ON THE PROJECTIVE GEOMETRY OF HYPERSURFACES

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Introduction

Projective differential geometry studies the properties of submanifolds of projective space that are invariant under the projective group. Its beginning goes back to the last century (see [14] for a brief history) and the systematic study was made first by E.J. Wilczynski and G. Fubini and E. Čech. (Refer [22], [23], [10], [11] as well as [4] by E. Cartan). The results up to 1960's are accumulated in G. Bol's books [2] with huge amount of references. The purpose of this note is to reformulate the projective geometry of hypersurface. This reformulation provides us an efficient way to develop an invariant theory on convex domains in a projective view point (cf. [17]). Before explaining contents, I want to add some remarks. As we can see in Bol's book, there had been only a few studies of submanifolds of codimension greater than one except the theory of curves (see [6], [12]). This is partly because the projective group is too large to yield fruitful local and global results. This reason also explains that the projective geometry does not attract much interest nowadays.

I do think however that there remain several subjects, for instance W-congruences, Laplace sequences ....., which should be refined and generalized. Note that these subjects are related with a geometric theory of linear partial differential equations of some kind.

In this note I have tried to make the theory of hypersurfaces in euclidean spaces as far as possible. Indeed on hypersurfaces which satisfy some non-degeneracy condition, we define the "second fundamental form" and construct a normal conformal connection. The method we use here owes much to two investigations; one is by H. Flanders [9] and S.S. Chern [7] on the affine differential geometry of hypersurfaces (see also [3]). The other is by N. Tanaka [20] on the geometry of CR-hypersurfaces (see also [21] and [8]).

For the sake of simplicity, we assume here that a hypersurface is given by a local expression

$$y = \frac{1}{2}\sum (x^i)^2 + \frac{1}{6}\sum a_{ijk}x^i x^j x^k + \frac{1}{24}\sum a_{ijkl}x^i x^j x^k x^l + 0(x^5),$$

$$1 \leq i,j,\ldots \leq n.$$

By using an appropriate projective change of coordinates, we normalize this expression as $\sum a_{iij} = \sum a_{iijj} = 0$ (§ 4.2). Then the form $\phi_2 = \sum a_{ij}dx^i dx^j$ and $\phi_3 = \sum a_{ijk}dx^i dx^j dx^k$ have some invariant properties. In fact, in § 1 after presenting these forms by use of the moving frame method, we prove that the form ϕ_2 is conformally invariant. The scalar $F = \sum (a_{ijk})^2$, which is called the Fubine-Pick invariant, turns out to be a relative invariant. In § 2 we construct a normal conformal connection associated with the form ϕ_2 in case $n \geq 3$. At the same time we obtain the second fundamental form by using the form ϕ_3 and auxiliary invariants. We next derive the Gauß equation and the Codazzi-Minardi equation and prove in terms of these equations the fundamental theorem of hypersurfaces (Theorem 2.8). In § 3 we assume in addition that F does not vanish as in most literatures. In this case, the problem reduces to Riemannian geometry. Some remarks are given in § 4. We compare in § 4.1 our normalization and classical ones by Fubini and by Wilczynski for $n=2$. In § 4.2 we explain how to normalize a defining equation of a hypersurface and in § 4.3 an explicit calculation of invariants is given by use of this normalization. The case $n=1$ is remarked in § 4.4.

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§ 1. Fundamental invariants of a non-degenerate hypersurface

Let P^{n+1} be a real projective space of dimension $n+1$ with homogeneous coordinate system $(\chi^0, \chi^1, \ldots, \chi^{n+1})$. The group of projective transformations $G=PL(n+1,\mathbf{R})$ is by definition the group $SL(n+2,\mathbb{R})$ modulo its center. The action of $SL(n+2,\mathbb{R})$ on $\mathbb{R}^{n+2}$ induces the action of G on P^{n+1}. The isotropy group is denoted by H. Thus $G \longrightarrow P^{n+1}=G/_H$ is a H-principal bundle. Fix a basis $e^0=\{e_0{}^0, \ldots, e_{n+1}{}^0\}$ of $\mathbb{R}^{n+2}$. Then each element g of G defines another basis $e=ge^0$ which we call a projective frame. In the following discussion we may assume each projective frame $e=\{e_0, \ldots, e_{n+1}\}$ satisfies

(1.1) $$\det (e_0, \ldots\ldots, e_{n+1}) = 1 .$$

We define forms $\omega_\alpha{}^\beta$ on the group G by

(1.2) $$de_\alpha = \omega_\alpha{}^\beta e_\beta .$$

Then the equation of Maurer-Cartan is

(1.3) $$d\omega_\alpha{}^\beta = \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta .$$

The condition (1.1) implies

(1.4) $$\omega_\alpha{}^\alpha = 0 .$$

Here we use the summation convention and the index range is assumed to be $0 \leq \alpha, \beta, \ldots \leq n+1$. The indices $i, j, \ldots$ used below are assumed to run from 1 to n. Let M be a piece of hypersurface in the projective space P^{n+1}. The aim of this section is to define some fundamental invariants of the hypersurface M. Let e be a local projective frame field over M. It is a local section on M of the bundle $G \longrightarrow P^{n+1}$ due to the above identification. We write the induced form $e^*\omega$ of ω by e also by ω. Restrict our consideration to a local frame e which satisfies the condition

$$(1.5) \qquad \omega_0^{n+1} = 0, \quad \{\omega_0^1, \ldots, \omega_0^n\} \text{ are linearly independent.}$$

