On the maximum principle

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Abstract: The particular form taken by the maximum principle for a class of geometric problems is examined. A number of applications are made.

1. Let Ω be an open set in $\mathbb{R}^n$, denote by Σ its (ideal) boundary. Let $u(x)$ satisfy

$$\Delta u = 0 \tag{1}$$

in Ω, and suppose

$$\limsup_{x \to \Sigma} u(x) \leq 0 \tag{2}$$

in the sense that (2) holds for any sequence x_j that leaves any compact subset of Ω. The classical maximum principle now assures us that either $u < 0$ or $u \equiv 0$ in Ω. Neglecting some details of rigor, we may construct a proof by observing that if the result were false, the set $\Omega_\varepsilon = \{x \in \Omega : x > \varepsilon\}$ would be non null for small enough ε and have compact closure in Ω. Thus the function

$$w(x) = \begin{cases} u-\varepsilon \ , & u(x) > \varepsilon \\ 0 \ , & u(x) \leq \varepsilon \end{cases}$$

would be non constant in Ω_ε , and

$$0 = \int_\Omega w\Delta u dx = - \int_{\Omega_\varepsilon} |\nabla u|^2 dx \ ,$$

a contradiction.

There are two features of the result that may not have received sufficient attention in the literature.

i) Insensitivity to the configuration of $\partial\Omega$. The proof works for any open set Ω .

ii) Extreme sensitivity to the completeness of $\partial\Omega$. If a single point is deleted, the result can fail.

The second point is simply illustrated with the function (in polar coordinates in the plane)

$$u(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

which satisfies (1) in $\Omega : r < 1$ and (2) on $\Sigma = \partial\Omega$ except for the single point $(1,0)$, and for which $u > 0$ in Ω .

2. The above considerations apply -- despite the possible nonlinearity -- to the difference $u-v$ of two solutions of any elliptic equation

$$\operatorname{div} A(Du) = \varphi(u) \tag{3}$$

for which

$$\varphi'(u) \geq 0 \, . \tag{4}$$

Here $A(Du)$ is a vector function of Du. To i) and ii) above we may therefore add

(iii) Intensitivity to the ellipticity constants of (3).

That is, no account need be taken of possible degeneration outside compact subsets of Ω.

In what follows, we concern ourselves chiefly with ii). It turns out that the boundary completeness sensitivity can be greatly ameliorated by restricting attention to equations (3) with particular nonlinearities. Because of their geometric and physical significance for minimal surfaces, H-graphs, capillarity, gas dynamics and plasticity we consider elliptic equations of the form (3) subject to (4) and the single additional restriction

$$|A(Du)| < 1 \tag{5}$$

for all Du that can appear. (We note that the form (5) can be achieved as an equivalent relation in any situation for which $|A(Du)|$ is bounded.) Our particular interest will be directed to the minimal surface equation

$$\operatorname{div} Tu = 0 \, , \quad Tu = \frac{1}{\sqrt{1+|Du|^2}} Du \tag{6}$$

and the capillarity equation

$$\operatorname{div} Tu = \kappa u + nH \tag{7}$$

in the "non-negative gravity" case $\kappa \geq 0$.

Definition 1: A set $\Sigma_0 \subset \Sigma$ will be said to satisfy Hypothesis H if for any $\varepsilon_0 > 0$, Σ_0 can be covered by a countable number of balls B_{δ_i} of radius δ_i , such that $\Sigma\delta_i^{n-1} < \varepsilon_0$.

The hypothesis is equivalent to the requirement that Σ_0 have $(n-1)$ dimensional Hausdorff measure zero.

Theorem 1: Suppose (5) is satisfied by an elliptic $A(Du)$, (4) is satisfied by $\varphi(u)$. Let $\Sigma \equiv \Sigma_\alpha \cup \Sigma_0$, where Σ_0 satisfies Hypothesis H . Set

$$Nu \equiv \operatorname{div} A\,(Du) - \varphi(u) \tag{8}$$

and suppose $Nu \geq Nv$ for functions u,v in Ω . Suppose there exists $A \geq 0$ such that for every $\varepsilon_0 > 0$ there holds $\limsup\,(u-v) \leq A$ for any sequence $x_j \subset \Omega_0 = \Omega \setminus \cup B_{\delta_i}$, for which $x_j \to \Sigma_\alpha$. Then $u \leq v + A$ in Ω .

