pi)$ by Lemma 3.3.

Observe that since the bundle map u in the definition of an Uhlenbeck neighborhood respects the SO(3) action on frames, there is a global action of SO(3) on the space $\bar{M}_\kappa^{\diamond,s}(S^4)$. This action is free on all strata except $[\Theta] \times \mathrm{Sym}^\kappa(S^4)$, where the background connection is trivial. In addition, the SO(4) action on $M_\kappa^{\diamond,s}(S^4)$ extends over $\bar{M}_\kappa^{\diamond,s}(S^4)$.

3.2.2. *Gluing data bundles.* We now define bundles of gluing data, $\mathrm{Gl}_\kappa^w(X, \mathcal{U}_0, \Sigma, \lambda_0)$, associated to a choice of bundle data (κ, w), smooth stratum $\Sigma \subset \mathrm{Sym}^\ell(X)$, subset $C \subset \mathrm{GL}(T^*X)$, open neighborhood $\mathcal{U}_0 \subset M_{\kappa_0}^w(X, C)$, metric g on TX, and positive constant λ_0. Here,

$C \subset \mathrm{GL}(T^*X)$ is a compact, connected, smooth submanifold-with-boundary containing the identity: the pair (C, g) defines a compact, connected, smooth submanifold-with-boundary of $\mathrm{Met}(X)$ which we also denote, when no confusion can arise, by C. Our definition is motivated by those of Friedman-Morgan [25, §§3.4.2–3.4.4], Kotschick-Morgan [36, §4], and Taubes [55], [56].

Choose $\kappa \in \frac{1}{4}\mathbb{Z}$ and $w \in H^2(X;\mathbb{Z})$ according to condition (2.1) and let E be the corresponding Hermitian, rank-two vector bundle over X defined by condition (2.4). Let ℓ be an integer such that $1 \leq \ell \leq \lfloor \kappa \rfloor$ and let $\Sigma \subset \mathrm{Sym}^\ell(X)$ be a smooth stratum, defined by an integer partition $\kappa_1 \geq \cdots \geq \kappa_m \geq 1$ of ℓ such that $\kappa_1 + \cdots + \kappa_m = \ell$. Let E_0 be the Hermitian, rank-two vector bundle over X with $\det E_0 = \det E$ and $c_2(E_0) = c_2(E) - \ell$. Let $\mathrm{Fr}(TX)$ denote the principal $\mathrm{SO}(4)$ bundle of oriented, g-orthonormal frames for TX. Suppose $\tilde{\mathcal{U}}_0 \subset \mathcal{A}_{E_0}(X) \times C$ projects to an open subset $\mathcal{U}_0 \subset M_{E_0}(X, C)$. Observe that the fibered product

$$\tilde{\mathcal{U}}_0 \times_{\mathcal{G}_{E_0}} \prod_{i=1}^m (\mathrm{Fr}(\mathfrak{g}_{E_0}) \times_X \mathrm{Fr}(TX)) \to \mathcal{U}_0 \times \prod_{i=1}^m X \tag{3.8}$$

is a principal $\prod_{i=1}^m (\mathrm{SO}(3) \times \mathrm{SO}(4))$ bundle. Let $\Delta \subset \prod_{i=1}^m X$ be the full diagonal, that is, the subset of m-tuples of points in X where some point occurs with multiplicity at least two. Let $\mathfrak{S} = \mathfrak{S}_\Sigma = \mathfrak{S}(\kappa_1, \dots, \kappa_m)$ be the largest subgroup of the symmetric group on m symbols which preserves the multiplicities. The group $\mathfrak{S}(\kappa_1, \dots, \kappa_m)$ acts freely on $\prod_{i=1}^m X - \Delta$ and we have an identification

$$\left(\prod_{i=1}^m X - \Delta\right) / \mathfrak{S}(\kappa_1, \dots, \kappa_m) = \Sigma \subset \mathrm{Sym}^\ell(X).$$

The group $\mathfrak{S}(\kappa_1, \dots, \kappa_m)$ acts on the fibered product bundle (3.8) to give a principal bundle

$$\mathbf{Fr}_{E_0}(X, \mathcal{U}_0, \Sigma) \to \mathcal{U}_0 \times \Sigma, \tag{3.9}$$

with total space $\mathbf{Fr}^w_\kappa(X, \mathcal{U}_0, \Sigma) = \mathbf{Fr}_{E_0}(X, \mathcal{U}_0, \Sigma)$, where

$$\mathbf{Fr}_{E_0}(X, \mathcal{U}_0, \Sigma) := \left(\tilde{\mathcal{U}}_0 \times_{\mathcal{G}_{E_0}} \prod_{i=1}^m (\mathrm{Fr}(\mathfrak{g}_{E_0}) \times_X \mathrm{Fr}(TX))\right)|_{\prod_{i=1}^m X - \Delta} \tag{3.10}$$

and structure group

$$\mathbf{G}(\Sigma) = \mathbf{G}(\kappa_1, \dots, \kappa_m) := \left(\prod_{i=1}^m (\mathrm{SO}(3) \times \mathrm{SO}(4))\right) \rtimes \mathfrak{S}(\kappa_1, \dots, \kappa_m). \tag{3.11}$$

Let S^4 have its standard round metric of radius one, let $E_1, \dots, E_m$ be Hermitian, rank-two vector bundles over S^4 with $c_2(E_j) = \kappa_j$ for $j = 1, \dots, m$, and let $\mathfrak{g}_{E_j} \equiv \mathfrak{su}(E_j)$ be the corresponding $\mathrm{SO}(3)$ bundles with $-\frac{1}{4}p_1(\mathfrak{g}_{E_j}) = \kappa_j$. Denote $\mathbb{R}^+ := (0, \infty)$. The smooth manifold

$$\mathcal{Z}(\Sigma) = \mathcal{Z}(\kappa_1, \dots, \kappa_m) := \prod_{i=1}^m \left(\tilde{M}^\diamond_{E_i} \times_{\mathcal{G}_{E_i}} \mathrm{Fr}(\mathfrak{g}_{E_i})|_s \times \mathbb{R}^+\right) \tag{3.12}$$

carries a natural action of $\mathbf{G}(\kappa_1, \dots, \kappa_m)$ coming from the action of $\mathrm{SO}(4)$ rotating S^4 and the action of $\mathrm{SO}(3)$ on the choice of frame for $\mathfrak{g}_{E_i}|_s$ in each factor, and the action of

$\mathfrak{S}(\kappa_1,\dots,\kappa_m)$ permuting the factors whose underlying bundles are isomorphic. The bundle $\mathrm{Fr}(TX)$ has its usual right $\mathrm{SO}(4)$ action given by

$$\mathrm{SO}(4)\times \mathrm{Fr}(TX)\to \mathrm{Fr}(TX),\quad (u,f)\mapsto fu^{-1}. \tag{3.13}$$

We form the fibered product $\mathrm{Gl}^w_\kappa(X,\mathcal{U}_0,\Sigma)=\mathrm{Gl}_E(X,\mathcal{U}_0,\Sigma)$, where

$$\mathrm{Gl}_E(X,\mathcal{U}_0,\Sigma):=\mathbf{Fr}_{E_0}(X,\mathcal{U}_0,\Sigma)\times_{\mathbf{G}(\Sigma)}\mathcal{Z}(\Sigma), \tag{3.14}$$

together with the natural projection map

$$\pi:\mathrm{Gl}_E(X,\mathcal{U}_0,\Sigma)\to\mathcal{U}_0\times\Sigma. \tag{3.15}$$

This is a locally trivial fiber bundle with structure group $\mathbf{G}(\kappa_1,\dots,\kappa_m)$. We let $\mathcal{Z}(\Sigma,\lambda_0)\subset\mathcal{Z}(\Sigma)$ denote the open subspace obtained by replacing $\mathbb{R}^+$ by $(0,\lambda_0)$ in the definition (3.12) and let $\mathrm{Gl}^w_\kappa(X,\mathcal{U}_0,\Sigma,\lambda_0)=\mathrm{Gl}_E(X,\mathcal{U}_0,\Sigma,\lambda_0)$ denote the corresponding gluing data bundle.

3.2.3. *Compactification of the gluing data.* The gluing-data bundle $\mathrm{Gl}_{E_0}(X,\mathcal{U}_0,\Sigma)$ has two sources of non-compactness, the non-compact fiber $Z(\Sigma)$ and the non-compact base, Σ. We now describe a compactification of $\mathrm{Gl}_{E_0}(X,\mathcal{U}_0,\Sigma)$. It is a trivial matter to replace the precompact neighborhood $\mathcal{U}_0\subset M_{E_0}(X,C)$ by its compactification $\bar{\mathcal{U}}_0$, so we assume that $\mathcal{U}_0$ is itself compact.

We first describe the closure of the fibers. The definition of $\mathcal{Z}(\Sigma)$ uses the space $M^{\diamond,s}_{E_i}(S^4)$ and an dilation parameter lying in $\mathbb{R}^+=(0,\infty)$. The dilation action (3.5) maps points in $M^{\diamond,s}_{E_i}\times(0,\infty)$ to framed connections on $\mathfrak{g}_{E_i}$. This action extends naturally to $\bar{M}^{\diamond,s}_{E_i}(S^4)\times(0,\infty)$. The Uhlenbeck limit of these connections as the dilation parameter goes to zero is given by $[\Theta,\mathbf{x}]$ where Θ is a trivial connection and $\mathbf{x}\in\mathrm{Sym}^\kappa(S^4)$ is represented by $(n,n,\dots,n)$, where n is the north pole. Thus, we define a ccompletion of

$$\bar{M}^{\diamond,s}_{E_i}(S^4)\times(0,\infty),$$

by the cone $c(\bar{M}^{\diamond,s}_{E_i}(S^4))$, where the cone point corresponds to zero dilation parameter. Then, let

$$\bar{\mathcal{Z}}(\Sigma)=\bar{\mathcal{Z}}(\kappa_1,\dots,\kappa_m):=\prod_{i=1}^m c(\bar{M}^{\diamond,s}_{E_i}(S^4)).$$

The fibers of the gluing data bundle are then compactified if we work with the space:

$$\bar{\mathrm{G}}\mathrm{l}_E(X,\mathcal{U}_0,\Sigma):=\mathbf{Fr}_{E_0}(X,\mathcal{U}_0,\Sigma)\times_{\mathbf{G}(\Sigma)}\bar{\mathcal{Z}}(\Sigma). \tag{3.16}$$

As the only difference between this space and $\mathrm{Gl}_E(X,\mathcal{U}_0,\Sigma)$ is given by the change in the fiber from $\mathcal{Z}(\Sigma)$ to $\bar{\mathcal{Z}}(\Sigma)$, the projection (3.15) extends to

$$\pi:\bar{\mathrm{G}}\mathrm{l}_E(X,\mathcal{U}_0,\Sigma)\to\mathcal{U}_0\times\Sigma,$$

and this extension is still a locally trivial fiber bundle with the same structure group. We again let $\bar{\mathrm{G}}\mathrm{l}_E(X,\mathcal{U}_0,\Sigma,\lambda_0)$ be the open subspace given by requiring all dilation parameters to be less than λ_0.

We deal with the non-compact base, Σ, by replacing $\mathbf{Fr}_{E_0}(X,\mathcal{U}_0,\Sigma)$ with

$$\mathbf{Fr}_{E_0}(X,\mathcal{U}_0,\bar{\Sigma}):=\tilde{\mathcal{U}}_0\times_{\mathcal{G}_{E_0}}\prod_{i=1}^m\left(\mathrm{Fr}(\mathfrak{g}_{E_0})\times_X\mathrm{Fr}(TX)\right). \tag{3.17}$$

The quotient $X^m/\mathfrak{S}_\Sigma$ can be identified with $\bar\Sigma$, the closure of Σ in $\mathrm{Sym}^\ell(X)$, and so there is a projection

$$\mathbf{Fr}_{E_0}(X,\mathcal{U}_0,\bar\Sigma) \to \mathcal{U}_0 \times \bar\Sigma.$$

However, because the action of $\mathfrak{S}_\Sigma$ is not free over $\bar\Sigma\backslash\Sigma$, this extension is not a fiber bundle.

The splicing and gluing maps are defined on a subspace of the following extension of the gluing data bundle,

$$\bar{\mathrm{G}}\mathrm{l}_E(X,\mathcal{U}_0,\bar\Sigma) := \mathbf{Fr}_{E_0}(X,\mathcal{U}_0,\bar\Sigma) \times_{\mathbf{G}(\Sigma)} \bar{\mathcal{Z}}(\Sigma), \tag{3.18}$$

as we shall see in §3.3.3.

3.3. **Splicing connections over** X**.** In this section we describe the *splicing construction* for SO(3) connections on $\mathfrak{g}_E$ via a partition of unity on X and define the *splicing maps* $\boldsymbol{\gamma}'$ for connections. Our description is largely motivated by those of Taubes in [52], [53], [54], [55, §4], [56]. We keep the notation of §3.2.

3.3.1. *Cutting and splicing connections.* Fix a smooth stratum $\Sigma \subset \mathrm{Sym}^\ell(X)$, let $\mathbf{x} \in \Sigma$, and choose $r_0 = r_0(\mathbf{x}) > 0$ to be one half the smaller of

- The minimum geodesic distance between distinct points x_i, x_j of the representative $(x_1,\dots,x_m)$ of $\mathbf{x}$, and
- The injectivity radius of the Riemannian manifold (X,g).

A choice of L^2_k connection A_0 on $\mathfrak{g}_{E_0}$ and SO(3) frames $p_i \in \mathrm{Fr}(\mathfrak{g}_{E_0})|_{x_i}$ define L^2_{k+1} local sections of $\mathrm{Fr}(\mathfrak{g}_{E_0})$,

$$\sigma_i = \sigma_i(A_0,p_i) : B(x_i,r_0) \to \mathrm{Fr}(\mathfrak{g}_{E_0}), \quad i = 1,\dots,m,$$

using parallel transport along radial geodesics emanating from x_i together with smoothing for sections of $\mathfrak{g}_{E_0}$ using the heat kernel $\exp(-td^*_{A_0}d_{A_0})$ for small t, as described in [21, §A.1]; see also the remarks in [56, p. 177]. Let

$$\mathfrak{g}_{E_0}|_{B(x_i,r_0)} \simeq B(x_i,r_0) \times \mathrm{SO}(3) \tag{3.19}$$

be the L^2_{k+1} local trivializations defined by the sections σ_i.

Define a smooth cutoff function $\chi_{x_0,\varepsilon} : X \to [0,1]$ by setting

$$\chi_{x_0,\varepsilon}(x) := \chi(\mathrm{dist}(x,x_0)/\varepsilon) \tag{3.20}$$

where $\chi : \mathbb{R} \to [0,1]$ is a smooth function such that $\chi(t) = 1$ for $t \ge 1$ and $\chi(t) = 0$ for $t \le 1/2$. Thus, we have

$$\chi_{x_0,\varepsilon}(x) = \begin{cases} 1 & \text{for } x \in X - B(x_0,\varepsilon), \\ 0 & \text{for } x \in B(x_0,\varepsilon/2). \end{cases}$$

We define a cut-off connection on the bundle $\mathfrak{g}_{E_0}$ over X by setting

$$\chi_{\mathbf{x},4\sqrt{\boldsymbol{\lambda}}}A_0 := \begin{cases} A_0 & \text{over } X - \cup_{i=1}^m B(x_i,4\sqrt{\lambda_i}), \\ \Gamma + \chi_{x_i,4\sqrt{\lambda_i}}\sigma_i^* A_0 & \text{over } \Omega(x_i,2\sqrt{\lambda_i},4\sqrt{\lambda_i}), \\ \Gamma & \text{over } B(x_i,2\sqrt{\lambda_i}), \end{cases} \tag{3.21}$$

where Γ denotes the product connection on $B(x_i,r_0)\times\mathrm{SO}(3)$, while $\mathbf{x} = (x_1,\dots,x_m)$ and $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_m)$.

Let A_i be centered L^2_k connections on the bundles $\mathfrak{g}_{E_i}$ over S^4. Let

$$\tau_i = \tau_i(A_i, q_i) : S^4 \setminus \{n\} \to \mathrm{Fr}(\mathfrak{g}_{E_i}), \quad i = 1, \dots, m,$$

be the analogously-defined L^2_{k+1} local sections induced by the connections $A_i \in \mathcal{A}_{E_i}$ and SO(3) frames $q_i \in \mathrm{Fr}(\mathfrak{g}_{E_i})|_s$. Let

$$\mathfrak{g}_{E_i}|_{S^4 \setminus \{n\}} \simeq S^4 \setminus \{n\} \times \mathrm{SO}(3) \tag{3.22}$$

be the L^2_{k+1} local trivializations defined by the τ_i. Let $\delta_\lambda : S^4 \to S^4$ be the conformal diffeomorphism of S^4 defined by

$$\delta_\lambda := \varphi_n \circ \lambda \circ \varphi_n^{-1}, \tag{3.23}$$

where $\lambda : \mathbb{R}^4 \to \mathbb{R}^4$ is the dilation given by $y \mapsto y/\lambda$, so δ_λ is a conformal diffeomorphism of S^4 which fixes the north and south poles. We define cut-off, rescaled connections on the bundles $\mathfrak{g}_{E_i}$ over S^4, with mass centers at the north pole, by setting

$$(1 - \chi_{x_i,\sqrt{\lambda_i}/2})\delta^*_{\lambda_i} A_i := \begin{cases} \delta^*_{\lambda_i} A_i & \text{over } \varphi_n(B(0, \tfrac{1}{4}\sqrt{\lambda_i})), \\ \Gamma + (1 - \chi_{x_i,\sqrt{\lambda_i}/2})\tau_i^*\delta^*_{\lambda_i} A_i & \text{over } \varphi_n(\Omega(0, \tfrac{1}{4}\sqrt{\lambda_i}, \tfrac{1}{2}\sqrt{\lambda_i})), \\ \Gamma & \text{over } S^4 - \varphi_n(B(0, \tfrac{1}{2}\sqrt{\lambda_i})), \end{cases} \tag{3.24}$$

where Γ denotes the product connection on $(S^4 - \{n\}) \times \mathrm{SO}(3)$. Note that $\tau_i^*\delta^*_{\lambda_i} A_i = \delta^*_{\lambda_i}\tau_i^* A_i$, since the section τ_i is defined by parallel translation from the south pole via the connection A_i.

A choice of SO(4) frames $f_i \in \mathrm{Fr}(TX)|_{x_i}$ defines coordinate charts

$$\varphi_i^{-1} : B(x_i, r_0) \subset X \to \mathbb{R}^4, \quad i = 1, \dots, m,$$

via the exponential maps $\exp_{f_i} : B(0, r_0) \subset TX|_{x_i} \to X$. The orientation-preserving diffeomorphism $\varphi_i \circ \varphi_n^{-1}$ identifies the annulus $\varphi_n(\Omega(0, \frac{1}{2}\sqrt{\lambda_i}, 2\sqrt{\lambda_i}))$ in S^4,

$$\Omega\left(0, \tfrac{1}{2}\sqrt{\lambda_i}, 2\sqrt{\lambda_i}\right) := \varphi_n\left(\left\{x \in \mathbb{R}^4 : \tfrac{1}{2}\sqrt{\lambda_i} < |x| < 2\sqrt{\lambda_i}\right\}\right) \subset \mathbb{R}^4$$

with the annulus in X,

$$\Omega\left(x_i, \tfrac{1}{2}\sqrt{\lambda_i}, 2\sqrt{\lambda_i}\right) := \left\{x \in X : \tfrac{1}{2}\sqrt{\lambda_i} < \mathrm{dist}_g(x, x_i) < 2\sqrt{\lambda_i}\right\} \subset X.$$

We define a glued-up SO(3) bundle $\mathfrak{g}_E$ over X by setting

$$\mathfrak{g}_E = \begin{cases} \mathfrak{g}_{E_0} & \text{over } X - \cup_{i=1}^m B(x_i, \tfrac{1}{2}\sqrt{\lambda_i}), \\ \mathfrak{g}_{E_i} & \text{over } B(x_i, 2\sqrt{\lambda_i}). \end{cases} \tag{3.25}$$

The bundles $\mathfrak{g}_{E_0}$ and $\mathfrak{g}_{E_i}$ are identified over the annuli $\Omega(x_i, \frac{1}{2}\sqrt{\lambda_i}, 2\sqrt{\lambda_i}\})$ in X via the isomorphisms of SO(3) bundles defined by the orientation-preserving diffeomorphisms $\varphi_i \circ \varphi_n^{-1}$, identifying the annuli $\Omega(x_i, \frac{1}{2}\sqrt{\lambda_i}, 2\sqrt{\lambda_i}\})$ with the corresponding annuli in S^4 and the SO(3) bundle maps defined by the trivializations (3.19) and (3.22). Define a spliced L^2_k connection A on $\mathfrak{g}_E$ by setting

$$A := \begin{cases} A_0 & \text{over } X - \cup_{i=1}^m B(x_i, 4\sqrt{\lambda_i}), \\ \Gamma + \chi_{x_i,4\sqrt{\lambda_i}} A_0 + (1 - \chi_{x_i,\sqrt{\lambda_i}/2})\delta^*_{\lambda_i} A_i & \text{over } \Omega(x_i, \tfrac{1}{4}\sqrt{\lambda_i}, 4\sqrt{\lambda_i}), \\ \delta^*_{\lambda_i} A_i & \text{over } B(x_i, \tfrac{1}{4}\sqrt{\lambda_i}),\ i = 1, \dots, m, \end{cases} \tag{3.26}$$

where the cut-off connections are defined as above; the bundle and annulus identifications are understood. Also implicit in definition (3.26) are the following constraints on the scales, given an m-tuple of distinct points, $(x_1, \dots, x_m)$,

$$8\sqrt{\lambda_i} + 8\sqrt{\lambda_j} < \operatorname{dist}_g(x_i, x_j), \quad i \neq j, \tag{3.27}$$

representing a multiset $\mathbf{x} \in \Sigma$. The *pre-splicing map*, $\boldsymbol{\gamma}''$, is defined on the open subset of the gluing-data bundle, $\mathrm{Gl}_\kappa^w(X, \mathcal{U}_0, \Sigma)$ in definition (3.14), defined by

$$\mathrm{Gl}_\kappa^{w,+}(X, \mathcal{U}_0, \Sigma) := \{\pi^{-1}([A_0], \mathbf{x}) \in \mathrm{Gl}_\kappa^w(X, \mathcal{U}_0, \Sigma) : (\boldsymbol{\lambda}, \mathbf{x}) \text{ obeys } (3.27)\}. \tag{3.28}$$

If we replace Σ in $\mathrm{Gl}_\kappa^w(X, \mathcal{U}_0, \Sigma)$ by a compact subset $K \Subset \Sigma$, so that there is a uniform positive lower bound for $\operatorname{dist}_g(x_i, x_j)$, $i \neq j$, for all $\mathbf{x} \in K$, then we can plainly choose a positive constant $\lambda_0 = \lambda_0(K, g)$ such that

$$\mathrm{Gl}_\kappa^w(X, \mathcal{U}_0, K, \lambda_0) \subset \mathrm{Gl}_\kappa^{w,+}(X, \mathcal{U}_0, \Sigma),$$

as the m-tuples of scales $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ present in $\mathrm{Gl}_\kappa^w(X, \mathcal{U}_0, K, \lambda_0)$ satisfy the constraints (3.27).

3.3.2. *Splicing maps.* It remains to summarize what the splicing construction has achieved. The construction, thus far, yields a pre-splicing map giving a smooth embedding

$$\boldsymbol{\gamma}'' : \mathrm{Gl}_\kappa^{w,+}(X, \mathcal{U}_0, \Sigma) \to \mathcal{B}_\kappa^{w,*}(X) \times \mathrm{Met}(X) \tag{3.29}$$

The fact that the map is a smooth embedding is a fairly straightforward consequence of the definitions; it will follow from the stronger result in [14] that the gluing map $\boldsymbol{\gamma}$ gives a smooth embedding of the gluing data. Similarly, the fact that the image of $\boldsymbol{\gamma}''$ contains only irreducible connections is also a fairly easy consequence of the definitions; it is proved in [14].

The simple pre-splicing map we have described in §3.3.1 is not the the map we shall ultimately use to produce almost anti-self-dual connections, however, for the reasons we mentioned in the Introduction. Specifically, we have assumed that the connections on the different copies of S^4 are anti-self-dual with respect to the standard round metric g_0 of radius one. In practice, we shall need to first use Theorem 1.1 to produce diffeomorphisms from the spaces $M_{\kappa_i}^s(S^4, g_0)$ to $M_{\kappa_i}^s(S^4, \tilde{g}_i)$, where $\tilde{g}_i$ is an almost round metric on S^4 which is conformally equivalent to g near the points $x_i \in X$. More precisely, we first define families of almost flat metrics g_i on $\mathbb{R}^4 = (TX)_{x_i}$, depending on the metric g on X, gluing site x_i, and scale λ_i, by setting

$$g_i := \begin{cases} \chi_{8\sqrt{\lambda_i}}\delta + (1 - \chi_{8\sqrt{\lambda_i}}) \exp_{f_i}^* g & \text{on } B(0, 8\sqrt{\lambda_i}), \\ & \text{on } (TX)_{x_i} - B(0, 8\sqrt{\lambda_i}), \end{cases} \tag{3.30}$$

where $\chi_{8\sqrt{\lambda_i}}$ is given by definition (3.20). Here, δ is simply the Euclidean metric on $(TX)_{x_i}$ defined by the metric g on TX, noting that $(\exp_{f_i}^* g)(0) = \delta$. We then obtain a family of conformally equivalent metrics $\tilde{g}_i$ on $S^4 = (TX)_{x_i} \cup \{\infty\}$ which is close to the standard metric g_0 by setting

$$\varphi_n^* \tilde{g}_i(x) := \frac{4}{(1 + |x|)^2} g_i(\lambda_i x), \quad x \in \mathbb{R}^4, \tag{3.31}$$

where we recall that $\varphi_n : \mathbb{R}^4 \to S^4 - \{s\}$ is inverse to stereographic projection. The definition of geodesic, normal (or 'Gaussian') coordinates implies that on a small ball around the points $x_i \in X$, the metric g_i is C^1-close to g and that the difference between g and g_i is C^2-bounded, uniformly with respect to the ball radius [3].

Lemma 3.4. *Continue the above notation. Then $\tilde{g}_i$ converges to g_0 in $C^\infty(S^4 \setminus s)$ as λ_i converges to zero.*

Hence, the true splicing map,

$$\boldsymbol{\gamma}' : \mathrm{Gl}_\kappa^{w,+}(X, \mathcal{U}_0, \Sigma) \to \mathcal{B}_\kappa^{w,*}(X) \times \mathrm{Met}(X) \tag{3.32}$$

will be a composition of the diffeomorphisms,

$$M^s_{\kappa_i}(S^4, g_0) \cong M^s_{\kappa_i}(S^4, \tilde{g}_i),$$

followed by the pre-splicing map $\boldsymbol{\gamma}''$ in definition (3.29) via the cut-and-paste operation described in §3.3.1.

3.3.3. *Extension of the splicing and gluing maps.* It remains to define an extension of the pre-splicing map $\boldsymbol{\gamma}''$, splicing map $\boldsymbol{\gamma}'$, and gluing map $\boldsymbol{\gamma}$ from $\mathrm{Gl}_\kappa^{w,+}(X, \mathcal{U}_0, \Sigma, \lambda_0)$ to $\bar{\mathrm{Gl}}_\kappa^{w,+}(X, \mathcal{U}_0, \bar{\Sigma}, \lambda_0)$. Thus, given a point $\mathbf{p}$ in $\bar{\mathrm{Gl}}_\kappa^{w,+}(X, \mathcal{U}_0, \bar{\Sigma}, \lambda_0)$, we obtain a spliced, ideal connection $\boldsymbol{\gamma}'(\mathbf{p}) = [A, \mathbf{x}] \in \mathcal{B}^w_{\kappa-\ell}(X) \times \mathrm{Sym}^\ell(X)$ by the following procedure:

- The connection A is obtained by splicing the background connection A_0 and the connections A_i on S^4 for which $\lambda_i > 0$, exactly as in §3.3.2.
- The multiset $\mathbf{x} \in \mathrm{Sym}^\ell(X)$ is the union of
  - All points $x_i \in \pi_2(\mathbf{p})$, where π_2 is the projection from $\bar{\mathrm{Gl}}_\kappa^{w,+}(X, \mathcal{U}_0, \bar{\Sigma}, \lambda_0)$ onto the second factor, $\bar{\Sigma}$, for which the corresponding scale λ_i is zero.
  - All points $\exp_{f_i}(\lambda_i \varphi_n^{-1}(y))$ such that $\lambda_i \neq 0$, $y \in \mathbf{y}_i$, and $[A_i, \mathbf{y}_i, q_i] \in \bar{M}^{\diamond,s}_{\kappa_i}(S^4)$, where $\mathbf{y}_i$ is non-empty.

The requisite domain of the splicing and gluing maps is given by $\bar{\mathrm{Gl}}^+_E(X, \mathcal{U}_0, \bar{\Sigma}, \lambda_0)$, where the $+$ indicates that the pairs $(\mathbf{x}, \boldsymbol{\lambda})$ must satisfy the constraints (3.27) (with equality now allowed), essentially mimicking the definition (3.28) of $\mathrm{Gl}^+_E(X, \mathcal{U}_0, \Sigma, \lambda_0)$.

The pre-splicing map (3.29) has an obvious extension to a map

$$\boldsymbol{\gamma}'' : \bar{\mathrm{Gl}}_\kappa^{w,+}(X, \mathcal{U}_0, \Sigma) \to \bar{\mathcal{B}}_\kappa^{w,*}(X) \times \mathrm{Met}(X), \tag{3.33}$$

and similarly for the splicing map $\boldsymbol{\gamma}'$ and the gluing map $\boldsymbol{\gamma}$, which we define in §7.5. One of the principal assertions of our main result, Theorem 1.3, is that our choice of extensions of these maps is the right one, as the resulting extensions are shown to be continuous.

4. Regularity of solutions to quasi-linear systems

Let A be a C^∞, orthogonal connection on the SO(3) bundle $\mathfrak{g}_E = \mathfrak{su}(E)$, where E is a Hermitian, rank-two vector bundle over X. Based on the approach we gave in [21] for the PU(2) monopole equations, we give the regularity theory — which we shall need both in the present article and its sequels [14], [15], [16] — for solutions $a \in L^2_1(\Lambda^1 \otimes \mathfrak{g}_E)$ to the quasi-linear elliptic system consisting of a quasi-linear equation, with a quadratic non-linearity, and the inhomogeneous Coulomb-gauge equation,

$$\begin{aligned} d_A^+ a + (a\wedge a)^+ &= w, \\ d_A^* a &= \zeta, \end{aligned} \tag{4.1}$$

where $(\zeta, w) \in L_k^p(\mathfrak{g}_E) \oplus L_k^p(\Lambda^+ \otimes \mathfrak{g}_E)$. We use the isomorphism $\mathrm{ad} : \mathfrak{g}_E \to \mathfrak{so}(\mathfrak{g}_E)$ to view sections of $\mathfrak{so}(\mathfrak{g}_E)$ as sections of $\mathfrak{g}_E$. A systematic treatment is given in [21, §3] and we merely summarize the main results here; some special cases appear in [8] and [24]. As in [52], [56], we use $a = d_A^* v$, where $v \in L_2^2(X, \Lambda^+ \otimes \mathfrak{g}_E)$, to rewrite the first-order quasi-linear equation for a in terms of w in (4.1) as

$$d_A^+ d_A^* v + (d_A^* v \wedge d_A^* v)^+ = w, \tag{4.2}$$

to give a second-order, quasi-linear elliptic equation for v in terms of w; the Coulomb-gauge equation reduces to $d_A^* d_A^* v = (F_A^+)^* v$.

Though our applications of the regularity results of this section will often be to solutions a to the anti-self-equation, $F^+(A + a) = 0$, namely equation (4.1) with $w = -F_A^+$, we shall also need to consider the regularity theory for more general equations, such as the various forms of the *extended anti-self-dual equation* (and their solutions, defining *extended anti-self-dual connections*) introduced in [7], [25], [54], [56] — see equations (7.4), (7.6), and (7.7), for example. These more general equations can be viewed as special cases of (4.1) or (4.2) obtained by a suitable choice of w, such as that given in equation (8.5). Therefore, by considering the slightly more general equations (4.1) and (4.2), we can address the regularity theory simultaneously for all such equations.

Remark 4.1. The precise dependence of the constants C on the connection A in this section is not something we track carefully, as we are primarily interested in local L_k^2 estimates of solutions a to equations (4.1) or of solutions v to equations (4.2) when $k \geq 2$ or $k \geq 3$. For this range of the Sobolev indices, the constants no longer depend on simply $\|F_A\|_{L^2(X)}$ or $\|F_A^+\|_{L^\infty(X)}$, for example. Rather, they depend on higher-order covariant derivatives of the connection A. In general, even when A is not anti-self-dual, we can always bound L_{k+1}^2 norms of local connection one-forms in Coulomb gauge by $L_{k,A}^2$ norms of the curvature, following the method of proof for [8, Lemma 2.3.11]. If A is anti-self-dual, the dependence on A reduces to the L^2 norm of F_A. Thus, throughout this section, constants depending on A can be gauge-invariantly given as constants depending on $L_{k,A}^2$ norms of F_A.

4.1. **Regularity for L_1^2 solutions to the inhomogeneous Coulomb-gauge and quasi-linear equations.** We apply the more general results of [21, §3] — concerning regularity theory for the PU(2) monopole equations — to systems (4.1) and (4.2). Briefly, an L_1^2 solution a to the anti-self-dual and Coulomb-gauge equations (4.1), with an L_k^2 inhomogeneous term (with $k \geq 1$) is in L_{k+1}^2. Thus, if the inhomogeneous term is in C^∞ then a is in C^∞.

Fix an exponent $1 \leq p \leq \infty$ and an integer $k \geq 0$ such that $(k+1)p > 4$. Let A be an orthogonal L_k^p connection on $\mathfrak{g}_E$. For any a in $C^\infty(\Lambda^1 \otimes \mathfrak{g}_E)$, with $1 \leq q \leq \infty$ an exponent and $\ell \geq$ an integer such that $L_k^p \subset L_\ell^q$ (see [2, Theorem 5.4]), set

$$\|a\|_{L_{\ell,A}^q(X,g)} := \left(\sum_{i=0}^{\ell} \|\nabla_A^i a\|_{L^q(X,g)}^q\right)^{1/q},$$

and let $L^p_k(\Lambda^1 \otimes \mathfrak{g}_E)$ be the completion of $\Omega^1(\mathfrak{g}_E)$ with respect to this norm. Define the Sobolev norms and Banach-space completions of the spaces $\Omega^0(\mathfrak{g}_E)$ and $\Omega^+(\mathfrak{g}_E)$ in the analogous manner.

Proposition 4.2. *(see [21, Proposition 3.2]) Let X be a closed, oriented, Riemannian four-manifold with metric g and let E be a Hermitian, rank-two bundle over X. Let A be a C^∞ connection on the C^∞ bundle $\mathfrak{g}_E$ over X and let $2 \le p < 4$. Then there are positive constants $\varepsilon = \varepsilon(A,p)$ and $C = C(A,p)$ with the following significance. Suppose that $a \in L^2_1(X, \Lambda^1 \otimes \mathfrak{g}_E)$ is a solution to the elliptic system (4.1) over X, where (ζ, w) is in L^p. If $\|a\|_{L^4(X)} < \varepsilon$ then a is in L^p_1 and*

$$\|a\|_{L^p_{1,A}(X)} \le C \left(\|(\zeta,w)\|_{L^p(X)} + \|a\|_{L^2(X)}\right).$$

Proposition 4.3. *(see [21, Proposition 3.3]) Continue the notation of Proposition 4.2. Let $k \ge 1$ be an integer and let $2 < p < \infty$. Let A be a C^∞ connection on the C^∞ bundle $\mathfrak{g}_E$ over X. Suppose that $a \in L^p_1(X, \Lambda^1 \otimes \mathfrak{g}_E)$ is a solution to the elliptic system (4.1) over X, where (ζ, w) is in L^2_k. Then a is in L^2_{k+1} and there is a universal polynomial $Q_k(x,y)$, with positive real coefficients, depending at most on A, k, such that $Q_k(0,0) = 0$ and*

$$\|a\|_{L^2_{k+1,A}(X)} \le Q_k\left(\|(\zeta,w)\|_{L^2_{k,A}(X)}, \|a\|_{L^p_{1,A}(X)}\right).$$

In particular, if (ζ, w) is in C^∞ then a is in C^∞ and if $(\zeta, w) = 0$, then

$$\|a\|_{L^2_{k+1,A}(X)} \le C\|a\|_{L^p_{1,A}(X)}.$$

By combining Propositions 4.2 and 4.3 we obtain the desired regularity result for L^2_1 solutions to the inhomogeneous Coulomb-gauge and anti-self-dual equations:

Corollary 4.4. *(see [21, Corollary 3.4]) Continue the notation of Proposition 4.2. Let A be a C^∞ connection on the C^∞ bundle $\mathfrak{g}_E$ over X. Then there is a positive constant $\varepsilon = \varepsilon(A)$ such that the following hold. Suppose that $a \in L^2_1(X, \Lambda^1 \otimes \mathfrak{g}_E)$, is a solution to the elliptic system (4.1) over X, where (ζ, w) is in L^2_k and $\|a\|_{L^4(X)} < \varepsilon$ and $k \ge 0$ is an integer. Then a is in L^2_{k+1} and there is a universal polynomial $Q_k(x,y)$, with positive real coefficients, depending at most on A, k, such that $Q_k(0,0) = 0$ and*

$$\|a\|_{L^2_{k+1,A}(X)} \le Q_k\left(\|(\zeta,w)\|_{L^2_{k,A}(X)}, \|a\|_{L^2(X)}\right).$$

In particular, if (ζ, w) is in C^∞ then a is in C^∞ and if $(\zeta, w) = 0$, then

$$\|a\|_{L^2_{k+1,A}(X)} \le C\|a\|_{L^2(X)}.$$

4.2. **Regularity for L^2_2 solutions to the inhomogeneous, second-order quasi-linear equation.** It remains to consider the regularity properties of the solution v to the second-order equation (4.2), where $a = d^{+,*}_A v$ and thus

$$d^+_A d^{+,*}_A v = d^+_A a.$$

The Laplacian $d^+_A d^{+,*}_A$ is elliptic, with C^∞ coefficients, and thus standard regularity theory implies that if $a \in L^p_k(X, \Lambda^1 \otimes \mathfrak{g}_E)$ then $d^+_A a \in L^p_{k-1}(X, \Lambda^1 \otimes \mathfrak{g}_E)$ and so $v \in L^p_{k+1}(X, \Lambda^1 \otimes \mathfrak{g}_E)$.

It will prove convenient to consider the following generalization of the second-order system (4.2):

$$d_A^+ d_A^* v + (d_A^* v \wedge d_A^* v)^+ + \{\alpha, d_A^* v\} = w, \tag{4.3}$$

where $v \in L_2^2(X, \Lambda^+ \otimes \mathfrak{g}_E)$, $w \in L_k^p(X, \Lambda^+ \otimes \mathfrak{g}_E)$, and $\alpha \in C^\infty(X, \Lambda^1 \otimes \mathfrak{g}_E)$. Here,

$$\{\ ,\ \} : \Gamma(\Lambda^1 \otimes \mathfrak{g}_E) \times \Gamma(\Lambda^1 \otimes \mathfrak{g}_E) \to \Gamma(\Lambda^+ \otimes \mathfrak{g}_E),$$

denotes the linear map defined by

$$\{\alpha, \beta\} := (\alpha \wedge \beta)^+ + (\beta \wedge \alpha)^+.$$

Then the proofs of the regularity results of §4.1 adapt with essentially no change to give the following regularity results for equation (4.3).

Proposition 4.5. *(see* [21, Proposition 3.2]*) Let X be a closed, oriented, Riemannian four-manifold with metric g and let E be a Hermitian, rank-two bundle over X. Let A be a C^∞ connection on the C^∞ bundle $\mathfrak{g}_E$ over X and let $2 \le p < 4$. Suppose $\alpha \in C^\infty(X, \Lambda^1 \otimes \mathfrak{g}_E)$. Then there are positive constants $\varepsilon = \varepsilon(A, \alpha, p)$ and $C = C(A, \alpha, p)$ with the following significance. Suppose that $v \in L_2^2(X, \Lambda^+ \otimes \mathfrak{g}_E)$ is a solution to the elliptic system* (4.3) *over X, where w is in $L^p(X, \Lambda^+ \otimes \mathfrak{g}_E)$. Further, if $\|d_A^* v\|_{L^4(X)} < \varepsilon$ then v is in L_2^p and*

$$\|v\|_{L^p_{2,A}(X)} \le C\left(\|w\|_{L^p(X)} + \|v\|_{L^2(X)}\right).$$

Proposition 4.6. *(see* [21, Proposition 3.3]*) Continue the notation of Proposition 4.2. Let $k \ge 1$ be an integer and let $2 < p < \infty$. Let A be a C^∞ connection on the C^∞ bundle $\mathfrak{g}_E$ over X. Suppose $\alpha \in C^\infty(X, \Lambda^1 \otimes \mathfrak{g}_E)$. Suppose that $v \in L_2^p(X, \Lambda^+ \otimes \mathfrak{g}_E)$ is a solution to the elliptic system* (4.3) *over X, where w is in L_k^2. Then v is in L_{k+2}^2 and there is a universal polynomial $Q_k(x, y)$, with positive real coefficients, depending at most on A, α, k, such that $Q_k(0,0) = 0$ and*

$$\|v\|_{L^2_{k+2,A}(X)} \le Q_k\left(\|w\|_{L^2_{k,A}(X)}, \|v\|_{L^p_{2,A}(X)}\right).$$

In particular, if w is in C^∞ then v is in C^∞ and if $w = 0$, then

$$\|v\|_{L^2_{k+2,A}(X)} \le C\|v\|_{L^p_{2,A}(X)}.$$

By combining Propositions 4.5 and 4.6 we obtain the desired regularity result for L_2^2 solutions to the inhomogeneous general second-order anti-self-dual equation:

Corollary 4.7. *(see* [21, Corollary 3.4]*) Continue the notation of Proposition 4.5. Let A be a C^∞ connection on the C^∞ bundle $\mathfrak{g}_E$ over X. Suppose $\alpha \in C^\infty(X, \Lambda^1 \otimes \mathfrak{g}_E)$. Then there is a positive constant $\varepsilon = \varepsilon(A, \alpha)$ such that the following hold. Suppose that $a \in L_1^2(X, \Lambda^1 \otimes \mathfrak{g}_E)$, is a solution to the elliptic system* (4.3) *over X, where w is in L_k^2 and $\|d_A^* v\|_{L^4(X)} < \varepsilon$ and $k \ge 0$ is an integer. Then v is in L_{k+2}^2 and there is a universal polynomial $Q_k(x, y)$, with positive real coefficients, depending at most on A, α, k, such that $Q_k(0,0) = 0$ and*

$$\|v\|_{L^2_{k+2,A}(X)} \le Q_k\left(\|w\|_{L^2_{k,A}(X)}, \|v\|_{L^2(X)}\right).$$

In particular, if w is in C^∞ then v is in C^∞ and if $w = 0$, then

$$\|v\|_{L^2_{k+2,A}(X)} \le C\|v\|_{L^2(X)}.$$

4.3. **Local regularity and interior estimates for L^2_1 solutions to the inhomogeneous Coulomb-gauge and quasi-linear equations.** In this section we specialize the results of §4.1 to the case where the reference connection is a C^∞ flat connection, so $A=\Gamma$ on the bundle $\mathfrak{g}_E$ over an open subset $\Omega\subset X$.

We continue to assume that X is a closed, oriented four-manifold with C^∞ Riemannian metric g and Hermitian, rank-two bundle E extending that on $\Omega\subset X$. We then have the following local versions of Propositions 4.2 and 4.3 and Corollary 4.4:

Proposition 4.8. *(see* [21, Proposition 3.9]*) Continue the notation of the preceding paragraph. Let $\Omega'\Subset\Omega$ be a precompact open subset and let $2\le p<4$. Then there are positive constants $\varepsilon=\varepsilon(\Omega,p)$ and $C=C(\Omega',\Omega,p)$ with the following significance. Suppose that a is an $L^2_1(\Omega)$ solution to the elliptic system* (4.1) *over Ω, with $A=\Gamma$ and where (ζ,w) is in $L^p(\Omega)$. If $\|a\|_{L^4(\Omega)}<\varepsilon$ then a is in $L^p_1(\Omega')$ and*

$$\|a\|_{L^p_{1,\Gamma}(\Omega')}\le C\left(\|(\zeta,w)\|_{L^p(\Omega)}+\|a\|_{L^2(\Omega)}\right).$$

Proposition 4.9. *(see* [21, Proposition 3.10]*) Continue the notation of Proposition 4.8. Let $k\ge 1$ be an integer, and let $2<p<\infty$. Suppose that a is an $L^p_1(\Omega)$ solution to the elliptic system* (4.1) *over Ω with $A=\Gamma$, where (ζ,w) is in $L^2_k(\Omega)$. Then a is in $L^2_{k+1}(\Omega')$ and there is a universal polynomial $Q_k(x,y)$, with positive real coefficients, depending at most on k, Ω', Ω, such that $Q_k(0,0)=0$ and*

$$\|a\|_{L^2_{k+1,\Gamma}(\Omega')}\le Q_k\left(\|(\zeta,w)\|_{L^2_{k,\Gamma}(\Omega)},\|a\|_{L^p_{1,\Gamma}(\Omega)}\right).$$

If (ζ,w) is in $C^\infty(\Omega)$ then a is in $C^\infty(\Omega')$ and if $(\zeta,w)=0$, then

$$\|a\|_{L^2_{k+1,\Gamma}(\Omega')}\le C\|a\|_{L^p_{1,\Gamma}(\Omega)}.$$

Corollary 4.10. *(see* [21, Corollary 3.11]*) Continue the notation of Proposition 4.9. Then there is a positive constant $\varepsilon=\varepsilon(\Omega)$ with the following significance. Suppose that a is an $L^2_1(\Omega)$ solution to the elliptic system* (4.1) *over Ω with $A=\Gamma$, where (ζ,w) is in $L^2_k(\Omega)$ and $\|a\|_{L^4(\Omega)}<\varepsilon$. Then a is in $L^2_{k+1}(\Omega')$ and there is a universal polynomial $Q_k(x,y)$, with positive real coefficients, depending at most on k, Ω', Ω, such that $Q_k(0,0)=0$ and*

$$\|a\|_{L^2_{k+1,\Gamma}(\Omega')}\le Q_k\left(\|(\zeta,w)\|_{L^2_{k,\Gamma}(\Omega)},\|a\|_{L^2(\Omega)}\right).$$

If (ζ,w) is in $C^\infty(\Omega)$ then a is in $C^\infty(\Omega')$ and if $(\zeta,w)=0$, then

$$\|a\|_{L^2_{k+1,\Gamma}(\Omega')}\le C\|a\|_{L^2(\Omega)}.$$

Corollary 4.10 thus yields a sharp local elliptic regularity result for L^2_1 anti-self-dual connections A which are given to us in Coulomb gauge relative to Γ.

Proposition 4.11. *(see* [21, Proposition 3.12]*) Continue the notation of Corollary 4.10. Then there is a positive constant $\varepsilon=\varepsilon(\Omega)$ and, if $k\ge 1$ is an integer, there is a positive constant $C=C(\Omega',\Omega,k)$ with the following significance. Suppose that A is an L^2_1 solution to the g-anti-self-dual equation over Ω, which is in Coulomb gauge over Ω relative to Γ, so $d^*_\Gamma(A-\Gamma)=0$, and obeys $\|A-\Gamma\|_{L^4(\Omega)}<\varepsilon$. Then $A-\Gamma$ is in $C^\infty(\Omega')$ and for any $k\ge 1$,*

$$\|A-\Gamma\|_{L^2_{k,\Gamma}(\Omega')}\le C\|A-\Gamma\|_{L^2(\Omega)}.$$

4.4. **Local regularity and interior estimates for L^2_2 solutions to the general, inhomogeneous, second-order quasi-linear equation.** In this section we specialize the results of §4.2 to the case of an open subset $\Omega \subset X$.