Then another such a frame $\tilde{e}$ is given by

$$(1.6) \qquad \tilde{e} = ge$$

where

$$g = \begin{pmatrix} \lambda & 0 & 0 \\ b & a & 0 \\ \mu & c & \nu \end{pmatrix}, \quad \det g = 1$$

$$a = (a_i^j), \; b = (b_i), \; c = (c^i).$$

The induced form $\tilde{e}^{*}\omega$ being denoted by $\tilde{\omega}$, the equation of a frame change is

$$(1.7)\qquad \tilde{\omega} = dg.g^{-1} + g\omega g^{-1} .$$

Now (1.5) implies

$$0 = d\omega_0^{n+1} = \omega^i \wedge \omega_i^{n+1}$$

and ω_i^{n+1} can be written as

$$(1.8)\qquad \omega_i^{n+1} = h_{ij}\omega^j \ , \ h_{ij} = h_{ji} .$$

Here, for shortage of notation, we write ω^α for ω_0^α. We define

$$(1.9)\qquad \phi_2 = h_{ij}\omega^i\omega^j \ , \ h = (h_{ij}) \ \text{ and } \ H = \det h .$$

In order to see the frame dependence of these quantities, we use a component-wise writing of the equation (1.7):

$$(1.10)\qquad \left\{\begin{aligned} \tilde{\omega}^0 &= \omega^0 + d\log\lambda - b_i A^i_j \omega^j \\ \tilde{\omega}^i &= \lambda A^i_j \omega^j \\ \tilde{\omega}_i^{n+1} &= \nu^{-1} a_i^{\ k} \omega_k^{n+1} \\ \tilde{\omega}_i^{k} &= da_i^{\ j} A_j^{\ k} + a_i^{\ l}\omega_l^{\ j} A_j^{\ k} + b_i \omega^j A_j^{\ k} - \nu^{-1} a_i^{\ l} \omega_l^{n+1} c^j A_j^{\ k} \end{aligned}\right.$$

$$\left\{\begin{array}{l}\tilde{\omega}_{n+1}{}^{n+1} = \omega_{n+1}{}^{n+1} + d\log\nu + \nu^{-1}c^i\omega_j{}^{n+1} \\ \tilde{\omega}_{n+1}{}^i + c^j A^i{}_j \tilde{\omega}_{n+1}{}^{n+1} = (dc^j + c^k\omega_k{}^j + \mu\omega^j + \nu\omega_{n+1}{}^j)A_j{}^i \\ \lambda\tilde{\omega}_i{}^0 + \tilde{\omega}_i{}^j b_j + \mu\tilde{\omega}_i{}^{n+1} = a_i{}^j\omega_j{}^0 + db_i + b_i\omega^0 \\ \lambda\tilde{\omega}_{n+1}{}^0 + b_i\tilde{\omega}_{n+1}{}^i + \mu\tilde{\omega}_{n+1}{}^{n+1} = \nu\omega_{n+1}{}^0 + d\mu + \mu\omega^0 + c^i\omega_j{}^0 \ ,\end{array}\right.$$

where $(A_i{}^j)$ means the inverse matrix of $(a_i{}^j)$. From now on the quantities corresponding to the frame $\tilde{e}$ are written with tildes $\tilde{}$. Then from the equation (1.10) follow the next formulas.

$$\tilde{h} = (\lambda\nu)^{-1} a h {}^t a \ , \tag{1.11}$$

$$\tilde{H} = (\det a)^{n+2} H \ , \tag{1.12}$$

$$\frac{1}{n+2}\, d\log\tilde{H} + \tilde{\omega}^0 + \tilde{\omega}_{n+1}{}^{n+1} = \frac{1}{n+2}\, d\log H + \omega^0 + \omega_{n+1}{}^{n+1} + (\nu^{-1}c^j h_{jk} - b_j A_k{}^j)\omega^k \ . \tag{1.13}$$

We now make a basic

<u>Assumption</u> 1.1. The matrix $h=(h_{ij})$ is throughout non-degenerate,

which turns out to be independent of the choice of frames due to (1.11). Then, from (1.12) and (1.13), we can see that a frame may be supposed to satisfy further restricions

$$|H| = 1 \ , \tag{1.14}$$

$$\omega^0 + \omega_{n+1}{}^{n+1} = 0 \ . \tag{1.15}$$

From these restrictions follows that $|\det a| = 1$ and $|\lambda\nu| = 1$. We now assume that h and $\tilde{h}$ have the same signature and non-negative index's. Then $\lambda\nu = 1$ unless index $h = 0$. Even when index $h = 0$, we treat only the case $\lambda\nu = 1$ to simplify writings. Then the matrix a is a special orthogonal transformation with respect to h and $\tilde{h}$. The frame change is now restricted to

$$(1.16) \qquad b_i = \lambda c^i h_{jk} a_i{}^k, \ \lambda\nu = 1, \ \det a = 1 .$$

We next take an exterior derivation of (1.8) and use (1.7) to get

$$(dh_{ij} - h_{ik}\omega^k{}_j - h_{jk}\omega^k{}_i) \wedge \omega^j = 0 .$$

Then it is possible to put

$$(1.17) \qquad dh_{ij} - h_{ik}\omega_j{}^k - h_{jk}\omega^k{}_i = h_{ijk}\omega^k ;$$

(h_{ijk}) is symmetric.

By this definition

$$d\log H = h^{ij} dh_{ij} = h^{ij} h_{ijk}\omega^k + 2\omega_j{}^j ,$$

where (h^{ij}) is the inverse of (h_{ij}). So the conditions (1.14) and (1.15) imply

$$(1.18) \qquad h^{ij} h_{ijk} = 0 .$$

This is called the *apolarity* condition on (h_{ijk}). The straightforward calculation by use of (1.10) and (1.16) shows

<u>Lemma</u> 1.2. $\lambda\tilde{h}_{ijk} = h_{pqr}a_i{}^p a_j{}^q a_k{}^r$.