The proof follows the general lines of § 1. If the theorem were false for some u,v , then there would exist $\varepsilon,\varepsilon_0,M > 0$

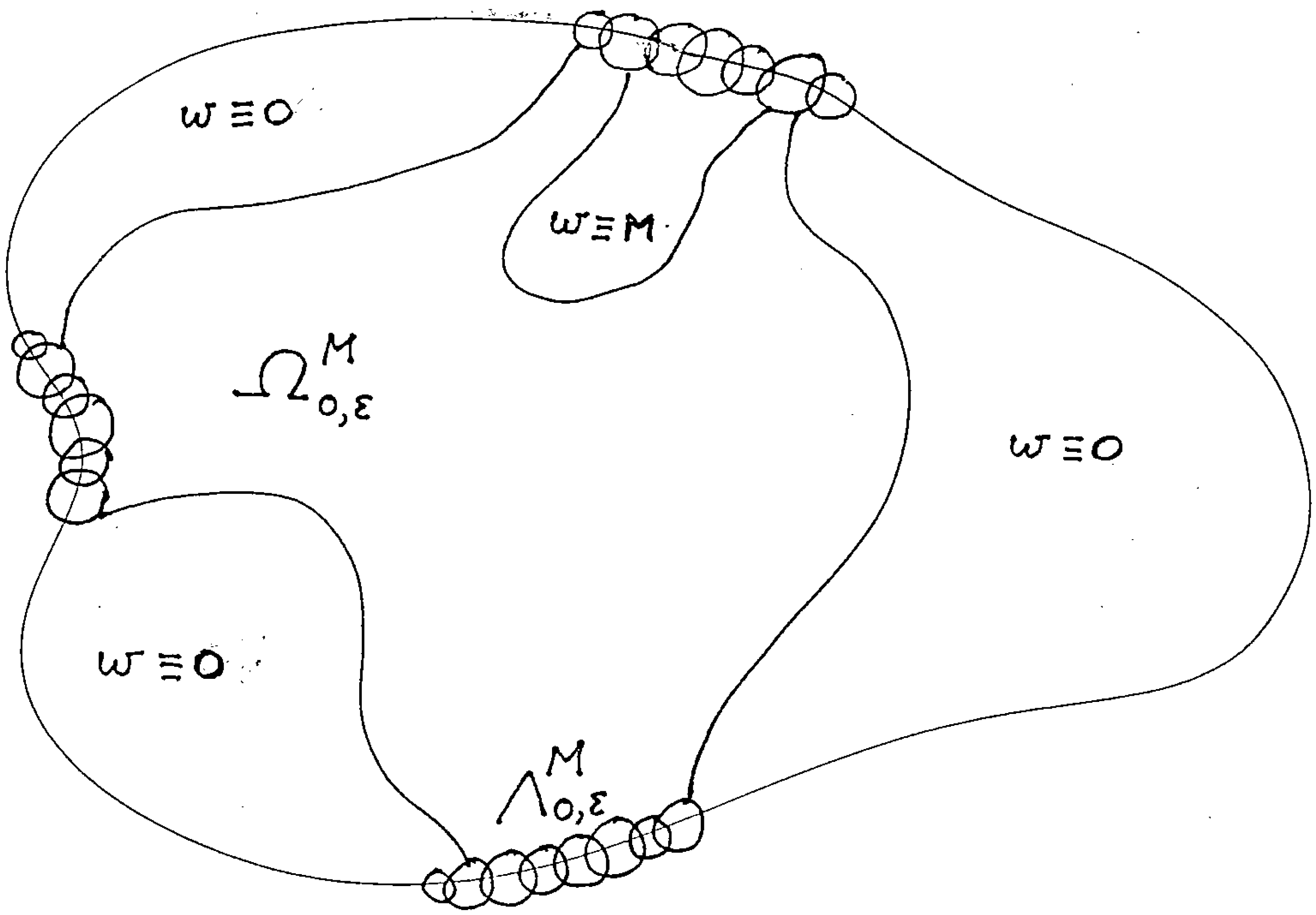

Figure 1: Proof of Theorem 1.