We continue to assume that X is a closed, oriented four-manifold with C^∞ Riemannian metric g and Hermitian, rank-two bundle E extending that on $\Omega \subset X$. We then have the following local versions of Propositions 4.5 and 4.6 and Corollary 4.7:

Proposition 4.12. *(see* [21, Proposition 3.9]*) Continue the notation of the preceding paragraph. Suppose $\alpha \in C^\infty(X, \Lambda^1 \otimes \mathfrak{g}_E)$. Let $\Omega' \Subset \Omega$ be a precompact open subset and let $2 \le p < 4$. Then there are positive constants $\varepsilon = \varepsilon(A, \alpha, \Omega, p)$ and $C = C(A, \alpha, \Omega', \Omega, p)$ with the following significance. Suppose that v is an $L^2_2(\Omega)$ solution to the elliptic system* (4.3) *over Ω, where w is in $L^p(\Omega)$. If $\|d_A^* v\|_{L^4(\Omega)} < \varepsilon$ then v is in $L^p_2(\Omega')$ and*

$$\|v\|_{L^p_{2,A}(\Omega')} \le C\left(\|w\|_{L^p(\Omega)} + \|v\|_{L^2(\Omega)}\right).$$

Proposition 4.13. *(see* [21, Proposition 3.10]*) Continue the notation of Proposition 4.12. Let $k \ge 1$ be an integer, and let $2 < p < \infty$. Suppose that v is an $L^p_2(\Omega)$ solution to the elliptic system* (4.3) *over Ω, where w is in $L^2_k(\Omega)$. Then v is in $L^2_{k+2}(\Omega')$ and there is a universal polynomial $Q_k(x,y)$, with positive real coefficients, depending at most on A, α, k, Ω', Ω, such that $Q_k(0,0) = 0$ and*

$$\|v\|_{L^2_{k+2,A}(\Omega')} \le Q_k\left(\|w\|_{L^2_{k,A}(\Omega)}, \|v\|_{L^p_{2,A}(\Omega)}\right).$$

If w is in $C^\infty(\Omega)$ then v is in $C^\infty(\Omega')$ and if $w = 0$, then

$$\|v\|_{L^2_{k+2,A}(\Omega')} \le C\|v\|_{L^p_{2,A}(\Omega)}.$$

Corollary 4.14. *(see* [21, Corollary 3.11]*) Continue the notation of Proposition 4.13. Then there is a positive constant $\varepsilon = \varepsilon(A, \alpha, \Omega)$ with the following significance. Suppose that v is an $L^2_2(\Omega)$ solution to the elliptic system* (4.3) *over Ω, where w is in $L^2_k(\Omega)$ and $\|d_A^* v\|_{L^4(\Omega)} < \varepsilon$. Then v is in $L^2_{k+2}(\Omega')$ and there is a universal polynomial $Q_k(x,y)$, with positive real coefficients, depending at most on A, α, k, Ω', Ω, such that $Q_k(0,0) = 0$ and*

$$\|v\|_{L^2_{k+2,A}(\Omega')} \le Q_k\left(\|w\|_{L^2_{k,A}(\Omega)}, \|v\|_{L^2(\Omega)}\right).$$

If w is in $C^\infty(\Omega)$ then v is in $C^\infty(\Omega')$ and if $w = 0$, then

$$\|v\|_{L^2_{k+2,A}(\Omega')} \le C\|v\|_{L^2(\Omega)}.$$

4.5. **Estimates for extended anti-self-dual connections in a good local gauge.** It remains to combine the local regularity results and estimates of §4.3, for extended anti-self-dual connections A where the connection A is assumed to be in Coulomb gauge relative to the product SO(3) connection Γ, with Uhlenbeck's local, Coulomb gauge-fixing theorem. We then obtain regularity results and estimates for anti-self-dual connections A with small curvature F_A (see Theorem 2.3.8 and Proposition 4.4.10 in [8]).

In order to apply Corollary 4.10 we need Uhlenbeck's Coulomb gauge-fixing result [60, Theorem 2.1 & Corollary 2.2]. Let B (respectively, $\bar{B}$) be the open (respectively, closed) unit ball centered at the origin in $\mathbb{R}^4$ and let G be a compact Lie group. In order to provide universal constants we assume $\mathbb{R}^4$ has its standard metric, though the results of this subsection naturally hold for any C^∞ Riemannian metric, with comparable constants for metrics which are suitably close.

Theorem 4.15. *There are positive constants c and ε with the following significance. If $2 \le p < 4$ is a constant and $A \in L^p_1(B, \Lambda^1 \otimes \mathfrak{g}) \cap L^p_1(\partial B, \Lambda^1 \otimes \mathfrak{g})$ is a connection matrix whose curvature satisfies $\|F_A\|_{L^p(B)} < \varepsilon$, then there is a gauge transformation $u \in L^p_2(B, G) \cap L^p_2(\partial B, G)$ such that $u(A) := uAu^{-1} - (du)u^{-1}$ satisfies*

$$d^* u(A) = 0 \text{ on } B, \tag{1}$$

$$\frac{\partial}{\partial r} \lrcorner u(A) = 0 \text{ on } \partial B, \tag{2}$$

$$\|u(A)\|_{L^p_1(B)} \le c\|F_A\|_{L^p(B)}. \tag{3}$$

If A is in $L^p_k(B)$, for $k \ge 2$, then u is in $L^p_{k+1}(B)$. The gauge transformation u is unique up to multiplication by a constant element of G.

Remark 4.16.
1. The L^p_2 gauge transformation u is obtained by solving the second-order quasi-linear elliptic equation $d^*A^u = 0$, that is
$$d^* du = - * (duu^{-1} \wedge *du) - d^*Au + *(*A \wedge du) - *(duu^{-1} \wedge *Au),$$
given A in L^p_1. For any open subset $U \Subset B$, there is a positive constant $\varepsilon_0(U)$ such that if A in $L^p_2(B)$ satisfies $\|A\|_{L^p_2(B)} < \varepsilon_0$ then u is in $L^p_3(U)$ by the implicit function theorem. In particular, if A is in $C^\infty(B)$ then standard elliptic theory and the Sobolev multiplication theorem imply that u is in $C^\infty(U)$ [8, Proposition 2.3.4].
2. Uhlenbeck's methods [60] imply that the gauge transformation u_A depends analytically on the connection A.
3. If G is abelian then the requirement that $\|F_A\|_{L^p(B)} < \varepsilon$ can be omitted.

It is often useful to rephrase Theorem 4.15 in two other slightly different ways. Suppose A is an L^2_k connection on a C^∞ principal G bundle P over B with $k \ge 2$ and $\|F_A\|_{L^2(B)} < \varepsilon$. Then the assertions of Theorem 4.15 are equivalent to each of the following:

- There is an L^2_{k+1} trivialization $\tau : P \to B \times G$ such that (i) $d^*_\Gamma(\tau(A) - \Gamma) = 0$, where Γ is the product connection on $B \times G$, (ii) $\frac{\partial}{\partial r} \lrcorner (\tau(A) - \Gamma) = 0$, and (iii) $\|(\tau(A) - \Gamma)\|_{L^2_1(B)} \le c\|F_A\|_{L^2(B)}$.
- There is an L^2_{k+1} flat connection Γ on P such that (i) $d^*_\Gamma(A - \Gamma) = 0$, (ii) $\frac{\partial}{\partial r} \lrcorner (A - \Gamma) = 0$, and (iii) $\|(A - \Gamma)\|_{L^2_1(B)} \le c\|F_A\|_{L^2(B)}$, and an L^2_{k+1} trivialization $P|_B \simeq B \times G$ taking Γ to the product connection.

We can now combine Theorem 4.15 with Corollary 4.10 or Proposition 4.11 to give (see [8, Theorem 2.3.8], [24, Proposition 8.3]):

Theorem 4.17. *Let $B \subset \mathbb{R}^4$ be the open unit ball with center at the origin, let $U \Subset B$ be an open subset, and let Γ be the product connection on $B \times \mathrm{SO}(3)$. Then there is a positive constant ε and if $k \ge 1$ is an integer, there is a positive constant $C(k, U)$ with the following significance. Suppose that A is an L^2_1 solution to the* extended anti-self-dual connection *on $\mathfrak{g}_E$ over B, namely*

$$F^+_A = w,$$

for some $w \in L^2_{k-1}(B, \Lambda^+ \otimes \mathfrak{g}_E)$*, and that the curvature of the* $\mathrm{SO}(3)$ *connection matrix* A *obeys*

$$\|F_A\|_{L^2(B)} < \varepsilon.$$

Then there is an L^2_{k+1} *gauge transformation* $u : B \to \mathrm{SU}(2)$ *such that* $u(A) - \Gamma$ *is in* $L^2_k(U)$ *with* $d^*(u(A) - \Gamma) = 0$ *over* B *and*

$$\|u(A) - \Gamma\|_{L^2_{k,\Gamma}(U)} \le Q_k \left(\|u(w)\|_{L^2_{k-1,\Gamma}(B)}, \|F_A\|_{L^2(B)}\right).$$

If $u(w)$ *is in* $C^\infty(B)$*, then* $u(A) - \Gamma$ *is in* $C^\infty(U)$*. If* $w = 0$*, so that* A *is an anti-self-dual connection, then*

$$\|u(A) - \Gamma\|_{L^2_{k,\Gamma}(U)} \le C\|F_A\|_{L^2(B)}.$$

Remark 4.18. The L^2_k estimates for A obtained from Theorem 4.17 are uniform over all metrics in a small C^k neighborhood of a fixed metric [24, p. 120].

Again, it is often useful to rephrase Theorem 4.17 in the two other slightly different ways. Suppose $\ell \ge 1$ and that A is an L^2_k anti-self-dual connection on $\mathfrak{g}_E$ over the unit ball $B \subset \mathbb{R}^4$ with $k \ge \max\{2, \ell\}$, $\|F_A\|_{L^2(B)} < \varepsilon$ and $U \Subset B$. Then the assertions of Theorem 4.17 are equivalent to each of the following:

- There is a C^∞ trivialization $\tau : \mathfrak{g}_E|_B \to B \times \mathfrak{su}(2)$ and a L^2_{k+1} determinant-one, unitary bundle automorphism u of $E|_B$ such that, with respect to the product connection Γ on $B \times \mathfrak{su}(2)$, we have (i) $d^*_\Gamma(\tau u(A) - \Gamma) = 0$, and (ii) $\|(\tau u(A) - \Gamma, \tau u\Phi)\|_{L^2_{\ell,\Gamma}(U)} \le C\|F_A\|_{L^2(B)}$.
- There is an L^2_{k+1} flat connection Γ on $\mathfrak{g}_E|_B$ such that (i) $d^*_\Gamma(A - \Gamma) = 0$, and (ii) $\|A - \Gamma\|_{L^2_{\ell,\Gamma}(U)} \le c\|F_A\|_{L^2(B)}$, and an L^2_{k+1} trivialization $\mathfrak{g}_E|_B \simeq B \times \mathfrak{su}(2)$ taking Γ to the product connection.

We will need interior estimates for anti-self-dual connections in a good local gauge over more general simply-connected regions than the open balls considered in Theorem 4.17. Specifically, recall that a domain $\Omega \subset X$ is *strongly simply-connected* if it has an open covering by balls $D_1, \dots, D_m$ (not necessarily geodesic) such that for $1 \le r \le m$ the intersection $D_r \cap (D_1 \cup \dots \cup D_{r-1})$ is connected. We recall (see [8, Proposition 2.2.3] or [33, Proposition I.2.6]):

Proposition 4.19. *If* Γ *is a* C^∞ *flat connection on a principal* G *bundle* P *over a simply-connected manifold* Ω*, then there is a* C^∞ *isomorphism* $P \simeq \Omega \times G$ *taking* Γ *to the product connection on* $\Omega \times G$*.*

More generally, if A is C^∞ connection on a G bundle P over a simply-connected manifold-with-boundary $\bar\Omega = \Omega \cup \partial\Omega$ with L^p-small curvature (with $p > 2$), then Uhlenbeck's theorem implies that A is L^p_2-gauge equivalent to a connection which is L^p_1-close to an L^p_1 flat connection on P (see [61, Corollary 4.3] or [8, p. 163]). We recall the following *a priori* interior estimate (see also [21, Proposition 3.18] and [8, Proposition 4.4.10]).

Proposition 4.20. *Let* X *be a closed, oriented, Riemannian four-manifold and let* $\Omega \subset X$ *be a strongly simply-connected open subset. Then there is a positive constant* $\varepsilon(\Omega)$ *with the following significance. For* $\Omega' \Subset \Omega$ *a precompact open subset and an integer* $\ell \ge 1$*,* *there is a constant $C(\ell,\Omega',\Omega)$ such that the following holds. Suppose A is an L^2_k* extended anti-self-dual connection *on $\mathfrak{g}_E$ over Ω, namely*

$$F^+_A = w,$$

for some $w \in L^2_{k-1}(\Omega, \Lambda^+ \otimes \mathfrak{g}_E)$, with $k \geq \max\{2,\ell\}$ such that

$$\|F_A\|_{L^2(\Omega)} < \varepsilon.$$

Then there is an L^2_{k+1} flat connection Γ on $\mathfrak{g}_E|_{\Omega'}$ such that

$$\|A-\Gamma\|_{L^2_{\ell,\Gamma}(\Omega')} \leq Q_k\left(\|w\|_{L^2_{k-1,\Gamma}(\Omega)}, \|F_A\|_{L^2(\Omega)}\right),$$

and an L^2_{k+1} trivialization $\mathfrak{g}_E|_{\Omega'} \simeq \Omega' \times \mathfrak{su}(2)$ taking Γ to the product connection. If w is in $C^\infty(\Omega)$, then A, Γ are in $C^\infty(\Omega')$. If $w=0$, so that A is an anti-self-dual connection, then

$$\|A-\Gamma\|_{L^2_{\ell,\Gamma}(\Omega')} \leq C\|F_A\|_{L^2(\Omega)}.$$

For some local patching arguments over simply-connected open regions $Y \subset X$ it is very useful to be able to lift gauge transformations in $\mathcal{G}_{\mathfrak{su}(E)}(Y)$ to gauge transformations in $\mathcal{G}_E(Y)$. The following result tells us that this is always possible when Y is simply connected; it is an extension of Theorem IV.3.1 in [44] from SU(2) to U(2) bundles.

Proposition 4.21. [21, Proposition 2.16] *Let E be a rank-two, Hermitian bundle over a connected manifold X. Then there is an exact sequence* [5]

$$1 \to \{\pm \mathrm{id}_E\} \to \mathcal{G}_E \to \mathcal{G}_{\mathfrak{su}(E)} \to H^1(X;\mathbb{Z}/2\mathbb{Z}).$$

4.6. **Regularity of Taubes' solutions to extended anti-self-dual equations.** Finally, we come to one of the main results of this section.

Proposition 4.22. *Let X be a closed, oriented, C^∞ four-manifold with C^∞ Riemannian metric g and let E be a Hermitian, rank-two bundle over X. Let A be a C^∞ connection on the C^∞ bundle $\mathfrak{g}_E$ over X. Suppose that $a \in L^2_1(X,\Lambda^1 \otimes \mathfrak{g}_E)$ is a solution to an extended anti-self-dual equation over X,*

$$F^{+,g}(A+a) = w,$$

where $a = d_A^{+,*}v$, for some $v \in (C^0 \cap L^2_2)(X,\Lambda^+ \otimes \mathfrak{g}_E)$, and $w \in C^\infty(X,\Lambda^+ \otimes \mathfrak{g}_E)$. Then a is in $C^\infty(X,\Lambda^1 \otimes \mathfrak{g}_E)$ and $v \in C^\infty(X,\Lambda^+ \otimes \mathfrak{g}_E)$.

Remark 4.23. The reader may wonder why we cannot apply Corollary 4.4 to the resulting elliptic system (4.4) and deduce straightaway that the solution a is in C^∞. The difficulty, of course, is that the hypotheses of Corollary 4.4 are not necessarily satisfied due to the possibly unfavorable dependence of the constant $\varepsilon(A)$ on the connection A. For example, by combining results of §5 and §7, we can show that $\|a\|_{L^4(X)} \leq C\lambda$ when A_λ is a spliced connection (produced by the alogorithm of §3), λ is a parameter which can be made arbitrarily small, and C is an essentially universal constant, depending at most on $\|F^+_{A_\lambda}\|_{L^\infty(X)}$ and $\|F_{A_\lambda}\|_{L^2}$: in particular, C is independent of λ. Now, as we make λ smaller to try to satisfy the constraint of Corollary 4.4, we will in general find that $\varepsilon(A_\lambda)$ also becomes smaller (as it will, for example, if $\varepsilon(A_\lambda)$ depends on an L^p_ℓ norm of F_{A_λ} with $p>2$, $\ell=0$ or $\ell \geq 1$,

[5] It can also be shown that the map $\mathcal{G}_{\mathfrak{su}(E)} \to H^1(X;\mathbb{Z}/2\mathbb{Z})$ is surjective.

$p = 2$). Thus, as we do not necessarily have a lower bound for $\varepsilon(A)$ which is uniform with respect to A, we are left 'chasing our tail' and Corollary 4.4 will not apply.

Proof of Proposition 4.22. Since $a = d_A^{+,*}v$, the extended anti-self-dual and Coulomb-gauge equations become

$$\begin{aligned} d_A^+ a + (a \wedge a)^+ &= w - F_A^+, \\ d_A^* a &= (F_A^+)^* v, \end{aligned} \tag{4.4}$$

where $F_A^+ \in C^\infty(\Lambda^+ \otimes \mathfrak{g}_E)$ and $(F_A^+)^* v \in (C^0 \cap L_2^2)(\Lambda^+ \otimes \mathfrak{g}_E)$. Observe that

$$(\zeta, w') := ((F_A^+)^* v, w - F_A^+) \in L_2^2(\mathfrak{g}_E) \oplus L_2^2(\Lambda^+ \otimes \mathfrak{g}_E).$$

From Remark 4.23 we note that we *cannot* take the apparently obvious route and apply Corollary 4.4. Instead, we fix a point $x \in X$ and a geodesic ball $B(x,\delta)$ centered at x, fix a C^∞ trivialization $\mathfrak{g}_E|_{B(x,\delta)} \simeq B(x,\delta) \times \mathfrak{su}(2)$, and let Γ be the resulting product connection. Then

$$\begin{aligned} d^+ b + (b \wedge b)^+ &= w - F_\Gamma^+ = w, \\ d^* b &= *((A - \Gamma) \wedge *a) + d^*(A - \Gamma) + (F_A^+)^* v, \end{aligned} \tag{4.5}$$

where $b := A - \Gamma + a \in L_1^2(B(x,\delta), \Lambda^1 \otimes \mathfrak{g}_E)$ and the first equation is the expansion of $F^+(\Gamma + b) = w$. In particular, the right-hand side of equation (4.5) is in $L_1^2(B(x,\delta))$.

In Corollary 4.10, let $\Omega = B \subset \mathbb{R}^4$ be the unit ball, let $\varepsilon = \varepsilon(\Omega)$ be the corresponding positive constant, and let $\Omega' = B'$ be a slightly smaller concentric ball. Since $a \in L^4(X, \Lambda^1 \otimes \mathfrak{g}_E)$, we know that $\|a\|_{L^4(U)} \to 0$ as $\mathrm{Vol}(U) \to 0$, where U is any measurable subset of X. In particular, for a small enough ball $B(x,\delta)$, we have $\|b\|_{L^4(B(x,\delta))} < \varepsilon$. We now rescale the metric on the ball $B(x,\delta)$ to give a ball $\tilde{B}$ with unit radius, center x, and metric which is approximately flat. Note that the equations (4.5) are scale-equivariant and that

$$\|b\|_{L^4(\tilde{B})} = \|b\|_{L^4(B(x,\delta))} < \varepsilon,$$

by the scale-invariance of the L^4 norm on one-forms. Corollary 4.10 now implies that b — and thus a — is in L_2^2 on the slightly smaller rescaled ball $\tilde{B}'$. Since $d_A^+ d_A^{+,*} v = d_A^+ a$ is now in $L_1^2(\tilde{B}')$, we have $v \in L_3^2(\tilde{B}')$ by standard elliptic regularity theory for a Laplacian, $d_A^+ d_A^{+,*}$, with C^∞ coefficients [32]. Hence, the right-hand side of equation (4.5) is in $L_2^2(\tilde{B}')$. Repeating the above argument shows that a is in C^∞ on the ball $\tilde{B}_{1/2} \subset \tilde{B}$ with center x and radius $\frac{1}{2}$. Hence, a is in C^∞ on $B(x,\delta/2)$ and so, as the point $x \in X$ was arbitrary, we see that $a \in C^\infty(X, \Lambda^1 \otimes \mathfrak{g}_E)$ and $v \in C^\infty(X, \Lambda^+ \otimes \mathfrak{g}_E)$. □

5. Decay estimates for curvature and connections in Coulomb and radial gauges

Our splicing and unsplicing constructions for connections requires estimates for local connection one-forms in suitable gauges, usually Coulomb or radial, over small balls, over their complements, and over annuli. The most useful estimates bound Sobolev or Hölder norms of local connection one-forms in terms of the L^2 norms of their curvatures and the radii defining the given regions, with estimate constants which are essentially universal. These are the kinds of estimates we derive in this section. Some of these bounds can be derived from the curvature decay estimates due, in various forms, to Donaldson, Groisser-Parker, Råde, Taubes, and Uhlenbeck. Perhaps the most useful and powerful statement is due to Donaldson, Groisser-Parker and Råde; see the discussion in [14]. However, we prefer to use methods which are as simple as possible. Furthermore, our chosen methods have the the following advantages:

- They extend easily to the case of solutions to *extended* anti-self-dual equations, such as equations (7.1) or (7.3).
- They extend easily to the case of PU(2) *monopoles* [18].

In contrast, the proofs of the elegant, pointwise decay estimates of Donaldson, Groisser-Parker, and Råde — for the curvature of anti-self-dual or Yang-Mills connections — requires a delicate application of the Chern-Simons functional and considerably more work is needed to extend them to the case of PU(2) monopoles, though decay estimates of the desired type for Seiberg-Witten monopoles are described in [47]. It is likely that their estimates can be extended to cover the case of extended anti-self-dual connections, but again some work is required to show this. The disadvantage of our approach is that we must separately consider the cases of (i) balls, (ii) complements of balls in $\mathbb{R}^4$, and (iii) annuli, while the decay estimate theorem of Donaldson, Groisser-Parker, and Råde would give a more uniform treatment of these different regions.

In all cases, the estimates of this section continue to hold in situations where the Riemannian metric is uniformly C^r close enough to the standard metric. Thus, for the sake of notational simplicity, we shall assume that our connections are defined on regions in $\mathbb{R}^4$ equipped with its standard Riemannian metric g. We let $B(r)$ denote the open ball $\{x \in \mathbb{R}^4 : |x| < r\}$ of radius r and center at the origin and let $\bar{B}(r)$ denote the closed ball $\{x \in \mathbb{R}^4 : |x| \leq r\}$.

5.1. **Estimates for anti-self-dual connections over balls.** The basic case is that of ball in $\mathbb{R}^4$: estimates for connections over the complements of balls and over annuli follow from this. Theorem 4.17 yields an interior L^∞ bound for the curvature F_A of an anti-self-dual connection over a ball in terms of the L^2 norm of F_A; compare Proposition 8.3 in [24] and Lemma 9.1 in [55].

Lemma 5.1. *There are positive constants ε and, given a constant $1 \leq p \leq \infty$ and an integer $k \geq 0$, a positive constant $c(k,p)$ with the following significance. Let G be a compact Lie group. Suppose $r_1 > 0$ is a constant and that A is an anti-self-dual connection on the product bundle $P = B(r_1) \times G$, where $B(r_1)$ is a ball centered at the origin in $\mathbb{R}^4$, with its Euclidean metric, such that*

$$\|F_A\|_{L^2(B(r_1))} < \varepsilon,$$

then

$$\|\nabla_\Gamma^k F_A\|_{L^p(B(r_1/2))} \leq c(k,p) r_1^{(4/p)-k-2} \|F_A\|_{L^2(B(r_1))}.$$

Moreover, if Γ is the product connection on $\mathbb{R}^4 \times G$ and $d_\Gamma^*(A - \Gamma) = 0$, then

$$\|\nabla_\Gamma^k (A - \Gamma)\|_{L^p(B(r_1/2))} \leq c(k,p) r_1^{(4/p)-k-1} \|F_A\|_{L^2(B(r_1))}.$$

Finally, the estimates for F_A also hold when ∇_Γ is replaced by ∇_A.

Proof. We define a rescaled metric $\tilde{g}$ on $\mathbb{R}^4$ by setting $\tilde{g}_{\mu\nu} = r_1^{-2} g_{\mu\nu}$, so $\tilde{g}^{\mu\nu} = r_1^2 g^{\mu\nu}$ and $B_g(r_1) = B_{\tilde{g}}(1)$. Then,

$$
\begin{aligned}
|\nabla_\Gamma^k (A-\Gamma)|_g &= r_1^{-k-1} |\nabla_\Gamma^k (A-\Gamma)|_{\tilde{g}}, \\
|\nabla_A^k F_A|_g &= r_1^{-k-2} |\nabla_A^k F_A|_{\tilde{g}}.
\end{aligned}
\tag{5.1}
$$

Since $\|F_A\|_{L^2(B(1),\tilde{g})} < \varepsilon$ and ε is sufficiently small, Theorem 4.15 provides a gauge transformation u over $B_{\tilde{g}}(1)$ such that $d_\Gamma^*(u(A) - \Gamma) = 0$, so without loss of generality we may assume that A is already in Coulomb gauge, as we suppose in the second assertion of the Lemma. Theorem 4.17, noting that its proof extends from $p = 2$ to the general case in a straightforward manner, then yields

$$
\|A - \Gamma\|_{L^p_{k,\Gamma}(B(1/2),\tilde{g})} \le c(k,p) \|F_A\|_{L^2(B(1),\tilde{g})},
$$

and the desired Sobolev L^p_k estimates follow, since

$$
\begin{aligned}
\|\nabla_\Gamma^k (A-\Gamma)\|_{L^p(B(r_1/2),g)} &= r_1^{(4/p)-k-1} \|\nabla_\Gamma^k (A-\Gamma)|_{L^2(B(1/2),\tilde{g})}, \\
\|\nabla_A^k F_A\|_{L^p(B(r_1/2),g)} &= r_1^{(4/p)-k-2} \|\nabla_A^k F_A\|_{L^2(B(1/2),\tilde{g})}.
\end{aligned}
$$

This completes the proof. □

In our treatment of 'ungluing' in [14] we shall need L^2_1 estimates for covariant derivatives of radial-gauge connection one-forms, while L^p estimates are needed throughout our article. Therefore, suppose that the connection one-form $A - \Gamma$ is in radial gauge with respect to the origin, $x^\mu A_\mu(x) = 0$, so [59]

$$
A_\nu(x) = \int_0^1 t x^\mu (F_A)_{\mu\nu}(ty)\, dt,
\tag{5.2}
$$

where $A_\mu = (A - \Gamma)_\mu$. We then have the following estimates for the local connection one-forms and their derivatives (compare Taubes' Lemmas 9.1 and 9.2 in [55]):

Lemma 5.2. *Continue the hypotheses and notation of Lemma 5.1. Suppose that the connection one-form $A - \Gamma$ is in radial gauge with respect to the origin. Then, for any $1 \le p \le \infty$,*

$$
\|\nabla_\Gamma^k (A-\Gamma)\|_{L^p(B(r_1/2))} \le c(k,p) r_1^{(4/p)-k-1} \|F_A\|_{L^2(B(r_1))}.
$$

Proof. Denote $A - \Gamma$ simply by A. We confine our attention to the case $k = 0, 1$ (which are the only ones we shall need) and leave the general case to the reader. From equation (5.2) we have

$$
\begin{aligned}
|A|(x) &\le |x| \cdot |F_A|(x), \quad x \in \mathbb{R}^4, \\
|\nabla_\Gamma A|(x) &\le |x| \cdot |\nabla_\Gamma F_A|(x) + |F_A|(x).
\end{aligned}
\tag{5.3}
$$

Lemma 5.1 then yields the following Sobolev estimates. First,

$$
\|A\|_{L^p(B(r_1/2)} \le c r_1^{(4/p)-1} \|F_A\|_{L^2(B(r_1))},
$$

which gives the desired bound when $k=0$. For $k=1$, we have

$$\begin{aligned}\|\nabla_\Gamma A\|_{L^p(B(r_1/2)} &\leq r_1\|\nabla_\Gamma F_A\|_{L^p(B(r_1/2))} + \|F_A\|_{L^p(B(r_1/2))} \\ &\leq c r_1^{4/p}(r_1\cdot r_1^{-3} + r_1^{-2})\|F_A\|_{L^2(B(r_1))} \\ &= c r_1^{(4/p)-2}\|F_A\|_{L^2(B(r_1))}.\end{aligned}$$

The cases $k\geq 2$ are similar. □

Remark 5.3. The estimates in Lemmas 5.1 and 5.2 agree in the case $k=0$ for F_A with that obtained by integrating the decay estimate of Donaldson et al. when $r_0=0$. The estimates for F_A and $A-\Gamma$ agree with those obtained by integrating the pointwise decay estimates of [55, Lemma 9.1].

5.2. **Estimates for anti-self-dual connections over the complement of a ball.** We now consider the case where A is an anti-self-dual connection with small curvature on the complement of a ball in $\mathbb{R}^4$.

Our first task is to note that given a C^∞ anti-self-dual connection A on $\mathbb{R}^4 - B(r_0)$, with finite energy, there is no loss in generality — thanks to the removable singularities theorem of K. Uhlenbeck [8, Theorem 4.4.12], [59, Theorem 4.1] — if we assume that A is a C^∞ connection on $S^4 - \varphi_n(B(r_0))$, where the origin of $\mathbb{R}^4$ is identified with the north pole of S^4.

Theorem 5.4. *Let A be a C^∞ anti-self-dual connection on a G bundle P over the punctured unit ball $B\setminus\{0\}$ in $\mathbb{R}^4$, with its Euclidean metric. If*

$$\|F_A\|_{L^2(B)} < \infty,$$

there is a C^∞ anti-self-dual connection $\bar A$ on a C^∞ bundle $\overline{P}$ over B and a C^∞ bundle isomorphism $u:\overline{P}|_{B\setminus\{0\}}\to P|_{B\setminus\{0\}}$ such that $u(\bar A)=A$ over $B\setminus\{0\}$.

We now consider the problem of estimating the L^p_k norms of F_A and $A-\Gamma$ (in radial gauge) on $\mathbb{R}^4 - B(r_0)$.

Lemma 5.5. *There are positive constants ε and, given an integer $k\geq 0$ and a constant $1\leq p\leq\infty$, a positive constant $c(k,p)$ with the following significance. Let G be a compact Lie group. Suppose $r_0>0$ is a constant and that A is an anti-self-dual connection on the product bundle $P=(\mathbb{R}^4-B(r_0))\times G$, where $B(r_0)$ is a ball centered at the origin in $\mathbb{R}^4$, with its Euclidean metric, such that*

$$\|F_A\|_{L^2(\mathbb{R}^4-B(r_0))} < \varepsilon,$$

then, provided $k+2-4/p\geq 0$,

$$\|\nabla_A^k F_A\|_{L^p(\mathbb{R}^4-B(2r_0))} \leq c(k,p) r_0^{(4/p)-k-2}\|F_A\|_{L^2(\mathbb{R}^4-B(r_0))}.$$

Moreover, if Γ is the product connection on $(\mathbb{R}^4-B(r_0))\times G$ and $A-\Gamma$ is in radial gauge with respect to ∞ then, for $k+1-4/p\geq 0$,

$$\|\nabla_\Gamma^k(A-\Gamma)\|_{L^p(\mathbb{R}^4-B(2r_0))} \leq c(k,p) r_0^{(4/p)-k-1}\|F_A\|_{L^2(\mathbb{R}^4-B(r_0))}.$$

Proof. Identifying $\mathbb{R}^4$ with the quaternions $\mathbb{H}$, we recall that $y = \iota^*(y) = \varphi_s^{-1} \circ \varphi_n(x) = x^{-1}$ for $x \in \mathbb{R}^4 - \{0\}$, where $\varphi_s : S^4 - \{n\} \to \mathbb{R}^4$ and $\varphi_n : S^4 - \{s\} \to \mathbb{R}^4$ are stereographic projections. The inversion map is conformal, so $\iota^* A$ is an anti-self-dual connection on the punctured ball $\bar{B}(1/r_0) - \{0\}$, with curvature bound

$$\|F_{\iota^* A}\|_{L^2(B(1/r_0))} = \|F_A\|_{L^2(\mathbb{R}^4 - B(r_0))} < \varepsilon.$$

As $\|F_{\iota^* A}\|_{L^2(B(1/r_0))} < \infty$, Theorem 5.4 implies that there is a C^∞ gauge transformation u of $\bar{B}(1/r_0) \times G$ such that the connection one-form $u(\iota^* A)$ extends to a C^∞ connection over the ball $B(1/r_0)$. Thus, we may consider A to be a C^∞ connection on the product bundle $P = (S^4 - \varphi_n(B(r_0))) \times G$, with product connection Γ.

From the identity $|dx| = |y|^{-2}|dy|$, we obtain

$$\begin{aligned}
|\varphi_n^* \nabla_A^k F_A|(x) &= |y|^{-2(k+2)} \cdot |\varphi_s^* \nabla_A^k F_A|(y), \\
|\varphi_n^* \nabla_\Gamma^k (A - \Gamma)|(x) &= |y|^{-2(k+1)} \cdot |\varphi_s^* \nabla_\Gamma^k (A - \Gamma)|(y).
\end{aligned}$$

Combining these observations and noting that $d^4 x = |y|^{-8} d^4 y$, we see that (for $2(k+2) - 8/p \geq 0$ and $2(k+1) - 8/p \geq 0$, respectively)

$$\begin{aligned}
\|\varphi_n^* \nabla_A^k F_A\|_{L^p(\mathbb{R}^4 - B(2r_0))} &= \||y|^{2(k+2)-8/p} \varphi_s^* \nabla_A^k F_A\|_{L^p(B(1/2r_0))} \\
&\leq (2r_0)^{-2(k+2)+8/p} \|\varphi_s^* \nabla_A^k F_A\|_{L^p(B(1/2r_0))}, \\
\|\varphi_n^* \nabla_\Gamma^k (A - \Gamma)\|_{L^p(\mathbb{R}^4 - B(2r_0))} &= \||y|^{2(k+1)-8/p} \varphi_s^* \nabla_\Gamma^k (A - \Gamma)\|_{L^p(B(1/2r_0))} \\
&\leq (2r_0)^{-2(k+1)+8/p} \|\varphi_s^* \nabla_\Gamma^k (A - \Gamma)\|_{L^p(B(1/2r_0))}.
\end{aligned}$$

Applying Lemmas 5.1 and 5.2 to the connection $\varphi_s^* A$ on $B(1/r_0)$, in radial gauge with respect to the origin, we see that

$$\begin{aligned}
\|\varphi_s^* \nabla_A^k F_A\|_{L^p(B(1/2r_0))} &\leq c(k,p) r_0^{-(4/p)+k+2} \|F_A\|_{L^2(B(1/r_0))}, \\
\|\varphi_s^* \nabla_\Gamma^k (A - \Gamma)\|_{L^p(B(1/2r_0))} &\leq c(k,p) r_0^{-(4/p)+k+1} \|F_A\|_{L^2(B(1/r_0))}.
\end{aligned}$$

Combining the preceding two pairs of estimates then yields the desired bounds. □

Remark 5.6. The estimates in Lemma 5.5 agree in the case $k = 0$ for F_A with that obtained by integrating the decay estimate of Donaldson et al. when $r_1 = \infty$. The estimates for F_A and $A - \Gamma$ agree with those obtained by integrating the pointwise decay estimates of [55, Lemma 9.2].

5.3. **Estimates for anti-self-dual connections over annuli.** The case of an annulus requires a slightly more sophisticated approach and a complete treatment — where both inner and outer radius can be specified independently — requires the decay estimate of Donaldson et al. For positive constants $r_0 < r_1$, let

$$\Omega(r_0, r_1) := \{x \in \mathbb{R}^4 : r_0 < |x| < r_1\}.$$

We then have the following consequence of Proposition 4.20.

Lemma 5.7. *There are positive constants ε and, given an integer $k \geq 0$ and a constant $1 \leq p \leq \infty$, a positive constant $c(k,p)$ with the following significance. Let G be a compact*

Lie group. Suppose $r_1 > 0$ is a constant and that A is an C^∞ anti-self-dual connection on a G bundle P over the annulus $\Omega = \Omega(\frac{1}{4}r_1, 4r_1)$ in $\mathbb{R}^4$, with its Euclidean metric, such that

$$\|F_A\|_{L^2(\Omega(\frac{1}{4}r_1,4r_1))} < \varepsilon.$$

Then there is an C^∞ flat connection on $P|_{\Omega'}$ such that

$$\|\nabla_\Gamma^k(A-\Gamma)\|_{L^p(\Omega(\frac{1}{2}r_1,2r_1))} \le c(k,p)r_1^{(4/p)-k-1}\|F_A\|_{L^2(\Omega(\frac{1}{4}r_1,4r_1))},$$
$$\|\nabla_\Gamma^k F_A\|_{L^p(\Omega(\frac{1}{2}r_1,2r_1))} \le c(k,p)r_1^{(4/p)-k-2}\|F_A\|_{L^2(\Omega(\frac{1}{4}r_1,4r_1))},$$

and a C^∞ bundle isomorphism $u : P|_{\Omega'} \to \Omega' \times G$ taking Γ to the product connection. Finally, if Γ is the product connection on $\Omega \times G$, $P = \Omega \times G$, and $A - \Gamma$ is in radial gauge on Ω with respect to the origin, then the preceding Sobolev bounds for $A - \Gamma$ continue to hold.

Proof. We proceed much as in the proof of Lemma 5.1. Define a rescaled metric $\tilde g$ on $\mathbb{R}^4$ by setting $\tilde g_{\mu\nu} = r_1^{-2}g_{\mu\nu}$, so $\tilde g^{\mu\nu} = r_1^2 g^{\mu\nu}$ and $\Omega_g(\frac{1}{4}r_1, 4r_1) = \Omega_{\tilde g}(\frac{1}{4}, 4)$. Since $\|F_A\|_{L^2(\Omega(\frac{1}{4},4),\tilde g)} < \varepsilon$ and ε is sufficiently small, Proposition 4.20 provides a C^∞ flat connection Γ and a C^∞ bundle isomorphism $P|_{\Omega'} \to \Omega' \times G$, taking Γ to the product connection, such that (noting that its proof extends from $p = 2$ to the general case in a straightforward manner)

$$\|A-\Gamma\|_{L^p_{k,\Gamma}(\Omega_{\tilde g}(\frac{1}{2},2))} \le c(k,p)\|F_A\|_{L^2(\Omega_{\tilde g}(\frac{1}{4},4))},$$
$$\|F_A\|_{L^p_{k,\Gamma}(\Omega_{\tilde g}(\frac{1}{2},2))} \le c(k,p)\|F_A\|_{L^2(\Omega_{\tilde g}(\frac{1}{4},4))}.$$

The rescaling behavior of pointwise norms (5.1) implies that

$$\|\nabla_\Gamma^k(A-\Gamma)\|_{L^p(\Omega(\frac{1}{2}r_1,2r_1),g)} = r_1^{(4/p)-k-1}\|\nabla_\Gamma^k(A-\Gamma)|_{L^2(\Omega(\frac{1}{2},2),\tilde g)},$$
$$\|\nabla_A^k F_A\|_{L^p(\Omega(\frac{1}{2}r_1,2r_1),g)} = r_1^{(4/p)-k-2}\|\nabla_A^k F_A\|_{L^2(\Omega(\frac{1}{2},2),\tilde g)}.$$

Therefore, the desired Sobolev L^p_k estimates follow by combining the preceding two pairs of estimates.

Lastly, if $A - \Gamma$ is in radial gauge with respect to the origin, that is $x^\mu A_\mu(x) = 0$ for all $x \in \Omega$, then (5.2) yields

$$A_\nu(x) = \int_{r_1/4}^{|x|} t\theta^\mu (F_A)_{\mu\nu}(t\theta)\,dt, \quad x \in \Omega,$$

where $\theta = x/|x| \in S^3$. The rest of the proof now mimics that of Lemma 5.2. □

5.4. **Estimates for cut-off functions.** For any $N > 1$ and $\lambda > 0$, Lemma 7.2.10 in [8] provides a clever construction of a cut-off function β on $\mathbb{R}^4$ such that $\beta(x) = 0$ for $|x| \le N^{-1}\sqrt{\lambda}$, $\beta(x) = 1$ for $|x| \ge N\sqrt{\lambda}$, and satisfying the crucial estimate

$$\|\nabla\beta\|_{L^4} \le c(\log N)^{-3/4},$$

where c is a constant independent of N and λ. We will need the following refinement of this result; for the definition and properties of the $L^\sharp$ and related families of Sobolev norms, see [12, §4].

Lemma 5.8. *There is a positive constant c such that the following holds. For any $N > 4$ and $\lambda > 0$, there is a C^∞ cut-off function $\beta = \beta_{N,\lambda}$ on $\mathbb{R}^4$ such that*

$$\beta(x) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{2}\sqrt{\lambda}, \\ 0 & \text{if } |x| \leq N^{-1}\sqrt{\lambda}, \end{cases}$$

and satisfying the following estimates:

$$\|\nabla\beta\|_{L^4} \leq c(\log N)^{-3/4}, \tag{1}$$

$$\|\nabla^2\beta\|_{L^2} \leq c(\log N)^{-1/2}, \tag{2}$$

$$\|\nabla\beta\|_{L^{2\sharp}} \leq c(\log N)^{-1/2}, \tag{3}$$

$$\|\nabla\beta\|_{L^2} \leq c(\log N)^{-1}\sqrt{\lambda}, \tag{4}$$

$$\|\nabla^2\beta\|_{L^{4/3}} \leq c(\log N)^{-1}\sqrt{\lambda}. \tag{5}$$

Proof. We fix a C^∞ cut-off function $\kappa : \mathbb{R} \to [0,1]$ such that $\kappa(t) = 1$ if $t \geq 1$ and $\kappa(t) = 0$ if $t \leq 0$. Now define a C^∞ cut-off function $\alpha = \alpha_N : \mathbb{R} \to [0,1]$, depending on the parameter N, by setting

$$\alpha(t) := \kappa\left(\frac{\log N + t}{\log N - \log 2}\right).$$

Therefore,

$$\alpha(t) = \begin{cases} 1 & \text{if } t \geq -\log 2, \\ 0 & \text{if } t \leq -\log N, \end{cases}$$

and there is a constant c independent of N such that

$$\left|\frac{d\alpha}{dt}\right| \leq \frac{c}{\log N} \quad \text{and} \quad \left|\frac{d^2\alpha}{dt^2}\right| \leq \frac{c}{(\log N)^2}.$$

Now define the cut-off function $\beta = \beta_{N,\lambda} : \mathbb{R} \to [0,1]$, depending on the parameters N and λ, by setting

$$\beta(x) := \alpha(\log|x| - \log\sqrt{\lambda}), \qquad x \in \mathbb{R}^4,$$

and observe that β satisfies the following pointwise bounds on $\mathbb{R}^4$:

$$|\nabla\beta(x)| \leq \frac{c}{r\log N} \quad \text{and} \quad |\nabla^2\beta(x)| \leq \frac{c}{r^2\log N}, \tag{5.4}$$

where $r = |x|$. Consequently, using $dV = r^3\,dr\,d\theta$,

$$\int_{\mathbb{R}^4} |\nabla\beta|^4\,dV \leq \frac{c}{(\log N)^4}\int_{N^{-1}\sqrt{\lambda}}^{\frac{1}{2}\sqrt{\lambda}} \frac{dr}{r} \leq c\frac{\log N - \log 2}{(\log N)^4} \leq \frac{c}{(\log N)^3},$$

and so this gives (1). Next, by the pointwise estimate (5.4) we have

$$\int_{\mathbb{R}^4} |\nabla^2\beta|^2\,dV \leq \frac{c}{(\log N)^2}\int_{N^{-1}\sqrt{\lambda}}^{\frac{1}{2}\sqrt{\lambda}} \frac{dr}{r} \leq c\frac{\log N - \log 2}{(\log N)^2} \leq \frac{c}{\log N},$$

which gives (2). Lastly, [12, Lemma 5.2] implies that

$$\|\nabla\beta\|_{L^{2\sharp}} \leq c(\|\nabla^2\beta\|_{L^2} + \|\nabla\beta\|_{L^4}),$$

and so (3) follows from (1) and (2). Next, the pointwise estimate (5.4) yields

$$\int_{\mathbb{R}^4} |\nabla\beta|^2\, dV \le \frac{c}{(\log N)^2}\int_{N^{-1}\sqrt{\lambda}}^{\frac{1}{2}\sqrt{\lambda}} r\,dr \le c\frac{\lambda(1-N^{-2})}{(\log N)^2} \le \frac{c\lambda}{(\log N)^2},$$

and so this gives (4). Finally, the pointwise estimate (5.4) yields

$$\int_{\mathbb{R}^4} |\nabla\beta|^{4/3}\, dV \le \frac{c}{(\log N)^{4/3}}\int_{N^{-1}\sqrt{\lambda}}^{\frac{1}{2}\sqrt{\lambda}} r^{1/3}dr \le c\frac{\lambda^{2/3}(1-N^{-4/3})}{(\log N)^{4/3}} \le \frac{c\lambda^{2/3}}{(\log N)^{4/3}},$$

and so this gives (5). □

5.5. **Estimate for the self-dual curvature of cut-off connections.** An important application of our local estimates for anti-self-dual connections is to bound the self-dual curvature of approximately anti-self-dual connections produced by splicing.

Before we proceed to estimate F_A^+, where A is connection produced by splicing, it is useful to consider a slightly more general situation than that envisaged in the definition of the splicing map $\boldsymbol{\gamma}'$ in §3: this will, for example, allow us to apply our self-dual curvature estimates in situations where the gluing maps are iterated and the component connections are generally only anti-self-dual on open sets (where no cutting off occurs). Specifically, we require that the curvatures of the connections to be cut off are L^2-small and anti-self-dual near the cut-off points:

Definition 5.9. Let (X,g) be a C^∞, closed, oriented, Riemannian manifold, let $\ell<\kappa$ be positive integers, and let $w\in H^2(X,\mathbb{Z})$. Let $Y\subset X$ be a precompact open subset, let $\Sigma\subset \mathrm{Sym}^\ell(X)$ be a smooth stratum, let ε be a positive constant which is less than or equal to the constants of Lemmas 5.1 and 5.5, and let $0<\varrho\le 1$ be a constant. We call a connection A *suitable for gluing* if it has been produced by the splicing construction of §3 (see definition (3.26), for example). The background connection A_0 and the four-sphere connections A_i need *not* be anti-self-dual but are required to obey the following constraints. Suppose $\mathbf{x}\in\mathrm{Sym}^\ell(X)$ is represented by the m-tuple $(x_1,\dots,x_m)$ with multiplicities κ_i and $\cup_{x\in\mathbf{x}}B(x,\varrho)\subset Y$. Then:

1. The background connection, A_0, satisfies
   - $F_{A_0}^{+,g}=0$ on Y, and
   - $\sup_{x\in Y}\|F_A\|_{L^2(B(x,\varrho))}<\varepsilon$.
2. Require that ϱ be small enough that $\varrho\sqrt{8\pi^2\kappa_i}<\varepsilon$, for all $i\ge 1$. Let δ_i be a positive constant, let f_i be an orthonormal frame for $TX|_{x_i}$, for each point $x_i\in\mathbf{x}$, and let g_i be the corresponding metric on $\mathbb{R}^4=S^4-\{s\}$ defined in §3.3.2. Let λ_i be the metric-rescaling constant and let $\tilde g_i=\lambda_i^{-2}g_i$ be the rescaled metric on $\mathbb{R}^4=S^4-\{s\}$. The four-sphere connections, A_i, $i\ge1$, satisfy
   - $F_{A_i}^{+,\tilde g_i}=0$ on $\mathbb{R}^4-B(\varrho^{-1})$,
   - $z[A_i,\tilde g_i]=0$ and $\lambda[A_i,\tilde g_i]=1$.
3. The scales satisfy $\sqrt{\lambda}\le\varrho/16$, where $\lambda:=\max_{1\le i\le m}\lambda_i$.