Let us define

(1.19)
$$\phi_3 = h_{ijk}\omega^i\omega^j\omega^k ,$$
$$F = h_{ijk}\, h_{pqr}\, h^{ip}h^{iq}h^{kr} .$$

The cubic form ϕ_3 is called the *Fubini-Pick form* and the scalar F , which might be thought of the norm of ϕ_3 , is called the *Fubini-Pick invariant*. These invariants together with ϕ_2 are the fundamental invariants of a hypersurface. From Lemma 1.2 and the formula (1.11) one has

<u>Proposition</u> 1.3. (1) By a frame change there hold

$$\tilde{\phi}_2 = \lambda^2\phi_2 , \quad \tilde{\phi}_3 = \lambda^2\phi_3 , \quad \lambda^2\tilde{F} = F .$$

(2) $F\phi_2$ is invariant.

We call this invariant form $F\phi_2$ the <u>projective metric</u>, although degenerate when $F=0$. The associated area element $|F|^{n/2}\,dA$, where $dA = \sqrt{|H|}\,\omega^1 \wedge \ldots\ldots \wedge \omega^n$, defines an area functional. A criterial hypersurface with respect to this functional is called *projectively minimal* ([1], [2]). Several interesting local results are known for a projectively minimal surface of hyperbolic type; see vol. 2 of [2]. Higher dimensional examples and a global result for a surface will be given in [18].

We further continue to restrict frames. Take an exterior derivative of (1.15) to get

$$(1.20) \qquad (h_{ij}\omega_{n+1}{}^{j} - \omega_{i}{}^{0}) \wedge \omega^{i} = 0 .$$

This enables us to put

$$(1.21) \qquad h_{ij}\omega^{i}{}_{n+1} - \omega_{i}{}^{0} = L_{ij}\omega^{i};\ L_{ij} = L_{ji} ,$$

and define L by

$$(1.22) \qquad L = L_{ij}h^{ij}/n .$$

Then a lenghty calculation by use of (1.10) shows

$$\lambda\tilde{L}_{ij} = \nu L_{pq}a_{i}{}^{p}a_{j}{}^{q} + (2\mu - \lambda c^{k}c^{l}h_{kl})\tilde{h}_{ij} - c^{p}h_{pqr}a_{i}{}^{q}a_{j}{}^{r} .$$

Hence

$$\lambda\tilde{L} = \nu L + (2\mu - \lambda c^{k}c^{l}h_{kl}) .$$

Consequently, one can assume renaming frames that the frame is so chosen that the condition

(1.23) $$L = 0$$

is always satisfied and that the frame change satisfy

(1.24) $$\mu = \frac{1}{2}\lambda c^i c^i h_{ij} .$$

With this restriction the L_{ij} transforms as

(1.25) $$\lambda^2 \tilde{L}_{ij} = (L_{kl} - \lambda h_{klm} c^m) a_i{}^k a_j{}^l .$$

In summary we have

<u>Proposition</u> 1.4. (1) For a given hypersurface which satisfies Assumption 1.1, there is a local projective frame field satisfying

$\omega^{n+1}=0$, $\{\omega^1, \ldots, \omega^n\}$ are linearly independent,

$|H|=1$, $\omega_0{}^0+\omega_{n+1}{}^{n+1}=0$ and $L=0$.

(2) Another frame satisfying this condition is given by a transformation of the form

$$\begin{pmatrix} \lambda & & \\ b & a & \\ \mu & c & \lambda^{-1} \end{pmatrix}, \quad \begin{aligned} b_i &= \lambda a_i{}^k h_{kj} c^j \\ \mu &= \tfrac{1}{2}\lambda c^i c^j h_{ij} . \end{aligned}$$

If a frame change is restricted in this way, formulas in (1.10) become simpler. The last three of (1.10), in fact, are rewritten as in

$$\begin{aligned} &\lambda\tilde{\omega}_i{}^0 = a_i{}^k\omega_k{}^0 + \{d(b_j A^j{}_k) - b_j A_l{}^j \omega_k{}^l\} a_i{}^k + b_i \omega^0 \\ &\qquad + \lambda\mu a_i{}^l \omega_l{}^{n+1} - b_i b_j A_k{}^j \omega^k , \\ (1.26)\qquad &\tilde{\omega}_{n+1}{}^i a_i{}^j = \lambda^{-1}\omega_{n+1}{}^j + dc^j + c^k\omega_k{}^j + c^j(\omega^0 + d\log\lambda) \\ &\qquad + \mu\omega^j - \lambda c^j c^k \omega_k{}^{n+1} , \\ &\tilde{\omega}_{n+1}{}^0 = \lambda^{-2}\omega_{n+1}{}^0 - (\lambda^{-1} c^i L_{ik} - \tfrac{1}{2} c^i c^j h_{ijk})\omega^k . \end{aligned}$$

We put here

$$(1.27) \qquad \omega_{n+1}{}^0 = -\gamma_j \omega^j .$$

Then the last of (1.26) implies

$$(1.28) \qquad \tilde{\gamma}_j = \lambda^{-2}\gamma_j + \lambda^{-1} c^i L_{ij} - \tfrac{1}{2} c^i c^k h_{ijk} .$$

This formula and the formula (1.25) say that $\{L_{ij}\}$ and $\{\gamma_j\}$ are also invariants on a hypersurface in a certain sense, which will be explained in the next section.