such that the set

$$\Omega^M_{0,\varepsilon} = \{x\in\Omega \setminus \cup B_{\delta_i} : A+\varepsilon<u-v<M+A+\varepsilon\} \tag{9}$$

is non-null. Setting

$$w(x) = \begin{cases} M , & u-v \geq M+A+\varepsilon \\ u-v-(A+\varepsilon) , & A+\varepsilon<u-v<M+A+\varepsilon \\ 0 , & u-v \leq A+\varepsilon \end{cases} \tag{10}$$

we find from the divergence theorem

$$\begin{aligned} 0 \leq \int_{\Omega_0} w(Nu-Nv)\,dx &= \int_{\Lambda^M_{0,\varepsilon}} w\ \nu\cdot A\ d\sigma \\ &-\int_{\Omega_0} w(\varphi(u)-\varphi(v))\,dx - \int_{\Omega^M_{0,\varepsilon}} Dw\cdot[\mathcal{A}(Du)-\mathcal{A}(Dv)]\,dx \ . \end{aligned} \tag{11}$$

Here $\Lambda^M_{0,\varepsilon} = \Omega^M_{0,\varepsilon} \cap \partial(\cup B_{\delta_i})$, ν = unit normal, see Figure 1.

The last integral on the right in (11) is positive in view of the assumed ellipticity. The preceding integral is non negative in view of the hypothesis (4). The crucial new point is that because of (5) we have $|\nu\cdot A| < 1$ on $\Lambda^M_{0,\varepsilon}$. By Hypothesis H , $|\Lambda^M_{0,\varepsilon}| < C\varepsilon_0$. Since $0 \leq w \leq M$, we obtain a contradiction by letting $\varepsilon_0 \to 0$.

The same reasoning shows that <u>if</u> $Nu \leq Nv$ and if $u-v \geq -A$ <u>on</u> Σ_α <u>in the sense indicated</u>, <u>then</u> $u-v \geq -A$ in Ω . We may use the result to prove

<u>Theorem 2</u> (cf. [1]): <u>The solutions of an elliptic equation</u> (3) <u>for which</u> (4) <u>and</u> (5) <u>hold admit no isolated singularities</u>.

For if a solution $u(x)$ had such a singularity at $p \in \Omega$, then for small enough h the solutions $u(x)$ and $v(x) = u(x-h)$ would be defined and smooth in a common domain $\mathcal{D}$ with two points p, $p+h$ deleted. Letting $\Sigma_\alpha \subset \mathcal{D}$ be a fixed smooth surface that encloses the two points for all small enough h, and setting $\Sigma_0 = p \cup (p+h)$, we find from the above result that $u^h = \frac{u(x)-u(x-h)}{h}$ is uniformly bounded interior to Σ_α (since it is bounded on Σ_α) as $h \to 0$ in any way, hence $|Du|$ is bounded at p; the remainder of the proof follows easily from general properties of elliptic equations.

For other applications of Theorem 1, see, e.g., [2,3,4,5,6].

3. In physical problems arising in capillarity, it is necessary to prescribe the angle γ formed between the solution surface and a given support surface (bounding walls of the container). For a cylindrical support surface (capillary tube) we obtain the equation (7) over a domain Ω, with the condition (see, e.g. [7] Chapter 1)

$$\nu \cdot Tu = \cos \gamma, \quad 0 \leq \gamma \leq \pi \tag{12}$$

on a subset $\Sigma_\beta \subset \Sigma$; analogously, we consider the formal condition

$$\nu \cdot A = \cos \gamma \tag{13}$$

on Σ_β, corresponding to a solution of (3) in Ω. Since these conditions involve explicitly the normal ν, it is natural to assume some smoothness of Σ_β. We also assume at first that