Remark 5.10. The requirements of Definition 5.9 are satisfied by connections A produced by the splicing map $\boldsymbol{\gamma}'$ described in §3, with domain $\mathrm{Gl}_\kappa^w(X,\mathcal{U}_0,\Sigma,\lambda_0)$. The background connections A_0 are g-anti-self-dual over X and the four-sphere connections A_i are $\tilde g_i$-anti-self-dual via the deformation described in §9.

We briefly recall some elementary facts concerning metrics, connections, and rescaling. Suppose, to begin, that g is a Euclidean metric on $\mathbb{R}^4$ and so there are coordinates $\{x^\mu\}$ on $\mathbb{R}^4$ with respect to which g has the standard form, $g(x) = \delta_{\mu\nu}dx^\mu \otimes dx^\nu = dx^\mu \otimes dx^\mu$. Given a positive constant λ, let $\tilde{g} = \lambda^{-2}g$ denote the rescaled metric, so $\tilde{g}(x) = \lambda^{-2}dx^\mu \otimes dx^\mu$. With respect to the coordinates $y = x/\lambda$, the metric $\tilde{g}$ becomes $\tilde{g}(y) = dy^\mu \otimes dy^\mu$. Analogous remarks apply to metrics which are not necessarily Euclidean.

Let g be a metric on $\mathbb{R}^4$ and let $\lambda[A,g]$ be the scale of a unitary connection A over $\mathbb{R}^4$ with energy $8\pi^2\kappa$ and center of mass at the origin,

$$\lambda[A,g]^2 := \frac{1}{8\pi^2\kappa}\int_{\mathbb{R}^4} \operatorname{dist}_g(0,\cdot)^2|F_A|_g^2\,dV_g.$$

With respect to the rescaled metric, $\tilde{g} = \lambda^{-2}g$, we see that

$$\lambda[A,\tilde{g}]^2 := \frac{1}{8\pi^2\kappa}\int_{\mathbb{R}^4} \operatorname{dist}_{\tilde{g}}(0,\cdot)^2|F_A|_{\tilde{g}}^2\,dV_{\tilde{g}} = 1,$$

since $\operatorname{dist}_{\tilde{g}}(0,\cdot) = \lambda\operatorname{dist}_g(0,\cdot)$ and $|F_A|_{\tilde{g}}^2\,dV_{\tilde{g}} = |F_A|_g^2\,dV_g$.

Returning to the second part of Definition 5.9, the Chebychev inequality (see Lemma 3.3) implies that

$$\|F_{A_i}\|_{L^2(\mathbb{R}^4-B(0,1/\varrho),\tilde{g}_i)} = \|F_{A_i}\|_{L^2(\mathbb{R}^4-B(0,\lambda_i/\varrho),g_i)} \le \varrho\sqrt{8\pi^2\kappa_i} < \varepsilon,$$

since $\varrho\sqrt{8\pi^2\kappa_i} < \varepsilon$, by definition of ϱ, and as

$$\lambda[A_i,g_i]^2 = \frac{1}{8\pi^2\kappa}\int_{\mathbb{R}^4} \operatorname{dist}_{g_i}(0,\cdot)^2|F_{A_i}|_{g_i}^2\,dV_{g_i} = \lambda_i^2.$$

The following proposition gives the key estimate we need:

Proposition 5.11. *There is a positive constant c with the following significance. Let A be a spliced connection on $\mathfrak{g}_E$ which is* suitable for gluing*, as in Definition 5.9. Then,*

$$\|F_A^{+,g}\|_{L^p(X,g)} \le c\varepsilon\lambda^{2/p}\varrho^{-2} + c\|F_{A_0}^{+,g}\|_{L^p(X,g)} + c\sum_{i=1}^m \|F_{A_i}^{+,g_i}\|_{L^p(S^4,g_i)}.$$

If A_0 is g-anti-self-dual over X, the A_i are g_i-anti-self-dual over S^4, and $c' := c\varepsilon\varrho^{-2}$,

$$\|F_A^{+,g}\|_{L^p(X)} \le c'\lambda^{2/p}. \tag{5.5}$$

Proof. Recall from definition (3.26) that

$$A := \begin{cases} A_0 & \text{on } X - \cup_{i=1}^m B(x_i,4\sqrt{\lambda_i}), \\ \Gamma + \chi_{x_i,4\sqrt{\lambda_i}}\sigma_i^*A_0 & \text{on } \Omega(x_i,2\sqrt{\lambda_i},4\sqrt{\lambda_i}), \\ \Gamma & \text{on } \Omega(x_i,\sqrt{\lambda_i}/2,2\sqrt{\lambda_i}), \\ \Gamma + (1-\chi_{x_i,\sqrt{\lambda_i}/2})(\varphi_n\circ\varphi_i^{-1})^*\tau_i^*A_i & \text{on } \Omega(x_i,\sqrt{\lambda_i}/4,\sqrt{\lambda_i}/2), \\ (\varphi_n\circ\varphi_i^{-1})^*A_i & \text{on } B(x_i,\sqrt{\lambda_i}/4), \end{cases} \tag{5.6}$$

where the cut-off functions are defined in equation (3.20). Over the region $X - \cup_{i=1}^m B(x_i,4\sqrt{\lambda_i})$ we have $F_A = F_{A_0}$, while over the annuli $\Omega(x_i,\sqrt{\lambda_i}/2,2\sqrt{\lambda_i})$ we have $F_A = 0$, and over the balls $B(x_i,\sqrt{\lambda_i}/4)$ we have $F_A = F_{A_i}$. Thus, it remains to consider the two annuli where cutting off occurs.

The hypotheses imply that $F^{+,g}_{A_0} = 0$ on each ball $B(x_i,\varrho)$. Over the outer annulus $\Omega(x_i, 2\sqrt{\lambda_i}, 4\sqrt{\lambda_i})$ we see that (5.6) yields

$$F_A = \chi_i F_{A_0} + d\chi_i \wedge \sigma_i^* A_0 + (\chi_i^2 - \chi_i)\sigma_i^* A_0 \wedge \sigma_i^* A_0, \tag{5.7}$$

where we set $\chi_i = \chi_{x_i, 4\sqrt{\lambda_i}}$ for convenience. According to Lemma 5.1, we have the L^∞ curvature estimate and by Lemma 5.2 , since the connection one-form $\sigma_i^* A_0$ is in radial gauge with respect to the point x_i, the L^p connection one-form estimate,

$$\begin{aligned}
\|F_{A_0}\|_{L^\infty(B(x_i,\varrho/2))} &\le c\varrho^{-2}\|F_{A_0}\|_{L^2(B(x_i,\varrho))}, \\
\|\sigma_i^* A_0\|_{L^p(B(x_i,4\sqrt{\lambda_i}))} &\le c\lambda_i^{(2/p)-1/2}\|F_{A_0}\|_{L^2(B(x_i,8\sqrt{\lambda_i}))}.
\end{aligned}$$

The lemmas are applicable because of the choice of ε in Definition 5.9. Because $\|F_{A_0}\|_{L^2(B(x_i,8\sqrt{\lambda_i}))} \le c\lambda_i\|F_{A_0}\|_{L^\infty(B(x_i,8\sqrt{\lambda_i}))}$, we can combine the preceding two estimates to get the L^p estimate, for any $1 \le p \le \infty$,

$$\|\sigma_i^* A_0\|_{L^p(B(x_i,4\sqrt{\lambda_i}))} \le c\lambda_i^{(2/p)+1/2}\varrho^{-2}\|F_{A_0}\|_{L^2(B(x_i,\varrho))}, \quad \sqrt{\lambda_i} \le \varrho/16. \tag{5.8}$$

Given $1 \le p \le \infty$, choose $4/3 \le q \le \infty$ by setting $1/p = 1/4 + 1/q$. From the expression (5.7) for F_A over the annulus $\Omega(x_i, 2\sqrt{\lambda_i}, 4\sqrt{\lambda_i})$ and the integral estimates (5.8) for $\sigma_i^* A_0$ and that of Lemma 5.8 for $d\chi_i$, we obtain an estimate for F_A^+,

$$\begin{aligned}
\|F_A^+\|_{L^p(\Omega(x_i,2\sqrt{\lambda_i},4\sqrt{\lambda_i}))} &\le \|F_{A_0}^+\|_{L^p} + \|d\chi_i\|_{L^4}\|\sigma_i^* A_0\|_{L^q(\Omega(x_i,2\sqrt{\lambda_i},4\sqrt{\lambda_i}))} \\
&\quad + \|\sigma_i^* A_0\|^2_{L^{2p}(\Omega(x_i,2\sqrt{\lambda_i},4\sqrt{\lambda_i}))} \\
&\le \|F_{A_0}^+\|_{L^p} + c\varrho^{-2}(\lambda_i^{2/p} + \lambda_i^{(2/p)+1})\|F_{A_0}\|_{L^2(B(x_i,\varrho))}.
\end{aligned} \tag{5.9}$$

This completes the required estimates for F_A^+ over the outer annuli.

Next we turn to the estimate of F_A^+ over the inner annulus $\Omega(x_i, \sqrt{\lambda_i}/4, \sqrt{\lambda_i}/2)$. The hypotheses imply that $F^{+,g_i}_{A_i} = 0$ on $\mathbb{R}^4 - B(\lambda_i/\varrho)$. This time, our expression (5.6) for the connection A yields

$$F_A = \chi_i F(\tau_i^* A_i) + d\chi_i \wedge \tau_i^* A_i + (\chi_i^2 - \chi_i)\tau_i^* A_i \wedge \tau_i^* A_i, \tag{5.10}$$

where we now abbreviate $\chi_i = 1 - \chi_{x_i,\sqrt{\lambda_i}/2}$ and $\tau_i^* A_i = (\varphi_n \circ \varphi_i^{-1})^* \tau_i^* A_i$.

According to Lemma 5.1, we have the L^∞ curvature estimate and by Lemma 5.5, since the connection one-form $\tau_i^* A_i$ is in radial gauge with respect to the south pole s, the L^p connection one-form estimate,

$$\begin{aligned}
\|F_{A_i}\|_{L^\infty(\varphi_s(B(\varrho/2)),\tilde g_i)} &\le c\varrho^{-2}\|F_{A_i}\|_{L^2(\varphi_s(B(\varrho)),\tilde g_i)}, \\
\|\tau_i^* A_i\|_{L^p(\varphi_n(\mathbb{R}^4 - B(\sqrt{\lambda_i}/4)),g_i)} &\le c\lambda_i^{(2/p)-1/2}\|F_{A_i}\|_{L^2(\varphi_n(\mathbb{R}^4 - B(x_i,\sqrt{\lambda_i}/8)),g_i)}.
\end{aligned}$$

Again, the lemmas are applicable because of the choice of ε in Definition 5.9. Since

$$\begin{aligned}
\|F_{A_i}\|_{L^2(\varphi_s(B(\varrho)),\tilde g_i)} = \|F_{A_i}\|_{L^2(\varphi_n(\mathbb{R}^4 - B(1/\varrho)),\tilde g_i)} &= \|F_{A_i}\|_{L^2(\varphi_n(\mathbb{R}^4 - B(\lambda_i/\varrho)),g_i)}, \\
\|F_{A_i}\|_{L^2(\varphi_s(B(\sqrt{\lambda_i}/8)),\tilde g_i)} &\le c\lambda_i\|F_{A_i}\|_{L^\infty(\varphi_s(B(\sqrt{\lambda_i}/8)),\tilde g_i)}, \\
\|F_{A_i}\|_{L^2(\varphi_n(\mathbb{R}^4 - B(\sqrt{\lambda_i}/8)),g_i)} = \|F_{A_i}\|_{L^2(\varphi_n(\mathbb{R}^4 - B(1/8\sqrt{\lambda_i})),\tilde g_i)} &= \|F_{A_i}\|_{L^2(\varphi_s(B(8\sqrt{\lambda_i})),\tilde g_i)},
\end{aligned}$$

we can combine the two preceding sets of estimates to get the L^p bound, for any $1 \le p \le \infty$,

$$\|\tau_i^* A_i\|_{L^p(\varphi_n(\mathbb{R}^4 - B(\sqrt{\lambda_i}/4)),g_i)} \le c\lambda_i^{(2/p)+1/2}\varrho^{-2}\|F_{A_i}\|_{L^2(\varphi_s(B(\varrho)),\tilde g_i)}, \quad \sqrt{\lambda_i} \le \varrho/16. \tag{5.11}$$

From the expression (5.10) for F_A over the inner annulus $\Omega(x_i, \sqrt{\lambda_i}/4, \sqrt{\lambda_i}/2)$ and the integral estimates (5.11) for $\varphi_n^*\tau_i^*A_i$ and that of Lemma 5.8 for $d\chi_i$, we obtain an estimate for F_A^+,

$$
\begin{aligned}
(5.12) \qquad & \|F_A^+\|_{L^p(\Omega(x_i,\sqrt{\lambda_i}/4,\sqrt{\lambda_i}/2))} \\
&\le \|F_{A_i}^+\|_{L^p(S^4,g_i)} + \|d\chi_i\|_{L^4}\|\tau_i^*A_i\|_{L^q(\Omega(n,\sqrt{\lambda_i}/4,\sqrt{\lambda_i}/2),g_i)} \\
&\quad + \|\tau_i^*A_i\|^2_{L^{2p}(\Omega(n,\sqrt{\lambda_i}/4,\sqrt{\lambda_i}/2),g_i)} \\
&\le \|F_{A_i}^+\|_{L^p(S^4,g_i)} + c\varrho^{-2}(\lambda_i^{2/p} + \lambda_i^{(2/p)+1})\|F_{A_i}\|_{L^2(\Omega(n,\lambda_i/\varrho,\infty),g_i)}.
\end{aligned}
$$

This completes the required estimate for F_A^+ over the inner annulus.

The required bound for F_A^+ now follows by combining the estimates (5.9) and (5.12) for F_A^+ over the annuli where cutting off occurs, together with the trivial estimates for F_A^+ on the complement of these annuli. □

Remark 5.12. A slightly different approach to constructing pointwise estimates for local connection one-forms on annuli (via temporal gauge on cylinders) is described in [8, §7.3.4].

Corollary 5.13. *Continue the hypotheses and notation of Proposition 5.11. Then, for any constant* $2 < p \le \infty$*,*

$$
(5.13) \qquad \|F_A^{+,g}\|_{L^{\sharp,2}(X)} \le c'\lambda^{2/p} \quad \text{and} \quad \|F_A^{+,g}\|_{L^\infty(X)} \le c'.
$$

where the constant c' *now also depends on the choice of* p.

Proof. This follows from the estimate (5.5) of Proposition 5.11, together with the embedding $L^p(X) \subset L^\sharp(X)$ of [12, Lemma 4.1 (1)]. □

6. Eigenvalue estimates for Laplacians on self-dual two-forms

We need to analyze the behavior of the small eigenvalues of the Laplacian $d_A^+d_A^{+,*}$ on $L^2(X, \Lambda^+ \otimes \mathfrak{g}_E)$ as the point $[A]$ in $\mathcal{B}_E(X)$ converges, in the Uhlenbeck topology, to a point $([A_0], \mathbf{x})$ in $\mathcal{B}_{E_0}(X) \times \mathrm{Sym}^\ell(X)$. Our strategy is modelled on the method used by Taubes to describe the small eigenvalues of $d_A^+d_A^{+,*}$ [54, Lemma 4.8], where A is an approximately anti-self-dual connection constructed by splicing $|\mathbf{x}| \le \ell$ one-instantons onto the background product connection Γ. However, the upper and lower bounds for the small eigenvalues we obtain here are more precise and more general than those of [54] or the analogous arguments are used by Mrowka in the proof of Theorem 6.1.0.0.4 [49] in order to establish his 'main eigenvalue estimate'. Our derivation in [14] of the eigenvalue bounds for the Laplacian $d_A^*d_A$ on $L^2(X, \mathfrak{g}_E)$ is closely modelled on the arguments given here for the Laplacian $d_A^+d_A^{+,*}$ on $L^2(X, \Lambda^+ \otimes \mathfrak{g}_E)$.

In the present article, our main application of the eigenvalue bounds of this section will be to deriving an estimate for the partial right inverse $P_{A,\mu} = d_A^{+,*}(d_A^+d_A^{+,*})^{-1}\Pi_{A,\mu}^\perp$ for d_A^+, as given in Proposition 7.3. We have the following analogue of Proposition 4.6 in [54].

Theorem 6.1. *Let* M *be a positive constant and let* (X, g) *be a closed, oriented, Riemannian four-manifold. Then there are positive constants* $c = c(g)$*,* $C = C(g, M)$*, and small enough* $\lambda_0 = \lambda_0(C)$ *such that for all* $\lambda \in (0, \lambda_0]$*, the following holds. Let* E*,* E_0 *be Hermitian, rank-two vector bundles over* X *with* $\det E = \det E_0$*,* $c_2(E) = c_2(E_0) + \ell$*, for* *some integer $\ell \geq 0$, and suppose $\frac{1}{4}p_1(\mathfrak{g}_E) \leq M$. Let $[A_0] \subset \mathcal{B}_{E_0}(X)$ be a point such that $\|F^{+,g}_{A_0}\|_{L^\infty(X,g)} < M$. Suppose $\mathbf{x} \in \mathrm{Sym}^\ell(X)$ and define a precompact open subset of $X \setminus \mathbf{x}$ by setting $U := X - \cup_{x\in\mathbf{x}} \bar{B}(x, 4\sqrt{\lambda})$. Let A be a connection representing a point in the open subset of $\mathcal{B}_E(X)$ defined by the following constraints:*

- $\|F^{+,g}_A\|_{L^\infty(X,g)} < M$,
- $\|A - A_0\|_{L^2(U)} < c\sqrt{\lambda}$ *and* $\|A - A_0\|_{L^2_{1,A_0}(U)} < M$, *(noting that $E|_{X\setminus\mathbf{x}} \cong E_0|_{X\setminus\mathbf{x}}$).*

Let $\nu_2[A_0]$ be the least positive eigenvalue *of the Laplacian $d^+_{A_0}d^{+,*}_{A_0}$ on $L^2(\Lambda^+ \otimes \mathfrak{g}_{E_0})$ and suppose* $\max\{\nu_2[A_0], \nu_2[A_0]^{-1}, \|F^{+,g}_{A_0}\|_{L^\infty(X,g)}\} \leq M$. *Assume that the kernel of $d^+_{A_0}d^{+,*}_{A_0}$ has dimension $n \leq M$. Let $\{\mu_i[A]\}_{i\geq 1}$ denote the eigenvalues, repeated according to their multiplicity, of the Laplacian $d^+_A d^{+,*}_A$ on $L^2(\Lambda^+ \otimes \mathfrak{g}_E)$. Then*

$$\mu_i[A] \leq \begin{cases} C\lambda, & \text{if } i = 1,\dots,n, \\ \nu_2[A_0] + C\sqrt{\lambda}, & \text{if } i = n+1, \end{cases}$$

$$\mu_i[A] \geq \nu_2[A_0] - C\sqrt{\lambda}, \quad \text{if } i \geq n+1.$$

Remark 6.2. 1. Recall that by [8, §2.1.4]

$$\kappa = -\tfrac{1}{4}p_1(\mathfrak{g}_E) = \frac{1}{8\pi^2}(\|F^-_A\|^2_{L^2} - \|F^+_A\|^2_{L^2}) = \frac{1}{8\pi^2}(\|F_A\|^2_{L^2} - 2\|F^+_A\|^2_{L^2}),$$

and therefore,

$$\|F_A\|^2_{L^2} \leq 2M^2 + 8\pi^2\kappa.$$

Thus, we have a universal energy bound for all points in $\mathcal{B}_E(X)$ with $\|F^{+,g}_A\|_{L^\infty(X,g)} < M$.

2. If a point $[A]$ in $\mathcal{B}_E(X)$ lies in an Uhlenbeck neighborhood (see Definition 3.1) of a point $([A_0], \mathbf{x})$ in $\mathcal{B}_{E_0}(X) \times \mathrm{Sym}^\ell(X)$, then the hypotheses of Theorem 6.1 are fulfilled.

Corollary 6.3. *Let (X, g) be a closed, oriented, Riemannian four-manifold. Suppose $[A_\alpha]$ is a sequence of points in $\mathcal{B}_E(X)$ which converges, in the Uhlenbeck topology, to a point $([A_0], \mathbf{x}) \in \mathcal{B}_{E_0}(X) \times \mathrm{Sym}^\ell(X)$ and suppose the kernel of $d^+_{A_0}d^{+,*}_{A_0}$ has dimension n. Then the first n eigenvalues (counted with multiplicity) of $d^+_{A_\alpha}d^{+,*}_{A_\alpha}$ converge to zero, while the $(n+1)$-st eigenvalue of $d^+_{A_\alpha}d^{+,*}_{A_\alpha}$ converges to $\nu_2[A_0]$, the least positive eigenvalue of $d^+_{A_0}d^{+,*}_{A_0}$.*

Remark 6.4. The statements of Theorem 6.1 and Corollary 6.3 extend without difficulty from the case of a single, fixed metric g and connection $[A_0]$ to the case of a family $\mathcal{U}_0 \subset \mathcal{B}_E(X) \times \mathrm{Met}(X)$ satisfying the constraints

- $\mathrm{Spec}\, d^{+,g}_{A_0}d^{+,g,*}_{A_0} \subset [0, t) \cup (2\mu, \infty)$, for some positive constants t, μ such that $t \leq \mu/2$ and all $[A_0, g] \in \mathcal{U}_0$,
- The number of eigenvalues, counted with multiplicity, of $d^{+,g}_{A_0}d^{+,g,*}_{A_0}$ which are less than μ is constant, say n, as $[A_0, g]$ varies over $\mathcal{U}_0$, and
- $\|F^{+,g}_{A_0}\|_{L^\infty(X,g)} < M$.

Thus, for example, in Corollary 6.3 we can replace g by a sequence $\{g_\alpha\}$ of smooth Riemannian metrics on X converging in C^r, $r \geq 1$, to a metric g_0. In the statements of Theorem 6.1 and Corollary 6.3, we replace $\nu_2[A_0]$ by μ. Then, in the proofs below, we replace $\nu_2[A_0]$

by μ, $\operatorname{Ker} d_{A_0}^{+,g} d_{A_0}^{+,g,*}$ by $\operatorname{Ran}\Pi_{A_0,\mu}$ and Π_{A_0} by $\Pi_{A_0,\mu}$, the projection from $L^2(\Lambda^+ \otimes \mathfrak{g}_{E_0})$ onto the span of the eigenvectors of $d_{A_0}^{+,g} d_{A_0}^{+,g,*}$ with eigenvalue less than μ. Thus, in the statement of Theorem 6.1, the upper bounds are replaced by $C(t+\lambda)$ and $\mu + C(\sqrt{t}+\sqrt{\lambda})$, respectively, while the lower bound is replaced by $\mu - C(\sqrt{t}+\sqrt{\lambda})$, for all $t < t_0(C)$ and $\lambda < \lambda_0(C)$. Similar modifications apply to Corollary 6.3, noting that λ is replaced by a sequence $\{\lambda_\alpha\}$ converging to zero.

The proof of Theorem 6.1 takes up the remainder of this section. While the argument is lengthy, the basic strategy is reasonably straightforward:

- Find upper bounds for the first $n+1$ eigenvalues of $d_A^+ d_A^{+,*}$;
- Construct an approximately L^2-orthonormal basis for $\operatorname{Ker} d_{A_0}^+ d_{A_0}^{+,*}$ by cutting off the eigenvectors corresponding to the first n eigenvalues of $d_A^+ d_A^{+,*}$;
- Compute a lower bound for the $(n+1)$-st eigenvalue of $d_A^+ d_A^{+,*}$.

We begin in the next subsection by disposing of a few technical lemmas.

6.1. **Technical preliminaries.** We begin with some elementary lemmas before proceeding to the different stages of the proof of Theorem 6.1. The following Kato-Sobolev inequality is well-known; see Lemma 4.6 in [52] for the case $p=2$.

Lemma 6.5. *Let X be a C^∞, closed, oriented, Riemannian, four-manifold and let $1 \le p < 4$. Then there is a positive constant c with the following significance. Let A be an orthogonal L_1^p connection on a Riemannian vector bundle V over X. Then, for any $a \in L_1^p(V)$, we have*

$$\|a\|_{L^q(X)} \le c\|a\|_{L^p_{1,A}(X)},$$

when $1/p = 1/q + 1/4$.

Proof. The estimate follows from the pointwise Kato inequality $|\nabla|a|| \le |\nabla_A a|$ (see [24, Equation (6.20)] and the Sobolev inequality

$$\|f\|_{L^q} \le c(\|\nabla f\|_{L^q} + \|f\|_p),$$

for any $f \in L_1^p(X)$ [2, Theorem 5.4]. ☐

We shall need the Bochner-Weitzenböck formulas,

$$d_A d_A^* + 2 d_A^{+,*} d_A^+ = \nabla_A^* \nabla_A + \{\mathrm{Ric}, \cdot\} - 2\{F_A^-, \cdot\}, \tag{6.1}$$

$$2 d_A^+ d_A^{+,*} = \nabla_A^* \nabla_A - 2\{W^+, \cdot\} + \frac{R}{3} + \{F_A^+, \cdot\}, \tag{6.2}$$

for the Laplacians on $\Gamma(\Lambda^1 \otimes \mathfrak{g}_E)$ and $\Gamma(\Lambda^+ \otimes \mathfrak{g}_E)$, respectively; see [24, p. 94]. We give $L^2_{1,A}$ estimates for v in terms of the L^2 norm of $d_A^{+,*}$; similar estimates are given in [52, Lemma 5.2] and in [54, Appendix A].

Lemma 6.6. *Let X be a C^∞, closed, oriented, Riemannian four-manifold. Then there is a positive constant c with the following significance. Let A be an L_4^2 orthogonal connection* *on an* $\mathrm{SO}(3)$ *bundle* $\mathfrak{g}_E$ *over* X. *Then, for all* $v \in L^{4/3}_2(\Lambda^+ \otimes \mathfrak{g}_E)$,

$$\|v\|_{L^2_{1,A}(X)} \le c(1 + \|F_A^+\|_{L^\infty(X)})^{1/2}(\|d_A^{+,*}v\|_{L^2(X)} + \|v\|_{L^2(X)}), \tag{1}$$

$$\|d_A^{+,*}v\|_{L^2(X)} \le (1 + \|F_A^+\|_{L^\infty(X)})^{1/2}\|d_A^+d_A^{+,*}v\|_{L^{4/3}(X)}. \tag{2}$$

Proof. The Bochner-Weitzenbock formula (6.2) and integration by parts yields

$$\begin{aligned}\|\nabla_A v\|^2_{L^2} &= (\nabla_A^*\nabla_A v, v)_{L^2} \\ &= 2(d_A^+d_A^{+,*}v, v)_{L^2} + (\{W^+, v\}, v)_{L^2} - \frac{1}{3}(Rv, v)_{L^2} - (\{F_A^+, v\}, v)_{L^2} \\ &\le 2\|d_A^{+,*}v\|^2_{L^2} + c(1 + \|F_A^+\|_{L^\infty})\|v\|^2_{L^2},\end{aligned}$$

which gives the first assertion. The second assertion follows by integrating by parts, Hölder's inequality, and Kato's inequality (Lemma 6.5),

$$\|d_A^{+,*}v\|^2_{L^2} = (d_A^+d_A^{+,*}v, v)_{L^2} \le \|d_A^+d_A^{+,*}v\|_{L^{4/3}}\|v\|_{L^4} \le c\|d_A^+d_A^{+,*}v\|_{L^{4/3}}\|v\|_{L^2_{1,A}},$$

together with the first inequality. □

The following consequence of [12, Lemma 5.9] is the principal tool we use to obtain the required eigenvector estimates.

Lemma 6.7. *Let* X *be a* C^∞, *closed, oriented, Riemannian four-manifold. Then there is a constant* c *with the following significance. Let* A *be an* L^2_4 *connection on* $\mathfrak{g}_E$ *and let* $\eta = \eta(A) \in L^2_3(\Lambda^+ \otimes \mathfrak{g}_E)$ *be an eigenvector of* $d_A^+d_A^{+,*}$ *with eigenvalue* $\mu[A]$. *Then*

$$\|\eta\|_{L^\infty(X)} + \|\eta\|_{L^2_{2,A}(X)} \le C[A]\|\eta\|_{L^2(X)},$$

where $C[A] := C([A], \mu[A])$ *with* $\mu = \mu[A]$ *in*

$$C([A], \mu) := c(1 + \|F_A\|_{L^2(X)})^2(1 + \|F_A^+\|_{L^\infty(X)})^{5/2}(1 + \mu)(1 + \sqrt{\mu}).$$

More generally, if $\eta \in L^2(\Lambda^+ \otimes \mathfrak{g}_E)$ *lies in the span of the eigenvectors of* $d_A^+d_A^{+,*}$ *with eigenvalues less than or equal to a positive constant* μ, *then*

$$\|\eta\|_{L^\infty(X)} + \|\eta\|_{L^2_{2,A}(X)} \le C([A], \mu)\|\eta\|_{L^2(X)}.$$

Proof. By [12, Lemma 5.9] (or Lemma 7.1 here),

$$\begin{aligned}&\|\eta\|_{L^\infty} + \|\eta\|_{L^2_{2,A}(X)} \\ &\quad\le c(1 + \|F_A\|_{L^2})^2(1 + \|F_A^+\|_{L^\infty})^2(\|\eta\|_{L^2} + \|d_A^+d_A^{+,*}\eta\|_{L^\sharp}) \\ &\quad= c(1 + \|F_A\|_{L^2})^2(1 + \|F_A^+\|_{L^\infty})^2(1 + \mu[A])\|\eta\|_{L^{\sharp,2}},\end{aligned}$$

using $d_A^+d_A^{+,*}\eta = \mu[A]\eta$ in the second line. On the other hand,

$$\begin{aligned}\|\eta\|_{L^{\sharp,2}} &\le c\|\eta\|_{L^4} \quad \text{(by [12, Lemma 4.1])} \\ &\le c\|\eta\|_{L^2_{1,A}} \quad \text{(by Lemma 6.5)} \\ &\le c(1 + \|F_A^+\|_{L^\infty})^{1/2}(\|\eta\|_{L^2} + \|d_A^{+,*}\eta\|_{L^2}) \quad \text{(by Lemma 6.6)} \\ &= c(1 + \|F_A^+\|_{L^\infty})^{1/2}(1 + \sqrt{\mu[A]})\|\eta\|_{L^2},\end{aligned}$$

using $\|d_A^{+,*}\eta\|_{L^2} = \sqrt{\mu[A]}\|\eta\|_{L^2}$ to obtain the final equality. Combining the preceding estimates completes the proof of the first inequality.

Next suppose that $\eta \in L^2(\Lambda^+ \otimes \mathfrak{g}_E)$ lies in the span of the n L^2-orthonormal eigenvectors $\eta_i(A)$ of $d_A^+ d_A^{+,*}$ with eigenvalues $\mu_i[A] \leq \mu$. Since $\eta = \sum_{i=1}^n (\eta, \eta_i)_{L^2}\eta_i$ and denoting $C' = (1 + \|F_A\|_{L^2})^2(1 + \|F_A^+\|_{L^\infty})^{5/2}$, we have

$$\begin{aligned}
\|\eta\|_{L^\infty} &\leq \sum_{i=1}^n |(\eta, \eta_i)_{L^2}| \cdot \|\eta_i\|_{L^\infty} \\
&\leq C' \sum_{i=1}^n |(\eta, \eta_i)_{L^2}| \cdot (1 + \mu_i)(1 + \sqrt{\mu_i})\|\eta_i\|_{L^2} \quad \text{(by the special case)}, \\
&\leq C'(1 + \mu)(1 + \sqrt{\mu}) \left(\sum_{i=1}^n (\eta, \eta_i)_{L^2}^2\right)^{1/2} \\
&= C'(1 + \mu)(1 + \sqrt{\mu})\|\eta\|_{L^2},
\end{aligned}$$

where we use the inequality $(\sum_{i=1}^n a_i)^2 \leq 2\sum_{i=1}^n a_i^2$, for $a_i \in \mathbb{R}$, in the second last line. This completes the proof. $\square$

The other basic tool we shall need is Courant's min-max principle for eigenvalues (see, for example, [4, pp. 16-17]).

Lemma 6.8. *Let T be a non-negative, self-adjoint operator on a Hilbert space, $\mathfrak{H}$, such that $(\mathrm{id} + T)^{-1}$ is compact. Let $0 \leq \mu_1 \leq \mu_2 \leq \cdots$ be the eigenvalues of T, repeated according to their multiplicity. Then, for any subspace $V_n \subset \mathfrak{H}$ with $\dim V_n = n$,*

$$\mu_{n+1} \geq \inf_{\eta \in V_n^\perp} \frac{(\eta, T\eta)}{\|\eta\|^2},$$

where $V_n^\perp$ is the orthogonal complement of V_n in $\mathfrak{H}$. If V_n is the span of the eigenvectors of T corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$, then equality holds above.

6.2. **Good sequences of cut-off functions.** In many of our arguments involving Uhlenbeck continuity of eigenvalues, eigenvectors, gluing maps, gluing-map differentials and so on, we shall need suitable sequences of cut-off functions to compare connections, and objects depending on them, which live on different bundles but which become isomorphic when restricted to an open subset $X \setminus \mathbf{x}$, for some $\mathbf{x} \in \mathrm{Sym}^\bullet(X)$. Since the construction of such sequences will arise many times, it is useful to extract the essential properties we shall need once and for all in the following definition-lemma:

Lemma 6.9. *Let X be a smooth Riemannian four-manifold and let $\mathbf{x} \in \mathrm{Sym}^\bullet(X)$. Then there is a sequence of cut-off functions $\{\chi_n\}_{n=1}^\infty$ such that the following hold:*

- $\cup_{n=1}^\infty U_n = X \setminus \mathbf{x}$, *where $U_n \Subset X \setminus \mathbf{x}$ is the open subset given by the interior of* $\operatorname{supp}\chi_n$,
- $\chi_m \leq \chi_n$ *for all $m \leq n$, so $U_m \subset U_n$ and* $\operatorname{supp}\chi_m \subset \operatorname{supp}\chi_n$,
- $\operatorname{supp}\chi_n \Subset X \setminus \mathbf{x}$,
- $\lim_{n\to\infty}(\|\nabla\chi_n\|_{L^2_1} + \|\nabla^2\chi_n\|_{L^\sharp}) = 0$,

We call any sequence of cut-off functions satisfying the above properties an $\mathbf{x}$-good sequence.

Proof. We may assume without loss that the multiset $\mathbf{x} \in \mathrm{Sym}^\bullet(X)$ is represented by a single point $x_0 \in X$. Recall from Lemma 5.8 that we can define a cut-off function $\beta_{N,\lambda} : X \to \mathbb{R}$ with the property that

$$\beta_{N,\lambda}(x) = \begin{cases} 1 & \text{if } \operatorname{dist}(x, x_0) \geq \frac{1}{2}\sqrt{\lambda}, \\ 0 & \text{if } \operatorname{dist}(x, x_0) \leq N^{-1}\sqrt{\lambda}, \end{cases}$$

and satisfying the estimates

$$\begin{aligned} \|\nabla \beta_{N,\lambda}\|_{L^4} &\leq c(\log N)^{-3/4}, \\ \|\nabla^2 \beta_{N,\lambda}\|_{L^2} &\leq c(\log N)^{-1/2}, \\ \|\nabla \beta_{N,\lambda}\|_{L^{2\sharp}} &\leq c(\log N)^{-1/2}, \\ \|\nabla \beta_{N,\lambda}\|_{L^2} &\leq c(\log N)^{-1}\sqrt{\lambda}. \end{aligned}$$

If we choose $N = n$ and $\lambda = n^{-4}$, for example, so $\chi_n := \beta_{n,n^{-4}}$ and $U_n = X \setminus \bar{B}(x_0, \frac{1}{2}n^{-2})$, then it is immediate that the sequences $\{\chi_n\}_{n=1}^\infty$ and $\{U_n\}_{n=1}^\infty$ have the desired properties. □

6.3. **An upper bound for the small eigenvalues.** We turn to the proof of Theorem 6.1. We define an "$\mathbf{x}$-good" cut-off function χ_0 on X (see Lemma 6.9) by setting $\chi_0 = \chi_{x_i, 8\sqrt{\lambda_i}}$ (see equation (3.20) for its definition) on each ball $B(x_i, 8\sqrt{\lambda_i})$, so χ_0 is equal to zero on the balls $B(x_i, 4\sqrt{\lambda_i})$, and setting $\chi_0 = 1$ on the complement of these balls, $X - \cup_{i=1}^m B(x_i, 8\sqrt{\lambda_i})$. We take

$$U := X - \bigcup_{i=1}^m B(x_i, 4\sqrt{\lambda_i}),$$

so $U \Subset X \setminus \mathbf{x}$ and U is the interior of $\operatorname{supp}\chi_0$.

Though it suffices to consider sequences of $\mathbf{x}$-good cut-off functions only satisfying the broad properties of Lemma 6.9, we assume for simplicity that χ_0 is defined as in the preceding paragraph and observe that $\operatorname{supp}\chi_0 \Subset X \setminus \mathbf{x}$. We may assume without loss that $E|_{X\setminus\mathbf{x}} = E_0|_{X\setminus\mathbf{x}}$ and thus, over $X \setminus \mathbf{x}$, we may write

$$a := A - A_0 \in L^2_{k,\mathrm{loc}}(X \setminus \mathbf{x}, \Lambda^1 \otimes \mathfrak{g}_{E_0}), \tag{6.3}$$

where the section a may also be viewed as an element of $L^2_{k,\mathrm{loc}}(X \setminus \mathbf{x}, \Lambda^1 \otimes \mathfrak{g}_E)$. Our initial task is to find an upper bound for the first $n+1$ eigenvalues of $d_A^+ d_A^{+,*}$.

Lemma 6.10. *Continue the hypotheses and notation of Proposition 6.1. Let $\nu_2[A_0]$ be the least positive eigenvalue of $d_{A_0}^+ d_{A_0}^{+,*}$ and let $\mu_1[A] \leq \cdots \leq \mu_{n+1}[A]$ be the first $n+1$ eigenvalues of $d_A^+ d_A^{+,*}$. Then there are positive constants $C = C(\|F_{A_0}\|_{L^2}, \|F_{A_0}^+\|_{L^\infty}, \nu_2[A_0])$ and $\lambda_0 = \lambda_0(C)$ such that for all $\lambda \in (0, \lambda_0]$,*

$$\mu_i[A] \leq \begin{cases} C\lambda, & \text{if } i = 1, \ldots, n, \\ \nu_2[A_0] + C\sqrt{\lambda}, & \text{if } i = n+1. \end{cases} \tag{6.4}$$

Proof. Let $\{\xi_i(A_0)\}_{i=1}^n \subset L^2(\Lambda^+ \otimes \mathfrak{g}_{E_0})$ be an L^2-orthonormal basis for $\operatorname{Ker} d_{A_0}^+ d_{A_0}^{+,*}$ and let $\xi_{n+1}[A_0] \in (\operatorname{Ker} d_{A_0}^+ d_{A_0}^{+,*})^\perp$ be an L^2 unit eigenvector corresponding to the least positive eigenvalue $\nu_2[A_0]$ of $d^+_{A_0}d^{+,*}_{A_0}$. We begin by constructing a set of approximate eigenvectors, $\{\xi'_i\}_{i=1}^{n+1}$, for the first $n+1$ eigenvalues of $d^+_A d^{+,*}_A$ by setting $\xi'_i := \chi_0\xi_i$ for each $i \in \{1,\dots,n+1\}$.

Claim 6.11. *Continue the above notation. Then there are positive constants $C = C(\|F_{A_0}\|_{L^2}, \|F^+_{A_0}\|_{L^\infty}, \nu_2[A_0])$ and $\lambda_0 = \lambda_0(C)$ such that for all $\lambda \in (0,\lambda_0]$,*

$$\|d^{+,*}_A \xi'_i\|_{L^2(X)} \le \begin{cases} C\sqrt{\lambda}, & \text{if } 1 \le i \le n, \\ \sqrt{\nu_2[A_0]} + C\sqrt{\lambda}, & \text{if } i = n+1, \end{cases} \tag{1}$$

$$\left|(\xi'_i, \xi'_j)_{L^2(X)} - \delta_{ij}\right| \le C\lambda^2, \quad 1 \le i,j \le n+1. \tag{2}$$

Proof. To verify the first assertion in Claim 6.11, note that $A = A_0 + a$ on $X \setminus \mathbf{x}$ and $d^{+,*}_A = *d_A$ on $\Gamma(\Lambda^+ \otimes \mathfrak{g}_E)$, so

$$\begin{aligned} d^{+,*}_A \xi'_i &= *d_A\xi'_i = *d_{A_0}(\chi_0\xi_i) + *(a \wedge \chi_0\xi_i) \\ &= *(d\chi_0 \wedge \xi_i) + \chi_0 d^{+,*}_{A_0}\xi_i + *(\chi_0 a \wedge \xi_i), \end{aligned} \tag{6.5}$$

while

$$\|d^{+,*}_{A_0}\xi_i\|_{L^2} = \begin{cases} 0, & \text{if } 1 \le i \le n, \\ \sqrt{\nu_2[A_0]}\|\xi_i\|_{L^2}, & \text{if } i = n+1. \end{cases}$$

By Lemma 5.8 (yielding $\|d\chi_0\|_{L^2} \le c\sqrt{\lambda}$), Lemma 6.7, the preceding eigenvalue estimate, and equation (6.5),

$$\begin{aligned} \|d^{+,*}_A\xi'_i\|_{L^2} &\le (\|d\chi_0\|_{L^2} + \|\chi_0 a\|_{L^2})\|\xi_i\|_{L^\infty} + \|d^{+,*}_{A_0}\xi_i\|_{L^2} \\ &\le c\sqrt{\lambda}\|\xi_i\|_{L^\infty} + \|d^{+,*}_{A_0}\xi_i\|_{L^2} \quad \text{(by Lemma 5.8 and hypothesis)} \\ &\le C\sqrt{\lambda}\|\xi_i\|_{L^2} + \|d^{+,*}_{A_0}\xi_i\|_{L^2} \quad \text{(by Lemma 6.7)} \\ &= C\sqrt{\lambda}\|\xi_i\|_{L^2} + \sqrt{\nu_2[A_0]}\|\xi_i\|_{L^2}, \quad 1 \le i \le n+1, \end{aligned}$$

and so the first assertion in Claim 6.11 follows. Using $\xi'_i = \chi_0\xi = \xi_i + (\chi_0 - 1)\xi_i$,

$$(\xi'_i, \xi'_j)_{L^2} = (\xi_i, \xi_j)_{L^2} + 2((\chi_0 - 1)\xi_i, \xi_j)_{L^2} + ((\chi_0 - 1)\xi_i, (\chi_0 - 1)\xi_j)_{L^2}.$$

But $(\xi_i, \xi_j)_{L^2} = \delta_{ij}$, while the second two terms on the right above are bounded by

$$\begin{aligned} &|2((\chi_0 - 1)\xi_i, \xi_j)_{L^2} + ((\chi_0 - 1)\xi_i, (\chi_0 - 1)\xi_j)_{L^2}| \\ &\quad \le c\lambda^2\|\xi_i\|_{L^\infty}\|\xi_j\|_{L^\infty} \le C\lambda^2, \end{aligned}$$

courtesy of Lemma 6.7. This gives the second assertion in Claim 6.11. □

Now $\|\xi'_i\|_{L^2} \ge 1 - C\lambda$ by the second assertion of Claim 6.11 and so its first assertion yields

$$\begin{aligned} \|d^{+,*}_A\xi'_i\|_{L^2} &\le C\sqrt{\lambda}(1 - C\lambda)^{-1}\|\xi'_i\|_{L^2} \\ &\le C\sqrt{\lambda}\|\xi'_i\|_{L^2}, \quad 1 \le i \le n \quad (\text{choosing } \lambda < C/2), \\ \|d^{+,*}_A\xi'_{n+1}\|_{L^2} &\le \left(\sqrt{\nu_2[A_0]} + C\sqrt{\lambda}\right)\|\xi'_{n+1}\|_{L^2}. \end{aligned}$$

Let $V_i := [\xi'_1, \dots, \xi'_i]$ denote the span of the vectors $\xi'_1, \dots, \xi'_i$ in $L^2(\Lambda^+ \otimes \mathfrak{g}_E)$ for $i = 1, \dots, n+1$. By applying Gram-Schmidt orthonormalization to the basis $\{\xi'_1, \dots, \xi'_i\}$ for V_i, the preceding estimates imply that

$$\|d_A^{+,*}\eta\|_{L^2} \le \begin{cases} C\sqrt{\lambda}\|\eta\|_{L^2}, & \text{if } \eta \in V_n, \\ \left(\sqrt{\nu_2[A_0]} + C\sqrt{\lambda}\right)\|\eta\|_{L^2}, & \text{if } \eta \in V_{n+1}. \end{cases} \tag{6.6}$$

Recall from the statement of Theorem 6.1 that $\mu_1[A] \le \cdots \le \mu_{n+1}[A]$ are the first $n+1$ eigenvalues of $d_A^+ d_A^{+,*}$; let $\eta_1(A), \dots, \eta_{n+1}(A)$ be the corresponding L^2-orthonormal eigenvectors. Let $[\eta_1, \dots, \eta_i]$ denote the span of the eigenvectors $\eta_1, \dots, \eta_i$ in $L^2(\Lambda^+ \otimes \mathfrak{g}_E)$ and let $[\eta_1, \dots, \eta_i]^\perp$ be its L^2-orthogonal complement in $L^2(\Lambda^+ \otimes \mathfrak{g}_E)$. Note that $\dim([\eta_1, \dots, \eta_i]^\perp \cap V_n) \ge n - i$. By the estimates (6.6) we have

$$\mu_i[A] = \inf_{\eta \in [\eta_1, \dots, \eta_{i-1}]^\perp} \frac{\|d_A^{+,*}\eta\|_{L^2}^2}{\|\eta\|_{L^2}^2} \le \inf_{\eta \in [\eta_1, \dots, \eta_{i-1}]^\perp \cap V_n} \frac{\|d_A^{+,*}\eta\|_{L^2}^2}{\|\eta\|_{L^2}^2} \le C\lambda,$$

for $i = 1, \dots, n$, which gives the first inequality in the stated eigenvalue bounds (6.4). Therefore, $d_A^+ d_A^{+,*}$ has at least n eigenvalues which are less than or equal to $C\lambda$. Similarly, the estimates (6.6) yield

$$\mu_{n+1}[A] = \inf_{\eta \in [\eta_1, \dots, \eta_n]^\perp} \frac{\|d_A^{+,*}\eta\|_{L^2}^2}{\|\eta\|_{L^2}^2} \le \inf_{\eta \in [\eta_1, \dots, \eta_n]^\perp \cap V_{n+1}} \frac{\|d_A^{+,*}\eta\|_{L^2}^2}{\|\eta\|_{L^2}^2} \le \left(\sqrt{\nu_2[A_0]} + C\sqrt{\lambda}\right)^2,$$

completing the proof of the second inequality in the eigenvalue bounds (6.4). □

6.4. **An approximately orthonormal basis of eigenvectors.** We next construct an approximately L^2-orthonormal basis for $\operatorname{Ker} d_{A_0}^+ d_{A_0}^{+,*}$ using the eigenvectors corresponding to the first n eigenvalues of $d_A^+ d_A^{+,*}$. Let Π_{A_0} be the L^2-orthogonal projection from $L^2(\Lambda^+ \otimes \mathfrak{g}_{E_0})$ onto $\operatorname{Ker} d_{A_0}^+ d_{A_0}^{+,*}$ and let $\Pi_{A_0}^\perp := 1 - \Pi_{A_0}$ be the L^2-orthogonal projection from $L^2(\Lambda^+ \otimes \mathfrak{g}_{E_0})$ onto $(\operatorname{Ker} d_{A_0}^+ d_{A_0}^{+,*})^\perp = \operatorname{Ran} d_{A_0}^+ d_{A_0}^{+,*}$.