We shall next briefly explain a relation of the above reduction with an affine description of a hypersurface in the affine space A^{n+1} of dimension $n+1$. Let $f = (f_1,\ldots\ldots,f_{n+1})$ be a basis of A^{n+1} with the property

$$\det (f_1,\ldots\ldots,f_{n+1}) = 1 .$$

This is called a *unimodular* affine frame. Let $\omega_\alpha{}^\beta$ $(0 \leq \alpha \leq n+1,$ $1 \leq \beta \leq n+1)$ be the Maurer-Cartan form of the unimodular affine group. We write shortly $\omega^\beta = \omega_0{}^\beta$ as before. When given a hypersurface in A^{n+1} , we associate a frame satisfying

$$\omega^{n+1} = 0 .$$

This condition enables us also to write $\omega_i{}^{n+1} = h_{ij}\omega^j$ and to define ϕ_2 in the same manner. Furthermore, under the assumption of non-degeneracy of $h = (h_{ij})$, it is able to restrict a local frame field so that $\det h = 1$ and $\omega_{n+1}{}^{n+1} = 0$. A cubic form ϕ_3 is defined by a similar reasoning. This fact implies that ϕ_3 is an affinely defined object. Let us here recall that the Fubini-Pick form ϕ_3 measures the difference of a hypersurface from a quadratic hypersurface (see, f. ex. [9] § 12; cf. Corollary 2.11). Since there is now no $\omega_\alpha{}^0$, we can put $h_{ij}\omega^j{}_{n+1} = l_{ij}\omega^j$, $l_{ij} = l_{ji}$. This tensor l_{ij} is called the affine mean curvature tensor and $l = l_{ij} h^{ij}/n$ is called the *affine mean curvature*. It is known that the last vector f_{n+1} of a frame is affinely invariant (up to ± 1 in case index $h=0$) and it is called an *affine normal*. (See [3], [7])

Let us now consider this hypersurface in the projective space. Assume A^{n+1} is contained in P^{n+1} as an open set defined by the first coordinate $x^0 \neq 0$. Let $e=(1, 0, \ldots, 0)$ be a fixed vector and f_0 be the tantological vector on the hypersurface: $f_0(x)=x$. We define a projective frame $\bar{e}$ by

$$\bar{e}_0 = e + f_0 ,$$

$$\bar{e}_\alpha = f_\alpha , \ \alpha \neq 0 .$$

Then with respect to this frame the Maurer-Cartan form looks like

$$(1.29) \qquad \begin{pmatrix} 0 & \omega^j & 0 \\ 0 & \omega_i{}^j & \omega_i{}^{n+1} \\ 0 & \omega^j{}_{n+1} & 0 \end{pmatrix} .$$

Perform a transformation by

$$\begin{pmatrix} 1 & & \\ & I & \\ -\frac{1}{2} & & 1 \end{pmatrix} .$$

Then the frame obtained satisfies the condition (1) of Proposition 1.4: namely the corresponding coframe is given by

$$(1.30)\qquad \begin{pmatrix} 0 & \omega^i & 0 \\ \frac{1}{2}\omega_i{}^{n+1} & \omega_i{}^j & \omega_i{}^{n+1} \\ -\frac{1}{2}\,dl & -\frac{1}{2}\omega^i+\omega^j{}_{n+1} & 0 \end{pmatrix},$$

so that

$$(1.31)\qquad \begin{aligned} L_{ij} &= l_{ij} - l\,h_{ij} \\ \gamma_j &= \frac{1}{2}\,l_j \quad \text{where} \quad dl = l_j\omega^j\,. \end{aligned}$$

<u>Example</u> 1.5. It is known that the equation $l_{ij}-lh_{ij}=0$ characterizes an affine hypersphere and that in this case l is constant ([1]). Because a quadric is one of affine hyperspheres, (1.31) tells that $L_{ij}=\gamma_j=0$ in the above frame (for a direct verification see § 2). Combining this with the fact $\phi_3=0$, it holds $L_{ij}=\gamma_j=0$ always for a quadric.

We close this section by giving a corollary of the projective invariance of $F\phi_2$. Assume $n\geq 2$ and the hypersurface M is closed and strictly convex, i.e. ϕ_2 is definite. Then, on the set N where F does not vanish, the form $F\phi_2$ defines a Riemannian metric. Since any projective transformation which leave M invariant is an isometry with respect to this metric and keeps the set N , Theorem 3.2 in [13, chapter I] implies

Corollary 1.6 ([22],[25]). Let M be a closed strictly convex hypersurface in P^{n+1}, $n\geq 2$. Then the connected component of the group of projective transformations which leave M invariant is compact unless F vanishes everywhere, namely unless M is a quadric.

The above argument is due to Obata [16], Proposition 4.1. Note that the above proof is simple but needs some smoothness condition on M. Under a less restrictive condition this corollary was already known by Vinberg and Kats in [22] and by J. Benzécri in [25] with a close look on projective transformations.

§ 2. A normal conformal connection of a non-degenerate hypersurface

Because of the conformal invariance of a fundamental two form ϕ_2, one can derive an associated normal conformal connection on a hypersurface satisfying Assumption 1.1. The aim of this section is to construct this connection and, by use of this connection, to formulate the fundamental theorem of hypersurfaces in the projective space P^{n+1}. We assume the dimension $n \geq 3$ throughout this section.

Let $h=(h_{ij})$ be a non-degenerate $n\times n$ symmetric matrix and put

$$I = \begin{pmatrix} & & -1 \\ & h & \\ -1 & & \end{pmatrix}$$

Let G_1 denote the orthogonal group with respect to I moduls its center: $G_1 = \{g \in GL(n+2, \mathbb{R});\ gI{}^tg = I\}/\text{center}$. Its Lie algebra is denoted by $\mathfrak{g}_1$. Let H_1 be the subgroup of G_1 whose element has a representative

$$\begin{pmatrix} \lambda & & \\ b & a & \\ \mu & c & \lambda^{-1} \end{pmatrix} ; \qquad \begin{aligned} ah{}^ta &= h \\ b &= \lambda ah{}^tc \\ \mu &= \tfrac{1}{2}\lambda ch{}^tc \ . \end{aligned}$$

Then the homogeneous space ${}^{G}1/H_1$ is a quadric which is canonically embedded in P^{n+1}. The derivation of a normal conformal connection is now stated as follows: to find canonically a $\mathfrak{g}_1$-valued 1-form π satisfying a certain curvature condition. Since a general process how to obtain π is well-known (see [13] Theorem 4.2 in p. 135 ~ 136), it is simply necessary to relate π using ω. Assume π has the following form with an unknown 1-form τ.