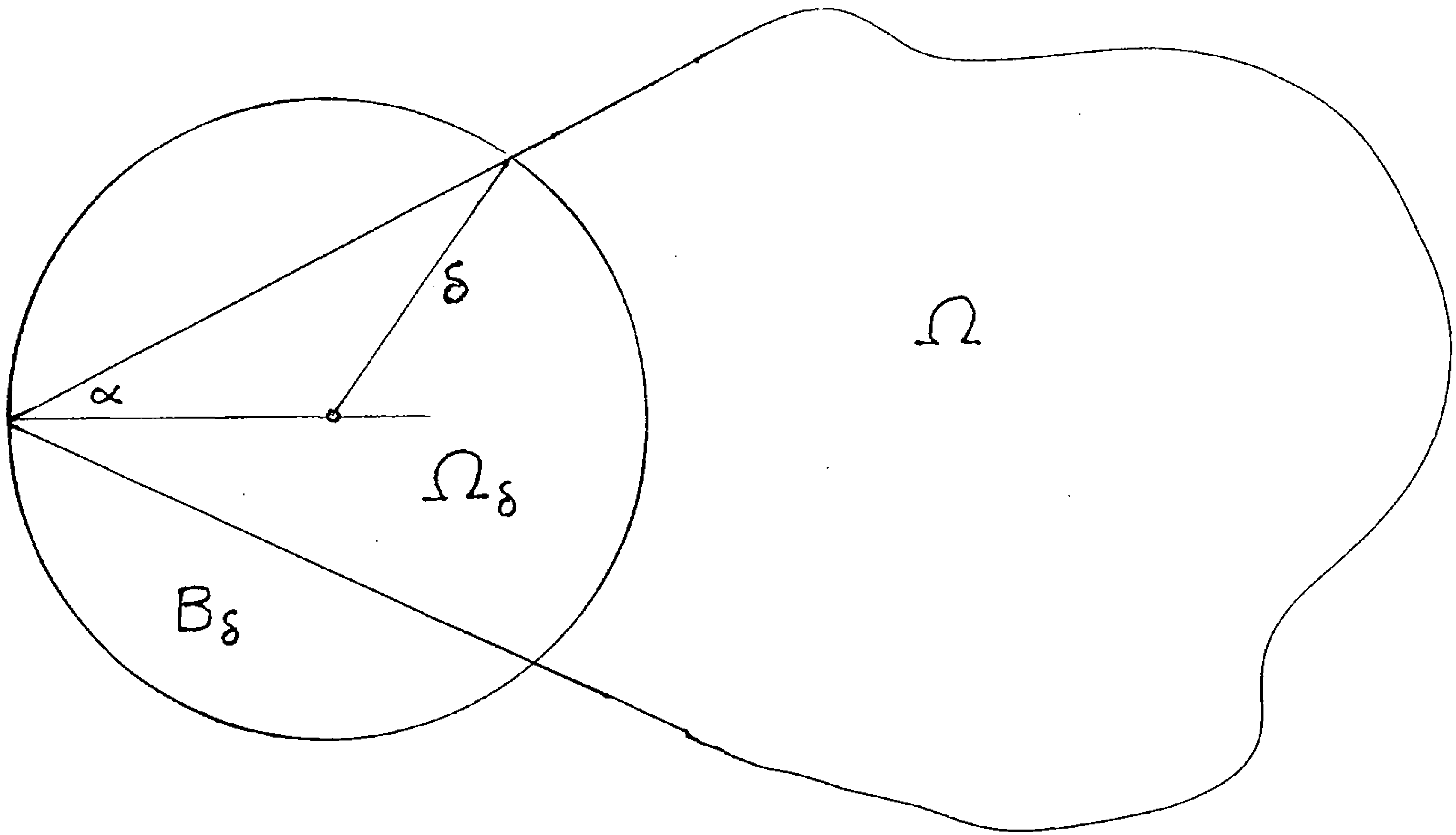

Figure 2: Configuration for a priori bound.

for each $p \in \Sigma_\beta$ with normal ν_p , $\nu_p \cdot A$ exists as a continuous limit as p is approached from within Ω ; note that this does not imply continuity or even boundedness of Du up to Σ_β . We obtain

Theorem 3: *Let $u(x)$, $v(x)$ be solutions of an elliptic equation (3) in Ω , for which (4) and (5) hold. With designations as above, let $\Sigma = \partial\Omega = \Sigma_0 \cup \Sigma_\alpha \cup \Sigma_\beta$, and suppose $u \le v + A$ on Σ_α , $\nu \cdot A(Du) \le \nu \cdot A(Dv)$ on Σ_β in the senses indicated above. Then*

i) *if $\varphi(u) \not\equiv \varphi(v)$ in Ω or if $\Sigma_\alpha \neq \phi$, then $u \le v + A$ in Ω ;*

ii) *if $\Sigma_\alpha = \phi$ and $\varphi(u) \equiv \varphi(v)$, then $u(x) \equiv v(x) + \text{const.}$ in Ω .*

The proof is a fairly routine extension of the method used for Theorem 1, and we do not give it in detail.