Recall from the proof of Lemma 6.10 that $\{\eta_1[A], \dots, \eta_{n+1}[A]\}$ were defined to be the L^2-orthonormal eigenvectors associated with the eigenvalues $\mu_1[A] \le \cdots \le \mu_{n+1}[A]$ of $d_A^+ d_A^{+,*}$ and construct an approximately L^2-orthonormal subset $\{\eta'_i\}_{i=1}^{n+1} \subset L^2(\Lambda^+ \otimes \mathfrak{g}_{E_0})$ by setting $\eta'_i := \chi_0 \eta_i$ for $i = 1, \dots, n+1$.

Lemma 6.12. *Continue the hypotheses and notation of Theorem 6.1. Then there are positive constants $C = C(\|F_A\|_{L^2}, \|F_A^+\|_{L^\infty}, \|F_{A_0}\|_{L^2}, \|F_{A_0}^+\|_{L^\infty}, \nu_2[A_0])$ and $\lambda_0 = \lambda_0(C)$ such that for all $\lambda \in (0, \lambda_0]$,*

$$\|\Pi_{A_0}^\perp \eta'_i\|_{L^2} \le C\sqrt{\lambda}, \tag{1}$$

$$\left| \left(\Pi_{A_0}\eta'_i, \Pi_{A_0}\eta'_j\right)_{L^2(X)} - \delta_{ij} \right| \le C\lambda, \quad 1 \le i, j \le n, \tag{2}$$

$$\|\Pi_{A_0}\eta'_{n+1}\|_{L^2(X)} \le C\sqrt{\lambda} \quad \textit{and} \quad \|\Pi_{A_0}^\perp \eta'_{n+1}\|_{L^2(X)} \ge 1 - C\sqrt{\lambda}. \tag{3}$$

Proof. The following estimates are the key to the proof of Lemma 6.12.

Claim 6.13. *Continue the above notation. Then there are positive constants* $C = C(\|F_A\|_{L^2}, \|F_A^+\|_{L^\infty}, \|F_{A_0}\|_{L^2}, \|F_{A_0}^+\|_{L^\infty}, \nu_2[A_0])$ *and* $\lambda_0 = \lambda_0(C)$ *such that for all* $\lambda \in (0, \lambda_0]$,

$$\|d_{A_0}^{+,*}\eta_i'\|_{L^2(X)} \le C\sqrt{\lambda}, \quad 1 \le i \le n, \tag{1}$$

$$\|d_{A_0}^+ d_{A_0}^{+,*}\eta_i'\|_{L^{4/3}(X)} \le C\sqrt{\lambda}, \quad 1 \le i \le n, \tag{2}$$

$$\left|(\eta_i', \eta_j')_{L^2(X)} - \delta_{ij}\right| \le C\lambda^2, \quad 1 \le i, j \le n+1. \tag{3}$$

Proof. To verify the first assertion in Claim 6.13, note that $d_{A_0}^{+,*} = *d_{A_0}$ on $\Gamma(\Lambda^+ \otimes \mathfrak{g}_{E_0})$, and so

$$\begin{aligned} d_{A_0}^{+,*}\eta_i' &= d_A^{+,*}(\chi_0\eta_i) - *(a \wedge \chi_0\eta_i) \\ &= *(d\chi_0 \wedge \eta_i) + \chi_0 d_A^{+,*}\eta_i - *(\chi_0 a \wedge \eta_i), \quad i = 1, \dots, n+1, \end{aligned} \tag{6.7}$$

where we use the fact that $A = A_0 + a$ on $X \setminus \mathbf{x}$. Thus, by the estimates in (6.4) (see Lemma 6.10) and Lemmas 5.8 and 6.7, our hypotheses, and the identity (6.7),

$$\begin{aligned} \|d_{A_0}^{+,*}\eta_i'\|_{L^2} &\le (\|d\chi_0\|_{L^2} + \|\chi_0 a\|_{L^2})\|\eta_i\|_{L^\infty} + \|d_A^{+,*}\eta_i\|_{L^2} \\ &\le c(\sqrt{\lambda}\|d\chi_0\|_{L^4} + \|\chi_0 a\|_{L^2})\|\eta_i\|_{L^\infty} + \sqrt{\mu_i[A]} \\ &\le C\sqrt{\lambda}, \quad 1 \le i \le n, \end{aligned}$$

using $\|d_A^{+,*}\eta_i\|_{L^2} = \sqrt{\mu_i[A]}\|\eta_i\|_{L^2}$; this proves the first assertion in Claim 6.13. Next, observe that schematically (and again using the fact that $A = A_0 + a$ on $X \setminus \mathbf{x}$)

$$\begin{aligned} d_{A_0}^+ d_{A_0}^{+,*}\eta_i' &= d_{A-a}^+(*(d\chi_0 \wedge \eta_i) + \chi_0 d_A^{+,*}\eta_i - *(\chi_0 a \wedge \eta_i)) \\ &= (\nabla^2\chi_0 + \chi_0\nabla_{A_0}a + \chi_0 a \otimes a) \otimes \eta_i \\ &\quad + \nabla\chi_0 \otimes (a \otimes \eta_i + \nabla_A\eta_i) + \chi_0 d_A^+ d_A^{+,*}\eta_i, \end{aligned}$$

and so, using Lemmas 5.8, 6.5, 6.7, and 6.10, we get

$$\begin{aligned} \|d_{A_0}^+ d_{A_0}^{+,*}\eta_i'\|_{L^{4/3}} &\le \big(\|\nabla^2\chi_0\|_{L^{4/3}} + \|\chi_0\nabla_{A_0}a\|_{L^{4/3}} \\ &\quad + \|\nabla\chi_0\|_{L^2}\|a\|_{L^4(\operatorname{supp}\chi_0)} + \|\sqrt{\chi_0}a\|_{L^2}\|\sqrt{\chi_0}a\|_{L^4}\big)\,\|\eta_i\|_{L^\infty} \\ &\quad + c\|\nabla\chi_0\|_{L^2}\|\nabla_A\eta_i\|_{L^4} + \|d_A^+ d_A^{+,*}\eta_i\|_{L^2} \\ &\le C\sqrt{\lambda}\|\eta_i\|_{L^\infty \cap L^4_{1,A}} + \mu_i[A] \\ &\le C\sqrt{\lambda}, \quad 1 \le i \le n. \end{aligned}$$

Hence, the second assertion in Claim 6.13 follows from Lemma 5.8 and our hypotheses. Using $\eta_i' = \chi_0\eta_i = \eta_i + (\chi_0 - 1)\eta_i$,

$$(\eta_i', \eta_j')_{L^2} = (\eta_i, \eta_j)_{L^2} + 2((\chi_0 - 1)\eta_i, \eta_j)_{L^2} + ((\chi_0 - 1)\eta_i, (\chi_0 - 1)\eta_j)_{L^2}.$$

Now $(\eta_i, \eta_j)_{L^2} = \delta_{ij}$, while the second two terms on the right above are bounded by

$$\begin{aligned} &|2((\chi_0 - 1)\eta_i, \eta_j)_{L^2} + ((\chi_0 - 1)\eta_i, (\chi_0 - 1)\eta_j)_{L^2}| \\ &\quad \le c\lambda^2\|\eta_i\|_{L^\infty}\|\eta_j\|_{L^\infty} \le C\lambda^2, \end{aligned}$$

and this gives the third assertion in Claim 6.13. $\square$

By Lemmas 6.5 and 6.6, integration by parts, and Claim 6.13 we have, for any $\xi \in (\operatorname{Ker} d_{A_0}^+ d_{A_0}^{+,*})^\perp$,

$$\begin{aligned}
(\eta_i', d_{A_0}^+ d_{A_0}^{+,*}\xi)_{L^2} &= (d_{A_0}^+ d_{A_0}^{+,*}\eta_i', \xi)_{L^2} \le \|d_{A_0}^+ d_{A_0}^{+,*}\eta_i'\|_{L^{4/3}}\|\xi\|_{L^4} \\
&\le C\sqrt{\lambda}\left(\|\xi\|_{L^2} + \|d_{A_0}^{+,*}\xi\|_{L^2}\right) \\
&\le C\sqrt{\lambda}\left(\|\xi\|_{L^2} + \|d_{A_0}^+ d_{A_0}^{+,*}\xi\|_{L^2}\right) \\
&\le C(\nu_0[A_0]^{-1} + 1)\sqrt{\lambda}\|d_{A_0}^+ d_{A_0}^{+,*}\xi\|_{L^2},
\end{aligned}$$

and therefore we obtain

$$(\eta_i', d_{A_0}^+ d_{A_0}^{+,*}\xi)_{L^2} \le C\sqrt{\lambda}\|d_{A_0}^+ d_{A_0}^{+,*}\xi\|_{L^2}, \quad \xi \in (\operatorname{Ker} d_{A_0}^+ d_{A_0}^{+,*})^\perp, \quad i = 1, \dots, n. \tag{6.8}$$

Since $\operatorname{Ran}\Pi_{A_0}^\perp = \operatorname{Ran} d_{A_0}^+ d_{A_0}^{+,*}$, we may choose $\rho_i \in L_2^2(\Lambda^+ \otimes \mathfrak{g}_{E_0})$ such that $\Pi_{A_0}^\perp \eta_i' = d_{A_0}^+ d_{A_0}^{+,*}\rho_i$. If θ_i denotes the angle between η_i' and $\Pi_{A_0}^\perp \eta_i'$ in $L^2(\Lambda^+ \otimes \mathfrak{g}_{E_0})$, then

$$\|\Pi_{A_0}^\perp \eta_i'\|_{L^2} = \|\eta_i'\|_{L^2}\cos\theta_i = \frac{\left(\eta_i', d_{A_0}^+ d_{A_0}^{+,*}\rho_i\right)_{L^2}}{\|d_{A_0}^+ d_{A_0}^{+,*}\rho_i\|_{L^2}}.$$

Together with equation (6.8), this yields for $i = 1, \dots, n$,

$$\|\Pi_{A_0}^\perp \eta_i'\|_{L^2} \le C\sqrt{\lambda}. \tag{6.9}$$

By Claim 6.13 we have, for $1 \le i, j \le n$,

$$\begin{aligned}
\left|\left(\Pi_{A_0}\eta_i', \Pi_{A_0}\eta_j'\right)_{L^2} - \delta_{ij}\right| &\le \left|\left(\Pi_{A_0}\eta_i', \Pi_{A_0}\eta_j'\right)_{L^2} - \left(\eta_i', \eta_j'\right)_{L^2}\right| + \left|\left(\eta_i', \eta_j'\right)_{L^2} - \delta_{ij}\right| \\
&\le \left|\left(\Pi_{A_0}^\perp\eta_i', \Pi_{A_0}^\perp\eta_j'\right)_{L^2}\right| + C\lambda^2 \\
&\le \|\Pi_{A_0}^\perp\eta_i'\|_{L^2}\|\Pi_{A_0}^\perp\eta_j'\|_{L^2} + C\lambda^2 \\
&\le C(\lambda + \lambda^2) \quad \text{(from equation (6.9))},
\end{aligned}$$

and so

$$\left|\left(\Pi_{A_0}\eta_i', \Pi_{A_0}\eta_j'\right)_{L^2} - \delta_{ij}\right| \le C\lambda, \quad 1 \le i, j \le n. \tag{6.10}$$

Thus, for small enough λ, the vectors $\{\Pi_{A_0}\eta_i'\}_{i=1}^n$ form an approximately L^2-orthonormal basis for $\operatorname{Ker} d_{A_0}^+ d_{A_0}^{+,*}$.

Recall from the proof of Lemma 6.10 that $\{\xi_i(A_0)\}_{i=1}^n \subset L^2(\Lambda^+ \otimes \mathfrak{g}_{E_0})$ denotes an ordered, L^2-orthonormal basis for $\operatorname{Ker} d_{A_0}^+ d_{A_0}^{+,*}$. We can therefore assume without loss of generality that $\{\xi_i\}_{i=1}^n$ is the basis for $\operatorname{Ker} d_{A_0}^+ d_{A_0}^{+,*}$ obtained by applying Gram-Schmidt orthonormalization to the basis $\{\Pi_{A_0}\eta_i'\}_{i=1}^n$:

$$\tilde\xi_\ell = \Pi_{A_0}\eta_\ell' - \sum_{i=1}^{\ell-1}(\Pi_{A_0}\eta_\ell', \xi_i)_{L^2}\xi_i \quad \text{and} \quad \xi_\ell = \frac{\tilde\xi_\ell}{\|\tilde\xi_\ell\|_{L^2}}, \quad 1 \le \ell \le n. \tag{6.11}$$

By Claim 6.13 and the orthogonal-complement estimates (6.9) we have

$$
\begin{aligned}
|(\eta'_{n+1}, \Pi_{A_0}\eta'_i)_{L^2}| &= |(\eta'_{n+1}, \eta'_i)_{L^2} - (\eta'_{n+1}, \Pi^{\perp}_{A_0}\eta'_i)_{L^2}| \\
&\le C\lambda^2 + C(1+\lambda)\sqrt{\lambda} \le C\sqrt{\lambda}.
\end{aligned}
\tag{6.12}
$$

Consequently, by the Gram-Schmidt equations (6.11) together with the inner-product estimates (6.12), and the fact that

$$\Pi_{A_0}\eta'_{n+1} = \sum_{i=1}^{n} \left(\eta'_{n+1}, \xi_i\right)_{L^2} \xi_i,$$

we have

$$\|\Pi_{A_0}\eta'_{n+1}\|_{L^2} \le C\sqrt{\lambda} \quad \text{and} \quad \|\Pi^{\perp}_{A_0}\eta'_{n+1}\|_{L^2} \ge 1 - C\sqrt{\lambda}. \tag{6.13}$$

Thus, inequalities (6.9), (6.10), and (6.13) give the required estimates for $\{\Pi_{A_0}\eta'_i\}_{i=1}^{n+1}$ and completes the proof of Lemma 6.12. □

6.5. **A lower bound for the first non-small eigenvalue.** Lastly, we compute a lower bound for the $(n+1)$-st eigenvalue of $d_A^+ d_A^{+,*}$. Recall that $\eta_{n+1} = \eta_{n+1}(A)$ is the L^2-unit eigenvector associated to the $(n+1)$-st eigenvalue $\mu_{n+1}[A]$ of $d_A^+ d_A^{+,*}$.

Lemma 6.14. *Continue the above notation. Then there are positive constants $C = C(\|F_A\|_{L^2}, \|F_A^+\|_{L^\infty}, \|F_{A_0}\|_{L^2}, \|F_{A_0}^+\|_{L^\infty}, \nu_2[A_0])$ and $\lambda_0 = \lambda_0(C)$ such that for all $\lambda \in (0, \lambda_0]$,*

$$\mu_{n+1}[A] \ge \nu_2[A_0] - C\sqrt{\lambda}.$$

Proof. Recall that $\eta'_{n+1} = \chi_0\eta_{n+1} \in \Gamma(\Lambda^+ \otimes \mathfrak{g}_{E_0})$. Therefore, via the identity (6.7), Lemma 5.8, our hypotheses, and the fact that $\|d_A^{+,*}\eta_{n+1}\|_{L^2} = \sqrt{\mu_{n+1}[A]}\|\eta_{n+1}\|_{L^2}$, we get

$$
\begin{aligned}
\|d_{A_0}^{+,*}\eta'_{n+1}\|_{L^2} &\le (\|d\chi_0\|_{L^2} + \|\chi_0 a\|_{L^2})\,\|\eta_{n+1}\|_{L^\infty} + \|d_A^{+,*}\eta_{n+1}\|_{L^2} \\
&\le c\sqrt{\lambda}\|\eta_{n+1}\|_{L^\infty} + \sqrt{\mu_{n+1}[A]}\|\eta_{n+1}\|_{L^2}.
\end{aligned}
$$

According to Lemma 6.7 we have

$$
\begin{aligned}
\|\eta_{n+1}\|_{L^\infty} &\le c(1+\|F_A\|_{L^2})^2(1+\|F_A^+\|_{L^\infty})^{5/2}(1+\mu_{n+1}[A])(1+\sqrt{\mu_{n+1}[A]})\|\eta_{n+1}\|_{L^2} \\
&\le C\|\eta_{n+1}\|_{L^2},
\end{aligned}
$$

using the upper bound for $\mu_{n+1}[A]$ in Lemma 6.10. Hence, we obtain

$$\|d_{A_0}^{+,*}\eta'_{n+1}\|_{L^2} \le C\sqrt{\lambda} + \sqrt{\mu_{n+1}[A]}.$$

In the other direction, we have the eigenvalue estimate $\|d_{A_0}^{+,*}v\|_{L^2} \ge \sqrt{\nu_2[A_0]}\|v\|_{L^2}$ for all $v \in (\operatorname{Ker} d_{A_0}^+ d_{A_0}^{+,*})^\perp \subset L^2(\Lambda^+ \otimes \mathfrak{g}_{E_0})$, and so by the orthogonal projection estimates (6.13),

$$\|d_{A_0}^{+,*}\eta'_{n+1}\|_{L^2} \ge \sqrt{\nu_2[A_0]}\|\Pi^{\perp}_{A_0}\eta'_{n+1}\|_{L^2} \ge \sqrt{\nu_2[A_0]}(1 - C\lambda).$$

Combining these upper and lower L^2 bounds for $d_{A_0}^{+,*}\eta'_{n+1}$ yields

$$\sqrt{\nu_2[A_0]}\left(1 - C\sqrt{\lambda}\right) \le C\sqrt{\lambda} + \sqrt{\mu_{n+1}[A]},$$

and this gives the lower bound on $\mu_{n+1}[A]$ for small enough λ. □

Proof of Theorem 6.1. Combine Lemmas 6.10 and 6.14. □

6.6. **Variation of eigenvalues with the connection.** Our final task is to estimate the variation in the eigenvalues $\mu_i[A]$ of $d_A^+ d_A^{+,*}$ with the connection A.

Lemma 6.15. *Let X be a C^∞, closed, oriented, Riemannian four-manifold. Then there is a positive constant c with the following significance. Let E be a rank-two, Hermitian vector bundle over X, let A be an orthogonal L^2_4 connection on $\mathfrak{g}_E$, and let $0 \le \mu_1[A] \le \mu_2[A] \le \cdots$ be the eigenvalues of $d_A^+ d_A^{+,*}$ on $L^2(\Lambda^+ \otimes \mathfrak{g}_E)$, repeated according to their multiplicity. Then, for any integer $i \ge 1$ and $a \in L^2_4(\Lambda^1 \otimes \mathfrak{g}_E)$,*

$$\mu_i[A+a] \le \mu_i[A](1 + C\|a\|_{L^4(X)} + C\|a\|^2_{L^4(X)}) + C\|a\|_{L^4(X)}(1 + \|a\|_{L^4(X)}), \tag{1}$$

$$\mu_i[A] \le \mu_i[A+a](1 + C\|a\|_{L^4(X)} + C\|a\|^2_{L^4(X)}) + C\|a\|_{L^4(X)}(1 + \|a\|_{L^4(X)}), \tag{2}$$

where $C[A] := c(1 + \|F_A^+\|_{L^\infty})$.

Proof. Recall that by Lemma 6.8

$$\mu_i[A+a] = \inf_{\eta \in [\eta_1[A+a], \dots, \eta_{i-1}[A+a]]^\perp} \frac{\|d^{+,*}_{A+a}\eta\|^2_{L^2}}{\|\eta\|^2_{L^2}}, \quad i \in \mathbb{N},$$

where the infimum is taken over all $\eta \in L^2_1(\Lambda^+ \otimes \mathfrak{g}_E)$ which are L^2 orthogonal to the span of $\eta_1[A+a], \dots, \eta_{i-1}[A+a]$ in $L^2(\Lambda^+ \otimes \mathfrak{g}_E)$, the first $i-1$ eigenvectors of $d^+_{A+a} d^{+,*}_{A+a}$ on $L^2(\Lambda^+ \otimes \mathfrak{g}_E)$. Since $d^{+,*}_{A+a}\eta = d^{+,*}_A \eta + *(a \wedge \eta)$ we have, for any $\eta \in L^2_1(\Lambda^+ \otimes \mathfrak{g}_E)$,

$$\begin{aligned}\|d^{+,*}_{A+a}\eta\|^2_{L^2} &= \|d^{+,*}_A\eta\|^2_{L^2} + \|*(a\wedge\eta)\|^2_{L^2} + 2(d^{+,*}_A\eta, *(a\wedge\eta))_{L^2} \\ &\le \|d^{+,*}_A\eta\|^2_{L^2}(1 + \|a\|_{L^4}) + (1 + \|a\|_{L^4})\|a\|_{L^4}\|\eta\|^2_{L^4}.\end{aligned}$$

But Lemmas 6.5 and 6.6 imply that

$$\|\eta\|_{L^4} \le c\|\eta\|_{L^2_{1,A}} \le c(1 + \|F_A^+\|_{L^\infty})^{1/2}(\|d^{+,*}_A\eta\|_{L^2} + \|\eta\|_{L^2}),$$

for any $\eta \in L^2_1(\Lambda^+ \otimes \mathfrak{g}_E)$ and so

$$\begin{aligned}\|d^{+,*}_{A+a}\eta\|^2_{L^2} &\le \|d^{+,*}_A\eta\|^2_{L^2}(1 + \|a\|_{L^4}) + C[A](1 + \|a\|_{L^4})\|a\|_{L^4}(\|d^{+,*}_A\eta\|_{L^2} + \|\eta\|_{L^2})^2 \\ &\le \|d^{+,*}_A\eta\|^2_{L^2}(1 + C[A]\|a\|_{L^4} + C[A]\|a\|^2_{L^4}) + C[A](1 + \|a\|_{L^4})\|a\|_{L^4}\|\eta\|^2_{L^2},\end{aligned}$$

where $C[A] = c(1 + \|F_A^+\|_{L^\infty})$. By Lemma 6.8 and the preceding estimate we see that

$$\begin{aligned}\mu_i[A+a] &= \inf_{\eta \in [\eta_1[A+a], \dots, \eta_{i-1}[A+a]]^\perp} \frac{\|d^{+,*}_{A+a}\eta\|^2_{L^2}}{\|\eta\|^2_{L^2}} \\ &\le \inf_{\eta \in [\eta_1[A+a], \dots, \eta_{i-1}[A+a]]^\perp} \frac{\|d^{+,*}_A\eta\|^2_{L^2}}{\|\eta\|^2_{L^2}}(1 + C[A]\|a\|_{L^4} + C[A]\|a\|^2_{L^4}) \\ &\quad + C[A](1 + \|a\|_{L^4})\|a\|_{L^4} \\ &\le \mu_i[A](1 + C[A]\|a\|_{L^4} + C[A]\|a\|^2_{L^4}) + C[A](1 + \|a\|_{L^4})\|a\|_{L^4},\end{aligned}$$

and this gives the upper bound for $\mu_i[A+a]$ in terms of $\mu_i[A]$. Interchanging the roles of A and $A+a$ gives the upper bound for $\mu_i[A]$ in terms of $\mu_i[A+a]$. □

7. Existence of gluing maps and obstruction sections

In this section we prove the assertions of Theorem 1.3 concerning the existence and smoothness of the gluing map $\boldsymbol{\gamma}$ and obstruction section $\boldsymbol{\chi}$, but not the assertions concerning the existence of continuous extensions of these maps to the Uhlenbeck compactification of the space of gluing data, $\bar{\mathrm{Gl}}_\kappa^{w,+}(X,\mathcal{U}_0,\Sigma,\lambda_0)$; the latter Uhlenbeck continuity assertions are taken up in §8.

7.1. **The extended anti-self-dual equations.** As we saw in §2.6, we need to allow for the presence of cokernel obstructions to deformation in the parameterized moduli space of background connections contained in the gluing data and, thus, small-eigenvalue obstructions to gluing. In this subsection we set up a version of the 'extended anti-self-dual equations' which we shall use to take account of such obstructions. Though similar setups are described in [5], [7], [8], [25], [49], [54], and [56], they are not identical so we shall describe our version in some detail.

Let A be an orthogonal connection on an SO(3) bundle $\mathfrak{g}_E = \mathfrak{su}(E)$ over X, where E is a Hermitian, rank-two vector bundle over X. Our first task is to obtain existence and global estimates for solutions $a \in L^2_1(\Lambda^1 \otimes \mathfrak{g}_E)$ to the quasi-linear equation,

$$d_A^+ a + (a\wedge a)^+ = w \quad \text{and} \quad a = d_A^{+,*}v, \tag{7.1}$$

where $v \in (C^0 \cap L^{\sharp,2}_2)(\Lambda^+ \otimes \mathfrak{g}_E)$ and $w \in L^{\sharp,2}(\Lambda^+\otimes\mathfrak{g}_E)$. (For the definition and properties of the $L^\sharp$ and related families of Sobolev norms, see [12, §4].) As in [52], [54], [56], we use $a = d_A^{+,*}v$ to rewrite the first-order quasi-linear equation (7.1) for a in terms of w as

$$d_A^+ d_A^{+,*}v + (d_A^{+,*}v \wedge d_A^{+,*}v)^+ = w, \tag{7.2}$$

to give a second-order, quasi-linear elliptic equation for v in terms of w. For the reader familiar with the construction of gluing maps in [8], the current section gives a generalization of the treatment in [8, §7.2.2 & 7.2.3], where existence of solutions (in certain cases) to the anti-self-dual equation is established.

As in [54], [56], we allow for the possibility that the Laplacian $d_A^+d_A^{+,*}$ has small eigenvalues. From §6 we know that this occurs when $[A]$ is close, in the Uhlenbeck topology, to a point $[A_0,\mathbf{x}] \in \mathcal{B}_\kappa^w(X)\times \mathrm{Sym}^\ell(X)$, where $d_{A_0}^+d_{A_0}^{+,*}$ has a positive-dimensional kernel. Thus, in place of the anti-self-dual equation,

$$F^+(A + d_A^{+,*}v) = 0, \tag{7.3}$$

for $v \in L^2_{k+1}(\Lambda^+\otimes\mathfrak{g}_E)$, we follow [54], [56] and consider the pair of equations

$$\Pi_{A,\mu}^\perp \cdot F^+(A + d_A^{+,*}v) = 0, \tag{7.4}$$

$$\Pi_{A,\mu} \cdot F^+(A + d_A^{+,*}v) = 0, \tag{7.5}$$

for solutions $v \in L^2_{k+1}(\Lambda^+\otimes\mathfrak{g}_E)$ satisfying the constraint $\Pi_{A,\mu}v = 0$; equation (7.4) is called the *extended anti-self-dual equation*. Here, $\Pi_{A,\mu}$ is the L^2-orthogonal projection from $L^2(\Lambda^+\otimes\mathfrak{g}_E)$ onto the finite-dimensional subspace spanned by the eigenvectors of $d_A^+d_A^{+,*}$ with eigenvalues in $[0,\mu]$, where μ is a positive constant (an upper bound for the small eigenvalues), and $\Pi_{A,\mu}^\perp := \mathrm{id} - \Pi_{A,\mu}$. Because of the small-eigenvalue problem, we shall only explicitly solve equation (7.4); the true solutions to equation (7.3) are then simply the solutions to equation (7.4) which also satisfy equation (7.5).

Equation (7.4) may be rewritten as

$$\Pi_{A,\mu}^{\perp}(d_A^+ d_A^{+,*}v + (d_A^{+,*}v \wedge d_A^{+,*}v)^+) = -\Pi_{A,\mu}^{\perp} F_A^+, \tag{7.6}$$

where again $v \in L_{k+1}^2(\Lambda^+ \otimes \mathfrak{g}_E)$ satisfies the constraint $\Pi_{A,\mu}v = 0$. For the purposes of the regularity theory of §4, it is useful to rewrite equation (7.6) as

$$d_A^+ d_A^{+,*}v + (d_A^{+,*}v \wedge d_A^{+,*}v)^+ = \Pi_{A,\mu}(d_A^+ d_A^{+,*}v + (d_A^{+,*}v \wedge d_A^{+,*}v)^+) - \Pi_{A,\mu}^{\perp} F_A^+. \tag{7.7}$$

On the other hand, for the purposes of applying Banach-space fixed theory to solve the non-linear equation (7.6), it is convenient to write

$$v = G_{A,\mu}\xi, \quad \text{for some } \xi \in L_{k-1}^2(\Lambda^+ \otimes \mathfrak{g}_E),$$

where $G_{A,\mu} : L_{k-1}^2(\Lambda^+ \otimes \mathfrak{g}_E) \to L_{k+1}^2(\Lambda^+ \otimes \mathfrak{g}_E)$ is the Green's operator for the Laplacian

$$d_A^+ d_A^{+,*} : \Pi_{A,\mu}^{\perp} \cdot L_{k+1}^2(\Lambda^+ \otimes \mathfrak{g}_E) \to \Pi_{A,\mu}^{\perp} \cdot L_{k-1}^2(\Lambda^+ \otimes \mathfrak{g}_E).$$

Thus, $d_A^+ d_A^{+,*} G_{A,\mu} = \Pi_{A,\mu}^{\perp}$ and $\Pi_{A,\mu}v = \Pi_{A,\mu}G_{A,\mu}\xi = 0$ for any $\xi \in L_{k-1}^2(\Lambda^+ \otimes \mathfrak{g}_E)$. Therefore, it suffices to solve for ξ such that

$$\xi + (d_A^{+,*}G_{A,\mu}\xi \wedge d_A^{+,*}G_{A,\mu}\xi)^+ = -F_A^+, \tag{7.8}$$

because, noting that $\Pi_{A,\mu}^{\perp}\xi = \Pi_{A,\mu}^{\perp} d_A^+ d_A^{+,*} G_{A,\mu}\xi$, the resulting $v = G_{A,\mu}\xi$ solves equation (7.6), namely

$$\Pi_{A,\mu}^{\perp}(\xi + (d_A^{+,*}G_{A,\mu}\xi \wedge d_A^{+,*}G_{A,\mu}\xi)^+ + F_A^+) = 0,$$

as well as the constraint $\Pi_{A,\mu}v = 0$.

Noting that $\Pi_{A,\mu} d_A^+ d_A^{+,*} G_{A,\mu}\xi = 0$, the solution ξ to (7.8) solves equation (7.5), namely

$$\Pi_{A,\mu}((d_A^{+,*}G_{A,\mu}\xi \wedge d_A^{+,*}G_{A,\mu}\xi)^+ + F_A^+) = 0,$$

if and only if

$$\Pi_{A,\mu}\xi = 0. \tag{7.9}$$

For applications of Banach-space fixed-point theory, the integral form (7.8) of the extended anti-self-dual equation is a little more convenient, while for the purposes of regularity theory and localization, we use the "differential" form (7.7).

7.2. **Global estimates: linear theory.** The results we need are treated in detail in [12], so we just summarize the main conclusions we need here; the $L^\sharp$ and related families of Sobolev norms are defined in [12, §4].

Lemma 7.1. [12, Lemma 5.9] *Let X be a closed, oriented four-manifold with metric g. Then there is a constant c with the following significance. Let E be a Hermitian, rank-two vector bundle over X and let A be an orthogonal L_4^2 connection on $\mathfrak{g}_E$ with curvature F_A. Then the following estimate holds for any $v \in L_2^{\sharp,2}(\Lambda^+ \otimes \mathfrak{g}_E)$:*

$$\begin{aligned}\|v\|_{L_{2,A}^2(X)} + \|v\|_{C^0(X)} &\le c(1 + \|F_A\|_{L^2(X)})^2(1 + \|F_A^+\|_{C^0(X)})^2 \\ &\quad \times (\|d_A^+ d_A^{+,*}v\|_{L^{\sharp,2}(X)} + \|v\|_{L^2(X)}).\end{aligned}$$

Corollary 7.2. [12, Corollary 5.10] *Continue the hypotheses of Lemma 7.1. Then:*

$$\begin{aligned}\|d_A^{+,*}v\|_{L^2_{1,A}(X)} &\le c(1+\|F_A\|_{L^2(X)})^2(1+\|F_A^+\|_{C^0(X)})^2 \\ &\quad \times (\|d_A^+d_A^{+,*}v\|_{L^{\sharp,2}(X)}+\|v\|_{L^2(X)}).\end{aligned}$$

Note that if $a \in L^2_k(\Lambda^1\otimes\mathfrak{g}_E)$ is L^2-orthogonal to $\operatorname{Ker} d_A^+$, so that $a = d_A^{+,*}v$ for some $v \in L^2_{k+1}(\Lambda^+\otimes\mathfrak{g}_E)$, then the estimate of Corollary 7.2 can be written in the more familiar form (see [12, Eq. (5.10)])

$$\begin{aligned}\|a\|_{L^2_{1,A}(X)} &\le c(1+\|F_A\|_{L^2(X)})^2(1+\|F_A^+\|_{C^0(X)})^2 \\ &\quad \times (\|d_A^+a\|_{L^{\sharp,2}(X)}+\mu^{-1/2}\|a\|_{L^2(X)}),\end{aligned} \tag{7.10}$$

where $\mu = \nu_2[A] > 0$ and we make use of the eigenvalue estimate $\|v\|_{L^2} \le \mu^{-1/2}\|d_A^{+,*}v\|_{L^2}$; the term d_A^+a above can be replaced by $(d_A^{+,*}d_A^+)a$ without changing the estimate constants. More generally, if μ is any positive constant and $\Pi_{A,\mu}v = 0$, then the estimate (7.10) continues to hold.

Fix a positive constant μ. Given an L^2_4 orthogonal connection A on $\mathfrak{g}_E$, we define a partial right inverse

$$P_{A,\mu} := d_A^{+,*}\left(d_A^+d_A^{+,*}\right)^{-1}\cdot\Pi^{\perp}_{A,\mu} : L^{\sharp,2}(\Lambda^+\otimes\mathfrak{g}_E) \to L^2_1(\Lambda^1\otimes\mathfrak{g}_E),$$

to the operator

$$d_A^+ : L^2_1(\Lambda^1\otimes\mathfrak{g}_E) \to L^{\sharp,2}(\Lambda^+\otimes\mathfrak{g}_E).$$

We have the following estimates for $P_{A,\mu}$:

Proposition 7.3. *Let X be a closed, oriented, smooth four-manifold with Riemannian metric g. Let A be an L^2_4 connection on $\mathfrak{g}_E$ and let μ be a positive constant. Then there is a positive constant $C(\|F_A^+\|_{L^\infty},\|F_A^-\|_{L^2},\mu)$ such that for all $\xi \in L^{\sharp,2}(\Lambda^+\otimes\mathfrak{g}_E)$,*

$$\|P_{A,\mu}\xi\|_{L^2} \le C\|\xi\|_{L^{4/3}}, \tag{1}$$

$$\|P_{A,\mu}\xi\|_{L^2_{1,A}} \le C\|\xi\|_{L^{\sharp,2}}. \tag{2}$$

Proof. We set $v = G_{A,\mu}\xi$, so that $\Pi_{A,\mu}v = 0$ with

$$d_A^+d_A^{+,*}v = \Pi^{\perp}_{A,\mu}\xi \quad\text{and}\quad a = d_A^{+,*}v = P_{A,\mu}\xi.$$

We first consider the L^2 estimate for a. We integrate by parts to get

$$\begin{aligned}\|a\|^2_{L^2} = \|d_A^{+,*}v\|^2_{L^2} &= (d_A^+d_A^{+,*}v,v)_{L^2} = (\Pi^{\perp}_{A,\mu}\xi,v)_{L^2} = (\xi,v)_{L^2} \\ &\le \|\xi\|_{L^{4/3}}\|v\|_{L^4} \le c\|\xi\|_{L^{4/3}}\|v\|_{L^2_{1,A}}.\end{aligned}$$

From Lemma 6.6 and the eigenvalue estimate $\|v\|_{L^2} \le \mu^{-1/2}\|d_A^{+,*}v\|_{L^2}$, we have

$$\|v\|_{L^2_{1,A}} \le C(\|d_A^{+,*}v\|_{L^2}+\|v\|_{L^2}) \le C(\|d_A^{+,*}v\|_{L^2}+\mu^{-1/2}\|d_A^{+,*}v\|_{L^2}) \le C\|d_A^{+,*}v\|_{L^2}.$$

Therefore,

$$\|d_A^{+,*}v\|^2_{L^2} \le C\|\xi\|_{L^{4/3}}\|d_A^{+,*}v\|_{L^2},$$

and so

$$\|d_A^{+,*}v\|_{L^2} \le C\|\xi\|_{L^{4/3}},$$

which gives the desired L^2 estimate for $a = d_A^{+,*}v$. We can also combine the preceding estimates, noting that we can choose to either retain or drop the projection $\Pi_{A,\mu}^{\perp}$ as above, to give

$$\begin{aligned} \|v\|_{L^2_{1,A}} &\le C\|\Pi_{A,\mu}^{\perp}\xi\|_{L^{4/3}}, \\ \|v\|_{L^2_{1,A}} &\le C\|\xi\|_{L^{4/3}}, \end{aligned} \tag{7.11}$$

estimates we shall later need.

We next derive the $L^2_{1,A}$ estimate for a. Since $a = d_A^{+,*}v$, then

$$\|a\|_{L^2_{1,A}} \le \|d_A^{+,*}v\|_{L^2_{1,A}}.$$

Using integration by parts, we see that

$$\begin{aligned} \|d_A^{+,*}v\|_{L^2}^2 &= (d_A^{+,*}v, d_A^{+,*}v)_{L^2} = (d_A^+ d_A^{+,*}v, v)_{L^2} \\ &\le \|d_A^+ d_A^{+,*}v\|_{L^2}\|v\|_{L^2} \le \tfrac{1}{2}(\|d_A^+ d_A^{+,*}v\|_{L^2}^2 + \|v\|_{L^2}^2), \end{aligned}$$

and so

$$\|d_A^{+,*}v\|_{L^2} \le \frac{1}{\sqrt{2}}(\|d_A^+ d_A^{+,*}v\|_{L^2} + \|v\|_{L^2}). \tag{7.12}$$

By combining Corollary 7.2 with equation (7.12) and noting that $d_A^+ d_A^{+,*}v = \Pi_{A,\mu}^{\perp}\xi = \xi - \Pi_{A,\mu}\xi$, we obtain

$$\begin{aligned} \|a\|_{L^2_{1,A}} &\le C(\|d_A^+ d_A^{+,*}v\|_{L^{\sharp,2}} + \|v\|_{L^2}) \\ &= C(\|d_A^+ d_A^{+,*}v\|_{L^{\sharp,2}} + \mu^{-1}\|d_A^+ d_A^{+,*}v\|_{L^2}) \\ &\le C\|\Pi_{A,\mu}^{\perp}\xi\|_{L^{\sharp,2}} \\ &\le C\|\xi\|_{L^{\sharp,2}}, \quad \text{(by Lemma 6.7 and [12, Lemma 4.1])} \end{aligned}$$

which gives the required $L^2_{1,A}$ estimate for a. Of course, the preceding argument also yields

$$\begin{aligned} \|v\|_{C^0\cap L^2_{2,A}} &\le C\|\Pi_{A,\mu}^{\perp}\xi\|_{L^{\sharp,2}}, \\ \|v\|_{C^0\cap L^2_{2,A}} &\le C\|\xi\|_{L^{\sharp,2}}, \end{aligned} \tag{7.13}$$

estimates we shall later need. □

7.3. **Global estimates: non-linear theory.** We now extend our global estimates in §7.2 for the linear operator $d_A^+ d_A^{+,*}$ to estimates for the non-linear operator $d_A^+ d_A^{+,*} + (d_A^{+,*}(\cdot) \wedge d_A^{+,*}(\cdot))^+$. By adapting the arguments of [52], [55], [56] we first obtain global $C^0 \cap L^2_{2,A}$ estimates for solutions v to equation (7.2) and thus global $L^2_{1,A}$ estimates for solutions $a = d_A^{+,*}v$ to equation (7.1). These global estimates have the property that the constant C is *uniform* with respect to the point $[A] \in \mathcal{B}_E(X)$ in the sense that C depends on A only through $\|F_A^+\|_{L^\infty}$, $\|F_A\|_{L^2}$, and μ, a choice of positive, small-eigenvalue cutoff constant for the Laplacian $d_A^+ d_A^{+,*}$ on $L^2(\Lambda^+ \otimes \mathfrak{g}_E)$. We will make frequent use of the following *a priori* estimate for L^2_2 solutions to equation (7.2).

Lemma 7.4. *Let X be a closed, oriented, smooth four-manifold with Riemannian metric g, and let M,Λ be positive constants. Then there are positive constants ε, C such that the following holds. Let A be an L^2_4 connection on $\mathfrak{g}_E$ with $\|F_A^+\|_{L^\infty}\le M$, $\|F_A\|_{L^2}\le M$, and let μ be a positive constant such that $\Lambda^{-1}\le\mu\le\Lambda$. Let $v\in(C^0\cap L^2_2)(\Lambda^+\otimes\mathfrak{g}_E)$ and $w\in L^{\sharp,2}(\Lambda^+\otimes\mathfrak{g}_E)$ solve (7.2), with $\Pi_{A,\mu}v=0$, and suppose that $\|v\|_{L^2_{2,A}(X)}<\varepsilon$. Then*

$$\|v\|_{L^2_{1,A}(X)}\le C\|w\|_{L^{4/3}(X)}, \tag{1}$$

$$\|v\|_{L^2_{2,A}(X)}+\|v\|_{L^\infty(X)}\le C\|w\|_{L^{\sharp,2}(X)} \tag{2}$$

Proof. Note that estimate (7.11) yields

$$\|v\|_{L^2_{1,A}}\le C\|d_A^+d_A^{+,*}v\|_{L^{4/3}},$$

while from equation (7.2) and the Sobolev multiplications $L^2_{1,A}\times L^2\to L^{4/3}$ (with constant depending only on the metric) we have

$$\begin{aligned}\|d_A^+d_A^{+,*}v\|_{L^{4/3}}&\le\|(d_A^{+,*}v\wedge d_A^{+,*}v)^+\|_{L^{4/3}}+\|w\|_{L^{4/3}}\\&\le c\|d_A^{+,*}v\|_{L^2_{1,A}}\|d_A^{+,*}v\|_{L^2}+\|w\|_{L^{4/3}},\end{aligned}$$

and so,

$$\|v\|_{L^2_{1,A}}\le C\left(\|v\|_{L^2_{2,A}}\|v\|_{L^2_{1,A}}+\|w\|_{L^{4/3}}\right).$$

Thus, taking $\varepsilon\le 1/(2C)$, the first estimate follows by rearrangement.

Hölder's inequality and our elliptic estimate in Lemma 7.1 for the Laplacian $d_A^+d_A^{+,*}$ yields

$$\|v\|_{L^2_{2,A}}+\|v\|_{L^\infty}\le C\|d_A^+d_A^{+,*}v\|_{L^{\sharp,2}}+\|v\|_{L^2},$$

while from equation (7.2) and the Sobolev multiplications $L^2_{1,A}\times L^2_{1,A}\to L^{\sharp,2}$ (again with constant depending only on the metric) we have

$$\begin{aligned}\|d_A^+d_A^{+,*}v\|_{L^{\sharp,2}}&\le\|(d_A^{+,*}v\wedge d_A^{+,*}v)^+\|_{L^{\sharp,2}}+\|w\|_{L^{\sharp,2}}\\&\le c\|d_A^{+,*}v\|^2_{L^2_{1,A}}+\|w\|_{L^{\sharp,2}},\end{aligned}$$

and so,

$$\|v\|_{L^2_{2,A}}+\|v\|_{L^\infty}\le C\left(\|v\|^2_{L^2_{2,A}}+\|w\|_{L^{\sharp,2}}\right).$$

Thus, taking $\varepsilon\le 1/2C$, the second estimate follows by rearrangement. □

7.4. **Existence of solutions to the extended anti-self-dual equation.** We now turn to the question of existence and uniqueness of solutions to equation (7.4) or one of its equivalent forms, namely equations (7.6) or (7.7). We first recall the following elementary fixed-point result [8, Lemma 7.2.23].

Lemma 7.5. *Let $q:\mathfrak{B}\to\mathfrak{B}$ be a continuous map on a Banach space $\mathfrak{B}$ with $q(0)=0$ and*

$$\|q(x_1)-q(x_2)\|\le K\left(\|x_1\|+\|x_2\|\right)\|x_1-x_2\|$$

for some positive constant K and all x_1,x_2 in $\mathfrak{B}$. Then for each y in $\mathfrak{B}$ with $\|y\|<1/(10K)$ there is a unique x in $\mathfrak{B}$ such that $\|x\|\le 1/(5K)$ and $x+q(x)=y$.

With the aid of Lemma 7.5, we obtain existence and uniqueness for solutions to equation (7.4).

Proposition 7.6. *Let (X,g) be a closed, oriented, C^∞ Riemannian four-manifold, and let M,Λ be positive constants. Then there are positive constants ε, C such that the following holds. Let E be a Hermitian, rank-two vector bundle and let A be an L^2_4 orthogonal connection on $\mathfrak{g}_E$ with $\|F_A^+\|_{L^\infty} \le M$, $\|F_A\|_{L^2} \le M$, and let μ be a positive constant such that $\Lambda^{-1} \le \mu \le \Lambda$. Suppose that $\|F_A^+\|_{L^{\sharp,2}(X)} < \varepsilon$. Then there is a unique solution $v \in (C^0 \cap L^2_2)(\Lambda^+ \otimes \mathfrak{g}_E)$ to equation (7.4) such that $\Pi_{A,\mu} v = 0$ and*

$$\|v\|_{L^2_{1,A}(X)} \le C\|F_A^+\|_{L^{4/3}(X)}, \tag{1}$$

$$\|v\|_{L^2_{2,A}(X)} + \|v\|_{L^\infty(X)} \le C\|F_A^+\|_{L^{\sharp,2}(X)}. \tag{2}$$

Moreover, if A is an L^2_l connection, for any $l \ge 4$, then the solution v is contained in $L^2_{l+1}(\Lambda^+ \otimes \mathfrak{g}_E)$.

Proof. We try to solve equation (7.6) for solutions of the form $v = G_{A,\mu}\xi$, where $\xi \in L^{\sharp,2}(\Lambda^+ \otimes \mathfrak{g}_E)$ and $G_{A,\mu}$ is the Green's operator for the Laplacian $d_A^+ d_A^{+,*}$ on $\operatorname{Ran}\Pi_{A,\mu}^\perp$. Therefore, as explained in §7.1, we seek solutions ξ to equation (7.8). We now apply Lemma 7.5 to equation (7.8), with $\mathfrak{B} = L^{\sharp,2}(\Lambda^+ \otimes \mathfrak{g}_E)$, choosing $y = -F_A^+$ and $q(\xi) = (d_A^{+,*}G_{A,\mu}\xi \wedge d_A^{+,*}G_{A,\mu}\xi)^+$, so our goal is to solve

$$\xi + q(\xi) = -F_A^+ \quad \text{on } L^{\sharp,2}(\Lambda^+ \otimes \mathfrak{g}_E).$$

For any $\xi_1, \xi_2 \in L^{\sharp,2}(\Lambda^+ \otimes \mathfrak{g}_E)$, the estimate in Proposition 7.3 yields

$$\|d_A^{+,*}G_{A,\mu}\xi_i\|_{L^2_{1,A}} \le C\|\xi_i\|_{L^{\sharp,2}}.$$

Therefore, applying the preceding estimates, Hölder's inequality, Lemma 6.5, and the $L^\sharp$-family of embedding and multiplication results of [12, Lemmas 4.1 & 4.3], we obtain

$$\|q(\xi_1) - q(\xi_2)\|_{L^{\sharp,2}} \le K_2\left(\|\xi_1\|_{L^{\sharp,2}} + \|\xi_2\|_{L^{\sharp,2}}\right)\|\xi_1 - \xi_2\|_{L^{\sharp,2}},$$

where $K_2 = cZ_2$ with c a universal constant and Z_2 the constant of Proposition 7.3.