(2.1) $$\pi = \omega + \tau .$$

The curvature from Ω of π is defined by

(2.2) $$\Omega = d\pi - \pi\wedge\pi .$$

We want to determine τ so that $\Omega_i{}^j$ is written as

(2.3) $$\Omega_i{}^j = \frac{1}{2}C_i{}^j{}_{kl}\pi^k\wedge\pi^l , \quad C_i{}^j{}_{kl}+C_i{}^j{}_{lk}=0 ,$$

and has a property

(2.4) $$\sum_j C_i{}^j{}_{jl} = 0$$

From now on, the rule of raising and lowering indices with respect to h will be used. We first define

$$(2.5)\qquad \tau_0{}^{\alpha} = \tau_{\alpha}{}^{n+1} = 0 \ , \ \tau_0{}^{n+1} = -\omega_0{}^{n+1}$$

$$\tau_i{}^{j} = \frac{1}{2} h^{jk} h_{ikl}\omega^l = \frac{1}{2} h_i{}^{j}{}_{l}\omega^l \ .$$

From this definition follows

$$(2.6)\qquad \pi^i = \omega^i \ ,$$

$$dh_{ij} - h_{ik}\pi_j{}^k - h_{jk}\pi_i{}^k = 0 \ .$$

The remaining components of τ are tentatively supposed to have a form

$$(2.7)\qquad \tau_i{}^0 = M_{ij}\omega^j + L_{ij}\omega^j \ , \ M_{ij} = M_{ji}$$

$$\tau_{n+1}{}^i = h^{ij}M_{jk}\omega^k = M^i{}_k\omega^k \ .$$

Then π has values in $\mathfrak{g}_1$ relative to h . We next see

<u>Lemma</u> 2.1. $\Omega_0{}^{\beta} = \Omega_{\alpha}{}^{n+1} = \Omega_{n+1}{}^{0} = 0$.

<u>Proof</u>. These are shown by the symmetry of h_{ij} , h_{ijk} , L_{ij} and M_{ij} . As an illustration

$$\begin{aligned}\Omega_0{}^0 &= d\pi_0{}^0 - \pi_0{}^{\alpha} \wedge \pi_{\alpha}{}^0 \\ &= d\omega^0 - \omega_0{}^i \wedge (\omega_i{}^0 + L_{ij}\omega^j + M_{ij}\omega^j) \quad \text{(definition of } \pi)\end{aligned}$$

$$= -\omega^i \wedge (L_{ij} + M_{ij})\omega^j \quad \text{(structure equation of } \omega)$$

$$= 0 \qquad \text{(structure of } L_{ij} \text{ and } M_{ij})$$

Since others are similarly shown, we do not reproduce them here.

The tensor $\Omega_i{}^j$ is by definition

$$\begin{aligned}\Omega_i{}^j &= d\pi_i{}^j - \pi_i{}^\alpha \wedge \pi_\alpha{}^j \\ &= d\tau_i{}^j - \tau_i{}^k \wedge \omega_k{}^j - \omega_i{}^k \wedge \tau_k{}^j - \tau_i{}^k \wedge \tau_k{}^j \\ &\quad - \omega_i{}^{n+1} \wedge \tau^j - (L_{ik} + M_{ik})\,\omega^k \wedge \omega^j .\end{aligned}$$

Since a calculation shows

$$\begin{aligned}- d(h_{ijk}\omega^k) &= h_{ikl}\omega^l \wedge \omega^k{}_j + h_{jkl}\omega^l \wedge \omega^k{}_i + L_{jk}\omega_i{}^{n+1} \wedge \omega^k \\ &\quad + L_{ik}\omega_j{}^{n+1} \wedge \omega^k ,\end{aligned}$$

using this to compute $d\tau_i{}^j$ one has a formula

$$\begin{aligned}C_i{}^j{}_{kl} &= \tfrac{1}{4}\,(h_{ikm}h^{jm}{}_l - h_{ilm}h^{jm}{}_k) \\ &\quad + (M_{il} + \tfrac{1}{2}L_{il})\delta_{jk} - (M_{ik} + \tfrac{1}{2}L_{ik})\delta_j{}^l \\ &\quad + (M^j{}_k + \tfrac{1}{2}L^j{}_k)h_{il} - (M^j{}_l + \tfrac{1}{2}L^j{}_l)h_{ik}. \end{aligned} \tag{2.8}$$

Hence,

$$C_i{}^j{}_{jl} = \frac{1}{4} K_{il} + (n-2)(M_{il} + \frac{1}{2} L_{il}) + M^j{}_j h_{il} ,$$

where we have put

$$K_{il} = h_{ijk} h^{ik}{}_l . \tag{2.9}$$

Therefore the condition (2.4) is satisfied only when

$$M_{ij} = -\frac{1}{4(n-2)} K_{ij} - \frac{1}{2} L_{ij} + \frac{F}{8(n-2)(n-1)} h_{ij} . \tag{2.10}$$

We introduce a new invariant f_{ij} by

$$f_{ij} = -\frac{1}{4(n-2)} K_{ij} + \frac{F}{8(n-2)(n-1)} h_{ij} . \tag{2.11}$$

Then the definition (2.7) is rewritten as

$$\begin{aligned} \tau_i{}^0 &= (f_{ij} + \frac{1}{2} L_{ij})\omega^j , \\ \tau^j{}_{r+1} &= h^{ik}(f_{jk} - \frac{1}{2} L_{ik})\omega^i . \end{aligned} \tag{2.7'}$$