We apply Theorem 3 to the case of equation (7) with $\kappa > 0$. By a vertical translation, we can then achieve $H = 0$. We consider a domain Ω bounded in part by a symmetric cone K of half angle α , as indicated in Figure 2, and we suppose that the closure interior to K of the region Ω_δ cut off by an open ball B_δ as indicated in the figure lies interior to Ω . We suppose that on the smooth part K_δ of K cut off by B_δ , the boundary data γ satisfy $\alpha + \gamma \ge \pi/2$. No boundary data are assigned at the vertex V (that would not be possible, as there is no normal vector at V), nor is any growth hypothesis introduced on the solution $u(x)$ at V . *Nevertheless, we may show that*

<u>any solution in</u> Ω_δ <u>with data</u> γ <u>on</u> K_δ <u>satisfies</u>

$$|u(x)| < \frac{n}{\kappa\delta} + \delta \tag{14}$$

<u>throughout</u> Ω_δ .

To prove the assertion, we introduce a lower hemisphere $v(x)$ over B_δ and observe that $\operatorname{div} Tv \equiv n\delta^{-1}$. We add a constant to $v(x)$ so that its minimum height $v_0 = n/\kappa\delta$. Then $\operatorname{div} Tv = \kappa v_0 \leq \kappa v$ and thus $Nv \leq Nu$ in Ω_δ . We choose $\Sigma_0 = V \cup (\partial B_\delta \cap K)$, $\Sigma_\alpha = \phi$, $\Sigma_\beta = K_\delta \cup (\partial B_\delta \cap \Omega)$.

By hypothesis, $\nu \cdot Tu \leqq \nu \cdot Tv$ on K_δ ; also since $\nu \cdot Tv = 1$ on $\partial B_\delta \cap \Omega$ and $|Tu| < 1$ for any differentiable u , we have $\nu \cdot Tu \leqq \nu \cdot Tv$ on $\partial B_\delta \cap \Omega$. Since $\partial(B_\delta \cap \Omega) = \Sigma_0 \cup \Sigma_\alpha \cup \Sigma_\beta$ and $\kappa > 0$, we obtain from Theorem 3 that $u < v \leqq v_0 + \delta = \frac{n}{\kappa\delta} + \delta$ in $B_\delta \cap \Omega$, as was to be shown. The inequality for $|u|$ is obtained by repeating the reasoning with u replaced by its negative.

The result just shown continues to hold in the limiting case $\alpha + \gamma \equiv \pi/2$. <u>It is remarkable in that if</u> $\alpha + \gamma < \pi/2$, <u>then there exists</u> $A < \infty$ <u>such that, in terms of polar distance</u> r <u>and spherical angle</u> φ <u>measured from</u> V , <u>there exists</u> $f(\varphi) > 0$ <u>for which</u>

$$\left|u - \frac{1}{r} f(\varphi)\right| < A \tag{15}$$

<u>in</u> $B_\delta \cap \Omega$. <u>Thus, the solutions in such a domain depend discontinuously on the boundary data</u> γ .

The proof of (15) is obtained using the same maximum principle Theorem 3, cf. [8].

The existence of solutions, smooth up to the smooth parts of $\partial\Omega$, has been shown in both cases (see [10] or [7], Chapter 7).

4. We wish to apply the maximum principle to general configurations whose existence has been proved. For the equation (3) under the conditions (4), (5) with data (13) on at least part of the boundary, there is currently no existence theory available. Depending on properties of $\varphi(u)$ and on the domain, existence (or nonexistence) of a variational solution (see below) has been proved for the equation

$$\operatorname{div} Tu = \varphi(u) \ , \quad \varphi'(u) \geq 0 \tag{16}$$

with

$$\nu \cdot Tu = \cos \gamma \tag{17}$$

on Σ, see, e.g. [7], Chapters 6 and 7.