Thus, provided $\varepsilon \le (10K_2)^{-1}$, we have $\|F_A^+\|_{L^{\sharp,2}} < (10K_2)^{-1}$, and Lemma 7.5 implies that there is a unique solution $\xi \in L^{\sharp,2}(\Lambda^+ \otimes \mathfrak{g}_E)$ to equation (7.8) such that $\|\xi\|_{L^{\sharp,2}} \le (5K_2)^{-1}$. The $L^{4/3}$ and $L^{\sharp,2}$ estimates for ξ,

$$\|\xi\|_{L^{4/3}} \le 2\|F_A^+\|_{L^{4/3}} \quad \text{and} \quad \|\xi\|_{L^{\sharp,2}} \le 2\|F_A^+\|_{L^{\sharp,2}}, \tag{7.14}$$

follow from Proposition 7.3 and a standard rearrangement argument. To obtain the stated $L^2_{1,A}$ and $L^2_{2,A}$ estimates for v, observe that

$$d_A^+ d_A^{+,*} v = d_A^+ d_A^{+,*} G_{A,\mu}\xi = \Pi_{A,\mu}^\perp \xi, \tag{7.15}$$

so that the $L^2_{1,A}$ bound is obtained from

$$\begin{aligned}\|v\|_{L^2_{1,A}} &\le C\|\xi\|_{L^{4/3}} \quad \text{(by estimate (7.11))}\\ &\le C\|F_A^+\|_{L^{4/3}}, \quad \text{(by estimate (7.14))}.\end{aligned}$$

The $C^0 \cap L^2_{2,A}$ estimate similarly follows, as

$$\begin{aligned}\|v\|_{C^0\cap L^2_{2,A}} &\le C\|\xi\|_{L^{\sharp,2}} \quad \text{(by estimate (7.13))}\\ &\le C\|F^+_A\|_{L^{\sharp,2}} \quad \text{(by estimate (7.14))}.\end{aligned}$$

The final regularity assertion follows from Proposition 4.22, our regularity result for gluing solutions to the (extended) anti-self-dual equation. □

7.5. **Proof of main theorem, without Uhlenbeck continuity assertions.** At this stage, we are ready to complete the proof of at least part of Theorem 1.3, namely the existence of gluing and obstruction maps, but not the Uhlenbeck continuity assertions for these maps, which requires considerably more work and is taken up in §8. On the other hand, the proof of this first part of Theorem 1.3 is an essentially formal, almost immediate consequence of Proposition 7.6.

Proof of Theorem 1.3, without Uhlenbeck continuity assertions. According to Proposition 7.6 there are well-defined, $\mathcal{G}_E$-equivariant maps $\boldsymbol{\delta}$ and $\boldsymbol{\varphi}$ from the open subset of $\mathcal{A}^w_\kappa(X)\times C$,

$$\begin{aligned}\mathcal{A}^w_\kappa(X,C;M,\mu,\varepsilon) &:= \{(A,g)\in\mathcal{A}^w_\kappa(X)\times C :\\ &\qquad \operatorname{Spec}(d^{+,g}_A d^{+,g,*}_A)\subset[0,\tfrac{1}{2}\mu)\cup(\mu,\infty),\\ &\qquad \|F^{+,g}_A\|_{L^{\sharp,2}}<\varepsilon, \text{ and } \|F^{+,g}_A\|_{L^\infty}<M\},\\ \mathcal{B}^w_\kappa(X,C;M,\mu,\varepsilon) &:= \mathcal{A}^w_\kappa(X,C;M,\mu,\varepsilon)/\mathcal{G}_E,\end{aligned} \tag{7.16}$$

given by

$$\begin{aligned}\boldsymbol{\delta} &: (A,g)\mapsto (A+d^{+,g,*}_A v, g),\\ \boldsymbol{\varphi} &: (A,g)\mapsto \Pi_{A,\mu}F^{+,g}(A+d^{+,g,*}_A v),\end{aligned}$$

into $\mathcal{A}^w_\kappa(X)\times C$ and $L^2(X,\Lambda^{+,g}\otimes\mathfrak{g}_E)$, respectively. These descend to a map and a section,

$$\begin{aligned}\boldsymbol{\delta} &: \mathcal{B}^w_\kappa(X,C;M,\mu,\varepsilon)\to\mathcal{B}^w_\kappa(X)\times C,\\ \boldsymbol{\varphi} &: \mathcal{B}^w_\kappa(X,C;M,\mu,\varepsilon)\to\mathfrak{V}_\mu,\end{aligned} \tag{7.17}$$

where $\mathfrak{V}_\mu\to\mathcal{B}^w_\kappa(X,C;\mu)$ is the continuous, finite-rank, pseudovector bundle

$$\mathfrak{V}_\mu := \{(A,g,w) : (A,g)\in\mathcal{A}^w_\kappa(X,C;\mu) \text{ and } w\in\operatorname{Ran}\Pi_{A,g,\mu}\}/\mathcal{G}_E,$$

where we recall that $\operatorname{Ran}\Pi_{A,g,\mu}\subset L^2_{k-1}(X,\Lambda^{+,g}\otimes\mathfrak{g}_E)$. Here, the spaces $\mathcal{A}^w_\kappa(X,C;\mu)$ and $\mathcal{B}^w_\kappa(X,C;\mu)$ are defined by simply omitting the conditions in definition (7.16) involving M and ε. The map $\boldsymbol{\delta}$ and section $\boldsymbol{\varphi}$ have the desired property that

$$\boldsymbol{\delta}(\boldsymbol{\varphi}^{-1}(0))\subset M^w_\kappa(X,C),$$

and are smooth respect to the quotient L^2_k topology upon restriction to $\mathcal{B}^{w,*}_\kappa(X)$: smoothness follows by the same argument as used in the proof of smoothness in the case of the Banach-space inverse function theorem [1], where the existence of an inverse is established via a contraction-mapping argument, just as we employ here.

The gluing map, gluing-data obstruction bundle, and gluing-data obstruction section are then defined by

$$\begin{aligned}
\boldsymbol{\gamma} &= \boldsymbol{\delta} \circ \boldsymbol{\gamma}' : \mathrm{Gl}_\kappa^{w,+}(X,\mathcal{U}_0,\Sigma,\lambda_0) \to \mathcal{B}_\kappa^w(X) \times C, \\
\boldsymbol{\chi} &= \boldsymbol{\varphi} \circ \boldsymbol{\gamma} : \mathrm{Gl}_\kappa^{w,+}(X,\mathcal{U}_0,\Sigma,\lambda_0) \to \Xi, \\
\Xi &= \boldsymbol{\gamma}^* \mathfrak{V}_\mu,
\end{aligned}$$

where the open neighborhood $\mathcal{U}_0 \subset M_{\kappa_0}^w(X,C)$ is chosen according to Remark 2.11. The estimates in §5 and §6 ensure that the image of $\boldsymbol{\gamma}'$, when $\boldsymbol{\gamma}'$ is restricted to $\mathrm{Gl}_\kappa^{w,+}(X,\mathcal{U}_0,\Sigma,\lambda_0)$, is contained in $\mathcal{B}_\kappa^{w,*}(X,C;M,\mu,\varepsilon)$: in particular, that the *a priori* estimates on $F_A^{+,g}$ are obeyed (by Corollary 5.13) and that the spectrum of $d_{A_0}^{+,g} d_{A_0}^{+,*,g}$ obeys the required open constraints for a suitable μ (by Theorem 6.1). We can choose μ to be the infimum of $\frac{1}{2}\nu_2[A_0,g]$ (the least positive eigenvalue of the Laplacian $d_{A_0}^{+,g} d_{A_0}^{+,*,g}$), as $[A_0,g] \in \mathcal{B}_{\kappa_0}^w(X) \times C$ ranges over the space of gauge-equivalence classes of background connections (and metrics in C) contained in $\mathrm{Gl}_\kappa^{w,+}(X,\mathcal{U}_0,\Sigma,\lambda_0)$. □

Remark 7.7. The rank of $\mathfrak{V}_\mu$ varies over $\mathcal{B}_\kappa^w(X,C;M,\mu,\varepsilon)$, so that is why we only refer to it as being a 'pseudovector bundle'. However, the construction of $\mathrm{Gl}_\kappa^{w,+}(X,\mathcal{U}_0,\Sigma,\lambda_0)$ (via the choice of family $\mathcal{U}_0$ of background connections A_0) in §3 and Theorem 6.1 ensure that its image under $\boldsymbol{\gamma}'$ is contained in an open subset of $\mathcal{B}_\kappa^w(X,C;M,\mu,\varepsilon)$ over which $\mathfrak{V}_\mu$ has constant rank and is thus a vector bundle when restricted to this open subset.

8. Continuity of gluing maps and obstruction sections

Our goal in this section is to prove the assertions in Theorem 1.3 concerning the Uhlenbeck continuity of the gluing map $\boldsymbol{\gamma}$ and continuity of the Kuranishi obstruction map $\boldsymbol{\chi}$.

As we briefly discussed in §1, the statements of Theorems 3.4.10, 3.4.17, and 3.4.18 in [25] describe a gluing result similar to but less general than our main gluing theorem for anti-self-dual connections (see §1 here and [14, §1]). While proofs of [25, Theorems 3.4.10, 3.4.17, & 3.4.18] are attributed by Friedman and Morgan [25] to Taubes (specifically, [52] and [54]), it is nonetheless fair to point out that several important assertions of [25, Theorems 3.4.10, 3.4.17, & 3.4.18] are not proved in [52], [54] (or their sequels [55], [56]). In the preceding four papers of Taubes, the gluing map $\boldsymbol{\gamma}$ is shown — either directly or by routine arguments or modifications of existing arguments — to have the following properties:

- The gluing construction gives a map $\boldsymbol{\gamma}$ from $\mathrm{Gl}_\kappa^w(X,\mathcal{U}_0,K,\lambda_0)$ into $\bar{M}_\kappa^w(X,C)$, which is smooth upon restriction to smooth strata of the gluing-data bundle $\mathrm{Gl}_\kappa^w(X,\mathcal{U}_0,K,\lambda_0)$. (Here, $K \Subset \Sigma$ is a precompact subset of a smooth stratum Σ of the symmetric product $\mathrm{Sym}^\ell(X)$ and $C \subset \mathrm{Met}(X)$ is a compact subset.)
- Points in $M_\kappa^w(X,C)$ are shown to lie in the image of certain iterated gluing maps, whose definition is significantly different from that of $\boldsymbol{\gamma}$.

In the versions stated above we have assumed, for simplicity, that the "background connections" A_0 in $\mathrm{Gl}_\kappa^w(X,\mathcal{U}_0,K,\lambda_0)$ have non-vanishing $H_{A_0}^2$ as the issue of obstructions to gluing — though a complication — is not germane to the main concern of this section, namely Uhlenbeck continuity.

Contrary to the assertions of [25], it is not shown that $\boldsymbol{\gamma}$ is actually a homeomorphism from $\bar{\mathrm{Gl}}_\kappa^w(X,\bar{\mathcal{U}}_0,K,\lambda_0)$ onto an open subset of $\bar{M}_\kappa^w(X,C)$, nor is it shown to be a diffeomorphism upon restriction to smooth strata. Though desirable, these properties of $\boldsymbol{\gamma}$ are not obvious and do not follow by routine modifications of arguments contained in his papers or those, for example, of Donaldson [5], [8] or Mrowka [49]. The other difficulty, which we discussed in §1.1.1, concerns the implicit assumption in [25] that the Taubes perturbation method can *always* be used to perturb approximately anti-self-dual to anti-self-dual solutions when the standard round metric on S^4 near the north pole does not match the metric g on X near the gluing sites. While this not a problem when the subsets of the S^4 moduli spaces being glued in are compact (as in [8], [52], [54]), it becomes a serious issue when the subsets are non-compact (as in [56]) if one wishes — as we do and as is required by [25] — to both preserve the locally-trivial fiber-bundle structure of the gluing-data spaces *and* have a topological embedding of the gluing-data spaces (as opposed to a weaker relation, such as a homotopy equivalence as in [56]). This particular difficulty is discussed further in the introduction to §9.

Our goal in this section is to complete the proof of one assertion of [25], namely that the gluing maps extend continuously to the Uhlenbeck closure of the gluing data, $\bar{\mathrm{Gl}}_\kappa^{w,+}(X,\bar{\mathcal{U}}_0,\bar{\Sigma},\lambda_0)$, as discussed in §1.1.2. Note that this continuity assertion is considerably stronger than that of [25, Theorem 3.4.10], where Σ has been replaced by a precompact open subset $K\subset\Sigma$ and so the gluing sites are not allowed to approach each other. Our proof will also show that the gluing maps are continuous with respect to bubble-tree limits. A special case of the continuity assertion is treated by Donaldson and Kronheimer in [8, §7.2.8].

To show that our gluing maps are open, it suffices to consider their restrictions to smooth strata: openness then follows from dimension counting and the fact that the gluing maps are local diffeomorphisms onto their images after restriction to smooth strata. These and other properties are proved in [14].

8.1. **Statement of continuity results.** Let A, A_0 be connections on SO(3) bundles $\mathfrak{g}_E$, $\mathfrak{g}_{E_0}$, respectively, over X with $c_1(E)=c_1(E_0)$ and $c_2(E)-c_2(E_0)=:\ell>0$. Though we make no specific use of this fact for now, the case we have in mind is where the point $[A]\in\mathcal{B}_E(X,g)$ lies in an Uhlenbeck neighborhood of the point $[A_0,\mathbf{x}]\in\mathcal{B}_{E_0}(X,g)\times\mathrm{Sym}^\ell(X)$ (see Definition 3.1). Let $a\in L^2_k(X,\Lambda^1\otimes\mathfrak{g}_E)$ and $a_0\in L^2_k(X,\Lambda^1\otimes\mathfrak{g}_{E_0})$ be solutions to the quasi-linear equations (7.1) on the bundles $\mathfrak{g}_E$ and $\mathfrak{g}_{E_0}$, respectively,

$$d_A^+a+(a\wedge a)^+=w \quad\text{and}\quad a=d_A^{+,*}v, \tag{8.1}$$

$$d_{A_0}^+a_0+(a_0\wedge a_0)^+=w_0 \quad\text{and}\quad a_0=d_{A_0}^{+,*}v_0, \tag{8.2}$$

where $v,w\in\Gamma(X,\Lambda^+\otimes\mathfrak{g}_E)$ and $v_0,w_0\in\Gamma(X,\Lambda^+\otimes\mathfrak{g}_{E_0})$. The use of w, w_0 on the right-hand-sides of equations (8.1) and (8.2) allows us to give a common solution to a wider range of Uhlenbeck continuity problems than would be possible if we simply chose $w=-F_A^+$ or $w_0=F_{A_0}^+$.

We wish to compare the solutions a and a_0 to equations (8.1) and (8.2) when the connection A is close to $(A_0,\mathbf{x})$ in the Uhlenbeck topology and w is close to w_0 in $L^2_{k-1,\mathrm{loc}}$ on $X\setminus\mathbf{x}$. The purpose of this section is to show that a then approaches a_0 in $L^2_{k,\mathrm{loc}}$ on $X\setminus\mathbf{x}$; more precise bounds are given below.

As earlier we use $a(A) := d_A^{+,*}v$ and $a_0 = a(A_0) := d_{A_0}^{+,*}v_0$ to rewrite the first-order quasi-linear equations (8.1) and (8.2) for a and a_0 as

$$d_A^+ d_A^{+,*}v + (d_A^{+,*}v \wedge d_A^{+,*}v)^+ = w, \tag{8.3}$$

$$d_{A_0}^+ d_{A_0}^{+,*}v_0 + (d_{A_0}^{+,*}v_0 \wedge d_{A_0}^{+,*}v_0)^+ = w_0, \tag{8.4}$$

to give a pair of second-order, quasi-linear elliptic equations for v and v_0. In our applications, the terms w and w_0 will always depend on A and A_0 in such a way as to preserve gauge invariance. For example, when the Laplacian $d_A^+ d_A^{+,*}$ has no small eigenvalues, we set $w(A) := -F_A^+$ in equation (8.1) and so we recover the usual anti-self-dual equation. If the Laplacian $d_A^+ d_A^{+,*}$ does have small eigenvalues, we require that $\Pi_{A,\mu}v = 0$ and set

$$w(A) := \Pi_{A,\mu}((d_A^{+,*}v \wedge d_A^{+,*}v)^+ + F_A^+) - F_A^+, \tag{8.5}$$

and so recover the extended anti-self-dual equation (7.4) or, in detail, (7.6), where $\Pi_{A,\mu}$ is the L^2-orthogonal projection from $L^2(\Lambda^+ \otimes \mathfrak{g}_E)$ onto the finite-dimensional subspace spanned by the eigenvectors of $d_A^+ d_A^{+,*}$ with eigenvalues in $[0, \mu]$, where μ is a positive constant (an upper bound for the small eigenvalues), usually $\frac{1}{2}\nu_2[A_0]$: see the small-eigenvalue estimates supplied by Theorem 6.1. Similarly for the connection A_0.

We can now state our main results.

Theorem 8.1. *Let E, E_0 be Hermitian vector bundles over a closed, oriented, smooth four-manifold X with Riemannian metric g, such that $c_1(E_0) = c_2(E)$ and $c_2(E_0) < c_2(E)$. Let A_0 be an orthogonal L_k^2 connection on $\mathfrak{g}_{E_0}$ and suppose $\{A_\alpha\}$ is a sequence of L_k^2 connections on $\mathfrak{g}_E$ which converges to $(A_0, \mathbf{x}) \in \mathcal{A}_{E_0} \times \mathrm{Sym}^\ell(X)$ in the sense that*

- *The sequence of connections A_α converges to A_0 in $L^2_{k,A_0,\mathrm{loc}}$ on $X \setminus \mathbf{x}$, and*
- *The sequence of measures $|F_{A_\alpha}|^2$ converges to $|F_{A_0}|^2 + 8\pi^2 \sum_{x\in\mathbf{x}} \delta_x$ on X.*

We assume that $E|_{X\setminus\mathbf{x}} = E_0|_{X\setminus\mathbf{x}}$. Suppose $v_0 \in L^2_{k+1}(\Lambda^+ \otimes \mathfrak{g}_{E_0})$ is the solution to the second-order quasi-linear equation (8.3) on $\mathfrak{g}_{E_0}$, with right-hand side w_0, while $v_\alpha \in L^2_{k+1}(\Lambda^+ \otimes \mathfrak{g}_E)$ are the solutions to the second-order quasi-linear equation (8.3) on $\mathfrak{g}_E$, with right-hand side w_α. Suppose that the sequence $\{w_\alpha\}$ converges to w_0 in $L^2_{k-1,A_0,\mathrm{loc}}(X \setminus \mathbf{x})$. Then the following hold:

- *The sequence of solutions v_α to equation (8.3) converges to the solution v_0 to equation (8.4) in $L^2_{k+1,A_0,\mathrm{loc}}$ on $X \setminus \mathbf{x}$,*
- *The sequence of solutions $a_\alpha = d_{A_\alpha}^{+,*}v_\alpha$ to equation (8.1) converges to the solution $a_0 = d_{A_0}^{+,*}v_0$ to (8.2) in $L^2_{k,A_0,\mathrm{loc}}$ on $X \setminus \mathbf{x}$,*
- *The sequence of connections $A_\alpha + a_\alpha$ converges to $A_0 + a_0$ in $L^2_{k,A_0,\mathrm{loc}}$ on $X \setminus \mathbf{x}$, and*
- *The sequence of measures $|F(A_\alpha + a_\alpha)|^2$ converges to $|F(A_0 + a_0)|^2 + 8\pi^2 \sum_{x\in\mathbf{x}} \delta_x$ on X.*

The preceding result has the following consequence when applied to the extended anti-self-dual equation:

Theorem 8.2. *Let E, E_0 be Hermitian vector bundles over a closed, oriented, smooth four-manifold X with Riemannian metric g, such that $c_1(E_0) = c_2(E)$ and $c_2(E_0) < c_2(E)$. Let*

$M < \infty$, $\varepsilon > 0$ be constants, let A_0 be an orthogonal L^2_k connection on $\mathfrak{g}_{E_0}$ and suppose $\{A_\alpha\}$ is a sequence of L^2_k connections on $\mathfrak{g}_E$ such that

$$\sup_\alpha \|F^+_{A_\alpha}\|_{C^0(X)} \le M \quad \text{and} \quad \|F^+_{A_0}\|_{L^2(X)}, \quad \sup_\alpha \|F^+_{A_\alpha}\|_{L^2(X)} \le \varepsilon, \tag{8.6}$$

and which converges to $(A_0, \mathbf{x}) \in \mathcal{A}_{E_0} \times \mathrm{Sym}^\ell(X)$ in the sense that

- *The sequence of connections A_α converges to A_0 in $L^2_{k,A_0,\mathrm{loc}}$ on $X \setminus \mathbf{x}$, and*
- *The sequence of measures $|F_{A_\alpha}|^2$ converges to $|F_{A_0}|^2 + 8\pi^2 \sum_{x\in\mathbf{x}} \delta_x$ on X.*

We assume that $E|_{X\setminus\mathbf{x}} = E_0|_{X\setminus\mathbf{x}}$ and that $\varepsilon \ll 1$. Suppose $v_0 \in \Pi^\perp_{A_0,\mu} \cdot L^2_{k+1}(\Lambda^+ \otimes \mathfrak{g}_{E_0})$ is the solution to the extended second-order anti-self-dual equation (8.4) on $\mathfrak{g}_{E_0}$, while $v_\alpha \in \Pi^\perp_{A_\alpha,\mu} \cdot L^2_{k+1}(\Lambda^+ \otimes \mathfrak{g}_E)$ are the solutions to the extended second-order anti-self-dual equation (8.3) on $\mathfrak{g}_E$, so that

$$\begin{aligned} w_\alpha &:= w(A_\alpha) = \Pi_{A_\alpha,\mu}((d^{+,*}_{A_\alpha} v_\alpha \wedge d^{+,*}_{A_\alpha} v_\alpha)^+ + F^+_{A_\alpha}) - F^+_{A_\alpha}, \\ w_0 &:= w(A_0) = \Pi_{A_0,\mu}((d^{+,*}_{A_0} v_0 \wedge d^{+,*}_{A_0} v_0)^+ + F^+_{A_0}) - F^+_{A_0}. \end{aligned}$$

Then the following hold:

- *The sequence of solutions v_α to equation (8.3) converges to the solution v_0 to equation (8.4) in $L^2_{k+1,A_0,\mathrm{loc}}$ on $X \setminus \mathbf{x}$,*
- *The sequence of terms w_α converges to the term w_0 in $L^2_{k-1,A_0,\mathrm{loc}}$ on $X \setminus \mathbf{x}$,*
- *The sequence of solutions $a_\alpha = d^{+,*}_{A_\alpha} v_\alpha$ to equation (8.1) converges to the solution $a_0 = d^{+,*}_{A_0} v_0$ to (8.2) in $L^2_{k,A_0,\mathrm{loc}}$ on $X \setminus \mathbf{x}$,*
- *The sequence of connections $A_\alpha + a_\alpha$ converges to $A_0 + a_0$ in $L^2_{k,A_0,\mathrm{loc}}$ on $X \setminus \mathbf{x}$, and*
- *The sequence of measures $|F(A_\alpha + a_\alpha)|^2$ converges to $|F(A_0 + a_0)|^2 + 8\pi^2 \sum_{x\in\mathbf{x}} \delta_x$ on X.*

Theorem 8.2 has the following important consequence.

Proof of Uhlenbeck continuity assertions of Theorem 1.3. Let us first suppose that the metric g on X is flat near the gluing sites, so that the pre-splicing and splicing maps, $\boldsymbol{\gamma}'$ and $\boldsymbol{\gamma}''$, coincide. The assertions in Theorem 1.3 concerning the Uhlenbeck continuity of the gluing map $\boldsymbol{\gamma}$ and continuity of the Kuranishi obstruction map $\boldsymbol{\chi}$ follow immediately from the definition of the splicing map $\boldsymbol{\gamma}'$ and Theorem 8.2. For the general case, when the metric g is not necessarily flat near the gluing sites, the argument is the same except that we must now appeal to Theorem 1.1 in order to obtain the required preliminary homeomorphisms between $\bar M^{\diamond,s}_{\kappa_i}(S^4, g_0)$ and $\bar M^{\diamond,s}_{\kappa_i}(S^4, \tilde g_i)$. □

The remainder of this section is taken up with the proofs of Theorems 8.1 and 8.2.

8.2. Degeneration of solutions to the second-order quasi-linear equation: linear L^2_2 theory.

Prior to analyzing the asymptotic behavior of solutions to the full anti-self-dual equation, it is useful to first consider the asymptotic behavior of solutions $v \in L^2_{k+1}(X, \Lambda^+ \otimes \mathfrak{g}_E)$ to the linear part of the second order, elliptic anti-self-dual equation (7.2), so

$$d^+_A d^{+,*}_A v = w, \tag{8.7}$$

where $w \in L^2_{k-1}(X, \Lambda^+ \otimes \mathfrak{g}_E)$ and the point $[A] \in \mathcal{B}_E(X, g)$ converges to the point $[A_0, \mathbf{x}] \in \mathcal{B}_{E_0}(X, g) \times \mathrm{Sym}^\ell(X)$.

We need to compare the solutions v to equation (8.7) as A converges to A_0 and w converges to w_0 in $L^2_{k-1,A_0,\mathrm{loc}}$ on $X \setminus \mathbf{x}$. For this purpose we use an elementary Leibnitz formula for $d^+_A d^{+,*}_A$. We recall from (6.2) that $d^+_A d^{+,*}_A = \frac{1}{2}\nabla^*_A \nabla_A +$ zeroth-order terms on $L^2_{k-1}(X, \Lambda^+ \otimes \mathfrak{g}_E)$ and so we have, schematically, for any function $\psi_0 \in C^\infty(X)$,

$$d^+_A d^{+,*}_A(\psi_0 v) = \nabla^2\psi_0 \otimes v + \nabla\psi_0 \otimes \nabla_A v + \psi_0 d^+_A d^{+,*}_A v. \tag{8.8}$$

The 'excision' argument which allows us to compare solutions to the anti-self-dual equation on different bundles E and E_0 relies on a suitable choice of cutoff functions.

Over $X \setminus \mathbf{x}$ we may write $A = A_0 + (A - A_0)$ on $E|_{X\setminus\mathbf{x}} = E_0|_{X\setminus\mathbf{x}}$. Though we shall ultimately require a pair of sequences of $\mathbf{x}$-good cutoff functions satisfying the properties of Lemma 6.9, it suffices for now to choose smooth cutoff functions χ_0 and ψ_0 on X such that

- $\operatorname{supp}\chi_0 \Subset X \setminus \mathbf{x}$, $\operatorname{supp}\psi_0 \Subset X \setminus \mathbf{x}$,
- $\chi_0 = 1$ on $\operatorname{supp}\psi_0$,

and define open subsets of X by

$$\Omega := \{x \in X : (d\psi_0)_x \neq 0\} \quad \text{and} \quad U := \{x \in X : \psi_0(x) > 0\}.$$

We use the χ's to cut off and compare connections and the ψ's to cut off and compare solutions to the anti-self-dual equation on different bundles. Noting that $E|_{X\setminus\mathbf{x}} = E_0|_{X\setminus\mathbf{x}}$ and setting

$$b := \chi_0(A - A_0) \in L^2_k(X, \Lambda^1 \otimes \mathfrak{g}_{E_0}), \tag{8.9}$$

we have $A = A_0 + b$ over $\operatorname{supp}\psi_0 \subset X$ since $\chi_0 = 1$ on $\operatorname{supp}\psi_0$; the section b may also be viewed as an element of $L^2_k(X, \Lambda^1 \otimes \mathfrak{g}_E)$.

The following comparison of solutions to the linear part of the anti-self-dual equation is the basic prototype we later use to compare solutions to the full anti-self-dual equation on different bundles $\mathfrak{g}_E$ and $\mathfrak{g}_{E_0}$.

Lemma 8.3. *Let A, A_0 be connections on $\mathrm{SO}(3)$ bundles $\mathfrak{g}_E$, $\mathfrak{g}_{E_0}$, respectively, and assume that $\operatorname{Ker} d^+_{A_0} d^{+,*}_{A_0} = 0$. Suppose $U \subset X$ is an open subset such that $E|_U = E_0|_U$. Let ψ_0, χ_0 be cutoff functions on X such that $\operatorname{supp}\chi_0 \subset U$, $\operatorname{supp}\psi_0 \subset U$, and $\chi_0 = 1$ on $\operatorname{supp}\psi_0$. Given $w \in L^2_{k-1}(X, \Lambda^+ \otimes \mathfrak{g}_E)$ and $w_0 \in L^2_{k-1}(X, \Lambda^+ \otimes \mathfrak{g}_{E_0})$, suppose $v \in L^2_{k+1}(X, \Lambda^+ \otimes \mathfrak{g}_E)$ and $v_0 \in L^2_{k+1}(X, \Lambda^+ \otimes \mathfrak{g}_{E_0})$ satisfy*

$$d^+_A d^{+,*}_A v = w \quad \text{and} \quad d^+_{A_0} d^{+,*}_{A_0} v_0 = w_0.$$

Then,

$$\begin{aligned}
&\|\psi_0(v - v_0)\|_{L^\infty(X)} + \|\psi_0(v - v_0)\|_{L^2_{2,A_0}(X)} \\
&\le C\|\psi_0(w - w_0)\|_{L^{\sharp,2}} \\
&\quad + C(\|\nabla^2\psi_0\|_{L^{\sharp,2}} + \|b\|_{L^{2\sharp,4}} + \|\nabla_{A_0} b\|_{L^{\sharp,2}} + \|b\|^2_{L^{2\sharp,4}} \\
&\quad + \|\nabla\psi_0\|_{L^{2\sharp,4}}\|b\|_{L^{2\sharp,4}} + \|\nabla\psi_0\|_{L^{2\sharp,4}}) \\
&\qquad \times (\|v\|_{L^\infty} + \|\nabla_A v\|_{L^{2\sharp,4}} + \|v_0\|_{L^\infty} + \|\nabla_{A_0} v_0\|_{L^{2\sharp,4}}),
\end{aligned}$$

where $b := \chi_0(A - A_0) \in L^2_k(X, \Lambda^1 \otimes \mathfrak{g}_{E_0})$, *and* $C(\nu_2[A_0], \|F_{A_0}\|_{L^2}, \|F^+_{A_0}\|_{L^\infty})$ *is a constant, and* $\nu_2[A_0]$ *is the least positive eigenvalue of the Laplacian* $d^+_{A_0} d^{+,*}_{A_0}$.

Proof. We first compute a suitable $L^{\sharp,2}$ estimate for $d^+_{A_0} d^{+,*}_{A_0}\psi_0(v - v_0)$ and then apply our elliptic estimates for $d^+_{A_0} d^{+,*}_{A_0}$ given in Lemma 7.1. We have $A = A_0 + b$ over $\operatorname{supp}\psi_0 \subset X$, since $\chi_0 = 1$ on $\operatorname{supp}\psi_0$, and thus on $\operatorname{supp}\psi_0$, we see that

$$\begin{aligned} d^+_{A_0} d^{+,*}_{A_0} v &= -d^+_{A_0} * d_{A_0} v \\ &= -d^+_A * d_{A_0} v + (b \wedge * d_{A_0} v)^+ \\ &= -d^+_A * d_A v + d^+_A * (b \wedge v) + (b \wedge * d_A v)^+ - (b \wedge *(b \wedge v))^+, \end{aligned}$$

and so

$$d^+_{A_0} d^{+,*}_{A_0} v = d^+_A d^{+,*}_A v + d^+_A * (b \wedge v) + (b \wedge * d_A v)^+ - (b \wedge *(b \wedge v))^+. \tag{8.10}$$

According to our Leibnitz formula (8.8) we have

$$\begin{aligned} d^+_{A_0} d^{+,*}_{A_0}\psi_0(v - v_0) &= \nabla^2\psi_0 \otimes (v - v_0) + \nabla\psi_0 \otimes \nabla_{A_0}(v - v_0) \\ &\quad + \psi_0 d^+_{A_0} d^{+,*}_{A_0}(v - v_0). \end{aligned} \tag{8.11}$$

Therefore, combining (8.10) and (8.11), we obtain

$$\begin{aligned} d^+_{A_0} d^{+,*}_{A_0}\psi_0(v - v_0) &= \nabla^2\psi_0 \otimes (v - v_0) + \nabla\psi_0 \otimes \nabla_{A_0}(v - v_0) \\ &\quad + \psi_0(d^+_A d^{+,*}_A v - d^+_{A_0} d^{+,*}_{A_0} v_0) + \psi_0 d^+_A * (b \wedge (v - v_0)) \\ &\quad + \psi_0(b \wedge * d_A(v - v_0))^+ - \psi_0(b \wedge *(b \wedge (v - v_0)))^+. \end{aligned} \tag{8.12}$$

Consequently, as $d^+_A d^{+,*}_A v = w$ and $d^+_{A_0} d^{+,*}_{A_0} v_0 = w_0$, we have the schematic expression, employing tensor products to denote all bilinear products and using $\nabla_{A_0}(v - v_0) = \nabla_A v + b \otimes v - \nabla_{A_0} v_0$ in the identity (8.12),

$$\begin{aligned} d^+_{A_0} d^{+,*}_{A_0}\psi_0(v - v_0) &= \psi_0(w - w_0) + \nabla^2\psi_0 \otimes (v - v_0) \\ &\quad + \nabla\psi_0 \otimes (\nabla_A v - \nabla_{A_0} v_0 - b \otimes v_0)) \\ &\quad + \psi_0 \nabla_{A_0} b \otimes (v - v_0) \\ &\quad + \psi_0 b \otimes (\nabla_A v - \nabla_{A_0} v_0) + \psi_0 b \otimes b \otimes (v + v_0). \end{aligned} \tag{8.13}$$

We need an $L^{\sharp,2}$ bound on $d^+_{A_0} d^{+,*}_{A_0}\psi_0(v - v_0)$ in terms of the $L^{\sharp,2}$ norm of $\psi_0(w - w_0)$ and the lower order terms in the expression (8.13) for $d^+_{A_0} d^{+,*}_{A_0}\psi_0(v - v_0)$:

$$\begin{aligned} &\|d^+_{A_0} d^{+,*}_{A_0}\psi_0(v - v_0)\|_{L^{\sharp,2}} \\ &\le \|\psi_0(w - w_0)\|_{L^{\sharp,2}} + (\|\nabla^2\psi_0\|_{L^{\sharp,2}} + \|b\|_{L^{2\sharp,4}} + \|\nabla_{A_0} b\|_{L^{\sharp,2}} + \|b\|^2_{L^{2\sharp,4}} \\ &\qquad + \|\nabla\psi_0\|_{L^{2\sharp,4}}\|b\|_{L^{2\sharp,4}})(\|v\|_{L^\infty} + \|v_0\|_{L^\infty}) \\ &\quad + (\|\nabla\psi_0\|_{L^{2\sharp,4}} + \|b\|_{L^{2\sharp,4}})(\|\nabla_A v\|_{L^{2\sharp,4}} + \|\nabla_{A_0} v_0\|_{L^{2\sharp,4}}). \end{aligned} \tag{8.14}$$

From Lemma 7.1 and the eigenvalue bound,

$$\|\psi_0(v - v_0)\|_{L^2} \le \nu_2[A_0]^{-1}\|d^+_{A_0} d^{+,*}_{A_0}\psi_0(v - v_0)\|_{L^2},$$

we have the elliptic estimate

$$\|\psi_0(v-v_0)\|_{L^\infty} + \|\psi_0(v-v_0)\|_{L^2_{2,A_0}} \le C\|d_{A_0}^+ d_{A_0}^{+,*}\psi_0(v-v_0)\|_{L^{\sharp,2}}, \tag{8.15}$$

for a constant $C(\nu_2[A_0], \|F_{A_0}\|_{L^2}, \|F_{A_0}^+\|_{L^\infty})$. Combining the bound (8.15) with that of (8.14) then yields the desired estimate. □

Remark 8.4. Lemma 8.3 allows us to compare solutions $v \in L^2_{k+1}(X, \Lambda^+ \otimes \mathfrak{g}_E)$ to the second order, linear equation $d_A^+ d_A^{+,*}v = w$ with solutions $v_0 \in L^2_{k+1}(X, \Lambda^+ \otimes \mathfrak{g}_{E_0})$ to $d_{A_0}^+ d_{A_0}^{+,*}v_0 = w_0$ when the point $[A]$ lies in an Uhlenbeck neighborhood of a point $[A_0, \mathbf{x}]$. Otherwise said, it allows us to compare the second-order Green's operator for $d_A^+ d_A^{+,*}$ with the Green's operator for $d_{A_0}^+ d_{A_0}^{+,*}$ as the point $[A]$ approaches $[A_0, \mathbf{x}]$.

It is a simpler matter to bound the L^2_1 norm of the difference between v and v_0 over compact subsets of $X \setminus \mathbf{x}$:

Lemma 8.5. *Continue the notation and hypotheses of of Lemma 8.3. Then,*

$$\begin{aligned}
&\|\psi_0(v-v_0)\|_{L^2_{1,A_0}(X)} \\
&\le C(\|\psi_0(w-w_0)\|_{L^{4/3}} + \|\nabla^2\psi_0\|_{L^{4/3}}(\|v\|_{L^\infty} + \|v_0\|_{L^\infty})) \\
&\quad + C\|\nabla\psi_0\|_{L^2}(\|\nabla_A v\|_{L^4} + \|\nabla_{A_0}v_0\|_{L^4} + \|b\|_{L^4}\|v_0\|_{L^\infty}) \\
&\quad + C\|\nabla_{A_0}b\|_{L^{4/3}}(\|\psi_0(v-v_0)\|_{L^\infty} \\
&\quad + C(\|b\|_{L^2}(\|\nabla_A v\|_{L^4} + \|\nabla_{A_0}v_0)\|_{L^4} + \|b\|_{L^2}\|b\|_{L^4}(\|v\|_{L^\infty} + \|v_0\|_{L^\infty})).
\end{aligned}$$

where $b := \chi_0(A - A_0) \in L^2_k(X, \Lambda^1 \otimes \mathfrak{g}_{E_0})$, and $C(\nu_2[A_0], \|F_{A_0}\|_{L^2}, \|F_{A_0}^+\|_{L^\infty})$ is a constant, and $\nu_2[A_0]$ is the least positive eigenvalue of the Laplacian $d_{A_0}^+ d_{A_0}^{+,*}$.

Proof. According to the elliptic estimates of Lemma 6.6, we have

$$\|\psi_0(v-v_0)\|_{L^2_{1,A_0}} \le C\|d_{A_0}^+ d_{A_0}^{+,*}\psi_0(v-v_0)\|_{L^{4/3}}.$$

On the other hand, our schematic expression (8.13) for $d_{A_0}^+ d_{A_0}^{+,*}\psi_0(v-v_0)$ yields

$$\begin{aligned}
\|d_{A_0}^+ d_{A_0}^{+,*}\psi_0(v-v_0)\|_{L^{4/3}} &\le \|\psi_0(w-w_0)\|_{L^{4/3}} + \|\nabla^2\psi_0\|_{L^{4/3}}(\|v\|_{L^\infty} + \|v_0\|_{L^\infty}) \\
&\quad + \|\nabla\psi_0\|_{L^2}(\|\nabla_A v\|_{L^4} + \|\nabla_{A_0}v_0\|_{L^4} + \|b\|_{L^4}\|v_0\|_{L^\infty}) \\
&\quad + \|\nabla_{A_0}b\|_{L^{4/3}}\|\psi_0(v-v_0)\|_{L^\infty} \\
&\quad + \|b\|_{L^2}(\|\nabla_A v\|_{L^4} + \|\nabla_{A_0}v_0)\|_{L^4} \\
&\quad + \|b\|_{L^2}\|b\|_{L^4}(\|v\|_{L^\infty} + \|v_0\|_{L^\infty}).
\end{aligned}$$

Combining the preceding estimates gives the desired bound. □

It remains to give a bound for the difference $\psi_0(a - a_0)$.

Lemma 8.6. *Let A, A_0 be connections on* $\mathrm{SO}(3)$ *bundles $\mathfrak{g}_E$, $\mathfrak{g}_{E_0}$, respectively, over X. Suppose $U \subset X$ is an open subset such that $E|_U = E_0|_U$. Let ψ_0, χ_0 be cutoff functions on X such that* $\operatorname{supp}\chi_0 \subset U$, $\operatorname{supp}\psi_0 \subset U$, *and $\chi_0 = 1$ on* $\operatorname{supp}\psi_0$. *Given $w \in L^2_{k-1}(X, \Lambda^+ \otimes \mathfrak{g}_E)$*

and $w_0 \in L^2_{k-1}(X, \Lambda^+ \otimes \mathfrak{g}_{E_0})$*, suppose* $v \in L^2_{k+1}(X, \Lambda^+ \otimes \mathfrak{g}_E)$ *and* $v_0 \in L^2_{k+1}(X, \Lambda^+ \otimes \mathfrak{g}_{E_0})$*. Let* $a(A) := d_A^{+,*} v \in L^2_k(X, \Lambda^1 \otimes \mathfrak{g}_E)$ *and* $a_0 := a(A_0) = d_{A_0}^{+,*} v_0 \in L^2_k(X, \Lambda^1 \otimes \mathfrak{g}_{E_0})$*. Then*

$$\begin{aligned} \|\psi_0(a - a_0)\|_{L^2_{1,A_0}} \leq \|\psi_0(v - v_0)\|_{L^2_{2,A_0}} & \\ + (\|b\|_{L^2_{1,A_0}} + \|b\|_{L^4}\|\nabla\psi_0\|_{L^4} + \|\nabla\psi_0\|_{L^2_1})(\|v_0\|_{L^\infty} + \|v\|_{L^\infty}) & \\ + (\|\nabla\psi_0\|_{L^4} + \|b\|_{L^4})(\|\nabla_{A_0} v\|_{L^4} + \|\nabla_A v\|_{L^4} + \|b\|_{L^4}\|v\|_{L^\infty}). & \end{aligned}$$

where $b = \chi_0(A - A_0) \in L^2_k(X, \Lambda^1 \otimes \mathfrak{g}_{E_0})$*.*

Proof. Schematically, we have

$$\begin{aligned} \psi_0(a - a_0) &= \psi_0(\nabla_A v - \nabla_{A_0} v_0) \\ &= \nabla_A \psi_0(v - v_0) - \nabla\psi_0 \otimes (v - v_0) \\ &= \nabla_{A_0}\psi_0(v - v_0) + b \otimes \psi_0(v - v_0) - \nabla\psi_0 \otimes (v - v_0). \end{aligned} \tag{8.16}$$

Thus, we can bound the L^2_{1,A_0} norm of $\psi_0(a-a_0)$ in terms of the L^2_{1,A_0} norm of $\nabla_{A_0}(\psi_0(v - v_0))$ and the lower-order terms in the expression (8.16) for $\psi_0(a - a_0)$. For the L^2 norm we have

$$\begin{aligned} \|\psi_0(a - a_0)\|_{L^2} \leq \|\nabla_{A_0}\psi_0(v - v_0)\|_{L^2} & \\ + (\|b\|_{L^2} + \|\nabla\psi_0\|_{L^2})(\|v\|_{L^\infty} + \|v_0\|_{L^\infty}). & \end{aligned} \tag{8.17}$$

To compute the L^2 norm of $\nabla_{A_0}\psi_0(a - a_0)$, note that, schematically, the identity (8.16) yields

$$\begin{aligned} \nabla_{A_0}\psi_0(a - a_0) = \nabla^2_{A_0}\psi_0(v - v_0) &+ \nabla_{A_0} b \otimes \psi_0(v - v_0) \\ &+ b \otimes \nabla\psi_0 \otimes (v - v_0) + b \otimes \psi_0 \nabla_{A_0}(v - v_0) \\ &- \nabla^2\psi_0 \otimes (v - v_0) - \nabla\psi_0 \otimes \nabla_{A_0}(v - v_0), \end{aligned}$$

with $\nabla_{A_0} v = \nabla_A v - b \otimes v$. Therefore,

$$\begin{aligned} \|\nabla_{A_0}\psi_0(a - a_0)\|_{L^2} \leq \|\nabla^2_{A_0}\psi_0(v - v_0)\|_{L^2} & \\ + (\|\nabla_{A_0} b\|_{L^2} + \|b\|_{L^4}\|\nabla\psi_0\|_{L^4} + \|\nabla^2\psi_0\|_{L^2})(\|v_0\|_{L^\infty} + \|v\|_{L^\infty}) & \\ + (\|\nabla\psi_0\|_{L^4} + \|b\|_{L^4})(\|\nabla_{A_0} v\|_{L^4} + \|\nabla_A v\|_{L^4} + \|b\|_{L^4}\|v\|_{L^\infty}). & \end{aligned}$$

Combining this estimate with the L^2 bound (8.17) then gives the desired result. □

Again, the L^2 bound is simpler:

Lemma 8.7. *Continue the notation of Lemma 8.6. Then,*

$$\begin{aligned} \|\psi_0(a - a_0)\|_{L^2(X)} \leq \|\psi_0(v - v_0)\|_{L^2_{1,A_0}} & \\ + (\|b\|_{L^2} + \|\nabla\psi_0\|_{L^2})\|\chi_0(v - v_0)\|_{L^\infty}. & \end{aligned}$$

Proof. From the schematic expression (8.16) for $\psi_0(a - a_0)$ we see that

$$\begin{aligned} \|\psi_0(a - a_0)\|_{L^2} \leq \|\nabla_{A_0}\psi_0(v - v_0)\|_{L^2} & \\ + (\|b\|_{L^2} + \|\nabla\psi_0\|_{L^2})\|\chi_0(v - v_0)\|_{L^\infty}. & \end{aligned}$$

This completes the proof. □

It remains to bound the Sobolev norms of b with estimates for Sobolev norms of $A - A_0$ and χ_0.

Lemma 8.8. *Continue the above notation. Then the following estimates hold,*

$$\|b\|_{L^{2\sharp,4}(X)} \leq c\|b\|_{L^2_{1,A_0}(X)}, \tag{1}$$

$$\|b\|_{L^2_{1,A_0}(X)} \leq (1 + \|\nabla\chi_0\|_{L^2(X)})\|A - A_0\|_{L^2_{1,A_0}(U)}, \tag{2}$$

$$\|\nabla_{A_0} b\|_{L^{\sharp,2}(X)} \leq C(1 + \|\nabla\chi_0\|_{L^4(X)})\|A - A_0\|_{L^2_{2,A_0}(U)}, \tag{3}$$

where c depends only on the metric g, while C depends on U and g.

Proof. Assertion 1 follows immediately from [12, Lemma 4.1]. Next, using $\nabla_{A_0}\chi_0(A-A_0) = \nabla\chi_0 \otimes (A - A_0) + \chi_0\nabla_{A_0}(A - A_0)$, we see that

$$\|b\|_{L^2_{1,A_0}(X)} \leq (1 + \|\nabla\chi_0\|_{L^2(X)})\|A - A_0\|_{L^2(U)} + \|\nabla_{A_0}(A - A_0)\|_{L^2(U)},$$

as required for Assertion 2. Lastly, for any $2 < p < 4$ and $4 < q < \infty$ for which $1/p = 1/4 + 1/q$, the Sobolev embeddings and Hölder inequality yield

$$\begin{aligned}
\|\nabla_{A_0} b\|_{L^{\sharp}(X)} &\leq c\|\nabla_{A_0} b\|_{L^p(X)} \\
&\leq c\|\nabla\chi_0\|_{L^4(X)}\|A - A_0\|_{L^q(U)} + \|\nabla_{A_0}(A - A_0)\|_{L^p(U)}, \\
&\leq C\|\nabla\chi_0\|_{L^4(X)}\|A - A_0\|_{L^2_{2,A_0}(U)} + C\|\nabla_{A_0}(A - A_0)\|_{L^2_{1,A_0}(U)},
\end{aligned}$$

which completes the proof of Assertion 3. □

Remark 8.9. Although we assumed $\operatorname{Ker} d^+_{A_0} d^{+,*}_{A_0} = 0$ in Lemmas 8.3 and 8.5, we did not need to explicitly need this assumption in Lemmas 8.6 and 8.8.