<u>Proposition</u> 2.2. Let $\tilde{\tau}$ denote the form τ for a new frame ge . Then

$$\tilde{\tau} = g \tau g^{-1} .$$

<u>Proof.</u> By the definition (2.11) and by (1.10) it holds

$$\tilde{g}_{ij}\tilde{\omega}^i\tilde{\omega}^j = g_{ij}\omega^i\omega^j .$$

Using this identity and putting $\zeta_j = h_{jkl}\, c^l\omega^k$ for a moment, we can see following equalities from Lemma 1.2, (1.11), (1.25) and (1.26):

$$\tilde{\tau}_i{}^0 = \lambda^{-1}a_i{}^j\tau_j{}^0 - \frac{1}{2}\, a_i{}^j\zeta_j$$

$$\tilde{\tau}_{n+1}{}^j = \lambda^{-1}A^j{}_k\tau_{n+1}{}^k + \frac{1}{2}\, A^j{}_k\zeta^k$$

$$\tilde{\tau}_{n+1}{}^0 = \lambda^{-2}\tau_{n+1}{}^0 + \lambda^{-1}c^iL_{ij}\omega^j - \frac{1}{2}\, c^j\zeta_j .$$

The definition of $\tau_i{}^j$ gives a formula

$$\tilde{\tau}_i{}^j = a_i{}^k\tau_k{}^l A_l{}^j .$$

It is now immediate to see that these formulas together imply the result.

This proposition shows that if we define π by (2.1) making use of τ derived above, it has a following invariance by a chnage of frame:

(2.12) $$\tilde{\pi} = dg.g^{-1} + g\pi g^{-1} \quad \text{for} \quad g \in H_1.$$

This implies that π defines a normal conformal connection. To be more precise, we assume $h=(h_{ij})$ is a fixed constant non-degenerate matrix (cf.(1.11). Let P be the set of all frames satisfying the condition (1) of Proposition 1.6. Then the second part of this proposition means that P is a principal bundle over the hypersurface M with H_1 as a fibre group. Now the formula (2.12) says that there exists a $\mathfrak{g}_1$-valued 1-form on P whose restriction to every local section e is exactly π . We write again by π this form on P . On the other hand Proposition 2.2 implies that τ comes also from an intrinsic form on P , which we denote also by τ . This is a tensorial 1-form on P . So we have proved

<u>Proposition</u> 2.3. Let π and P be defined as above. Then π is a normal conformal connection on a hypersurface M . This is a Cartan connection on P of type G_1/H_1 .

As for a Cartan connection we refer to [13]. The bundle P is a subbundle of $G|_M \longrightarrow M$ defined in § 1. The relation (2.1) lifts to P .

The above reasoning enables us to say that the form τ is the second fundamental form in the projective case. The <u>Gauß equation</u> which expresses the curvature tensor by use of this form is given as follows.

<u>Proposition</u> 2.4. Let $\Omega_\alpha{}^\beta=\frac{1}{2}\, C_\alpha{}^\beta{}_{kl}\pi^k\wedge\pi^l$, $C_\alpha{}^\beta{}_{kl}+C_\alpha{}^\beta{}_{lk}=0$.

Then

(1) $$C_{ijkl} = h_{jm}\, C_i{}^m{}_{kl}$$
$$= \frac{1}{4}\,(h_{ikm}h_{lj}{}^m - h_{ilm}h_{jk}{}^m)$$
$$+ \frac{1}{4(n-2)}(h_{jl}K_{ik} - h_{jk}K_{il} + h_{ik}K_{jl} - h_{il}K_{jk})$$
$$+ \frac{1}{4(n-1)(n-2)}(h_{il}h_{jk} - h_{ik}h_{jl})F\ .$$

(2) $$C_{ijk} := C_i{}^0{}_{jk}$$
$$= f_{ik,j} - f_{ij,k} + \frac{1}{2}\,(h_{ij}{}^l L_{lk} - h_{ik}{}^l L_{lj})$$

where

(2.13) $$f_{ij,k}\pi^k = df_{ij} - f_{ik}\pi_j{}^k - f_{jk}\pi_i{}^k + 2f_{ij}\pi^0\ .$$

(3) $$\Omega^j{}_{n+1} = h^{ji}\Omega_i{}^0\ .$$

Proof. (1) is direct from (2.8) and (2.9). (3) is obvious by definition. For (2), we see by definition

$$\Omega_i{}^0 = d\tau_i{}^0 - \tau_i{}^j\wedge\tau_j{}^0 - \tau_i{}^0\wedge\pi^0 - \omega_i{}^{n+1}\wedge\tau_{n+1}{}^0$$
$$-\omega_i{}^j\wedge\tau_j{}^0 - \tau_i{}^0\wedge\omega_{n+1}{}^0\ .$$

First show

$$d\tau_i{}^0 = \frac{1}{2}(h_{ijk}\omega^k\wedge\omega^j{}_{n+1} - 2\omega_i{}^{n+1}\wedge\omega_{n+1}{}^0 + \omega_i{}^j\wedge L_{jk}\omega^k + L_{ik}\omega^k\wedge\omega^0)$$
$$+(df_{ik}\wedge\omega^k - f_{ij}\omega^j{}_k\wedge\omega^k + f_{ik}\omega^0\wedge\omega^k)\,,$$

and then insert this to the above equation. Several cancelations by use of idebtities defining τ and L will prove (2). Here note that the right handside of (2.13) is the covariant derivative of f_{ij} with respect to $\pi_i{}^j - \delta_i{}^j \pi^0$ and defines a tensorial form on M .

We next derive the Codazzi-Minardi equation. Define covariant derivatives of τ by

(2.14) $\quad D\tau = d\tau - \tau\wedge\pi - \pi\wedge\tau$,

then, from the structure equation (1.3)

(2.15) $\quad \Omega = D\tau + \tau\wedge\tau$.