Definition 2: A function $u(x)$ is a variational solution of (3), (13) in Ω if $u(x) \in H^{1,1}_{loc}(\Omega)$ and satisfies

$$\int_\Omega (D\eta \cdot A + \eta\varphi(u))dx = \oint_\Sigma \eta \cos \gamma \, d\sigma \tag{18}$$

for all $\eta \in H^{1,1}(\Omega)$.

In the case of (16), existence has been shown for very general domains when $\lim_{|u|\to\infty} |\varphi(u)| = \infty$; otherwise existence is known--for certain domains, nonexistence for others (see [7] Chapter 6 and 7). In both cases, if $|\cos\gamma| < 1$ on Σ the existence theory leads to $u(x) \in H^{1,1}(\Omega)$, and uniqueness in this class can be shown essentially by the preceding methods (simpler, as there is no need to introduce the set Λ !). But if $\cos\gamma = \pm 1$ on $\Sigma' \subset \Sigma$ the reasoning breaks down, as it can no longer be expected that $u(x) \in H^{1,1}(\Omega)$ and hence the difference $u-v$ of two variational solutions cannot in general serve as a test function.

Nevertheless information can still be obtained, at least in the case that Σ' is smooth.

Lemma 1: Let u be a variational solution of (16), (17) in Ω , and let $\Sigma' \subset\subset \Sigma''$, where Σ'' is a smooth subset of Σ . Then there is a neighbourhood $\Omega' \subset \Omega$ of Σ' for which

$$\int_{\Omega'} |\varphi(u)|dx < \infty \quad . \tag{19}$$

Proof: If $\varphi(u)$ is bounded apriori there is nothing to prove. Suppose $\lim_{u\to\infty} \varphi(u) = \infty$. Then the same procedure that was used in § 3 to prove (14) yields that $u(x)$ is bounded above near Σ' ; hence $\varphi(u)$ is bounded above. Similarly $\varphi(u)$ is bounded below.

Lemma 2: Let Σ' be a smooth piece of surface on Σ , with smooth boundary on Σ . Let $\Omega' \subset \Omega$ be bounded by Σ' and by a smooth surface $S' \subset \Omega$. Then for any variational solution $u(x)$ of (3), (13) in Ω there holds

$$\int_{\Omega'} \varphi(u)\,dx = \int_{\Sigma'} \cos\gamma\, d\sigma + \int_{S'} \nu \cdot A\, d\sigma \quad . \tag{20}$$

Proof: Choose $\eta = 1$ in Ω' , 0 in $\Omega \setminus \Omega'$, except for a transition strip region across S' in which η changes linearly across the normal to S' . A simple limiting procedure yields the result.

In view of Lemma 1 we obtain

Corollary 2: In the configuration as above, replace S' by a family of surfaces Σ'_n which converge weakly to Σ' in the sense that the corresponding $|\Omega'_n| \to 0$. Then for any variational solution of (16), (17) we have

$$\lim_{n\to\infty} \int_{\Sigma'_n} \nu \cdot Tu\, d\sigma = \int_{\Sigma'} \cos\gamma\, ds \quad . \tag{21}$$

Here we have taken for ν on Σ'_n the orientation of ν on Σ' .

Lemma 3. Let u be a variational solution of (16), (17) with $\cos\gamma \equiv 1$ $(\cos \equiv -1)$ on a smooth Σ' with smooth boundary on Σ . We write Σ' in local coordinates $x(\alpha,\beta)$, and we suppose a family of surfaces $\Sigma'_n : x_n(\alpha,\beta)$ can be introduced as above, such that $x_n \to x$, $Dx_n \to Dx$ uniformly on the parameter domain. Then, denoting by $\mu(S)$ the area of a smooth set $S \subset \Sigma'_n$, we have if $\gamma = 0$ on Σ'

$$\lim_{n\to\infty} \mu\{x \in \Sigma_n' : 1 - \nu \cdot Tu > \varepsilon\} = 0 \tag{22a}$$

<u>for any</u> $\varepsilon > 0$. <u>If</u> $\gamma = \pi$ <u>on</u> Σ' <u>then</u>

$$\lim_{n\to\infty} \mu\{x \in \Sigma_n' : 1 + \nu \cdot Tu > \varepsilon\} = 0 \quad . \tag{22b}$$