8.3. **Degeneration of solutions to the second-order quasi-linear equation: non-linear L^2_2 theory.** We now consider the difference of the non-linear terms in the anti-self-dual equation and, in conjunction with Lemma 8.3, we obtain the following comparison of solutions to the anti-self-dual equation.

Proposition 8.10. *Let A, A_0 be connections on $\mathrm{SO}(3)$ bundles $\mathfrak{g}_E$, $\mathfrak{g}_{E_0}$, respectively, over (X, g). Then there are positive constants $C = C(\nu_2[A_0], \|F_{A_0}\|_{L^2}, \|F^+_{A_0}\|_{L^\infty})$ and $\varepsilon = \varepsilon(C)$, $\nu_2[A_0]$ is the least positive eigenvalue of the Laplacian $d^+_{A_0} d^{+,*}_{A_0}$, such that the following holds. Suppose $U \subset X$ is an open subset such that $E|_U = E_0|_U$. Let ψ_0, χ_0 be cutoff functions on X such that $\operatorname{supp}\chi_0 \subset U$, $\operatorname{supp}\psi_0 \subset U$, and $\chi_0 = 1$ on $\operatorname{supp}\psi_0$. Given $w \in L^2_{k-1}(X, \Lambda^+ \otimes \mathfrak{g}_E)$ and $w_0 \in L^2_{k-1}(X, \Lambda^+ \otimes \mathfrak{g}_{E_0})$, suppose $v \in L^2_{k+1}(X, \Lambda^+ \otimes \mathfrak{g}_E)$ and $v_0 \in L^2_{k+1}(X, \Lambda^+ \otimes \mathfrak{g}_{E_0})$ satisfy equations* (8.3) *and* (8.4)*, respectively. If*

$$\max\{\|b\|_{L^{2\sharp,4}}, \|\nabla\psi_0\|_{L^{2\sharp,4}}, \|\nabla_{A_0} v_0\|_{L^{2\sharp,4}}, \|\nabla_A v\|_{L^{2\sharp,4}}\} < \varepsilon, \tag{8.18}$$

where $b = \chi_0(A - A_0) \in L^2_k(X, \Lambda^1 \otimes \mathfrak{g}_{E_0})$, *then*

$$\begin{aligned}
&\|\psi_0(v - v_0)\|_{L^\infty \cap L^2_{2,A_0}(X)} \\
&\le C(\|\psi_0(w - w_0)\|_{L^{\sharp,2}} + \|b\|_{L^{2\sharp,4}}\|v\|_{L^\infty}\|\nabla_A v\|_{L^{2\sharp,4}} + \|b\|^2_{L^{2\sharp,4}}\|v\|^2_{L^\infty}) \\
&\quad + C\|\nabla_{A_0}\psi_0(v - v_0)\|_{L^{2\sharp,4}}(\|\nabla_{A_0}v_0\|_{L^{2\sharp,4}} + \|\nabla_{A_0}v\|_{L^{2\sharp,4}} + \|b\|_{L^{2\sharp,4}}\|v\|_{L^\infty}) \\
&\quad + (\|\nabla^2\psi_0\|_{L^{\sharp,2}} + \|\nabla\psi_0\|_{L^4(X)} + \|b\|_{L^4} + \|\nabla_{A_0}b\|_{L^{\sharp,2}}) \\
&\quad \times (\|v\|_{L^\infty \cap L^2_{2,A}} + \|v_0\|_{L^\infty \cap L^2_{2,A_0}}).
\end{aligned} \tag{8.19}$$

Proof. From equations (8.3) and (8.4) we have

$$\begin{aligned}
d^+_A d^{+,*}_A v &= -(d^{+,*}_A v \wedge d^{+,*}_A v)^+ + w =: \tilde{w}, \\
d^+_{A_0} d^{+,*}_{A_0} v_0 &= -(d^{+,*}_{A_0} v_0 \wedge d^{+,*}_{A_0} v_0)^+ + w_0 =: \tilde{w}_0.
\end{aligned} \tag{8.20}$$

Therefore, applying Lemma 8.3 and the embedding $L^2_1 \subset L^{2\sharp,4}$ [12, Lemma 4.1] to the pair of equations (8.20) yields the estimate (when $\|b\|_{L^2_{1,A_0}} \le 1$)

$$\begin{aligned}
\|\psi_0(v - v_0)\|_{L^\infty \cap L^2_{2,A_0}} &\le C\|\psi_0(\tilde{w} - \tilde{w}_0)\|_{L^{\sharp,2}} \\
&\quad + C\left(\|\nabla^2\psi_0\|_{L^{\sharp,2}} + \|\nabla\psi_0\|_{L^4} + \|b\|_{L^4} + \|\nabla_{A_0}b\|_{L^{\sharp,2}}\right) \\
&\quad \times (\|v\|_{L^\infty \cap L^2_{2,A}} + \|v_0\|_{L^\infty \cap L^2_{2,A_0}}),
\end{aligned} \tag{8.21}$$

where b and the constant C are as in the statement of the proposition; this gives the first and last terms on the right of inequality (8.19).

It remains to bound the term $\psi_0(\tilde{w} - \tilde{w}_0)$ (giving the remaining terms in the bound (8.19)): this difference may be written schematically as

$$\psi_0(\tilde{w} - \tilde{w}_0) = \psi_0(w - w_0) + \psi_0\left(\nabla_A v \otimes \nabla_A v - \nabla_{A_0}v_0 \otimes \nabla_{A_0}v_0\right). \tag{8.22}$$

On $\operatorname{supp}\psi_0$ we have

$$\begin{aligned}
\nabla_A v \otimes \nabla_A v &= \nabla_{A_0}v \otimes \nabla_{A_0}v + b \otimes v \otimes \nabla_{A_0}v \\
&\quad + \nabla_{A_0}v \otimes b \otimes v + b \otimes v \otimes b \otimes v.
\end{aligned} \tag{8.23}$$

Now, using $v = (v - v_0) + v_0$ on $\operatorname{supp}\psi_0$ we observe that

$$\begin{aligned}
\nabla_{A_0}v \otimes \nabla_{A_0}v &= \nabla_{A_0}(v - v_0) \otimes \nabla_{A_0}(v - v_0) + \nabla_{A_0}v_0 \otimes \nabla_{A_0}(v - v_0) \\
&\quad + \nabla_{A_0}(v - v_0) \otimes \nabla_{A_0}v_0 + \nabla_{A_0}v_0 \otimes \nabla_{A_0}v_0.
\end{aligned} \tag{8.24}$$

Lastly, note that

$$\begin{aligned}
\psi_0\nabla_{A_0}(v - v_0) \otimes \nabla_{A_0}v_0 &= \nabla_{A_0}\psi_0(v - v_0) \otimes \nabla_{A_0}v_0 \\
&\quad - \nabla\psi_0 \otimes (v - v_0) \otimes \nabla_{A_0}v_0 \\
\psi_0\nabla_{A_0}v_0 \otimes \nabla_{A_0}(v - v_0) &= \nabla_{A_0}v_0 \otimes \nabla_{A_0}\psi_0(v - v_0) \\
&\quad - \nabla_{A_0}v_0 \otimes \nabla\psi_0 \otimes (v - v_0) \\
\psi_0\nabla_{A_0}(v - v_0) \otimes \nabla_{A_0}(v - v_0) &= \nabla_{A_0}\psi_0(v - v_0) \otimes \nabla_{A_0}(v - v_0) \\
&\quad - \nabla\psi_0 \otimes (v - v_0) \otimes \nabla_{A_0}(v - v_0).
\end{aligned} \tag{8.25}$$

Combining equations (8.22), (8.23), (8.24), and (8.25), we finally arrive at a schematic equation for the difference:

$$\begin{aligned}
&\psi_0(\tilde{w}-\tilde{w}_0)\\
&= \psi_0(w-w_0)\\
&\quad + \psi_0(b\otimes v\otimes \nabla_{A_0}v + \nabla_{A_0}v\otimes b\otimes v + b\otimes v\otimes b\otimes v)\\
&\quad + \nabla_{A_0}\psi_0(v-v_0)\otimes\nabla_{A_0}v_0 - \nabla\psi_0\otimes(v-v_0)\otimes\nabla_{A_0}v_0\\
&\quad + \nabla_{A_0}v_0\otimes\nabla_{A_0}\psi_0(v-v_0) - \nabla_{A_0}v_0\otimes\nabla\psi_0\otimes(v-v_0)\\
&\quad + \nabla_{A_0}\psi_0(v-v_0)\otimes\nabla_{A_0}(v-v_0) - \nabla\psi_0\otimes(v-v_0)\otimes\nabla_{A_0}(v-v_0).
\end{aligned} \tag{8.26}$$

Hence, from equation (8.26), Hölder's inequality, and the Sobolev multiplication $L^{2\sharp,4}\otimes L^{2\sharp,4}\to L^{\sharp,2}$ [12, Lemma 4.3], we have a bound

$$\begin{aligned}
&\|\psi_0(\tilde{w}-\tilde{w}_0)\|_{L^{\sharp,2}}\\
&\le C\left(\|\psi_0(w-w_0)\|_{L^{\sharp,2}} + \|b\|_{L^{2\sharp,4}}\|v\|_{L^\infty}\|\nabla_A v\|_{L^{2\sharp,4}} + \|b\|^2_{L^{2\sharp,4}}\|v\|^2_{L^\infty}\right.\\
&\quad + \|\nabla_{A_0}\psi_0(v-v_0)\|_{L^{2\sharp,4}}\left(\|\nabla_{A_0}v_0\|_{L^{2\sharp,4}} + \|\nabla_{A_0}v\|_{L^{2\sharp,4}} + \|b\|_{L^{2\sharp,4}}\|v\|_{L^\infty}\right)\\
&\quad \left.+\|\nabla\psi_0\|_{L^{2\sharp,4}}\|v-v_0\|_{L^\infty}\left(\|\nabla_{A_0}v_0\|_{L^{2\sharp,4}} + \|\nabla_A v\|_{L^{2\sharp,4}} + \|b\|_{L^{2\sharp,4}}\|v\|_{L^\infty}\right)\right).
\end{aligned} \tag{8.27}$$

For small enough ε, our hypotheses (8.18) implies that the terms $\|b\|_{L^{2\sharp,4}}$, $\|\nabla\psi_0\|_{L^{2\sharp,4}}$, $\|\nabla_{A_0}v_0\|_{L^{2\sharp,4}}$, and $\|\nabla_A v\|_{L^{2\sharp,4}}$ are small enough to allow us to rearrange the bound (8.21) and, together with the estimate (8.27), yield the desired bound (8.19). □

Remark 8.11. Just as in §8.2, Lemma 8.6 provides an L^2_{1,A_0} bound for the difference $\psi_0(a-a_0)$, while Lemma 8.8 gives a bound for the Sobolev norms of b in terms of estimates on the Sobolev norms of $A-A_0$ and χ_0.

Remark 8.12. To motivate the last paragraph of the proof of Proposition 8.10, we note that in our application A will be close to A_0 in $L^2_{k,A_0,\mathrm{loc}}$ on $X\setminus\mathbf{x}$, so the term $\|b\|_{L^{2\sharp,4}}$ will be close to zero. We can also choose ψ_0 such that $\|\nabla\psi_0\|_{L^{2\sharp,4}}$ is small. Since v and v_0 solve the second-order anti-self-dual equations, we may assume that the terms $\|\nabla_{A_0}v_0\|_{L^{2\sharp,4}}$ and $\|\nabla_A v\|_{L^{2\sharp,4}}$ are small, though not necessarily converging to zero if A converges to A_0 on on $X\setminus\mathbf{x}$.

8.4. **Degeneration of solutions to the second-order extended anti-self-dual equation: L^2_2 theory.** The new difficulty here is that we need to estimate $\|\psi_0(w-w_0)\|_{L^{\sharp,2}(X)}$ where w, w_0 are now explicitly defined by equation (8.5), so

$$\begin{aligned}
w(A) &:= \Pi_{A,\mu}((d_A^{+,*}v\wedge d_A^{+,*}v)^+ + F_A^+) - F_A^+,\\
w_0 &:= w(A_0) = \Pi_{A_0,\mu}((d_{A_0}^{+,*}v_0\wedge d_{A_0}^{+,*}v_0)^+ + F_{A_0}^+) - F_{A_0}^+,
\end{aligned} \tag{8.28}$$

and where v and v_0 must satisfy $\Pi_{A_0,\mu}v_0 = 0$ and $\Pi_{A,\mu}v = 0$. Typically, we choose $\mu = \frac{1}{2}\nu_2[A_0]$.

The obvious problem is that our resulting quasi-linear equation (8.3) is no longer local, that is, given purely by a partial differential equation, as we have included spectral projections. However, by virtue of our comparison in §6 of the eigenvectors of $d_A^+d_A^{+,*}$ with the approximate eigenvectors defined by cutting off those of $d_{A_0}^+d_{A_0}^{+,*}$, we can replace our new non-local quasi-linear equations with 'essentially local' ones plus error terms which converge to zero if A converges to $(A_0, \mathbf{x})$ in the Uhlenbeck sense. Given this broad outline, we proceed to the details.

Our task is to compare the solution v to equation (8.3) with the solution v_0 to the corresponding equation (8.4), where w and w_0 are defined by (8.28). Hence, the proof proceeds in much the same way as that of Proposition 8.10, the main additional step being that in order to estimate the $L^{\sharp,2}$ norms of the cut-off differences, $\psi_0(w-w_0)$, we need to also compare the L^2-orthogonal projections $\Pi_{A_0,\mu}$ and $\Pi_{A,\mu}$.

8.4.1. *Preliminary estimates.* With the aid of two elementary lemmas and our eigenvector estimates from §6, we can effectively reduce the problem of estimating the difference between solutions to the extended anti-self-dual equation to the simpler case of the ordinary anti-self-dual equation.

We begin with the more specific problem of estimating $v - v_0$ on the subset $\{x \in X : \psi_0(x) = 1\}$.

Lemma 8.13. *Let X be a closed, oriented, smooth, Riemannian four-manifold. Let A, A_0 be L^2_4 connections on* $\mathrm{SO}(3)$ *bundles $\mathfrak{g}_E$, $\mathfrak{g}_{E_0}$ over X, respectively. Suppose $U \subset X$ is an open subset such that $E|_U = E_0|_U$. Let ψ_0, χ_0 be cutoff functions on X such that* $\operatorname{supp}\chi_0 \subset U$, $\operatorname{supp}\psi_0 \subset U$, *and $\chi_0 = 1$ on* $\operatorname{supp}\psi_0$. *Let $v \in L^2_3(\Lambda^+ \otimes \mathfrak{g}_E)$, $v_0 \in L^2_3(\Lambda^+ \otimes \mathfrak{g}_{E_0})$, and suppose that $\Pi_{A,\mu} v = 0$ and $\Pi_{A_0,\mu} v_0 = 0$. Then,*

$$
\begin{aligned}
\|\psi_0(v-v_0)\|_{L^\infty \cap L^2_{2,A_0}(X)} &\le C\|d^+_{A_0} d^{+,*}_{A_0}\psi_0(v-v_0)\|_{L^{\sharp,2}(X)} \\
&\quad + C(\|1-\psi_0\|_{L^2(X)}(\|v_0\|_{L^\infty(X)} + \|v\|_{L^\infty(X)}) \\
&\quad + \|(\Pi_{A,\mu} - \Pi_{A_0,\mu})\psi_0 v\|_{L^2(X)}).
\end{aligned}
$$

where $C = C(\mu, \|F_{A_0}\|_{L^2}, \|F^+_{A_0}\|_{L^\infty})$.

Proof. Our estimate for $\|\psi_0(v-v_0)\|_{L^\infty \cap L^2_{2,A_0}}$ is initiated by

$$
\begin{aligned}
\|\psi_0(v-v_0)\|_{L^\infty \cap L^2_{2,A_0}} &\le \|\Pi^\perp_{A_0,\mu}\psi_0(v-v_0)\|_{L^\infty \cap L^2_{2,A_0}} \\
&\quad + \|\Pi_{A_0,\mu}\psi_0(v-v_0)\|_{L^\infty \cap L^2_{2,A_0}}.
\end{aligned}
\tag{8.29}
$$

The last term on the right in (8.29) is 'small' because $\Pi_{A,\mu} v = 0$ and $\Pi_{A_0,\mu} v_0 = 0$. Indeed,

$$
\begin{aligned}
\|\Pi_{A_0,\mu}\psi_0(v-v_0)\|_{L^\infty \cap L^2_{2,A_0}} &\le C\|\Pi_{A_0,\mu}\psi_0(v-v_0)\|_{L^2} \quad \text{(by Lemma 6.7)} \\
&\le C(\|\Pi_{A_0,\mu} v_0\|_{L^2} + \|\Pi_{A_0,\mu}(1-\psi_0)v_0\|_{L^2} \\
&\quad + \|\Pi_{A,\mu}\psi_0 v\|_{L^2} + \|(\Pi_{A,\mu} - \Pi_{A_0,\mu})\psi_0 v\|_{L^2}) \\
&\le C(\|\Pi_{A_0,\mu}(1-\psi_0)v_0\|_{L^2} + \|\Pi_{A,\mu} v\|_{L^2} \\
&\quad + \|\Pi_{A,\mu}(1-\psi_0)v\|_{L^2} + \|(\Pi_{A,\mu} - \Pi_{A_0,\mu})\psi_0 v\|_{L^2}),
\end{aligned}
$$

and thus

$$
\begin{aligned}
\|\Pi_{A_0,\mu}\psi_0(v-v_0)\|_{L^\infty \cap L^2_{2,A_0}} &\le C(\|1-\psi_0\|_{L^2}(\|v_0\|_{L^\infty} + \|v\|_{L^\infty}) \\
&\quad + \|(\Pi_{A,\mu} - \Pi_{A_0,\mu})\psi_0 v\|_{L^2}).
\end{aligned}
\tag{8.30}
$$

We estimate the second-last term on the right in (8.29) by

$$
\begin{aligned}
&\|\Pi_{A_0,\mu}^{\perp}\psi_0(v-v_0)\|_{L^\infty\cap L^2_{2,A_0}} \\
&= \|G_{A_0,\mu}d_{A_0}^+d_{A_0}^{+,*}\psi_0(v-v_0)\|_{L^\infty\cap L^2_{2,A_0}} \\
&\le C\|\Pi_{A_0,\mu}^{\perp}d_{A_0}^+d_{A_0}^{+,*}\psi_0(v-v_0)\|_{L^{\sharp,2}} \quad \text{by Lemma 7.1} \\
&\le C\|d_{A_0}^+d_{A_0}^{+,*}\psi_0(v-v_0)\|_{L^{\sharp,2}} + C\|\Pi_{A_0,\mu}d_{A_0}^+d_{A_0}^{+,*}\psi_0(v-v_0)\|_{L^{\sharp,2}} \\
&\le C\|d_{A_0}^+d_{A_0}^{+,*}\psi_0(v-v_0)\|_{L^{\sharp,2}} + C\|\Pi_{A_0,\mu}d_{A_0}^+d_{A_0}^{+,*}\psi_0(v-v_0)\|_{L^2} \\
&\le C\|d_{A_0}^+d_{A_0}^{+,*}\psi_0(v-v_0)\|_{L^{\sharp,2}} + C\mu\|\Pi_{A_0,\mu}\psi_0(v-v_0)\|_{L^2}.
\end{aligned}
\tag{8.31}
$$

The last term on the right in (8.31) can be estimated via inequality (8.30). The conclusion follows by combining inequalities (8.29), (8.30), and (8.31). □

Suppose $u \in \Gamma(\Lambda^+\otimes\mathfrak{g}_E)$ and $u_0 \in \Gamma(\Lambda^+\otimes\mathfrak{g}_{E_0})$. From equations (8.3), (8.4), (8.28) and Proposition 8.10, we see that we shall need at least $L^{\sharp,2}(X)$ estimates for terms of the general form $\psi_0(\Pi_{A,\mu}u - \Pi_{A_0,\mu}u_0)$.

Lemma 8.14. *Let X be a closed, oriented, smooth, Riemannian four-manifold. Let A, A_0 be L^2_4 connections on* SO(3) *bundles $\mathfrak{g}_E$, $\mathfrak{g}_{E_0}$ over X, respectively. Suppose $U\subset X$ is an open subset such that $E|_U = E_0|_U$. Let ψ_0, χ_0 be cutoff functions on X such that* $\operatorname{supp}\chi_0\subset U$, $\operatorname{supp}\psi_0\subset U$, *and $\chi_0 = 1$ on* $\operatorname{supp}\psi_0$. *Continue the notation of the preceding paragraph. Then, for any $1\le\ell\le k$,*

$$
\begin{aligned}
\|\psi_0(\Pi_{A,\mu}u - \Pi_{A_0,\mu}u_0)\|_{L^{\sharp,2}(X)} \le C(&\|\psi_0(u-u_0)\|_{L^2(X)} + \|\Pi_{A,\mu}(1-\psi_0)u\|_{L^2(X)} \\
&+ \|\Pi_{A_0,\mu}(1-\psi_0)u_0\|_{L^2(X)} \\
&+ \|(\Pi_{A,\mu}-\Pi_{A_0,\mu})\psi_0 u\|_{L^2(X)}),
\end{aligned}
$$

where $C = C(\mu, \|F_{A_0}\|_{L^2}, \|F_{A_0}^+\|_{L^\infty})$.

Proof. Observe that, via the embedding $L^4(X)\subset L^\sharp(X)$ of [12, Lemma 4.1] and $L^2_{1,A_0}(X)\subset L^4(X)$ of Lemma 6.5,

$$
\begin{aligned}
\|\psi_0(\Pi_{A,\mu}u - \Pi_{A_0,\mu}u_0)\|_{L^{\sharp,2}} &\le \|\Pi_{A,\mu}u - \Pi_{A_0,\mu}u_0\|_{L^{\sharp,2}} \\
&\le C\|\Pi_{A,\mu}u - \Pi_{A_0,\mu}u_0\|_{L^2} \quad \text{(by Lemma 6.7, above embeddings)} \\
&\le C(\|\Pi_{A,\mu}\psi_0 u - \Pi_{A_0,\mu}\psi_0 u_0\|_{L^2} + \|\Pi_{A,\mu}(1-\psi_0)u\|_{L^2} \\
&\quad + \|\Pi_{A_0,\mu}(1-\psi_0)u_0\|_{L^2}) \\
&\le C(\|\Pi_{A_0,\mu}\psi_0(u-u_0)\|_{L^2} + \|(\Pi_{A,\mu}-\Pi_{A_0,\mu})\psi_0 u\|_{L^2} \\
&\quad + \|\Pi_{A,\mu}(1-\psi_0)u\|_{L^2} + \|\Pi_{A_0,\mu}(1-\psi_0)u_0\|_{L^2}),
\end{aligned}
$$

and the conclusion follows. □

Remark 8.15. Recall from §6 that we can approximate $\Pi_{A,\mu}$ by L^2-orthogonal projection onto a space of vectors in $\Gamma(\Lambda^+\otimes\mathfrak{g}_E)$ with support contained in $\{x\in X : \psi_0(x) = 1\}$; similarly for $\Pi_{A_0,\mu}$.

With the aid of Proposition 8.10 and its proof, we obtain the following comparison result for solutions to the extended anti-self-dual equations.

Proposition 8.16. *Let A, A_0 be L^2_4 connections on* SO(3) *bundles $\mathfrak{g}_E$, $\mathfrak{g}_{E_0}$, respectively, over X. Suppose $U \subset X$ is an open subset such that $E|_U = E_0|_U$. Let ψ_0, χ_0 be cutoff functions on X such that $\operatorname{supp}\chi_0 \subset U$, $\operatorname{supp}\psi_0 \subset U$, and $\chi_0 = 1$ on $\operatorname{supp}\psi_0$. Suppose that $v \in L^2_3(X, \Lambda^+ \otimes \mathfrak{g}_E)$ and $v_0 \in L^2_3(X, \Lambda^+ \otimes \mathfrak{g}_{E_0})$ satisfy equations* (8.3) *and* (8.4), *respectively, with w, w_0 defined by equations* (8.28). *Then*

$$
\begin{aligned}
&\|\psi_0(v - v_0)\|_{L^\infty(X)} + \|\psi_0(v - v_0)\|_{L^2_{2,A_0}(X)} \\
&\le C\|\psi_0(F^+_A - F^+_{A_0})\|_{L^{\sharp,2}} \\
&\quad + C(\|1 - \psi_0\|_{L^2(X)}(\|v_0\|_{L^\infty(X)} + \|v\|_{L^\infty(X)}) + \|(\Pi_{A,\mu} - \Pi_{A_0,\mu})\psi_0 v\|_{L^2(X)}) \\
&\quad + C(\|\Pi_{A,\mu}(1 - \psi_0)u\|_{L^2(X)} + \|\Pi_{A_0,\mu}(1 - \psi_0)u_0\|_{L^2(X)} \\
&\quad + \|(\Pi_{A,\mu} - \Pi_{A_0,\mu})\psi_0 u\|_{L^2(X)}) \\
&\quad + \text{All terms on right of inequality (8.19) in Proposition 8.10,}
\end{aligned}
\tag{8.32}
$$

where u, u_0 are defined by (8.33), *$b = \chi_0(A - A_0) \in L^2_4(X, \Lambda^1 \otimes \mathfrak{g}_{E_0})$ and $C = C(\mu, \|F_{A_0}\|_{L^2}, \|F^+_{A_0}\|_{L^\infty})$ is a constant.*

Proof. According to Lemma 8.13, we have

$$\|\psi_0(v - v_0)\|_{L^\infty \cap L^2_{2,A_0}(X)} \le C\|d^+_{A_0} d^{+,*}_{A_0} \psi_0(v - v_0)\|_{L^{\sharp,2}(X)} + \text{Small terms,}$$

where the 'small terms' in question (those which tend to zero if A converges to $(A_0, \mathbf{x})$ in the Uhlenbeck topology) have been added to the right-hand side of (8.32) above. To estimate the term $\|d^+_{A_0} d^{+,*}_{A_0} \psi_0(v - v_0)\|_{L^{\sharp,2}(X)}$, we apply Proposition 8.10 to reduce the problem to one of estimating the term $\|\psi_0(w - w_0)\|_{L^{\sharp,2}}$, with w, w_0 defined by equations (8.28), again modulo small terms which have been added to the right-hand side of inequality (8.32).

To bound the term $\|\psi_0(w - w_0)\|_{L^{\sharp,2}}$, we apply Lemma 8.14, where we choose

$$
\begin{aligned}
u(A) &:= (d^{+,*}_A v \wedge d^{+,*}_A v)^+ + F^+_A, \\
u_0 &:= u(A_0) = (d^{+,*}_{A_0} v_0 \wedge d^{+,*}_{A_0} v_0)^+ + F^+_{A_0}.
\end{aligned}
\tag{8.33}
$$

The other term we need to estimate is $\|\psi_0(F^+_A - F^+_{A_0})\|_{L^{\sharp,2}(X)}$, which requires no further comment. From Lemma 8.14 we see that it suffices to estimate the term $\|\psi_0(u - u_0)\|_{L^2}$, as again the remaining terms from Lemma 8.14 are small and have been added to the right-hand side of inequality (8.32). Using the definition of u, u_0 in (8.33) we see from the proof of Proposition 8.10 that, starting from the definition of $\tilde{w}$, $\tilde{w}_0$ in (8.20) and using rearrangement, the term $\|\psi_0(u - u_0)\|_{L^2}$ is bounded by terms already appearing on the right-hand side of inequality (8.32). □

Proposition 8.16, in conjunction with the supplementary lemma below, shows that if $\{A_\alpha\}$ converges to $(A_0, \mathbf{x})$ in the Uhlenbeck sense, then indeed $v_\alpha = v(A_\alpha)$ converges to $v_0(A_0)$ in $L^2_{2,A_0,\mathrm{loc}}(X \setminus \mathbf{x})$ and $w_\alpha = w(A_\alpha)$ converges to $w_0(A_0)$ in $L^{\sharp,2}_{\mathrm{loc}}(X \setminus \mathbf{x})$.

Lemma 8.17. *Let the hypotheses be as in Theorem 8.2. Then the following hold:*

- *The sequence of solutions $\{v_\alpha\}$ to equation* (8.1) *converges to the solution v_0 to equation* (8.2) *in $L^\infty \cap L^2_{2,A_0,\mathrm{loc}}$ on $X \setminus \mathbf{x}$,*
- *The sequence of terms $\{w_\alpha\}$ converges to the term w_0 in $L^{\sharp,2}_{\mathrm{loc}}$ on $X \setminus \mathbf{x}$,*

- *The sequence of solutions* $a_\alpha = d^{+,*}_{A_\alpha} v_\alpha$ *to equation* (8.1) *converges to the solution* $a_0 = d^{+,*}_{A_0} v_0$ *to* (8.2) *in* $L^2_{1,A_0,\mathrm{loc}}$ *on* $X \setminus \mathbf{x}$,
- *The sequence of connections* $\{A_\alpha + a_\alpha\}$ *converges to* $A_0 + a_0$ *in* $L^2_{1,A_0,\mathrm{loc}}$ *on* $X \setminus \mathbf{x}$.

Proof. We first consider the terms contributed to the right-hand side of inequality (8.32) in Proposition 8.16 by Proposition 8.10. Let C be the constant of Proposition 8.10. Suppose $U \Subset X \setminus \mathbf{x}$ is a precompact open subset and that ε is a positive constant. By Lemma 6.9 we can choose a sequence of cutoff functions $\{\psi_n\}$ on X which are $\mathbf{x}$-good in the sense that

- $\psi_n = 1$ on U,
- $\operatorname{supp}\psi_n \subset V_n \Subset X \setminus \mathbf{x}$, where $\{V_n\}$ is a collection of open subsets such that $\cup V_n = X \setminus \mathbf{x}$,
- ψ_n converges pointwise to the constant 1 on $X \setminus \mathbf{x}$, and
- $\lim_{n\to\infty} \|\nabla\psi_n\|_{L^{2\sharp,4}(X)} + \|\nabla^2\psi_n\|_{L^{\sharp,2}(X)} = 0$.

We shall use the cutoff functions ψ_n in place of the ψ_0 appearing in the statement of Proposition 8.10. In particular, we may suppose that for large enough $n_0(\varepsilon, C, M)$ (letting M be the constant in the inequalities (8.37)),

$$\|\nabla^2\psi_n\|_{L^{\sharp,2}(X)} + \|\nabla\psi_n\|_{L^{2\sharp,4}(X)} \le \varepsilon/(1000CM^2), \quad n \ge n_0. \tag{8.34}$$

For any fixed n, the sequence $\{A_\alpha\}$ converges to A_0 in $L^2_{k,A_0}(V_n)$ and so, for large enough $\alpha_0 = \alpha_0(C, \varepsilon, n)$, we have

$$\|A_\alpha - A_0\|_{L^2_{k,A_0}(V_n)} \le \varepsilon/(1000CM^2), \quad \alpha \ge \alpha_0. \tag{8.35}$$

For the remainder of the section we employ the abbreviation

$$b_\alpha := A_\alpha - A_0 \quad \text{on } X \setminus \mathbf{x}. \tag{8.36}$$

Using $F(A_\alpha) = F(A_0 + b_\alpha) = F_{A_0} + d_{A_0} b_\alpha + b_\alpha \wedge b_\alpha$ on $X \setminus \mathbf{x}$ and the Sobolev embeddings of [12, Lemma 4.1], we see that

$$\begin{aligned}\|\psi_n(F^+_{A_\alpha} - F^+_{A_0})\|_{L^{\sharp,2}(X)} &\le \|\nabla_{A_0} b_\alpha\|_{L^{\sharp,2}(V_n)} + \|b_\alpha\|_{L^{2\sharp,4}(V_n)} \le c\|b_\alpha\|_{L^2_{2,A_0}(V_n)} \\ &\le c\|A_\alpha - A_0\|_{L^2_{2,A_0}(V_n)} \le \varepsilon/(1000CM^2).\end{aligned}$$

We note that Lemma 7.4 (with right-hand side data $-\Pi^\perp_{A_\alpha,\mu} F^+_{A_\alpha}$) provides uniform bounds M on v_α which only depend on $\sup_\alpha\{\|F^+_{A_\alpha}\|_{L^\infty}, \|F_{A_\alpha}\|_{L^2}\}$, μ, g, and similarly for v_0, so

$$\|v_\alpha\|_{L^\infty \cap L^2_{2,A_\alpha}(X)} \le M \quad \text{and} \quad \|v_0\|_{L^\infty \cap L^2_{2,A_0}(X)} \le M. \tag{8.37}$$

Since $U \subset \{\psi_n = 1\}$, by definition of ψ_n, we have

$$\|v_\alpha - v_0\|_{L^\infty \cap L^2_{2,A_0}(U)} \le \|\psi_n(v_\alpha - v_0)\|_{L^\infty \cap L^2_{2,A_0}(X)}.$$

Hence, for large enough n_0, the sum of the terms contributed to the right-hand side of inequality (8.32) in Proposition 8.16 by (8.19), is bounded by $\varepsilon/3$, for all $\alpha \ge \alpha_0$.

We now consider the terms on the right-hand side of inequality (8.32) which are contributed by Lemma 8.13. From the definition of the sequence of cutoff functions $\{\psi_n\}$, we have $\lim_{n\to\infty} \|1 - \psi_n\|_{L^2(X)} = 0$. By §6 we know that $\|\Pi_{A_\alpha,\mu}\psi_n - \Pi_{A_0,\mu}\psi_n\|$ converges to zero as $\alpha \to \infty$ (for any fixed n), where $\|\cdot\|$ denotes the operator norm in $\mathrm{End}(L^2(\Lambda^+ \otimes \mathfrak{g}_E))$. Therefore, for large enough n_0, the sum of the terms contributed to the right-hand side of inequality (8.32) in Proposition 8.16 by Lemma 8.13, may be bounded by $\varepsilon/3$ for all

$\alpha \geq \alpha_0$. (The term $d_{A_0}^+ d_{A_0}^{+,*}\psi_n(v_\alpha - v_0)$ is excluded, as it has already been accounted for by Proposition 8.10.)

Lastly, we consider the terms in inequality (8.32) contributed by Lemma 8.14. From the definition of $u_\alpha = u(A_\alpha) \in L^{\sharp,2}(\Lambda^+ \otimes \mathfrak{g}_E)$ in (8.33) and our $L^\infty \cap L^2_{2,A_\alpha}$ estimates for v_α in (8.37), we have a uniform L^2 bound on u_α. Hence, we can see that the term $\|\Pi_{A_\alpha,\mu}(1 - \psi_n)u_\alpha\|_{L^2(X)}$ converges to zero either by (i) approximating $\Pi_{A_\alpha,\mu}$ by L^2-orthogonal projection onto a subspace of $L^2(\Lambda^+ \otimes \mathfrak{g}_E)$ spanned by approximate eigenvectors of $d_{A_\alpha}^+ d_{A_\alpha}^{+,*}$ which are supported in $\{x \in X : 1 - \psi_n(x) = 0\}$, or (ii) applying the Lebesgue dominated convergence theorem to the sequence $(1-\psi_n)u_\alpha$, noting that the support of $1-\psi_n$ shrinks. The same remarks apply to the term $\|\Pi_{A_0,\mu}(1 - \psi_n)u_0\|_{L^2(X)}$. As with the corresponding term involving v_α in Lemma 8.13, the term $\|(\Pi_{A_\alpha,\mu} - \Pi_{A_0,\mu})\psi_n u_\alpha\|_{L^2(X)}$ also converges to zero as $\alpha \to \infty$, for any fixed n. Thus, the sum of the terms contributed to the right-hand side of inequality (8.32) in Proposition 8.10 by Lemma 8.14, may be bounded by $\varepsilon/3$, as

$$\lim_{n\to\infty}\lim_{\alpha\to\infty} \|\psi_n(\Pi_{A_\alpha,\mu}u_\alpha - \Pi_{A_0,\mu}u_0)\|_{L^2(X)} = 0. \tag{8.38}$$

Consequently, summing all three contributions to the right-hand side of inequality (8.32), and taking first n, then α sufficiently large yields

$$\|v_\alpha - v_0\|_{L^\infty \cap L^2_{2,A_0}(U)} \leq \varepsilon.$$

Hence, the sequence $\{v_\alpha\}$ converges to v_0 in $L^\infty \cap L^2_{2,A_0}(U)$. Lemma 8.6 then implies that $\{a_\alpha\}$ converges to a_0 in $L^2_{1,A_0}(U)$ and thus the sequence of connections $\{A_\alpha + a_\alpha\}$ converges to A_0 in $L^2_{1,A_0}(U)$. $\square$

8.5. **Uhlenbeck convergence of solutions to the anti-self-dual equation: $L^2_{k,\mathrm{loc}}$ convergence.** The estimates of the preceding subsection imply that $\psi_n a_\alpha$ converges to a_0 in $L^2_{1,A_0}(X)$ if the sequence of connections A_α converges to $(A_0,\mathbf{x})$ in $L^2_{k,A_0,\mathrm{loc}}(X \setminus \mathbf{x})$. It remains to show, with the aid of our higher-order interior elliptic estimates, that (i) the sequence of one-forms a_α converges to a_0 in $L^2_{\ell,A_0,\mathrm{loc}}$ over $X \setminus \mathbf{x}$ for all $2 \leq \ell \leq k$ and that (ii) if the sequence of measures $|F_{A_\alpha}|^2$ converges to $|F_{A_0}|^2 + 8\pi^2\sum_{x\in\mathbf{x}}\delta_x$ in the weak-* topology on measures, then the sequence of measures $|F_{A_\alpha+a_\alpha}|^2$ also converges to $|F_{A_0}|^2 + 8\pi^2\sum_{x\in\mathbf{x}}\delta_x$. Note that given A_α converges to $(A_0,\mathbf{x})$, we are not at liberty to pass to a subsequence in order to show that $A_\alpha + a_\alpha$ converges to $(A_0,\mathbf{x})$.

Lemma 8.18. *Suppose v, v_0 are solutions to equations* (8.3) *and* (8.4)*, respectively, defined by connections A on $\mathfrak{g}_E$ and A_0 on $\mathfrak{g}_{E_0}$, with w and w_0 defined by* (8.28)*. Suppose $U \subset X$ is an open subset such that $E|_U = E_0|_U$. Then $v' := v - v_0$ solves*

$$\begin{aligned} &d_{A_0}^+ d_{A_0}^{+,*}v' + (d_{A_0}^{+,*}v' \wedge d_{A_0}^{+,*}v')^+ + \{d_{A_0}^{+,*}v_0, d_{A_0}^{+,*}v'\} \\ &\quad = w' =: w - w_0 + b\otimes v\otimes d_{A_0}^{+,*}v + b\otimes b\otimes v\otimes v, \end{aligned} \tag{8.39}$$

where $b := (A - A_0)|_U$ and $\{\alpha,\beta\} := (\alpha\wedge\beta + \beta\wedge\alpha)^+$.

Proof. We work over the open set U where $E = E_0$. Using $d_A^{+,*}v = -*d_Av$ and $d_A^{+,*}v = -*d_{A_0}v - *(b\wedge v)$, we observe that

$$
\begin{aligned}
d_A^{+,*}v\wedge d_A^{+,*}v = d_{A_0}^{+,*}v\wedge d_{A_0}^{+,*}v - *(b\wedge v)\wedge d_{A_0}^{+,*}v - d_{A_0}^{+,*}v\wedge *(b\wedge v)\\
+ *(b\wedge v)\wedge *(b\wedge v).
\end{aligned}
\tag{8.40}
$$

By equation (8.10) we see that

$$
d_A^+d_A^{+,*}v = d_{A_0}^+d_{A_0}^{+,*}v - (d_{A_0}^+ *(b\wedge v)^+ - (b\wedge *d_{A_0}v)^+ + (b\wedge *(b\wedge v))^+. \tag{8.41}
$$

Noting that $v' = v - v_0$, we obtain the following difference between quadratic terms,

$$
\begin{aligned}
&d_{A_0}^{+,*}v\wedge d_{A_0}^{+,*}v - d_{A_0}^{+,*}v_0\wedge d_{A_0}^{+,*}v_0\\
&= d_{A_0}^{+,*}v'\wedge d_{A_0}^{+,*}v' + d_{A_0}^{+,*}v_0\wedge d_{A_0}^{+,*}v' + d_{A_0}^{+,*}v'\wedge d_{A_0}^{+,*}v_0.
\end{aligned}
\tag{8.42}
$$

Subtracting equation (8.4) from equation (8.3) and using the identities (8.40), (8.41), and (8.42), we obtain

$$
\begin{aligned}
&d_{A_0}^+d_{A_0}^{+,*}v' + (d_{A_0}^{+,*}v'\wedge d_{A_0}^{+,*}v')^+ + (d_{A_0}^{+,*}v_0\wedge d_{A_0}^{+,*}v')^+ + (d_{A_0}^{+,*}v'\wedge d_{A_0}^{+,*}v_0)^+\\
&= w - w_0 + d_{A_0}^+ *(b\wedge v) + (b\wedge *d_{A_0}v)^+ - (b\wedge *(b\wedge v))^+.
\end{aligned}
$$

This gives the desired result. □

We can now complete the proofs of our main gluing-map continuity results.

Proof of Theorem 8.1. The first step is to show that the sequence of connections $A_\alpha + a_\alpha$ converges to $A_0 + a_0$ in $L^2_{\ell,A_0,\mathrm{loc}}(X\setminus\mathbf{x})$, when $2\le\ell\le k$; Lemma 8.17 addresses the case $\ell = 1$.

Let $U''\Subset U'\Subset U\Subset X\setminus\mathbf{x}$ be precompact open subsets: we will show that $A_\alpha + a_\alpha$ converges to A_0+a_0 in $L^2_{\ell,A_0}(U'')$. Observe that $v'_\alpha := v_\alpha - v_0$ is a solution to equation (4.3) with A replaced by A_0, α replaced by $d_{A_0}^{+,*}v_0$, and w replaced by the right-hand side, say w'_α, of identity (8.39). Then applying Corollary 4.14 to equation (4.3), with the preceding replacements, provides the estimate

$$
\|v_\alpha - v_0\|_{L^2_{\ell+1,A_0}(U'')} \le Q_\ell(\|w'_\alpha\|_{L^2_{\ell-1,A_0}(U')}, \|v_\alpha - v_0\|_{L^2(U')}),\quad \ell\ge 2. \tag{8.43}
$$

The positive, real coefficients of the universal polynomial Q_ℓ depend at most on A_0, v_0, ℓ, U', U'', and the metric g on X; also, $Q(0,0) = 0$. We have already seen, from Lemma 8.17, that $\|v_\alpha - v_0\|_{L^2(U)}$ converges to zero. From equations (8.28) and (8.39),

$$
\begin{aligned}
&\|w'_\alpha\|_{L^2_{\ell-1,A_0}(U')}\\
&\le \|b_\alpha\otimes v_\alpha\otimes d_{A_0}^{+,*}v_\alpha\|_{L^2_{\ell-1,A_0}(U')} + \|b_\alpha\otimes b_\alpha\otimes v_\alpha\otimes v_\alpha\|_{L^2_{\ell-1,A_0}(U')}\\
&\quad + \|\Pi_{A_\alpha,\mu}u_\alpha - \Pi_{A_0,\mu}u_0\|_{L^2_{\ell-1,A_0}(U')} + \|F_{A_\alpha}^+ - F_{A_0}^+\|_{L^2_{\ell-1,A_0}(U')},
\end{aligned}
\tag{8.44}
$$

where u_α, u_0 are defined by equation (8.33) and b_α by (8.36). The Sobolev multiplication theorems and definition (8.36) imply that

$$\begin{aligned}
&\|b_\alpha \otimes v_\alpha \otimes d_{A_0}^{+,*} v_\alpha\|_{L^2_{\ell-1,A_0}(U')} + \|b_\alpha \otimes b_\alpha \otimes v_\alpha \otimes v_\alpha\|_{L^2_{\ell-1,A_0}(U')} \\
&\le \begin{cases} C\|A_\alpha - A_0\|_{L^2_3(U')}\|v_\alpha\|^2_{L^\infty \cap L^2_{2,A_0}(U')}, & \ell = 2, \\ C\|A_\alpha - A_0\|_{L^2_\ell(U')}\|v_\alpha\|^2_{L^2_{\ell,A_0}(U')}, & \ell \ge 3, \end{cases}
\end{aligned} \tag{8.45}$$

for large enough α (to bound quadratic terms in $\|A_\alpha - A_0\|_{L^2_\ell(U')}$ by $\|A_\alpha - A_0\|_{L^2_\ell(U')}$) while

$$\|F^+_{A_\alpha} - F^+_{A_0}\|_{L^2_{\ell-1,A_0}(U')} \le C\|A_\alpha - A_0\|_{L^2_\ell(U')}. \tag{8.46}$$

Using Lemma 6.9, choose a sequence of $\mathbf{x}$-good cutoff functions ψ_n on X, so

- $\operatorname{supp}\psi_n \Subset X \setminus \mathbf{x}$,
- $\psi_n = 1$ on U, and
- ψ_n converges pointwise to the constant 1 on $X \setminus \mathbf{x}$.

We now exploit the fact that, for all α sufficiently large, the sequence A_α is uniformly $L^2_{k,A_0}(U)$ close to A_0 over the precompact open subset U: with this observation, our standard estimates in Lemma 6.7 for eigenvectors of $d^+_{A_\alpha} d^{+,*}_{A_\alpha}$ and $d^+_{A_0} d^{+,*}_{A_0}$ yield the first inequality below (the second following by definition of the ψ_n),

$$\begin{aligned}
\|\Pi_{A_\alpha,\mu} u_\alpha - \Pi_{A_0,\mu} u_0\|_{L^2_{\ell-1,A_0}(U')} &\le C\|\Pi_{A_\alpha,\mu} u_\alpha - \Pi_{A_0,\mu} u_0\|_{L^2(U)} \\
&\le C\|\psi_n(\Pi_{A_\alpha,\mu} u_\alpha - \Pi_{A_0,\mu} u_0)\|_{L^2(X)},
\end{aligned} \tag{8.47}$$

where $u_\alpha = u(A_\alpha)$, $u_0 = u(A_0)$ are defined by (8.33), and $C = C([A_0], \mu, U, U', \ell, \alpha_0, g)$.

Combining estimates (8.43), (8.44), (8.45), (8.46), and (8.47) yields the bound, for $\ell \ge 2$,

$$\begin{aligned}
&\|v_\alpha - v_0\|_{L^2_{\ell+1,A_0}(U'')} \\
&\le Q_\ell \Big(\|v_\alpha - v_0\|_{L^2_{2,A_0}(U')} + \|A_\alpha - A_0\|_{L^2_{\ell,A_0}(U')} \\
&\qquad + \|\psi_n(\Pi_{A_\alpha,\mu} u_\alpha - \Pi_{A_0,\mu} u_0)\|_{L^2(X)}, \|v_\alpha - v_0\|_{L^2(U')} \Big).
\end{aligned} \tag{8.48}$$

Lemma 8.17 — which we use to get convergence-to-zero for the terms $\|v_\alpha - v_0\|_{L^2_{2,A_0}(U')}$ — and the limit result (8.38) (given our choice of cutoff functions ψ_n) imply that the right-hand side of (8.48) converges to zero as $\alpha, n \to \infty$. Hence, the sequence v_α converges to v_0 in $L^2_{\ell+1,A_0}(U'')$. The definition of the a_α and a_0 implies that the sequence a_α converges to a_0 in $L^2_{\ell,A_0}(U'')$. Consequently, the sequence of connections $A_\alpha + a_\alpha$ converges to $A_0 + a_0$ in $L^2_{\ell,A_0}(U'')$. We conclude that $A_\alpha + a_\alpha$ converges to $A_0 + a_0$ in $L^2_{k,A_0,\mathrm{loc}}(X \setminus \mathbf{x})$, as desired.