In order to write down explicitly this equation, we will introduce covariant derivatives of coefficients of τ by the following equations

$$(2.16)\quad \begin{aligned} h_{ijk,l}\pi^l &= dh_{ijk} - h_{ljk}\pi^l{}_i - h_{ilk}\pi^l{}_j - h_{ijl}\pi^l{}_k + h_{ijk}\pi^0 , \\ L_{ij,k}\pi^k &= dL_{ij} - L_{kj}\pi^k{}_i - L_{ik}\pi_j{}^k + 2L_{ij}\pi^0 + h_{ij}{}^l\pi_l{}^0 , \\ \gamma_{i,j}\pi^j &= d\gamma_i - \gamma_j\pi_i{}^j + 3\gamma_i\pi^0 + L_{ij}\pi^j{}_{n+1} . \end{aligned}$$

Because of the transformation rule of τ which is explicitly written in Lemma 1.2., (1.25) and (1.28), these definitions are natural in the sense that the right handsides become again tensorial forms. We now have

Proposition 2.5. (Codazzi-Minardi). The equation (2.15) is equivalent to the symmetry of h_{ijk} and L_{ij} and the equations

(1) $h_{ijk,l}-h_{ijl,k}=L_{il}h_{jk}-L_{ik}h_{jl}+L_{jl}h_{ik}-L_{jk}h_{il}$,

(2) $L_{ij,k}-L_{ik,j} \quad =h_{ij}{}^{l}f_{lk}-h_{ik}{}^{l}f_{lj}+2(h_{ik}{}^{\gamma}{}_{j}-h_{ij}{}^{\gamma}{}_{k})$,

(3) $\gamma_{i,j}-\gamma_{j,i} \quad =L_{il}f^{l}{}_{j}-L_{jl}f^{l}{}_{i}$.

Proof. The (0,0)-th component of the right handside of (2.15) is

$$\begin{aligned} d\tau_0{}^0 - \tau_0{}^{\alpha}\wedge\pi_{\alpha}{}^0 - \pi_0{}^{\alpha}\wedge\tau_{\alpha}{}^0 + \tau_0{}^{\alpha}\wedge\tau_{\alpha}{}^0 \\ = -\pi^i\wedge\tau_i{}^0 \\ = -(f_{ij}+\tfrac{1}{2}L_{ij})\pi^i\wedge\pi^j . \end{aligned}$$

Similarly the $(n+1, n+1)$-, $(0,i)$- and $(j, n+1)$-th components are $-(f_{ij}-\frac{1}{2}L_{ij})\pi^i\wedge\pi^j$, $-\frac{1}{2}h^i{}_{jk}\pi^j\wedge\pi^k$ and $-\frac{1}{2}h_{ijk}\pi^i\wedge\pi^k$ respectively. So the vanishing of $\Omega_0{}^{\alpha}$ and $\Omega_{\beta}{}^{n+1}$ implies the symmetry. To get (1) calculate first

$$h_{jk}D\tau^k{}_i=\{\tfrac{1}{2}h_{ijk,l}-(f_{il}+\tfrac{1}{2}L_{il})h_{jk}+(f_{jl}-\tfrac{1}{2}L_{jl})h_{ik}\}\pi^l\wedge\pi^k$$

and then use the formula (1) of Proposition 2.4. Similarly the $(i,0)$-th and the $(n+1,j)$-th components give the same equation (2) and the remaining $(n+1,0)$-th component gives (3).

Corollary 2.6. (1) $L_{ij} = -\frac{1}{n}h_{ijk,}{}^{k}$.

$$(2) \quad \gamma_i = \frac{1}{2(n-1)} L_{ij,}{}^{j} + \frac{1}{8(n-1)(n-2)} h_i{}^{jk} K_{jk} .$$

Proof. Take a contraction of (2.16) relative to h_{ij} and use the apolarity condition (1.18) and the trace condition (1.23) to get

$$h^{ij} h_{ijk,l} = h^{ij} L_{ij,k} = 0$$

Then the contraction of (1) and (2) of Proposition 2.5 gives the result.

The first relation of this corollary says that the curvature tensor C_{ijk} is expressible in terms of covariant derivatives of h_{ijk} . In special case we have

Corollary 2.7. $C^i{}_{ij} = -\frac{1}{4} h_j{}^{kl} L_{kl}$.

Proof. By the definition (2.13) of $f_{ij,k}$

$$f_i{}^i{}_{,j} = -\frac{1}{8(n-1)} F_{,j} \quad \text{where} \quad dF + 2F\pi^0 = F_{,j}\pi^j .$$

$$f^i{}_{j,i} = -\frac{1}{4(n-2)} (h^i{}_{pq,i} h_j{}^{pq} + h^i{}_{pq} h_j{}^{pq}{}_{,i}) + \frac{F_{,j}}{8(n-1)(n-2)} .$$

Then (1) of Proposition 2.5 shows

$$h^i{}_{pq} h_j{}^{pq}{}_{,i} = h^i{}_{pq}(h_j{}^{pq}{}_{,i} - h_i{}^{pq}{}_{,j}) + h^i{}_{pq} h_i{}^{pq}{}_{,j}$$

$$= 2h_j{}^{pq} L_{pq} + \frac{1}{2} F_{,j} .$$

Thus we have

$$f^i{}_{j,i} - f^i{}_{i,j} = \frac{1}{4} h_j{}^{kl} L_{kl} \, .$$

The previous corollary then implies the result.

The fundamental theorem of a hypersurface in the projective case can now be stated as follows.

<u>Theorem</u> 2.8. Let M be a n-dimensional manifold with a normal conformal connection π $(n \geq 3)$. Let τ be a tensorial 1-form with symmetric coefficients of the form as illustrated in (2.5) and (2.7). Assume that τ satisfies the covariant relation in Proposition 2.5 and the curvature tensor of π satisfies the relation in Lemma 2.1 and Proposition 2.4. Then, for a given point p of M , there exists a neighborhood of p which can be embedded as a non-degenerate hypersurface in the projective space of dimension $n+1$ so that π and τ are the connection and the invariant induced by this embedding respectively. This embedding is unique up to a projective transformation.