<u>Proof</u>: By Corollary 2, if $\gamma = 0$

$$\lim \int_{\Sigma_n'} \left\{\frac{d\sigma}{d\sigma_n} - \nu \cdot Tu\right\} d\sigma_n = \lim \int_{\Sigma_n'} \left\{\left(1 - \nu \cdot Tu\right) + \left(\frac{d\sigma}{d\sigma_n} - 1\right)\right\} d\sigma_n = 0 \ .$$

But

$$\int_{\Sigma_n'} \left(\frac{d\sigma}{d\sigma_n} - 1\right) d\sigma_n \to 0$$

and thus

$$\lim_{n\to\infty} \int_{\Sigma_n'} (1 - \nu \cdot Tu)\, d\sigma_n = 0 \quad .$$

Since $\nu \cdot Tu < 1$, the result follows. The case $\gamma = \pi$ on Σ' is analogous.

<u>Lemma 4</u>: <u>Let</u> u_1, u_2 <u>be variational solutions of</u> (16), (17) <u>with data</u> γ_1, γ_2 <u>for which</u> $\gamma_1 \equiv 0$ or $\gamma_2 \equiv \pi$ <u>on</u> $\Sigma' \subset \Sigma$ <u>as above. Then for any sequence</u> Σ_n' <u>as above there holds</u>

$$\liminf_{n\to\infty} \int_{\Sigma_n'} \eta(\nu \cdot Tu_1 - \nu \cdot Tu_2)\, d\sigma \ge 0$$

<u>for any non negative bounded</u> $\eta(x)$.

Proof: If $\gamma_1 = 0$ we write

$$\nu \cdot Tu_1 - \nu \cdot Tu_2 = (\nu \cdot Tu_1 - 1) + (1 - \nu \cdot Tu_2)$$
$$> \nu \cdot Tu_1 - 1$$

since $|\nu \cdot Tu_2| < 1$. Hence for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mu\{x \in \Sigma_n' : (\nu \cdot Tu_1 - \nu \cdot Tu_2) < -\varepsilon\} = 0 \tag{23}$$

by Lemma 3, from which the result follows.

5. With the aid of Lemma 4 and the previously described methods, it is not difficult to prove the following form of the maximum principle (cf. [9,10,11,12]):

Theorem 4: Suppose $\Sigma = \partial\Omega$ _admits a decomposition_ $\Sigma = \Sigma_0 \cup \Sigma_\alpha \cup \Sigma_\beta \cup \Sigma_\beta'$, _such that_ Σ_β , Σ_β' _are smooth and are bounded by smooth submanifolds on_ Σ , _and_ Σ_0 _satisfies hypothesis_ H . _Let_ u_1, u_2 _be variational solutions of_ (16), (17) _in_ Ω _relative to data_ γ_1, γ_2 _on_ Σ_β' , _with_ $\gamma_1 = 0$ _or_ $\gamma_2 = \pi$ _on each component of_ Σ_β' . _Suppose there is a sequence_ $\Sigma_\beta^{(n)}$ _of bounded length tending to_ Σ_β _weakly as in Corollary 2, on which_ $\nu \cdot Tu_1 \geq \nu \cdot Tu_2$ _in the sense of_ (23) _above. Suppose further that for some_ $A > 0$ _there holds_ $\liminf\,(u_1 - u_2) \geq -A$ _for any sequence_ $x_j \to \Sigma_\alpha$ _exterior to some covering of_ Σ_0 . _Then if_ $\Sigma_\alpha = \phi$ _and_ $\varphi(u_1) \equiv \varphi(u_2)$ _there holds_ $u_1 \equiv u_2 + C$ _for some constant_ C . _Otherwise_

$u_1 \geq u_2 - A$ in Ω .

From Theorem 4 in turn the following result can be proved for solutions of a (normalized) capillary equation in the absence of gravity [12]:

Theorem 5: Let $u(x)$ be a solution of

$$\operatorname{div} Tu = 2 \tag{24}$$

in $\Omega \subset \mathbb{R}^2$, and suppose $u(x)$ is a variational solution relative to boundary data γ , with $\gamma = 0$ on a smooth subarc $\Sigma' \subset \partial\Omega$. Then on any normal ν through $p \in \Sigma'$ at distance d to $\bar{p}$, there holds

$$|\nu \cdot Tu - 1| < C d \quad . \tag{25}$$

The constant C can be estimated explicitly in terms of elliptic integrals.

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