The second (and final) step is to show that the sequence of measures $|F(A_\alpha + a_\alpha)|^2$ converges to $|F(A_0 + a_0)|^2 + 8\pi^2 \sum_{x \in \mathbf{x}} \delta_x$ on X. Remark that, by hypothesis, the sequence of measures $|F(A_\alpha)|^2$ converges to $|F(A_0)|^2 + 8\pi^2 \sum_{x \in \mathbf{x}} \delta_x$ on X. Over $X \setminus \mathbf{x}$, the sequence of functions $|F(A_\alpha + a_\alpha)|^2$ converges in C^∞ to $|F(A_0 + a_0)|^2$ so, for any $f \in C^0(X)$ and $r > 0$,

$$\lim_{\alpha \to \infty} \frac{1}{8\pi^2} \int_{X \setminus \cup_i B(x_i, r)} f|F(A_\alpha + a_\alpha)|^2 \, dV = \frac{1}{8\pi^2} \int_{X \setminus \cup_i B(x_i, r)} f|F(A_0 + a_0)|^2 \, dV. \tag{8.49}$$

It remains to consider the corresponding limits over the balls $B(x_i, r)$.

We shall adapt the argument of [8, p. 165] and consider the Chern-Simons invariants of the restrictions of the extended anti-self-dual connections $A_0 + a_0$, and $A_\alpha + a_\alpha$ to three-spheres $S(x_i, r) = \partial B(x_i, r)$, where r is a small positive constant. Recall that if B is a connection on the product U(2) bundle over a three-manifold Y, then its Chern-Simons invariant is given by [8, p. 164]

$$\mathrm{CS}_Y(B) := \frac{1}{8\pi^2}\int_Y \mathrm{tr}\left(dB + \frac{2}{3}B\wedge B\wedge B\right).$$

If A is any extension of B to a connection over a four-manifold Z with boundary Y, then

$$\mathrm{CS}_Y(B) = -\frac{1}{8\pi^2}\int_Z \mathrm{tr}(F_A\wedge F_A) \pmod{\mathbb{Z}}. \tag{8.50}$$

Because $(|F_A^+|^2 - |F_A^+|^2)\,dV = -\mathrm{tr}(F_A\wedge F_A)$ for any connection A, we see that

$$\begin{aligned}
&\lim_{\alpha\to\infty}\frac{1}{8\pi^2}\int_{B(x_i,r)}\left(|F^-(A_\alpha+a_\alpha)|^2 - |F^+(A_\alpha+a_\alpha)|^2\right)\,dV\\
&= \lim_{\alpha\to\infty}\mathrm{CS}_{S(x_i,r)}(A_\alpha+a_\alpha) \pmod{\mathbb{Z}}\\
&= \mathrm{CS}_{S(x_i,r)}(A_0+a_0)\\
&= \frac{1}{8\pi^2}\int_{B(x_i,r)}\left(|F^-(A_0+a_0)|^2 - |F^+(A_0+a_0)|^2\right)\,dV \pmod{\mathbb{Z}}.
\end{aligned} \tag{8.51}$$

By construction (7.4) of the solutions $a_\alpha = d_{A_\alpha}^{+,*}v_\alpha$, we have $F^+(A_\alpha + a_\alpha) = \Pi_{A_\alpha,\mu}F^+(A_\alpha + a_\alpha)$ and so Lemma 6.7 implies that

$$\|F^+(A_\alpha+a_\alpha)\|_{C^0(X)} \le C\|F^+(A_\alpha+a_\alpha)\|_{L^2(X)},$$

where C is a constant independent of α by Lemma 6.7 and hypothesis (8.6). Therefore, the Lebesgue bounded convergence theorem yields

$$\lim_{\alpha\to\infty}\frac{1}{8\pi^2}\int_{B(x_i,r)}|F^+(A_\alpha+a_\alpha)|^2\,dV = \frac{1}{8\pi^2}\int_{B(x_i,r)}|F^+(A_0+a_0)|^2\,dV,$$

and so the preceding limit identity, the fact that $|F_A|^2 = |F_A^-|^2 + |F_A^+|^2$ for any connection A, and (8.51) imply that

$$\begin{aligned}
&\lim_{\alpha\to\infty}\frac{1}{8\pi^2}\int_{B(x_i,r)}|F(A_\alpha+a_\alpha)|^2\,dV\\
&\lim_{\alpha\to\infty}\frac{1}{8\pi^2}\int_{B(x_i,r)}\left((|F^-(A_\alpha+a_\alpha)|^2 - |F^+(A_\alpha+a_\alpha)|^2) + 2|F^+(A_\alpha+a_\alpha)|^2\right)\,dV\\
&= \frac{1}{8\pi^2}\int_{B(x_i,r)}\left((|F^-(A_0+a_0)|^2 - |F^+(A_0+a_0)|^2) + 2|F^+(A_0+a_0)|^2\right)\,dV\\
&= \frac{1}{8\pi^2}\int_{B(x_i,r)}|F(A_0+a_0)|^2\,dV \pmod{\mathbb{Z}}.
\end{aligned} \tag{8.52}$$

Note that $F(A_\alpha + a_\alpha) = F(A_\alpha) + d_{A_\alpha} a_\alpha + a_\alpha \wedge a_\alpha$, so we have the estimate

$$\begin{aligned}&\int_X \left||F(A_\alpha + a_\alpha)|^2 - |F(A_\alpha)|^2\right| \, dV \\ &\le 2\int_X (F(A_\alpha), d_{A_\alpha} a_\alpha + a_\alpha \wedge a_\alpha) \, dV + 2\int_X |d_{A_\alpha} a_\alpha + a_\alpha \wedge a_\alpha|^2 \, dV.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_X \left||F(A_\alpha + a_\alpha)|^2 - |F(A_\alpha)|^2\right| \, dV \\ &\quad \le 2(\|F(A_\alpha)\|_{L^2(X)} + \|a_\alpha\|_{L^2_{1,A_\alpha}(X)})\|a_\alpha\|_{L^2_{1,A_\alpha}(X)}.\end{aligned}$$

The terms $\|F(A_\alpha)\|^2_{L^2(X)}$ are bounded by a universal constant times $\|F(A_0)\|^2_{L^2(X)} + 8\pi^2\ell$. The terms $\|a_\alpha\|_{L^2_{1,A_\alpha}(X)}$ are bounded by a universal constant times $\|F^+(A_\alpha)\|_{L^{\sharp,2}(X)}$, by our *a priori* estimates of Lemma 7.4 for our solutions to the anti-self-dual equation (8.1). Similar remarks apply to the connections A_0 and $A_0 + a_0$, so we obtain

$$\begin{aligned}&\frac{1}{8\pi^2}\int_X \left||F(A_\alpha + a_\alpha)|^2 - |F(A_\alpha)|^2\right| \, dV \le \tfrac{1}{4}, \\ &\frac{1}{8\pi^2}\int_X \left||F(A_0 + a_0)|^2 - |F(A_0)|^2\right| \, dV \le \tfrac{1}{4},\end{aligned} \tag{8.53}$$

as our hypotheses (8.6) imply that the factors $\sup_\alpha \|F^+(A_\alpha)\|_{L^2(X)}$ and $\|F^+(A_0)\|_{L^2(X)}$ are small. Since

$$\lim_{\alpha\to\infty} \frac{1}{8\pi^2}\int_{B(x_i,r)} |F(A_\alpha)|^2 \, dV = \frac{1}{8\pi^2}\int_{B(x_i,r)} |F(A_0)|^2 \, dV + \kappa_i, \tag{8.54}$$

it follows from the limit (8.52), the estimates (8.53), and the limit (8.54) that

$$\begin{aligned}&\lim_{\alpha\to\infty} \frac{1}{8\pi^2}\int_{B(x_i,r)} |F(A_\alpha + a_\alpha)|^2 \, dV \\ &\quad = \frac{1}{8\pi^2}\int_{B(x_i,r)} \left(|F(A_0 + a_0)|^2 \, dV + 8\pi^2\kappa_i\delta_{x_i}\right).\end{aligned} \tag{8.55}$$

Finally, observe that for all α and $f \in C^0(X)$ we have

$$\frac{1}{8\pi^2}\int_{B(x_i,r)} |f(x_i) - f(x)||F(A_\alpha + a_\alpha)|^2 \, dV(x) \le \kappa \sup_{x\in B(x_i,r)} |f(x_i) - f(x)|, \tag{8.56}$$

where $\kappa = -\frac{1}{4}p_1(\mathfrak{g}_E)$. By choosing a small enough constant $r > 0$, we can make the right-hand side of (8.56) arbitrarily small, uniformly with respect to α. Hence, combining the limit identities (8.49) and (8.55) with the estimate (8.56) yields

$$\lim_{\alpha\to\infty} \frac{1}{8\pi^2}\int_X f|F(A_\alpha + a_\alpha)|^2 \, dV = \frac{1}{8\pi^2}\int_X f|F(A_0 + a_0)|^2 \, dV + \sum_{i=1}^m \kappa_i f(x_i),$$

as required, where $\mathbf{x} \in \mathrm{Sym}^\ell(X)$ is represented by the m-tuple $(x_1, \dots, x_m)$. The existence of the limit asserted above follows from its existence mod $\mathbb{Z}$ via the application of the Chern-Simons identity (8.50) in (8.51) and the curvature estimates (8.53). $\square$

9. Diffeomorphisms of moduli spaces defined by paths of metrics

Our goal in this section is to prove Theorem 1.1. defined by nearby metrics need not be diffeomorphic, no matter how close the metrics, it is worth pinpointing the precise nature of the difficulty. Thus, suppose $f : M \to [t_1, t_2]$ is a C^{r+1} map of compact manifolds with boundary, where f has no critical points and $f(\partial M) = \{t_1, t_2\}$. Then there is a C^r diffeomorphism $F : f^{-1}(t_1)\times[t_1, t_2] \cong M$ such that $f\circ F = \pi_2$, where $\pi_2 : f^{-1}(t_1)\times[t_1, t_2] \to [t_1, t_2]$ is the projection [31, Theorem 6.2.2]. The problem is that we cannot directly apply this standard result to a projection $\pi : M^{w,*}_\kappa(X, \gamma) \to [t_1, t_2]$, because $M^{w,*}_\kappa(X, \gamma)$ is not compact. Now Theorem 6.2.2 in [31] is true for non-compact manifolds M under the extra hypothesis that M has a *complete* Riemannian metric for which $|\operatorname{grad} f|$ is bounded; without this assumption there are simple counterexamples [31, Exercise 6.2.10]. In our situation, the L^2 metric might appear to be the obvious choice, but this is certainly not complete [29]. On the other hand we suspect that the L^2_1 metric is complete and, though we shall not prove this, our proof of Theorem 1.1 does indeed make essential use of L^2_1 distance functions at a crucial point.

9.1. **Vector fields and diffeomorphisms.** As the proof of Theorem 1.1 is rather involved, it is helpful to begin with the relatively simple underlying idea. Let Ξ be a C^∞ vector field on a manifold $\mathfrak{M}$. Let $D_\Psi \subset \mathfrak{M}\times\mathbb{R}$ be the set of points $(m, t) \in \mathfrak{M}\times\mathbb{R}$ such that there exists an integral curve $\Psi(m, \cdot) : I_m \to \mathfrak{M}$ of Ξ, where $I_m \subset \mathbb{R}$ is an open interval with $0 \in I_m$ and $\Psi(m, 0) = m$. Recall that Ξ is *complete* if $D_\Psi = \mathfrak{M}\times\mathbb{R}$; if $\mathfrak{M}$ is compact, then Ξ is necessarily complete. We shall make use of the following basic existence result for the flow of a vector field on a Banach manifold:

Theorem 9.1. [1, Theorem 4.1.13 & Proposition 4.1.17] *Let Ξ be a C^∞ vector field on a manifold $\mathfrak{M}$ and let $D_\Psi \subseteq \mathfrak{M}\times\mathbb{R}$ be as above. Then*

1. $\{0\}\times\mathfrak{M} \subset D_\Psi$.
2. *D_Ψ is an open subset of $\mathfrak{M}\times\mathbb{R}$.*
3. *There is a unique C^∞ map $\Psi : D_\Psi \to \mathfrak{M}$, $(m, t) \mapsto \Psi(m, t)$, such that $\Psi(m, \cdot)$ is an integral curve for Ξ through m, for all $m \in \mathfrak{M}$.*
4. *Suppose there are constants $T^- < 0 < T^+$ and an open interval $I = (T^+, T^-)$ such that $\mathfrak{M}\times I \subset D_\Psi$. If a map $\Psi_t : \mathfrak{M} \to \mathfrak{M}$ is defined by $m \mapsto \Psi(m, t)$, $t \in I$, then $\Psi_t(\mathfrak{M})$ is an open subset of $\mathfrak{M}$ and Ψ_t is a diffeomorphism from $\mathfrak{M}$ onto its image.*

Proof. The first three assertions simply restate the corresponding assertions of Proposition 4.1.17 in [1]. The last assertion follows from Theorem 4.1.13 and Definition 4.1.10 (iv) in [1]. □

Let $\mathcal{P}$ be a Banach manifold, let $\pi : \mathfrak{M} \to \mathcal{P}$ be a C^∞ map, and let $\gamma : I \to \mathcal{P}$ be a C^∞ curve with $\gamma(0) = p_0$ and $I \subseteq \mathbb{R}$ an open interval. Let $\gamma^{-1} : C = \gamma(I) \to I$ be the diffeomorphism $\gamma(t) \mapsto t$. Assume that every $p \in C \subset \mathcal{P}$ is a *regular value* of π, so that $\pi^{-1}(C) \subset \mathfrak{M}$ is a C^∞ submanifold. We thus obtain a function

$$\gamma^{-1}\circ\pi : \pi^{-1}(C) \subset \mathfrak{M} \to \mathbb{R}, \quad m \mapsto \gamma^{-1}(\pi(m)).$$

Require that $\mathfrak{M}$ have a Riemannian metric $\langle\,,\,\rangle$, so we can define a gradient vector field $\Xi := \operatorname{grad}(\gamma^{-1}\circ\pi)$ by

$$\langle\,\cdot\,, \operatorname{grad}(\gamma^{-1}\circ\pi)\rangle := d(\gamma^{-1}\circ\pi).$$

Let $\Psi : D_\Psi \subseteq \pi^{-1}(C) \times I \to \mathfrak{M}$ be the corresponding gradient flow and assume

$$D_\Psi \supset \pi^{-1}(p_0) \times I. \tag{9.1}$$

Then, *provided condition* (9.1) *holds*, a standard argument (see the proof of Theorem 6.2.2 in [31]) implies that the map

$$\Psi : \pi^{-1}(p_0) \times I \to \pi^{-1}(C), \quad (m,t) \mapsto \Psi(m,t), \tag{9.2}$$

is a diffeomorphism such that

$$(\gamma^{-1} \circ \pi) \circ \Psi = \pi_2, \tag{9.3}$$

where $\pi_2 : \pi^{-1}(p_0) \times I \to I$ is projection onto the second factor. The commutativity condition (9.3) implies that the map (9.2) restricts to a diffeomorphism

$$\Psi_t = \Psi(\cdot, t) : \pi^{-1}(p_0) \cong \pi^{-1}(p_t), \quad \text{for } t \in I, \text{ where } p_t = \gamma(t), \tag{9.4}$$

of the fibers of the map $\pi : \pi^{-1}(C) \subset \mathfrak{M} \to C \subset \mathcal{P}$. In summary:

Proposition 9.2. *Let $\mathfrak{M}$, $\mathcal{P}$ be C^∞ manifolds, let $\mathfrak{M}$ have a Riemannian metric, let $\pi : \mathfrak{M} \to \mathcal{P}$ be a C^∞ map, and let $\gamma : I \to \mathcal{P}$ be a C^∞ curve, where $I \subseteq \mathbb{R}$ is an open interval about the origin, $\gamma(I) = C \subset \mathcal{P}$, and $\gamma(0) = p_0$. If*

- *The map π is transverse to C, and*
- *The domain $D_\Psi \subseteq \pi^{-1}(C) \times I$ of the flow Ψ of the gradient vector field Ξ of $\gamma^{-1} \circ \pi|_{\pi^{-1}(C)} : \pi^{-1}(C) \to \mathbb{R}$ contains $\pi^{-1}(p_0) \times I$,*

then Ψ yields the diffeomorphisms (9.2) *and* (9.4).

9.2. **Existence of the diffeomorphism, I: transversality and definition of the gradient vector field.** We shall prove the first part of Theorem 1.1, which asserts the existence of the diffeomorphism $M_\kappa(S^4, g_0) \cong M_\kappa(S^4, g)$ for a metric g which is C^1-close enough to the standard metric g_0, by showing that we can satisfy the hypotheses of Proposition 9.2 when $X = S^4$, $\mathcal{P} \subset \mathrm{Met}(X)$ is a small enough C^1-open neighborhood of g_0, $\mathfrak{M} = M_\kappa(X, C)$ with the L^2 metric, π is the restriction of the projection $\mathcal{B}_\kappa(X) \times \mathcal{P} \to \mathcal{P}$, and $\gamma(t) = (1-t)g_0 + tg$, $t \in \mathbb{R}$. In this subsection we check the first condition of Proposition 9.2 (concerning transversality), while we verify the second one (concerning flow duration) in §9.3. In §9.4 and §9.5 we verify that the diffeomorphism of the fibers extends to a smoothly stratified diffeomorphism $\bar{M}_\kappa(S^4, g_0) \cong \bar{M}_\kappa(S^4, g)$ of their Uhlenbeck compactifications.

9.2.1. *Identification of the gradient vector field.* Let us initially relax the constraint that $(X, g) \cong (S^4, g_0)$ and allow (X, g) to be any closed, oriented, Riemannian, smooth four-manifold, let $\mathrm{Met}(X)$ be the Banach space of C^r Riemannian metrics on X, with $r \geq 1$, let $\gamma : I_g \to \mathrm{Met}(X)$ be a C^∞ path through g. By hypothesis, Ξ is the C^∞ gradient vector field $\mathrm{grad}(\gamma^{-1} \circ \pi)$: as we shall see below, Ξ can be taken to be the restriction of a C^∞ vector field defined on an open subset of $\mathcal{B}_\kappa^{w,*}(X) \times \mathrm{Met}(X)$. For each $([A], g)$ in the domain of Ξ, there is a unique *maximal open interval* $I_{([A],g)} \subset \mathbb{R}$ about the origin and a C^∞ *flow line* $\Psi([A], g, \cdot) : I_{([A],g)} \to \mathcal{B}_\kappa^{w,*}(X) \times \mathrm{Met}(X)$ such that

$$\frac{d\Psi}{dt} = \Xi(\Psi([A], g, t)), \quad t \in I_{([A],g)}, \quad \text{and} \quad \Psi([A], g, 0) = ([A], g). \tag{9.5}$$

We let $D_\Psi \subset \mathcal{B}^{w,*}_\kappa(X)\times\mathrm{Met}(X)\times\mathbb{R}$ be the open subset $\{([A],g,t)\in\mathcal{B}^{w,*}_\kappa(X)\times\mathrm{Met}(X)\times\mathbb{R}: t\in I_{([A],g)}\}$. The *flow* generated by Ξ is the C^∞ map $\Psi: D_\Xi\to\mathcal{B}^{w,*}_\kappa(X)\times\mathrm{Met}(X)$. The projection $\mathcal{A}^{w,*}_\kappa(X,g)\to\mathcal{B}^{w,*}_\kappa(X,g)$ is a principal $\mathcal{G}_E$ bundle, with a canonical connection defined by the L^2 metric induced by the Riemannian metric g on X [8, §5.4] [29], [30], so the vector field Ξ has a unique horizontal lift $\tilde\Xi$ on $\mathcal{A}^{w,*}_\kappa(X)\times\mathrm{Met}(X)$ [34, Proposition 2.1.2]. The integral curve $\tilde\Psi(A,g,\cdot)$ in $\mathcal{A}^{w,*}_\kappa(X)\times\mathrm{Met}(X)$ through (A,g) is the unique lift of the integral curve $\Psi([A],g,\cdot)$ in $\mathcal{B}^{w,*}_\kappa(X)\times\mathrm{Met}(X)$ through $([A],g)$ [34, Proposition 2.3.1]. When no confusion can arise, we shall not distinguish between vector fields and flows on the base and their lifts to the total space.

Let $P_+(g)=\frac{1}{2}(1+*_g):\Lambda^2(T^*X)\to\Lambda^{+,g}(T^*X)$ be orthogonal projection. The parameterized moduli space $\mathfrak{M}^{w,*}_\kappa(X):=\{([A],g)\in\mathcal{B}^{w,*}_\kappa(X)\times\mathrm{Met}(X): P_+(g)F_A=0\}$ is a C^∞ manifold by [8, Theorem 4.3.17] with tangent space at a point $([A],g)$ represented by

$$\{(b,h)\in\operatorname{Ker}d_A^{*,g}\oplus T_g\,\mathrm{Met}(X): P_+(g)d_Ab+(DP_+)_g(h)F_A=0\}. \tag{9.6}$$

We can now identify the gradient vector field Ξ on $M^{w,*}_\kappa(X,C):=\pi^{-1}(C)\subset\mathfrak{M}^{w,*}_\kappa(X)$ induced by the path $\gamma: I_g\to C\subset\mathrm{Met}(X)$.

Lemma 9.3. *Continue the above notation. Then for all $([A],g)\in M^{w,*}_\kappa(X)$,*

$$\Xi_{(A,g)}=\left(-d_A^{+,*,g}(d_A^{+,g}d_A^{+,*,g})^{-1}(DP_+)_g(h)F_A,h\right),\quad h=\dot g\in T_gC.$$

Proof. By definition of π, the differential

$$D_{(A,g)}(\gamma^{-1}\circ\pi): T_{(A,g)}M^{w,*}_\kappa(X,C)\to\mathbb{R},$$

is just the composition of

- The restriction of the linear projection $T_A\mathcal{B}^{w,*}_\kappa(X)\oplus T_g\mathcal{P}\to T_g\mathcal{P}$, $(b,h)\mapsto h$, to a projection $T_{(A,g)}M^{w,*}_\kappa(X,C)\to T_gC$, where $T_A\mathcal{B}^{w,*}_\kappa(X)=\operatorname{Ker}d_A^{*,g}\subset L^2_k(\Lambda^1\otimes\mathfrak{g}_E)$, and
- The differential $(D\gamma^{-1})_g: T_gC\cong\mathbb{R}$.

By definition of $\Xi:=\operatorname{grad}(\gamma^{-1}\circ\pi)$ we have, for all $(b,h)\in T_{(A,g)}M^{w,*}_\kappa(X,C)$,

$$D_{(A,g)}(\gamma^{-1}\circ\pi)(b,h)=\langle(b,h),(\Xi^1_{A,g},\Xi^2_{A,g})\rangle,$$

where we view $T_{(A,g)}M^{w,*}_\kappa(X,C)\subset\operatorname{Ker}d_A^{*,g}\oplus T_g\mathcal{P}$. Thus, since $(D\pi)_{(A,g)}$ is just projection onto the second factor,

$$\langle b,\Xi^1_{A,g}\rangle+\langle h,\Xi^2_{A,g}\rangle=\langle h,\Xi^2_{A,g}\rangle,\quad\text{for all }(b,h)\in T_{(A,g)}M^{w,*}_\kappa(X,C). \tag{9.7}$$

If we take $h=0$ in equation (9.6), we must have $b\in\operatorname{Ker}d_A^{+,g}$ and so the identity (9.7) implies that $\Xi^1_{(A,g)}\in(\operatorname{Ker}d_A^{+,g})^\perp$. Since we also have $\Xi^1_{(A,g)}\in\operatorname{Ker}d_A^{*,g}$, we see that $\Xi^1_{(A,g)}$ is uniquely determined by the condition (9.6). □

9.2.2. *Extension of the gradient vector field.* Though the gradient vector field Ξ is initially defined on $M^{w,*}_\kappa(X,C)$, we shall need to consider its extension to an open neighborhood $\mathcal{B}^{w,*}_\kappa(X,C,\mu)$ of $M^{w,*}_\kappa(X,C)$ in $\mathcal{B}^{w,*}_\kappa(X)\times C$. We use this extension in §9.5 when we consider the vector field Ξ and its induced flow Ψ for approximately anti-self-dual connections produced by cutting off: these considerations are required for the proof of the Uhlenbeck continuity assertion of Theorem 1.1.

For a Riemannian connection A on $\mathfrak{g}_E$ and Riemannian metric g on X, we recall from §6 that $\nu_2[A,g]$ is the least eigenvalue of the Laplacian $d_A^{+,g}d_A^{+,*}$ on $L^2(\Lambda^+\otimes\mathfrak{g}_E)$. Given a small positive constant μ, we define

$$\mathcal{B}_\kappa^{w,*}(X,C,\mu) := \{[A,g]\in\mathcal{B}_\kappa^{w,*}(X)\times C : \nu_2[A,g]>\mu\}, \tag{9.8}$$

Given our C^∞ curve $\gamma : I_g\subset\mathbb{R}\to C\subset\mathrm{Met}(X)$ through g at $t=0$, we can define an induced C^∞ vector field Ξ on $\mathcal{B}_\kappa^{w,*}(X,C,\mu)$ by setting $\Xi_{(A,g)}=(b,h)$, where $h:=\dot g\in S^2(T^*X)$, the Banach space of C^r, symmetric, rank-two, contravariant tensor fields on X, and

$$\begin{aligned} b(A,g) &:= d_A^{+,*,g}w(A,g) \quad \in L_k^2(\Lambda^1\otimes\mathfrak{g}_E). \\ w(A,g) &:= -(d_A^{+,g}d_A^{+,*,g})^{-1}(DP_+)_g(h)F_A \quad \in L_{k+1}^2(\Lambda^{+,g}\otimes\mathfrak{g}_E). \end{aligned} \tag{9.9}$$

Note that $d_A^{*,g}b(A,g)=0$ when $P_+(g)F_A=0$. The vector field Ξ can be regarded as a lift of the vector field $\dot g$ on $C\subset\mathrm{Met}(X)$. It has a C^∞ flow Ψ with domain an open subset $D_\Xi\subset\mathcal{B}_\kappa^{w,*}(X,C,\mu)\times\mathbb{R}$.

Provided we can choose a constant μ and curve C such that

$$M_\kappa^{w,*}(X,C)\subset\mathcal{B}_\kappa^{w,*}(X,C,\mu), \tag{9.10}$$

then the vector field Ξ on $\mathcal{B}_\kappa^{w,*}(X,C,\mu)$ restricts to the gradient field $\mathrm{grad}(\gamma^{-1}\circ\pi)$ on $M_\kappa^{w,*}(X,C)$. When no confusion can arise, we shall denote this extension of the gradient field by Ξ. Plainly, the extended vector field Ξ is *tangent* to $M_\kappa^{w,*}(X,C)$ and thus $M_\kappa^{w,*}(X,C)$ is an *invariant submanifold* of Ξ.

9.2.3. *Eigenvalue variation and the vector field domain.* We need to show that the curve C and constant μ can be chosen so that the condition (9.10) holds: without this condition we would not be able to obtain the necessary bounds on the vector field Ξ.

By definition of $\nu_2[A,g]$, there is a map

$$\nu_2 : \mathfrak{M}_\kappa^w(X)\to\mathbb{R},\quad [A,g]\mapsto\nu_2[A,g], \tag{9.11}$$

which extends continuously to the completion $\bar{\mathfrak{M}}_\kappa^w(X)$ — the closure of $\mathfrak{M}_\kappa^w(X)$ in $\cup_{\ell=0}^\kappa(\mathfrak{M}_\ell^w(X)\times \mathrm{Sym}^\ell(X))$ — by Theorem 6.1. The same holds for the map $\nu_0 : \mathfrak{M}_\kappa^w(X)\to\mathbb{R}$, assigning $[A,g]$ to the least eigenvalue $\nu_0[A,g]$ of the Laplacian $d_A^{*,g}d_A$ [14].

We call g a *regular value* of the projection $\pi_\ell : \mathfrak{M}_\ell^w(X)\to\mathrm{Met}(X)$ if $\mathrm{Ker}\, d_A^{+,g}d_A^{+,*}=0$ for all $[A,g]\in\mathfrak{M}_\ell^w(X)$, that is if $[A]$ is a *smooth point* for all $[A]\in M_\ell^w(X,g)$. We call g a *regular value* of the projection $\pi : \bar{\mathfrak{M}}_\kappa^w(X)\to\mathrm{Met}(X)$ if it is a regular value of the restrictions of π to $\mathfrak{M}_\ell^w(X)$, for each $\ell\in\{0,\dots,\kappa\}$.

If $\nu_2[A,g]>0$ for all $[A,\mathbf{x}]\in\bar M_\kappa^w(X,g)$ then, because $\bar M_\kappa^w(X,g)$ is compact and ν_2 is continuous, there is an open neighborhood of $\bar M_\kappa^w(X,g)$ in $\bar{\mathfrak{M}}_\kappa^w(X)$ on which ν_2 is strictly positive. Therefore, we may choose

- A positive constant μ, and
- A C^1-open neighborhood $\mathcal{P}$ of a regular value $g\in\mathrm{Met}(X)$ of $\pi : \bar{\mathfrak{M}}_\kappa^w(X)\to\mathrm{Met}(X)$,

such that

$$2\mu\le\inf_{[A,g]}\{\nu_2[A,g] : [A,\mathbf{x}]\in\bar M_\kappa^w(X,g)\text{ and }g\in\mathcal{P}\}. \tag{9.12}$$

Points in $M_\ell^w(X,g)$ representing connections which are flat, reducible, or twisted-reducible (see [38]) need not be smooth points, so we must ensure that they are not contained in

$\bar{M}^w_\kappa(X,g)$, noting that they cannot be excluded by simply choosing g to be generic as the hypotheses of the Freed-Uhlenbeck generic metrics theorem do not in general apply to flat or reducible connections [8], [24], [38]. Our main application will be to the case $X = S^4$, so we must allow $w = 0 \in H^2(X;\mathbb{Z})$: for the remainder of this section, we shall assume $w = 0$ and omit it from our notation. The trivial connection $[\Theta, \mathbf{x}] \in \bar{M}_\kappa(X,g)$ is a smooth point if and only if $H^+(X;\mathbb{R}) = 0$: thus, we assume X is simply-connected, so $[\Theta]$ is the only point represented by a flat connection, and assume $H^+(X;\mathbb{R}) = 0$ so that $[\Theta]$ is a smooth point. Reducibles in $M_\kappa(X,g)$ are in one-to-one correspondence with the set $\{\pm\alpha : \alpha \in H^2(X;\mathbb{Z})\}$ [8, Proposition 4.2.15] for any metric g, so we can exclude reducibles by requiring that $H^2(X;\mathbb{Z}) = 0$, that is, if $H^-(X;\mathbb{R}) = 0$. Thus, we are left with a closed, oriented, smooth four-manifold X for which $\pi_1(X) = 0$ and $H^2(X;\mathbb{Z}) = 0$, that is, a homotopy four-sphere. *For simplicity, we therefore assume for the remainder of this section that X is the four-sphere, S^4, and that $\kappa > 0$.*

9.2.4. *Bochner formulas and the vector field domain.* When $X = S^4$ with its standard round metric g of radius one we can alternatively use a simple Bochner argument to show that we can choose a positive constant μ and C^1-open neighborhood $\mathcal{P}$ of g_0 so that the conditions of (9.12) are satisfied.

Lemma 9.4. *Let M be a positive constant. Then there is a positive constant δ such that the following holds. For any metric g on S^4 such that $\|g - g_0\|_{C^2(S^4)} \le M$, where g_0 is the standard metric, and*

$$\int_{S^4} (|W_g^+|^2 + |R_g - 1|^2)\, dV_g \le \delta^2,$$

the Laplacian $d_A^{+,g} d_A^{*,g}$ has first eigenvalue greater than or equal to $1/12$ for all $[A] \in M_\kappa(S^4, g)$.

Proof. From the Bochner formula (6.2) we have $d_{A_0}^{+,g_0} d_{A_0}^{*,g_0} = \frac{1}{2}\nabla_{A_0}^{*,g}\nabla_{A_0} + \frac{1}{6}$, since $W_{g_0}^+ = 0$, $R_{g_0} = 1$, and $F_{A_0}^{+,g_0} = 0$, so $\nu_2[A_0, g_0] \ge \frac{1}{6}$ for all $[A_0] \in M_\kappa(S^4, g_0)$. For another metric g and connection A such that $F_A^{+,g} = 0$, we may write

$$\begin{aligned}(d_A^{+,g} d_A^{*,g} v, v)_{L^2} &= \tfrac{1}{2}(\nabla_A^{*,g}\nabla_A v, v)_{L^2} + \tfrac{1}{6}(v,v)_{L^2} \\ &\quad - (\{W_g^+, v\}, v)_{L^2} + \tfrac{1}{6}((R_g - 1)v, v)_{L^2}.\end{aligned} \tag{9.13}$$

Using Lemmas 6.5 and 6.6, we see that the second-last term in (9.13) is bounded by

$$\begin{aligned}|(\{W_g^+, v\}, v)_{L^2}| &\le \|W_g^+\|_{L^2}\|v\|_{L^4}^2 \le c\|W_g^+\|_{L^2}\|v\|_{L^2_{1,A}}^2 \\ &\le c\|W_g^+\|_{L^2}(\|d_A^{*,g} v\|_{L^2}^2 + \|v\|_{L^2}^2) \\ &= c\|W_g^+\|_{L^2}((d_A^{+,g} d_A^{*,g} v, v)_{L^2} + \|v\|_{L^2}^2).\end{aligned}$$

In the same vein, the last term in (9.13) is bounded by

$$|\tfrac{1}{6}((R_g - 1)v, v)_{L^2}| \le c\|R_g - 1\|_{L^2}((d_A^{+,g} d_A^{*,g} v, v)_{L^2} + \|v\|_{L^2}^2).$$

By combining these estimates we see that

$$\begin{aligned}(d_A^{+,g}d_A^{*,g}v,v)_{L^2} \geq \tfrac{1}{2}(\nabla_A^{*,g}\nabla_A v,v)_{L^2} + \tfrac{1}{6}(v,v)_{L^2} \\ - c(\|W_g^+\|_{L^2} + \|R_g - 1\|_{L^2})(d_A^{+,g}d_A^{*,g}v,v)_{L^2} \\ - c(\|W_g^+\|_{L^2} + \|R_g - 1\|_{L^2})(v,v)_{L^2},\end{aligned}$$

and so

$$\begin{aligned}\nu_2[A,g] &= \inf_{v\neq 0}\frac{(d_A^{+,g}d_A^{*,g}v,v)_{L^2}}{(v,v)_{L^2}} \\ &\geq \frac{\frac{1}{6} - c(\|W_g^+\|_{L^2} + \|R_g - 1\|_{L^2})}{1 + c(\|W_g^+\|_{L^2} + \|R_g - 1\|_{L^2})}.\end{aligned}$$

The conclusion follows for small enough δ. □

9.2.5. *Existence of the diffeomorphism.* We say that a Riemannian metric on X is *good* if the corresponding moduli spaces $M_\ell(X)$, $0 \leq \ell \leq \kappa$, contain no reducible connections or points $[A]$ with $H_A^2 \neq 0$. Plainly, the standard metric g_0 on S^4 is a good metric. Since $X = S^4$, the quotient space $\mathcal{B}_\kappa(X)$ contains no reducibles when $\kappa > 0$. For the remainder of this section, we fix a good metric g_0 on X and require that the curve $C \subset \mathrm{Met}(X)$ be contained in a neighborhood

$$\mathcal{P} := \{g \in \mathrm{Met}(X) : \|g - g_0\|_{C^1(X,g_0)} < \varepsilon \text{ and } \|g - g_0\|_{C^2(X,g_0)} < K\}, \tag{9.14}$$

where ε is small enough that $\mathcal{P}$ satisfies the condition (9.12) for some positive constant μ.

The proof of the existence of the diffeomorphism asserted by Theorem 1.1 is straightforward given the following key result:

Proposition 9.5. *Let $I_g \subset \mathbb{R}$ be the maximal open interval around the origin for which the curve $\gamma : I_g \to C \subset \mathrm{Met}(X)$ is defined. Let $([A], g) \in M_\kappa(X, C)$ and let $I_{([A],g)} \subset \mathbb{R}$ be the maximal open interval around the origin for which the flow-line through $([A], g)$ of the vector field Ξ on $\mathcal{B}_\kappa(X, C, \mu)$ is defined. Then $I_{([A],g)} = I_g$.*

Proof of Assertion (1) of Theorem 1.1, assuming Proposition 9.5. We need to check the hypotheses of Proposition 9.2 when $\mathfrak{M} = M_\kappa(X,C)$, $\gamma(0) = g_0$, and $\mathcal{P}$ is given by (9.14). By our choice of $\mathcal{P}$, the map π is transverse to C, while Proposition 9.5 ensures that the domain D_Ψ of the gradient flow Ψ contains $M_\kappa(X, g_0) \times I_{g_0}$. Then Proposition 9.2 implies that the map

$$\Psi_t : M_\kappa(X, g_0) \to M_\kappa(X, g_t), \quad [A] \mapsto \Psi_t([A], g_0) \equiv \Psi([A], g_0, t), \tag{9.15}$$

defines a C^∞ diffeomorphism for all $t \in I_{g_0}$, as desired. □

9.3. **Existence of the diffeomorphism, II: duration of the flow.** Our task in this subsection is to prove Proposition 9.5: the key point, as always, to derive suitable estimates on the flow.

We remark that it is not possible to obtain global L_1^p estimates on the flow-lines A_t, for the flow $(A_t, g_t) = \Psi(A, g, t)$ defined by the (lift of the) vector field Ξ in (9.9) when $p > 2$, although it is possible for the flow considered by Taubes in [53, §3], so the arguments here differ from those of [53]. Instead, we must make do with only global L_1^2 estimates on the flow-lines $A(t)$.

Given a point $[A,g]\in\mathcal{B}_\kappa(X,C,\mu)$, the maximal interval $I_{([A],g)}=(T^+_{([A],g)},T^-_{([A],g)})\subset\mathbb{R}$ on which the flow-line $\Psi([A],g,t)=(A_t,g_t)$ is defined is limited by the requirements that

- A_t is irreducible, so $d^{*,g_t}_{A_t}d_{A_t}$ has trivial kernel,
- $[A_t]$ is a smooth point, so $d^{+,g_t}_{A_t}d^{+,*,g_t}_{A_t}$ has trivial kernel,
- A_t does not bubble.

The possibility that the Laplacians $d^{*,g_t}_{A_t}d_{A_t}$ and $d^{+,g_t}_{A_t}d^{+,*,g_t}_{A_t}$ may acquire non-trivial kernels is addressed by choosing $X=S^4$ and the family of metrics $\mathcal{P}\subset\mathrm{Met}(X)$ containing the curve C to be a C^1-small neighborhood of the standard metric g_0, as discussed in §9.2. The remaining constraint, that the connections must not bubble, is addressed in this subsection. We first need to establish the existence of a universal estimate for the $L^{\sharp,2}$ norm of the curvature of a g-anti-self-dual connection.

Lemma 9.6. *Let (X,g) be a C^∞, closed, oriented, Riemannian four-manifold and let $\kappa\geq 0$ be an integer. Then there is a constant $c=c(X,g,\kappa)$ such that $\|F_A\|_{L^{\sharp,2}(X,g)}\leq c$ for all $[A]\in M^w_\kappa(X,g)$.*

Proof. By hypothesis and the usual Chern-Weil identity, we have $\|F_A\|_{L^2(X,g)}=\sqrt{8\pi^2\kappa}$, for any $[A]\in M^w_\kappa(X,g)$. The moduli space $M^w_\kappa(X,g)$ has a finite covering by Uhlenbeck neighborhoods of the type given in Definition 3.1, so it suffices to consider one such neighborhood $\mathcal{U}$. Following the notation of Definition 3.1, the compact subset $Y:=X-\cup_{i=1}^m B(x_i,\delta)$ of $X-\mathbf{x}$ has a finite covering by N balls $B(y_j,r)$ such that $B(y_j,4r)\subset Y':=X-\cup_{i=1}^m B(x_i,\tfrac12\delta)$, where $N=N(X,g,\kappa,\delta)$ and $r=r(\mathcal{U})$ is chosen so $\|F_A\|_{L^2(B(y_j,4r),g)}$ is small enough that the usual elliptic estimates hold [REF!!],

$$\|A-\Gamma_j\|_{L^3_{1,\Gamma}(B(y_j,2r),g)}\leq c\|F_A\|_{L^2(B(y_j,4r),g)}, \tag{9.16}$$

where $c=c(g,r)$ and Γ_j is a flat connection over $B(y_j,3r)$ such that $d^*_{\Gamma_j}(A-\Gamma_j)=0$. Thus,

$$\begin{aligned}\|F_A\|_{L^\sharp(B(y_j,r),g)}&\leq c\|F_A\|_{L^3(B(y_j,2r),g)}\quad\text{(Sobolev embedding [12, Lemma 4.1])}\\&\leq c\|A-\Gamma_j\|_{L^3_{\Gamma_j,1}(B(y_j,2r),g)}\\&\leq c\|F_A\|_{L^2(B(y_j,4r),g)}\quad\text{(by estimate (9.16))}\end{aligned}$$

Hence, combining the preceding estimates over each ball $B(y_j,r)$ covering Y, and noting that $B(y_j,4r)$ is contained in Y', for $j=1,\dots,N$, we see that

$$\|F_A\|_{L^\sharp(Y,g)}\leq c\|F_A\|_{L^2(Y',g)},$$

where $c=c(g,\mathcal{U})$.

We now consider each of the balls $B(x_i,2\delta)$ in turn and estimate $\|F_A\|_{L^\sharp(B(x_i,\delta),g)}$. The $L^\sharp$ norm on two-forms is invariant under rescalings of the metric, so we may assume without loss of generality that $\delta=1$. We can now apply the usual bubble-tree compactness argument [13, §4], [55, §5] to conclude that there is an Uhlenbeck neighborhood $\mathcal{U}_i$ of the type given in Definition 3.1 with X replaced by $B(x_i,2\delta)$, noting that no bubbling occurs in the annuli $\Omega(x_i,\tfrac34\delta,2\delta)$. We repeat the preceding argument to bound the $L^\sharp$ norm of F_A on a precompact subset of $B(x_i,2\delta)$ away from the (at most κ_i and at least two if $\kappa_i\geq 2$) points of curvature concentration. This process terminates after at most κ rescalings and so yields the desired $L^\sharp$ bound on F_A. □

We use the preceding universal bound on $\|F_A\|_{L^{\sharp,2}(X,g)}$ to obtain an estimate for the flow $\Psi([A],g,t)$.

Lemma 9.7. *Let (X,g) be a C^∞, closed, oriented, Riemannian four-manifold let $\kappa > 0$ be an integer, and let μ be the positive constant of (9.8). Then there is a positive constant C such that the following holds. Let $[A] \in M_\kappa(X,g)$ and let $\Psi([A],g,t)$ be a flow-line of Ξ on $\mathcal{B}_\kappa(X,C,\mu)$ through $([A],g)$ (at $t=0$). Then, for all $t_1,t_2 \in I_{([A],g)}$, and writing $\Psi([A],g,t) = ([A(t)],g(t)) = ([A_t],g_t)$, the following hold:*

$$\|A(t_2)-A(t_1)\|_{L^4(X,g)} \le C|t_2-t_1| \sup_{t\in[t_1,t_2]} \|\dot g(t)\|_{L^\infty(X,g)}, \tag{1}$$

$$\begin{aligned}\|A(t_2)-A(t_1)\|_{L^2_{1,A,g}(X)} &\le C\Big(1+|t_2-t_1| \sup_{t\in[t_1,t_2]} \|\dot g(t)\|_{L^\infty(X,g)}\Big)\\ &\quad\times |t_2-t_1| \sup_{t\in[t_1,t_2]} \|\dot g(t)\|_{L^\infty(X,g)}.\end{aligned} \tag{2}$$

Proof. Applying Proposition 7.3 to estimate our expression (9.9) for the connection component, $b(A,g)$, of the vector field $\Xi(A,g)$ yields

$$\|b(A,g)\|_{L^2_{1,A,g}(X)} \le C\|(DP_+)_g(h)F_A\|_{L^{\sharp,2}(X,g)}, \tag{9.17}$$

where $C = C(\kappa,\mu,g)$. We first derive an L^4 bound:

$$\begin{aligned}\|A(t_2)-A(t_1)\|_{L^4(X,g)} &\le \int_{t_1}^{t_2} \left\|\frac{dA}{dt}\right\|_{L^4(X,g)} dt\\ &= \int_{t_1}^{t_2} \|b(\Psi(A,g,t))\|_{L^4(X,g)}\,dt\\ &\le c\int_{t_1}^{t_2} \|b(\Psi(A,g,t))\|_{L^2_{1,A_t,g}(X)}\,dt \quad \text{(by Lemma 6.5)}\\ &\le C\int_{t_1}^{t_2} \|(DP_+)_{g_t}(\dot g)F_{A_t}\|_{L^{\sharp,2}(X,g)}\,dt \quad \text{(by (9.17))}\\ &\le C\int_{t_1}^{t_2} \|(DP_+)_{g_t}(\dot g)\|_{L^\infty(X,g)}\|F_{A_t}\|_{L^{\sharp,2}(X,g)}\,dt.\end{aligned} \tag{9.18}$$

Using Lemma 9.6 to bound the factor $\|F_{A_t}\|_{L^{\sharp,2}(X,g)}$ in the last line of (9.18) then yields the desired L^4 bound in Assertion (1). Next, we have

$$\begin{aligned}\|A(t_2)-A(t_1)\|_{L^2_{1,A,g}(X)} &\le \int_{t_1}^{t_2} \left\|\frac{dA}{dt}\right\|_{L^2_{1,A,g}(X)} dt\\ &= \int_{t_1}^{t_2} \|b(\Psi(A,g,t))\|_{L^2_{1,A,g}(X)}\,dt.\end{aligned} \tag{9.19}$$

Note that, applying Lemma 6.5 to obtain the second inequality below, yields

$$
\begin{aligned}
\|\nabla_A b(\Psi(A,g,t))\|_{L^2(X,g)} &\le \|\nabla_{A_t} b(\Psi(A,g,t))\|_{L^2(X,g)} \\
&\quad + \|A_t - A\|_{L^4(X,g)}\|b(\Psi(A,g,t))\|_{L^4(X,g)} \\
&\le \|\nabla_{A_t} b(\Psi(A,g,t))\|_{L^2(X,g)} \\
&\quad + c\|A_t - A\|_{L^4(X,g)}\|b(\Psi(A,g,t))\|_{L^2_{1,A_t,g}(X)} \\
&\le (1 + c\|A_t - A\|_{L^4(X,g)})\|b(\Psi(A,g,t))\|_{L^2_{1,A_t,g}(X)}.
\end{aligned} \tag{9.20}
$$

We now insert the vector-field estimate (9.20) into the preliminary flow estimate (9.19). We then continue exactly as in the proof (9.18) of Assertion (1) to obtain the desired $L^2_{1,A}$ bound in Assertion (2) of the lemma statement. □

We can now proceed to the proof of Proposition 9.5, which estimates the duration of the flow of Ξ through a point $([A],g) \in M_\kappa(X,C)$.

Proof of Proposition 9.5. Using (9.9), where we defined $\Xi(A,g) = (b,h) = (b,\dot g)$, the estimates of Proposition 7.3 and Lemma 9.6 give the universal bound

$$
\|b(A,g)\|_{L^2_{1,A,g}} \le C\|(DP_+)_g(\dot g)F_A\|_{L^{\sharp,2}(X,g)} \le C \sup_{t\in I_g} \|\dot g\|_{C^0(X,g)}, \tag{9.21}
$$

where $C = C(\mathcal{P}, M)$, with $M = \max\{1,\kappa,\mu,\mu^{-1}\}$. Note that $P_+(g(t))F(A(t)) = 0$ for all $t \in I_g$, where $\Psi([A],g,t) = ([A(t)],g(t))$, since $M_\kappa(X,C)$ is an invariant submanifold for Ξ.

Plainly $I_{([A],g)} \subset I_g$. We claim that $I_{([A],g)} \supset I_g$. Suppose $I_{([A],g)} \equiv (T^-_{([A],g)}, T^+_{([A],g)})$ and $I_g = (T^-_g, T^+_g)$, so

$$
T^-_g \le T^-_{([A],g)} < T^+_{([A],g)} \le T^+_g.
$$

Consider the case $T^+_{([A],g)} < T^+_g$. Let $\{t_\beta\} \subset I_{([A],g)}$ be a sequence such that $t_\beta \to t_\infty \equiv T^+_{([A],g)}$. For all $t_\alpha < t_\beta$, Lemma 9.7 yields

$$
\|A(t_\beta) - A(t_\alpha)\|_{L^2_{1,A,g}} \le C|t_\beta - t_\alpha|,
$$

where C is defined by Assertion (2) of Lemma 9.7 and in particular, is independent of α,β. Hence, the sequence $\{A(t_\beta)\}$ is $L^2_{1,A,g}$-Cauchy and so converges to some L^2_1 connection $\tilde A_\infty$ on $\mathfrak{g}_{E_\infty}$ such that $P_+(g_\infty)F(\tilde A_\infty) = 0$. Here, $g_\infty = \lim_{n\to\infty} g(t_\beta) = g(t_\infty)$, where the convergence is C^r. In particular, there is a sequence $\{u_\alpha\} \subset \mathcal{G}_E$ of L^2_{k+1} gauge transformations of E such that the sequence of L^2_k connections, $\{u_\alpha(A(t_\alpha))\}$ on $\mathfrak{g}_E$, converges strongly to an L^2_k connection A_∞ on $\mathfrak{g}_{E_\infty}$ such that $P_+(g_\infty)F(A_\infty) = 0$. The flow line $\Psi([A_\infty], g_\infty, \cdot + t_\infty)$ is defined on a open interval $I_{([A_\infty],g_\infty)}$ about the origin. Thus, $\Psi([A_\infty], g_\infty, \cdot)$ gives a C^∞ extension of the flow-line $\Psi([A],g,\cdot)$ from $(T^-_{([A],g)}, T^+_{([A],g)})$ to an open interval in $\mathbb{R}$ which contains $(T^-_{([A],g)}, T^+_{([A],g)}]$. This contradicts the maximality of $T^+_{([A],g)}$, so we must have $T^+_{([A],g)} = T^+_g$. Similarly, we must have $T^-_{([A],g)} = T^-_g$. This completes the proof. □

Remark 9.8. There is a possibly a simpler proof that the sequence $[A(t_\alpha), g(t_\alpha)] \in M_\kappa(X,C)$ converges to a limit $[A_\infty, g_\infty] \in M_\kappa(X,C)$ (where A_∞ is an L^2_k connection, anti-self-dual with respect to a C^r metric g_∞), as $\alpha \to \infty$. The argument is also used to prove Lemma

9.9, so we include it here. According to the extension of the usual Uhlenbeck compactness theorem given by Proposition 9.1.2 in [8], the sequence $[A(t_\alpha), g(t_\alpha)]$ converges to some $[A_\infty, \mathbf{x}, g_\infty] \in \bar{M}_\kappa(g_\infty) \times \{g_\infty\}$, where $\mathbf{x} \in \mathrm{Sym}^\ell(X)$ is represented by an m-tuple $(x_1, \dots, x_m)$ and the x_i have integer multiplicities $\kappa_i \geq 0$. We claim that $\ell = 0$, so $\kappa_i = 0$ for all i. Write $A_t = A + a_t$, so $F(A_t) = F_A + d_A a_t + a_t \wedge a_t = F_A + d_{A_t} a_t$. By Lemma 9.7 we have $\|a_t\|_{L^2_{1,A_t}(X)} \leq \varepsilon$, for some universal constant $\varepsilon \ll 1$ such that $C|t - t_0| \|\dot{g}\|_{C^0(X,g)} \leq \varepsilon$ for all $t \in I_g \cap [t_0, T_g^+)$, for t_0 close enough to T_g^+. In particular,

$$
\begin{aligned}
\|F(A_t)\|_{L^2(B(x_i,r))} &= \|F_A + d_{A_t} a_t\|_{L^2(B(x_i,r))} \\
&\leq \|F_A\|_{L^2(B(x_i,r))} + \|a_t\|_{L^2_{1,A_t}(B(x_i,r))} \\
&\leq cr^2 \|F_A\|_{L^\infty(B(x_i,r))} + \|a_t\|_{L^2_{1,A_t}(X)},
\end{aligned}
$$

and therefore,

$$
\|F(A_t)\|_{L^2(B(x_i,r))} \leq 2\varepsilon, \quad 0 < r < r_0[A], \quad t \in I_g \cap [t_0, T_g^+), \tag{9.22}
$$

provided $r_0[A]$ is chosen small enough that $cr_0^2 \|F_A\|_{L^\infty(B(x_i,r_0))} \leq \varepsilon$. On the other hand, by Uhlenbeck convergence,

$$
\lim_{\alpha \to \infty} \int_{B(x_i,r)} |F(A(t_\alpha))|^2 \, dV = \int_{B(x_i,r)} |F(A_\infty)|^2 \, dV + 8\pi^2 \kappa_i,
$$

where, as in the proof of Proposition 9.5, we suppose the sequence $\{t_\alpha\} \subset I_{([A],g)}$ converges to T_g^+. Now, for $r_0([A], [A_\infty])$ small enough that $cr_0^2 \|F(A_\infty)\|_{L^\infty(X)} \leq \varepsilon$,

$$
\|F(A_\infty)\|_{L^2(B(x_i,r))} \leq cr^2 \|F(A_\infty)\|_{L^\infty(X)} \leq \varepsilon, \quad 0 < r < r_0([A], [A_\infty]).
$$

Therefore, given $0 < r \leq r_0([A], [A_\infty])$, there is a positive constant $t_r < T_g^+$ such that for all $t \in [t_r, T_g^+)$,

$$
\|F(A_t)\|^2_{L^2(B(x_i,r))} \geq 8\pi^2 \kappa_i - \varepsilon^2,
$$

which contradicts the upper bound obtained in the inequality (9.22) unless $\kappa_i = 0$. Hence, the sequence $[A(t_\alpha), g(t_\alpha)]$ converges to a limit $[A_\infty, g_\infty]$, as desired.

9.4. **Uhlenbeck continuity, I: equality of the singular supports.** It remains to prove Assertion (2) of Theorem 1.1, namely that our diffeomorphism extends continuously to the Uhlenbeck compactification, namely that if $[A_\alpha] \in M_\kappa(X, g)$ is a sequence which converges to $[A_\infty, \mathbf{x}] \in M_{\kappa - \ell'}(X, g) \times \mathrm{Sym}^\ell(X)$, then $[A_\alpha(t)] \in M_\kappa(X, g(t))$ converges to $[A_\infty(t), \mathbf{x}] \in M_{\kappa - \ell}(X, g) \times \mathrm{Sym}^\ell(X)$. There are two parts to the proof, involving quite different arguments:

- Show that $A_\alpha(t)$ converges, modulo gauge transformations, in $L^2_{k,\mathrm{loc}}(X \setminus \mathbf{x})$ to $A_\infty(t)$, and hence that $A_\alpha(t)$ converges in the Uhlenbeck topology to $(A_\infty(t), \mathbf{x}(t))$ for some $\mathbf{x}(t) \in \mathrm{Sym}^\ell(X)$ and any $t \in I_g$;
- Show that $\mathbf{x}(t) = \mathbf{x}$ for any $t \in I_g$.

The second part is by far the easiest to prove, so we begin with that in this subsection, while the proof of the first part is given in §9.5.

We first show that the singular supports are preserved by the cobordism covering a path of metrics.

Lemma 9.9. *Continue the notation and hypotheses of Lemma 9.7. Suppose* $\Psi([A_\alpha],g,t)$ *is a sequence of* C^∞ *flow-lines through* $([A_\alpha],g)$*, at* $t=0$*, with* $[A_\alpha]\in M_\kappa(X,g)$ *converging to a limit* $[A_\infty,\mathbf{x}]\in M_{\kappa-\ell}(X,g)\times \mathrm{Sym}^\ell(X)$*. If* $\Psi([A_\alpha],g,t)=([A_\alpha(t)],g(t))$ *and* $[A_\alpha(t)]$ *converges to* $[B_t,\mathbf{x}_t]\in M_{\kappa-\ell'}(X,g)\times\mathrm{Sym}^{\ell'}(X)$*, then* $\ell'=\ell$ *and* $\mathbf{x}_t=\mathbf{x}$*, for all* $t\in I_g$*.*

Proof. Suppose $\mathbf{x}_t$, $\mathbf{x}$ are represented by the multisets $(x_{1t},\dots,x_{nt})$ and $(x_1,\dots,x_m)$, respectively, of points with positive integer multiplicities. We first claim that

$$\{x_{1t},\dots,x_{nt}\}\subseteq\{x_1,\dots,x_m\}. \tag{9.23}$$

Indeed, if not, there is some $x_{it}\notin\{x_1,\dots,x_m\}$. Suppose δ is a positive constant. Then I_g is covered by finitely many overlapping closed intervals $[t_0,t_0+\delta]$ of length δ and for all t_1,t_2 in such an interval, $[t_0,t_0+\delta]$,

$$\|A_\alpha(t_2)-A_\alpha(t_1)\|_{L^2_{1,A_\alpha(t_1),g}(X)}\le C\delta,$$

where C is a constant independent of α,t_1,t_2,δ. In particular, given any positive constant ε, we may choose δ small enough that $C\delta\le\varepsilon$. It suffices to show that $\mathbf{x}_t=\mathbf{x}$ for $t\in[0,\delta]$, the equality for all $t\in I_g$ following by induction.

We can now apply the argument of Remark 9.8 comparing the curvature of $A_\alpha(t)$, $t\in[0,\delta]$, with $A_\alpha\equiv A_\alpha(0)$ on $B(x_{it},r)$, where $\Psi([A_\alpha],g,t)=([A_\alpha(t)],g(t))$, and r is chosen small enough that the ball $B(x_{it},r)$ contains no points of $\mathbf{x}$. This argument implies that x_{it} has multiplicity zero, a contradiction. Hence, by relabeling the points if necessary, we have $n\le m$ and $x_{it}=x_i$ for $i=1,\dots,n$. By interchanging the roles of $A_\alpha(t)$ and A_α, we see that $m\le n$ and thus $n=m$.

Second, we claim that $\kappa(x_{it})=\kappa(x_i)\equiv\kappa_i$, for all i. We can again apply the argument of Remark 9.8 to deduce that, for all $t\in[0,\delta]$ and all α,

$$\|F(A_\alpha)\|_{L^2(B(x_i,r))}-\varepsilon\le\|F(A_\alpha(t))\|_{L^2(B(x_i,r))}\le\|F(A_\alpha)\|_{L^2(B(x_i,r))}+\varepsilon,$$

and $8\pi^2\kappa_i-\varepsilon\le\|F(A_\alpha)\|^2_{L^2(B(x_i,r))}\le 8\pi^2\kappa_i+\varepsilon$. It follows that $\kappa(x_{it})=\kappa(x_i)$. □

9.5. **Uhlenbeck continuity, II: convergence of the background connections.** The proof of the first part of Uhlenbeck continuity, convergence of the background connections, is slightly involved because of the long chains of estimates required, so we begin by providing a brief map of our strategy:

- Show that the $L^{2,\mathrm{loc}}(X\setminus\mathbf{x})$ difference between the connection components $b(\cdot,\cdot)$ of the vector fields $\Xi(\cdot,\cdot)$ at points $(\chi_0A_\alpha(t),g(t))$ and $(A_\alpha(t),g(t))$ is small, where $\operatorname{supp}\chi_0\subset X\setminus\mathbf{x}$ and $\chi_0=1$ on a large enough precompact open subset of $X\setminus\mathbf{x}$ (Lemmas 9.10 and 9.12).
- Show the L^2-distance between the vectors $\Xi(\Psi(\chi_0A_\alpha,g,t))$ and $\Xi(\Psi(A_\infty,g,t))$ is controlled by the L^2-distance between $\Psi(\chi_0A_\alpha,g,t)$ and $\Psi(A_\infty,g,t)$ (Lemma 9.13): this is is the subject of §9.5.1.
- Use Gronwall's inequality to show the L^2-distance between the flows $\Psi(\chi_0A_\alpha,g,t)$ and $\Psi(A_\infty,g,t)$ is controlled by the L^2-distance between the ODE's initial conditions χ_0A_α and A_∞, plus a small error term (Lemmas 9.14 and 9.15): this is is the subject of §9.5.2.

The concluding arguments are given in §9.5.4. Cut-off connections such as χ_0A_α are defined, given a choice of points in the fibers $\mathrm{Fr}(\mathfrak{g}_E)|_x$, $x\in\mathbf{x}$, by the recipe of §3.3.

9.5.1. *Comparison of gradient vector fields.* We begin by examining the continuity properties of the vector field Ξ with respect to Uhlenbeck limits.

Lemma 9.10. *Let (X,g) be a C^∞, closed, oriented, Riemannian four-manifold and let κ be an integer. Let A, B be L^2_4 g-anti-self-dual connections on $\mathrm{SO}(3)$ bundles $\mathfrak{g}_E$, $\mathfrak{g}_{E'}$, respectively, over X, where $p_1(\mathfrak{g}_E), p_1(\mathfrak{g}_{E'}) \geq -4\kappa$ and $\nu_2[A,g], \nu_2[B,g] \geq 2\mu$. Suppose $U \subset X$ is an open subset such that $E|_U = E'|_U$. Let ψ_0, χ_0 be cutoff functions on X such that $\operatorname{supp}\chi_0 \subset U$, $\operatorname{supp}\psi_0 \subset U$, and $\chi_0 = 1$ on $\operatorname{supp}\psi_0$. Then*

$$\begin{aligned}\|\psi_0(b(B,g) - b(A,g))\|_{L^2(X,g)} &\leq C\|\psi_0(DP_+)_g(h)(F_B - F_A)\|_{L^{4/3}(X,g)} \\ &\quad + p(\|\nabla\psi_0\|_{L^{\sharp,2}_1(X,g)}, \|\chi_0(B-A)\|_{L^2_{1,A,g}(X)}).\end{aligned}$$

where $p(x,y)$ is a polynomial, such that $p(0,0) = 0$, whose coefficients and the constant C depend only $(g, \|h(g)\|_{C^0(X,g)}, \mu^{-1}, \kappa)$.

Proof. The result is almost immediate from the expression (9.9) for $b(A,g)$, using Lemmas 8.5 and 8.7. We then use Lemmas 7.1 and 9.6 to give bounds for the norms

$$\|w(A,g\|_{L^\infty(X,g)} + \|w(A,g\|_{L^2_{2,A}(X,g)} \leq C\|(DP_+)(h)F_A\|_{L^{\sharp,2}(X,g)} \leq C',$$

where $C = C(g,M)$ and $M = \max\{1, \|h(g)\|_{C^0}, \kappa, \mu^{-1}\}$, and similarly for $w(B,g)$; these terms appear as coefficients in the expression provided by Lemmas 8.5 and 8.7. □

Remark 9.11. It is possible, just tedious, to explicitly write out the polynomial $p(\cdot,\cdot)$ in Lemma 9.10: one simply combines Lemmas 8.5 and 8.7.

We can use Lemma 9.10 to estimate the effect of cutting off connections on the vector field Ξ. Recall that cut-off connections such as $\chi_0 A$ are defined, given a choice of points in the fibers $\mathrm{Fr}(\mathfrak{g}_E)|_x$, $x \in \mathbf{x}$, by the recipe of §3.3.

Lemma 9.12. *Continue the notation of Lemma 9.10. Then,*

$$\|\psi_0(b(A,g) - b(\chi_0 A,g))\|_{L^2(X,g)} \leq p(\|\nabla\psi_0\|_{L^{\sharp,2}_1(X,g)}, \|\chi_0(1-\chi_0)A\|_{L^2_{1,A}(X,g)}), \tag{1}$$

$$\begin{aligned}\|\chi_0 b(A,g) - b(\chi_0 A,g)\|_{L^2(X,g)} &\leq p(\|\nabla\psi_0\|_{L^{\sharp,2}_1(X,g)}, \|\chi_0(1-\chi_0)A\|_{L^2_{1,A}(X,g)}) \\ &\quad + \|(1-\psi_0)(\chi_0 b(A,g) - b(\chi_0 A,g))\|_{L^2(X,g)}.\end{aligned} \tag{2}$$

where $p(x,y)$ is a polynomial, with coefficients depending only (g,μ,κ), such that $p(0,0) = 0$, and $\chi_0(1-\chi_0)A$ is regarded as an element of $L^2_k(X, \Lambda^1 \otimes \mathfrak{g}_E)$.

Proof. On $\operatorname{supp}\psi_0$ we have $\chi_0 = 1$ and thus, setting $B = \chi_0 A$ in Lemma 9.10, we have $\chi_0(B - A) = \chi_0(\chi_0 - 1)A$, and $F_A = F_{\chi_0 A}$ on $\operatorname{supp}\psi_0$. Hence, Assertion (1) follows from Lemma 9.10. Next, observe that

$$\begin{aligned}\|\chi_0 b(A,g) - b(\chi_0 A,g)\|_{L^2} &\leq \|\psi_0(\chi_0 b(A,g) - b(\chi_0 A,g))\|_{L^2} \\ &\quad + \|(1-\psi_0)(\chi_0 b(A,g) - b(\chi_0 A,g))\|_{L^2} \\ &= \|\psi_0(b(A,g) - b(\chi_0 A,g))\|_{L^2} \\ &\quad + \|(1-\psi_0)(\chi_0 b(A,g) - b(\chi_0 A,g))\|_{L^2}.\end{aligned}$$

Combining this with Assertion (1) yields the desired bound in Assertion (2). □

Lastly, we shall need the following version of the mean value theorem:

Lemma 9.13. *Let (X,g) be a C^∞, closed, oriented, Riemannian four-manifold. Let M, μ be positive constants and let A be an L^2_4 connection on $\mathfrak{g}_E$, with $\nu_2[A,g] \geq 2\mu$. Then there are positive constants $C = C([A],M,g)$ and $\delta = \delta([A],g)$ with the following significance. If A' is an L^2_2 connection on $\mathfrak{g}_E$, with $\|A'-A\|_{L^{\sharp,2}_{1,A}(X)} \leq M$ and $\|A'-A\|_{L^4(X)} \leq \delta$, then*

$$\|b(A',g) - b(A,g)\|_{L^2(X,g)} \leq C\|A'-A\|_{L^2(X,g)}.$$

Proof. Let G_A denote the Green's operator for the Laplacian $d_A^+ d_A^* \equiv d_A^{+,g} d_A^{+,*,g}$, which has least eigenvalue $\nu_2[A,g] > 0$. Since $\|A'-A\|_{L^4} \leq \delta$, Lemma 6.15 implies that we may choose $\delta > 0$ small enough that $d_{A'}^+ d_{A'}^*$ has least eigenvalue $\nu_2[A',g] \geq \frac{1}{2}\nu_2[A,g] > 0$. Writing $A' = A + a$, we need to bound the difference

$$b(A+a,g) - b(A,g) = d_{A+a}^{+,*} G_{A+a}(DP_+)_g(h)F_{A+a} - d_A^{+,*}G_A(DP_+)_g(h)F_A,$$

that is,

$$\begin{aligned} b(A+a,g) - b(A,g) = &- *(a \wedge G_{A+a}(DP_+)_g(h)F_{A+a}) \\ &+ d_A^{+,*}(G_{A+a} - G_A)(DP_+)_g(h)F_{A+a} \\ &+ d_A^{+,*}G_{A+a}(DP_+)_g(h)(d_A a + a\wedge a), \end{aligned} \tag{9.24}$$

where the difference of Green's operators is given by

$$\begin{aligned} G_{A+a} - G_A &= G_{A+a}(d_A^+ d_A^{+,*} - d_{A+a}^+ d_{A+a}^{+,*})G_A \\ &= G_A(P_+(g)a \wedge d_A^{+,*} - d_A^+ *(a\wedge\cdot))G_A. \end{aligned} \tag{9.25}$$

Recall that $\|a\|_{L^{\sharp,2}_{1,A}} := \|a\|_{L^2_{1,A}} + \|\nabla_A a\|_{L^\sharp}$ [12]. For any $\xi \in \Omega^+(\mathfrak{g}_E)$ we have

$$\begin{aligned} \|G_{A+a}\xi\|_{L^\infty} &\leq \|G_A\xi\|_{L^\infty} + \|G_A P_+(g)(a\wedge d_A^{+,*}G_A\xi)\|_{L^\infty} \\ &\quad + \|G_A d_A^+ *(a \wedge G_A\xi)\|_{L^\infty} \quad \text{(by identity (9.25))} \\ &\leq C\|\xi\|_{L^{\sharp,2}} + C\|a\wedge d_A^{+,*}G_A\xi\|_{L^{\sharp,2}} \\ &\quad + C\|d_A^+ *(a\wedge G_A\xi)\|_{L^{\sharp,2}} \quad \text{(by Corollary 7.2)} \\ &\leq C\|\xi\|_{L^{\sharp,2}} + C\|a\|_{L^2_{1,A}}\|d_A^{+,*}G_A\xi\|_{L^2_{1,A}} \\ &\quad + C(\|a\|_{L^{\sharp,2}_{1,A}}\|G_A\xi\|_{L^\infty} + \|a\|_{L^{2\sharp,4}}\|d_AG_A\xi\|_{L^{2\sharp,4}}). \end{aligned}$$

The second inequality also uses the simple eigenvalue estimate $\|v\|_{L^2} \leq \nu_2[A]^{-1}\|d_A^+ d_A^{+,*}v\|_{L^2}$. The last inequality above follows from the Hölder's inequality and Lemma 6.5, together with the Sobolev multiplication and embedding results of [12, Lemmas 4.1 & 4.3]. Therefore,

$$\|G_{A+a}\xi\|_{L^\infty} \leq C(1 + \|a\|_{L^{\sharp,2}_{1,A}})\|\xi\|_{L^{\sharp,2}}. \tag{9.26}$$

Using the estimate (9.26), the L^2 norm of the *first term* in the expression (9.24) for $b(A+a,g) - b(A,g)$ is thus bounded by

$$\begin{aligned} \|a\wedge G_{A+a}(DP_+)_g(h)F_{A+a}\|_{L^2} &\leq \|a\|_{L^2}\|G_{A+a}(DP_+)_g(h)F_{A+a}\|_{L^\infty} \\ &\leq C\|a\|_{L^2}(1 + \|a\|_{L^{\sharp,2}_{1,A}})\|F_{A+a}\|_{L^{\sharp,2}}. \end{aligned} \tag{9.27}$$

The L^2 norm of the *second term* in identity (9.24), courtesy of identity (9.25), is bounded by

$$\begin{aligned}
&\|d_A^{+,*}(G_{A+a}-G_A)(DP_+)_g(h)F_{A+a}\|_{L^2} \\
&= \|d_A^{+,*}G_A(P_+(g)a\wedge d_A^{+,*}-d_A^+ * (a\wedge\cdot))G_A(DP_+)_g(h)F_{A+a}\|_{L^2} \\
&\le C\|a\wedge d_A^{+,*}G_A(DP_+)_g(h)F_{A+a}\|_{L^{4/3}} + C\|a\wedge G_A(DP_+)_g(h)F_{A+a}\|_{L^2} \\
&\le C\|a\|_{L^2}(\|d_A^{+,*}G_A(DP_+)_g(h)F_{A+a}\|_{L^4} + \|G_A(DP_+)_g(h)F_{A+a}\|_{L^\infty}) \\
&\le C\|a\|_{L^2}\|F_{A+a}\|_{L^{\sharp,2}}.
\end{aligned} \tag{9.28}$$

The first inequality above follows from the estimates $\|\xi\|_{L^2}\le C\|d_A^{+,*}G_A\xi\|_{L^{4/3}}$ and $\|\xi\|_{L^2}\le C\|d_A^{+,*}G_Ad_A^+\xi\|_{L^2}$, for any $\xi\in\Gamma(\Lambda^+\otimes\mathfrak{g}_E)$. The inequality third above follows from the estimates $\|\xi\|_{L^2_{1,A}}\le C\|d_A^{+,*}G_A\xi\|_{L^2}$, while Corollary 7.2 and eigenvalue estimates are again used to give bounds $\|G_A\xi\|_{L^\infty}\le C\|\xi\|_{L^{\sharp,2}}$. Finally, the L^2 norm of the *third term* in (9.24) is bounded by

$$\begin{aligned}
&\|d_A^{+,*}G_{A+a}(DP_+)_g(h)(d_Aa+a\wedge a)\|_{L^2} \\
&\le \|d_A^{+,*}G_A(DP_+)_g(h)(d_Aa+a\wedge a)\|_{L^2} \\
&+ \|d_A^{+,*}(G_{A+a}-G_A)(DP_+)_g(h)(F_{A+a}-F_A)\|_{L^2} \\
&\le C'(\|a\|_{L^2}+\|a\wedge a\|_{L^{4/3}}) + C\|a\|_{L^2}\|(F_{A+a}-F_A)\|_{L^{\sharp,2}}, \\
&\le C'\|a\|_{L^2}(1+\|a\|_{L^4}) + C\|a\|_{L^2}(1+\|a\|_{L^2_{1,A}})\|a\|_{L^{\sharp,2}_{1,A}},
\end{aligned} \tag{9.29}$$

noting that $\|a\wedge a\|_{L^{4/3}}\le\|a\|_{L^4}\|a\|_{L^2}$. The estimate for the second term above follows in the same manner as the second term in (9.24). Combining the estimates (9.27), (9.28), and (9.29) for all terms in (9.24) (and (9.25))yields the desired conclusion. □

9.5.2. *Gronwall's inequality and comparison of flows.* Thus far we have compared the vector fields Ξ over precompact subsets of $X\setminus\mathbf{x}$. We now need to integrate these estimates with respect to time, t, and convert our vector-field estimates into estimates for the connection flows over precompact subsets of $X\setminus\mathbf{x}$. To show that the L^2-distance between the various flows is small when restricted such subsets, we shall use the fact that the 'initial conditions', A_α and A_∞, are close when restricted to precompact subsets. Hence, the key point is the dependence of the ordinary differential equations defining the flows on the initial conditions and for this, as usual, the main tool at our disposal is *Gronwall's inequality*:

Lemma 9.14. [1, Lemma 4.1.7] *Let $f,g:[t_1,t_2]\to\mathbb{R}$ be continuous and non-negative. Suppose there is a constant $M\ge 0$ such that*

$$f(t)\le M+\int_{t_1}^{t_2} f(s)g(s)\,ds.$$

Then, for all $t\in(t_1,t_2)$,

$$f(t)\le M\exp\left(\int_{t_1}^{t} g(s)\,ds\right).$$

We now come to the application of Gronwall's inequality.

Lemma 9.15. *Continue the notation and hypotheses of Lemma 9.10. Let $M < \infty$ be a constant. Given $[A], g$, there is a constant δ such the following holds. Require that the connections A, B be g-anti-self-dual with $\nu_2[A,g] \geq 2\mu$ and $\nu_2[B,g] \geq 2\mu$, that $\|\chi_0 B - A\|_{L^2_{1,A}} \leq \delta$. Let $K = K(\kappa,\mu)$ be the constant in Lemma 9.7 and require that the metric flow $g(t)$, $t \in I_g$ satisfy*

$$K|I_g| \sup_{t\in I_g} \|\dot g(t)\|_{L^\infty(X,g)} \leq \delta, \tag{9.30}$$

where $|I_g|$ is the length of the interval I_g. Then, for all $t \in I_g$, the following hold:

$$\begin{aligned}
&\|\chi_0\Psi(B,g,t) - \Psi(A,g,t)\|_{L^2_{1,A,g}(X)} \\
&\leq \|\chi_0 B - A\|_{L^2_{1,A,g}(X)} + Ct \sup_{s\in I_g} \mathfrak{p}(s) \qquad (1)\\
&\quad + C\int_0^t \|\chi_0\Psi(B,g,s) - \Psi(A,g,s)\|_{L^2(X,g)}\,ds,
\end{aligned}$$

$$\begin{aligned}
&\|\chi_0\Psi(B,g,t) - \Psi(A,g,t)\|_{L^2(X,g)} \qquad (2)\\
&\leq \left(\|\chi_0 B - A\|_{L^2(X,g)} + C|I_g| \sup_{s\in I_g} \mathfrak{p}(s)\right)\exp(-C|t|),
\end{aligned}$$

where $\mathfrak{p}$ is defined by the identity in (9.31) and $C = C([A],M,g)$, and cut-off connections such as $\chi_0 B$ are defined, given a choice of points in the fibers $\mathrm{Fr}(\mathfrak{g}_E)|_x$, $x \in \mathbf{x}$, by the recipe of §3.3.

Proof. Noting that $\Psi(B,g,t) = (B_t,g_t)$ and applying Assertion (2) of Lemma 9.12 yields, for all $t \in I_g$,

$$\begin{aligned}
\|\chi_0 b(B_t,g_t) - b(\chi_0 B_t,g_t)\|_{L^2} &\leq p(\|\nabla\psi_0\|_{L^{\sharp,2}_1}, \|\chi_0(1-\chi_0)B_t\|_{L^2_{1,B_t}}) \\
&\qquad + \|(1-\psi_0)(\chi_0 b(B_t,g_t) - b(\chi_0 B_t,g_t))\|_{L^2} \\
&:= \mathfrak{p}(\chi_0,\psi_0,B,g,t),
\end{aligned} \tag{9.31}$$

where $p(x,y)$ is a polynomial, with coefficients depending only (g,μ,κ), such that $p(0,0) = 0$, and $\chi_0(1-\chi_0)B_t$ is regarded as an element of $L^2_k(X,\Lambda^1\otimes\mathfrak{g}_E)$.

Recalling the form of Gronwall's inequality in Lemma (9.14), we define

$$f(t) := \|\chi_0\Psi(B,g,t) - \Psi(A,g,t)\|_{L^2}, \quad t \in I_g, \tag{9.32}$$

and observe that, as $\Psi(A,g,0) = (A,g)$ and $\Psi(B,g,0) = (B,g)$, the definition of the flow Ψ and the vector field Ξ, together with inequality (9.31), gives

$$\begin{aligned}
f(t) &= \left\|\chi_0 B - A + \int_0^t (\chi_0\Xi(\Psi(B,g,s)) - \Xi(\Psi(A,g,s)))\,ds\right\|_{L^2} \\
&\leq \|\chi_0 B - A\|_{L^2} + \int_0^t \|\chi_0 b(\Psi(B,g,s)) - b(\Psi(A,g,s))\|_{L^2}\,ds \\
&\leq \|\chi_0 B - A\|_{L^2} \\
&\quad + \int_0^t (\mathfrak{p}(s) + \|b(\chi_0\Psi(B,g,s)) - b(\Psi(A,g,s))\|_{L^2})\,ds.
\end{aligned} \tag{9.33}$$

To apply Gronwall's inequality, we need to bound $\|b(\chi_0\Psi(B,g,s))-b(\Psi(A,g,s))\|_{L^2}$ by a multiple of the term $f(s)=\|\chi_0\Psi(B,g,s)-\Psi(A,g,s)\|_{L^2}$. To achieve this, we shall apply Lemma 9.13 after checking that its hypotheses are satisfied in this case via the following claim:

Claim 9.16. *For the metric g, gauge-equivalence class $[A]$, and a positive constant K, there are positive constants ε, M such that the following holds. Continue the above notation. Suppose χ_0, B, A satisfy the bounds*

$$\|\chi_0\|_{L^2_1}, \|\chi_0(B-A)\|_{L^2_{1,A}} \le K \quad \text{and} \quad \sup_{t\in I_g}\|\dot g(t)\|_{L^\infty} < \varepsilon.$$

Then the following hold for all $t\in I_g$:

$$\|\chi_0\Psi(B,g,t)-\Psi(A,g,t)\|_{L^2_{1,A}} \le \delta, \tag{9.34}$$

$$\|\chi_0\Psi(B,g,t)-\Psi(A,g,t)\|_{L^{\sharp,2}_{1,A}} \le M. \tag{9.35}$$

Proof. We first observe that

$$\begin{aligned}
&\|\chi_0\Psi(B,g,t)-\Psi(A,g,t)\|_{L^2_{1,A}}\\
&\le \|\chi_0\Psi(B,g,t)-\chi_0 B\|_{L^2_{1,A}} + \|\chi_0 B - A\|_{L^2_{1,A}}\\
&\quad + \|A-\Psi(A,g,t)\|_{L^2_{1,A}},\\
&\le \|\chi_0\|_{L^2_1}\|\Psi(B,g,t)-B\|_{L^2_{1,B}} + \|\chi_0\nabla_B(\Psi(B,g,t)-B)\|_{L^2}\\
&\quad + \|\chi_0(B-A)\|_{L^4}\|\Psi(B,g,t)-B\|_{L^4}\\
&\quad + \|\chi_0 B-A\|_{L^2_{1,A}} + \|A-\Psi(A,g,t)\|_{L^2_{1,A}}.\\
&\le \left(1+\|\chi_0\|_{L^2_1}+\|\chi_0(B-A)\|_{L^4}\right)\|\Psi(B,g,t)-B\|_{L^2_{1,B}}\\
&\quad + \|\chi_0 B-A\|_{L^2_{1,A}} + \|A-\Psi(A,g,t)\|_{L^2_{1,A}}.
\end{aligned} \tag{9.36}$$

We use Lemma 9.7 to obtain bounds

$$\begin{aligned}
\|\Psi(B,g,t)-B\|_{L^2_{1,B}} &\le \delta,\\
\|\Psi(A,g,t)-A\|_{L^2_{1,A}} &\le \delta.
\end{aligned} \tag{9.37}$$

Consequently, substituting (9.37) into (9.36) gives the estimate (9.34).

For the inequality (9.35), the only term that might cause trouble is

$$\begin{aligned}
\|\nabla_A(\chi_0\Psi(B,g,t)-\Psi(A,g,t))\|_{L^\sharp} &\le \|\nabla_A(\chi_0(\Psi(B,g,t)-\Psi(A,g,t)))\|_{L^\sharp}\\
&\quad + \|\nabla_A((1-\chi_0)\Psi(A,g,t))\|_{L^\sharp}\\
&\le \|\nabla\chi_0\|_{L^{2\sharp}}\Big(\|\Psi(B,g,t)-\Psi(A,g,t)\|_{L^{2\sharp}(\operatorname{supp}\nabla\chi_0)}\\
&\quad + \|\Psi(A,g,t)\|_{L^{2\sharp}(\operatorname{supp}\nabla\chi_0)}\Big)\\
&\quad + \|(1-\chi_0)\nabla_A\Psi(A,g,t)\|_{L^\sharp(\operatorname{supp}(1-\chi_0))}\\
&\quad + \|\chi_0\nabla_A(\Psi(B,g,t)-\Psi(A,g,t))\|_{L^\sharp}.
\end{aligned}$$

(Via the cutting-off procedure and choice of trivializations given in §3.3, the terms $(1-\chi_0)\Psi(A,g,t)$, $\Psi(A,g,t)|_{\operatorname{supp}\nabla\chi_0}$, and $\Psi(A,g,t)|_{\operatorname{supp}(1-\chi_0)}$ can be regarded as $\mathfrak{g}_E$-valued one-forms over the given open subsets.) We now observe that over $\operatorname{supp}\chi_0$, the $g(t)$-anti-self-dual connections $\Psi(B,g,t)$ are $L^2_{1,A}$ close to the reference connections $\Psi(A,g,t)$, for all $t\in I_g$; the same is true for the difference between these connections over an open subset $U\subset X$ containing $\operatorname{supp}\chi_0$. Hence, our basic regularity theory from §4 yields an $L^3_{1,A}$ bound for the difference $\Psi(B,g,t)-\Psi(A,g,t)$ over $\operatorname{supp}\chi_0$, with constant depending on U — and thus χ_0: see, for example, our local elliptic estimates for anti-self-dual connections given by Proposition 4.20. The embedding $L^3\subset L^\sharp$ [12, Lemma 4.1], then completes the proof of the estimate (9.35) and, hence, completes the proof of the claim. □

Because we can satisfy its hypotheses using Claim 9.16, Lemma 9.13 yields, for $t\in I_g$,

$$\|b(\chi_0\Psi(B,g,t))-b(\Psi(A,g,t))\|_{L^2}\le C\|\chi_0\Psi(B,g,t)-\Psi(A,g,t)\|_{L^2}, \tag{9.38}$$

where C depends on $[A],M,g$. Therefore, combining inequalities (9.33) and (9.38), we see that

$$\begin{aligned} f(t) &\le \|\chi_0B-A\|_{L^2}+Ct\sup_{s\in[0,t]}\mathfrak{p}(s)\\ &\quad + C\int_0^t\|\chi_0\Psi(B,g,s)-\Psi(A,g,s)\|_{L^2}\,ds\\ &= \|\chi_0B-A\|_{L^2}+Ct\sup_{s\in[0,t]}\mathfrak{p}(s)+C\int_0^t f(s)\,ds. \end{aligned}$$

This yields Assertion (1). With f defined by (9.32) and g the constant,

$$g(t):=\|\chi_0B-A\|_{L^2_{1,A,g}(X)}+C|I_g|\sup_{s\in I_g}\mathfrak{p}(s),\quad t\in I_g,$$

we see that Assertion (2) follows from Assertion (1) and Gronwall's inequality, Lemma 9.14. □

9.5.3. *Convergence of the error terms.* Our application of Lemma 9.15 will, of course, be to the case where the connection B is replaced by a sequence of connections B_α converging to $(A,\mathbf{x})$ and the cutoff functions χ_0, ψ_0 are replaced by sequences χ_n, ψ_n of cutoff functions with supports in $X-\mathbf{x}$ and $\chi_n=1$ on $\operatorname{supp}\psi_n$. In this situation, we need to verify that the term $\|\chi_nB_\alpha-A\|_{L^2(X,g)}+Ct\|\mathfrak{p}_{n,\alpha}\|_{L^\infty([0,t])}$ in Lemma 9.15 converges to zero for $\mathbf{x}$-good sequences of cutoff functions.

Lemma 9.17. *Continue the notation and hypotheses of Lemma 9.15 and of the preceding paragraph. Let ε be a positive constant. Then for large enough n and $\alpha(n)$, we have $|\mathfrak{p}(\chi_n,\psi_n,B_\alpha,g,t)|<\varepsilon$ for all $t\in I_g$.*

Proof. Recall from inequality (9.31) that $\mathfrak{p}(t)=\mathfrak{p}(\chi_0,\psi_0,B,g,t)$ is defined by

$$\begin{aligned} \mathfrak{p}(\chi_0,\psi_0,B,g,t) &:= p(\|\nabla\psi_0\|_{L^2_1},\|\chi_0(1-\chi_0)B_t\|_{L^2_{1,B_t}})\\ &\quad + \|(1-\psi_0)(\chi_0b(B_t,g_t)-b(\chi_0B_t,g_t))\|_{L^2}, \end{aligned} \tag{9.39}$$

where $p(x,y)$ is a polynomial with $p(0,0)=0$ and coefficients depending only (g,μ,κ). We need to show that $\mathfrak{p}_{n,\alpha}(t) := \mathfrak{p}(\chi_n,\psi_n,B_\alpha,g,t)$ converges to zero for $\mathbf{x}$-good sequences of cutoff functions χ_n, ψ_n and connections B_α converging to $(A,\mathbf{x})$.

We first estimate the term $\|\chi_0(1-\chi_0)B_t\|_{L^2_{1,B_t}}$ appearing in the definition (9.39) of $\mathfrak{p}(t)$. Using $\nabla_{B_t} = \nabla_B + (B_t - B)$, Hölder's inequality, and the Sobolev embedding $L^2_{1,B} \subset L^4$ of Lemma 6.5, we see that

$$
\begin{aligned}
\|\chi_0(1-\chi_0)B_t\|_{L^2_{1,B_t}} &\le \|\chi_0(1-\chi_0)B_t\|_{L^2_{1,B}} + \|(B_t - B)\wedge\chi_0(1-\chi_0)B_t\|_{L^2} \\
&\le \|\chi_0(1-\chi_0)\|_{L^2_1}\|B_t\|_{L^2_{1,B}(\operatorname{supp}\chi_0(1-\chi_0))} \\
&\quad + \|\chi_0(1-\chi_0)\nabla_B B_t\|_{L^2} \\
&\quad + \|B_t - B\|_{L^4}\|\chi_0(1-\chi_0)B_t\|_{L^4},
\end{aligned} \tag{9.40}
$$

where

$$
\|\chi_0(1-\chi_0)\|_{L^2_1} \le \|\chi_0(1-\chi_0)\|_{L^2} + \|\nabla\chi_0\|_{L^4}(\|\chi_0\|_{L^4} + \|1-\chi_0\|_{L^4}). \tag{9.41}
$$

(By the cutting-off procedure and choice of trivializations given in §3.3, the terms $\chi_0(1-\chi_0)B_t$ and $B_t|_{\operatorname{supp}\chi_0(1-\chi_0)}$ can be regarded as $\mathfrak{g}_E$-valued one-forms over the given open subsets.) Lemma 6.9 implies that if we replace χ_0, above with an $\mathbf{x}$-good sequence, then the term $\|\chi_n(1-\chi_n)\|_{L^2_1}$ converges to zero as $n\to\infty$. On the other hand, Assertion (2) of Lemma 9.7 yields the bound

$$
\|B_t - B\|_{L^2_{1,B}} \le C, \quad t\in I_g, \tag{9.42}
$$

where $C = C(g,\|\dot g\|_{L^\infty(I_g)},\kappa,\mu)$. Thus, if B_t, B above are replaced by $B_{\alpha,t}\equiv\Psi(B_\alpha,g,t)$ and B_α, respectively, the terms $\|B_{\alpha,t}-B_\alpha\|_{L^2_{1,B_\alpha}}$ remain uniformly bounded with respect to α. Consequently, we see from from inequalities (9.40) and (9.41) that the terms $\|\chi_n(1-\chi_n)B_{\alpha,t}\|_{L^2_{1,B_{\alpha,t}}}$ on the left of inequality (9.40) converge to zero, for large enough n, $\alpha(n)$.

For the second term in (9.39), observe that

$$
\begin{aligned}
&\|(1-\psi_0)(\chi_0 b(\Psi(B,g,t)) - b(\chi_0\Psi(B,g,t)))\|_{L^2} \\
&\le \|(1-\psi_0)\|_{L^4}(\|b(\Psi(B,g,t))\|_{L^4} + \|b(\chi_0\Psi(B,g,t))\|_{L^4}).
\end{aligned} \tag{9.43}
$$

From (9.17), the embedding $L^2_1\subset L^4$ of Lemma 6.5, and the $L^{\sharp,2}$ curvature bound of Lemma 9.6, we have the estimates

$$
\begin{aligned}
\|b(\Psi(B,g,t))\|_{L^4} &\le C\|(DP_+)_g(\dot g)F(\Psi(B,g,t))\|_{L^{\sharp,2}} \le C, \\
\|b(\chi_0\Psi(B,g,t))\|_{L^4} &\le C\|(DP_+)_g(\dot g)F(\chi_0\Psi(B,g,t))\|_{L^{\sharp,2}} \le C.
\end{aligned} \tag{9.44}
$$

Here we use the fact, required by Proposition 7.3, that we have a uniform bound on the L^∞ norm of $P_+(g_t)F(\chi_0\Psi(B,g,t))$: this L^∞ bound follows from the general L^p estimates of Proposition 5.11 for the self-dual component of the curvature of cut-off or patched anti-self-dual connections, yielding a constant C depending only on (g,κ). Hence, from (9.44) we see that if we choose $\mathbf{x}$-good sequences $\{\psi_n\}$ and $\{\chi_n\}$ and if $\{B_\alpha\}$ converges to $(A,\mathbf{x})$, then the terms $\|b(\Psi(B_\alpha,g,t))\|_{L^4}$ and $\|b(\chi_n\Psi(B_\alpha,g,t))\|_{L^4}$ remain uniformly bounded with respect to α, n, while the terms $\|(1-\psi_n)\|_{L^4}$ converge to zero. Therefore the terms on the left in (9.43) converge to zero for large enough n, $\alpha(n)$.

Combining these convergence estimates for the left-hand terms in (9.40) and (9.43), we see that $\sup_{t\in I_g}\mathfrak{p}(\chi_n,\psi_n,B_\alpha,g,t)$ converges to zero for large enough n, $\alpha(n)$, as desired. $\square$

9.5.4. *Final arguments.* The assertion of continuity in Theorem 1.1 is an immediate consequence of the following result.

Proposition 9.18. *Let $\Psi([A_\alpha],g,\cdot)$ be a sequence of C^∞ flow-lines through $([A_\alpha],g)$, with $[A_\alpha]\in M_\kappa(X,g)$ converging to a limit $[A_\infty,\mathbf{x}]\in M_{\kappa-\ell}(X,g)\times\mathrm{Sym}^\ell(X)$. Then, the sequence of flow-lines $\Psi([A_\alpha],g,\cdot)$ in $\mathcal{B}_\kappa(X,C,\mu)$ converges (pointwise in $t\in I_g$) to the C^∞ flow-line $(\Psi([A_\infty],g,\cdot),\mathbf{x})$ in $\mathcal{B}_{\kappa-\ell}(X,C,\mu)\times\mathrm{Sym}^\ell(X)$.*

Proof. If A_∞ is a connection on g_{E_∞}, we may suppose without loss of generality that the connections A_α converge to A_∞ in $L^2_{k,\mathrm{loc}}$ on $X-\mathbf{x}$. For each $t\in I_g$, the sequence $\Psi([A_\alpha],g,t)\equiv([A_\alpha(t)],g(t))\in M_\kappa(X,g_t)$ converges to some $([A'_t,\mathbf{x}_t],g_t)\in\bar M_\kappa(X,g_t)$. We claim that $([A'_t,\mathbf{x}_t],g_t)=(\Psi([A_\infty],g,t),\mathbf{x})$, where $(\Psi([A_\infty],g,t),\mathbf{x})\equiv([A_\infty(t),\mathbf{x}],g_t)$, for each $t\in I_g$.

According to Lemma 9.9, we have $\mathbf{x}_t=\mathbf{x}$, so it remains to show that $[A'_t]=[A_\infty(t)]$. The standard Uhlenbeck compactness theorem [8, Theorem 4.4.3] implies that, after passing to a subsequence, the sequence of L^2_k connections $A_\alpha(t)$ converges to A'_t in $L^2_k(U)$, modulo L^2_{k+1} gauge transformations of E, for each precompact open subset $U\Subset X\setminus\mathbf{x}$. Let $\{\psi_n\}$, $\{\chi_n\}$ be $\mathbf{x}$-good sequences of cutoff functions on X, with $\chi_n=1$ on $\operatorname{supp}\psi_n$. Then Assertion (2) of Lemma 9.15 yields, for all $t\in I_g$,

$$\begin{aligned}&\|\chi_n\Psi(A_\alpha,g,t)-\Psi(A_\infty,g,t)\|_{L^2}\\&\le\left(\|\chi_nA_\alpha-A_\infty\|_{L^2}+Ct\sup_{t\in I_g}\mathfrak{p}(\chi_n,\psi_n,A_\alpha,g,t)\right)\exp(-C|t|).\end{aligned}$$

Clearly, the terms $\|\chi_nA_\alpha-A_\infty\|_{L^2}$ converge to zero and Lemma 9.17 implies that the terms $\sup_{t\in I_g}\mathfrak{p}(\chi_n,\psi_n,A_\alpha,g,t)$ converge to zero for large enough n and $\alpha(n)$. Therefore, over each $U\Subset X\setminus\mathbf{x}$, the sequence $A_\alpha(t)$ converges to $A_\infty(t)$ in $L^2(U)$, modulo L^2_{k+1} gauge transformations of E. Hence, $[A'_t]=[A_\infty(t)]$, as desired. $\square$

We can now conclude the proof of the second assertion of Theorem 1.1.

Proof of Assertion (2) of Theorem 1.1. This is an immediate consequence of Proposition 9.18. $\square$

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