<u>Proof</u>. Given π and τ , define $\omega=\pi-\tau$. The conditions on π and τ imply that $d\omega=\omega\wedge\omega$. This says that one can solve the differential equation (1.2):$de = \omega e$ and we have Theorem. The ambiguity depends on the choice of initial conditions.

<u>Corollary</u> 2.9. Assume $\phi_3 = 0$ and $n \geq 3$. Then the hypersurface

is locally a quadric.

Proof. From $\phi_3 = 0$ follows $\Omega = 0$ and $\tau = 0$. Then Theorem 2.10 implies the result since $\phi_3 = 0$ for a quadric.

The Bianchi identity is as usual given by differentiating the defining equation of the curvature tensor. That is,

$$(2.17) \qquad d\Omega = \pi\wedge\Omega - \Omega\wedge\pi \ .$$

We here define covariant derivatives of the curvature tensor by

$$(2.18) \qquad C_{ijkl,m}\pi^m = dC_{ijkl} - C_{mjkl}\pi_i{}^m - C_{imkl}\pi_j{}^m - C_{ijml}\pi_k{}^m - C_{ijkm}\pi_l{}^m + 2C_{ijkl}\pi^0 ,$$

$$(2.19) \qquad C_{ijk,l}\pi^l = dC_{ijk} - C_{ljk}\pi_i{}^l - C_{ilk}\pi_j{}^l - C_{ijl}\pi_k{}^l + 3C_{ijk}\pi^0 + C_i{}^l{}_{jk}\pi_l{}^0 \ .$$

Proposition 2.10. The Bianchi identity (2.17) implies

(1) $\quad S(jkl)C_{ijkl} = 0, \ S(ijk)C_{ijk} = 0,$

(2) $\quad S(klm)(C_{ijkl,m} - h_{im}C_{jkl} + h_{jm}C_{ikl}) = 0,$

$\quad S(jkl)C_{ijk,l} = 0,$

where $S(i\ldots j)$ means a operator to take a cyclic summation from i to j.

Proof. The identity (2.17) for indices (0,0) and (0,i) are

$$\pi^i\wedge\Omega_i{}^0 = \pi^j\wedge\Omega_j{}^i = 0 .$$

This implies (1) as usual. Components with indices $(n+1,n+1)$ and $(i,n+1)$ give the same result. The (i,j)-th component and the $(i,0)$-th component are respectively

$$d\Omega_i{}^j-\pi_i{}^k\wedge\Omega_k{}^j+\Omega_i{}^k\wedge\pi_k{}^j-\pi_i{}^{n+1}\wedge\Omega_{n+1}{}^j+\Omega_i{}^0\wedge\pi^j=0,$$

$$d\Omega_i{}^0-\pi_i{}^j\wedge\Omega_j{}^0+\Omega_i{}^j\wedge\pi_j{}^0-\pi_i{}^{n+1}\wedge\Omega_{n+1}{}^0+\Omega_i{}^0\wedge\pi^0=0.$$

Then (2.18) and (2.19) imply (2). The $(n+1,i)$-th component gives also the second of (2). The component with index $(0,n+1)$ is trivial and that with index $(n+1,0)$ is also trivial in view of the relation (3) of Proposition 2.4.

By taking contractions of (1) and (2), we have

Corollary 2.11. (1) $(n-3)C_{ijk}=C_{lijk,}{}^{l}-h_{ij}C^{l}{}_{lk}+h_{ik}C^{l}{}_{lj}$.

(2) When $n\geq 4$, $\Omega_i{}^j=0$ implies $\Omega_i{}^0=\Omega^j{}_{n+1}=0$.

Applying these calculations we will prove

<u>Theorem</u> 2.12. Assume that a hypersurface is strictly convex, i.e. h is definite, and that $\Omega_i{}^j=0$ in case $n\geq 4$ and $\Omega_i{}^0=0$ in case $n=3$. Then the hypersurface is locally a quadric.

<u>Proof</u>. The assumption says that the connection π is flat due to Corollary 2.11. Then, from Corollary 2.8, $h_j{}^{kl}L_{kl}=0$ for any choice of a normalized frame. However the formula (1.25) implies

$$\lambda^3\tilde{h}_j{}^{kl}\tilde{L}_{kl}= a_j{}^p h_p{}^{qr}L_{qr}- \lambda a_j{}^p K_{pq}C^q \ .$$

Hence $K_{pq}=0$ and $F=K_p{}^p=0$. This shows $\phi_3=0$ by the definiteness of h and Corollary 2.9 proves the theorem.

This theorem can be regarded as a rigidity theorem of a special kind. Note that this theorem and a theorem by Obata [16, Theorem I] together imply Corollary 1.6 for $n\geq 3$.

§ 3. The case $F \neq 0$

In this section we treat a hypersurface whose Fubini-Pick invariant does not vanish everywhere. This is the case that attracted classically much interest. As we shall soon see, the formulation becomes extremely simple in this case.

Recall first the identity $\lambda^2\tilde{F}=F$. Taking derivatives,

$$\mathrm{dlog}\,\tilde{F} = \mathrm{dlog}\,F - 2\,\mathrm{dlog}\lambda \ ,$$

hence the first formula of (1.10) is rewritten as

$$\tilde{\omega}^0 + \frac{1}{2}\,\mathrm{dlog}\,\tilde{F} = \omega^0 + \frac{1}{2}\,\mathrm{dlog}\,F - b_i A_j{}^i \omega^j \ .$$

This allows us to choose b_i so that the left handside is zero. Renaming a frame we may assume

(3.1) $$\omega^0 = -\frac{1}{2}\,\mathrm{dlog}\,F \ , \ b_i = 0 \ .$$

We next consider F as a function given in advance, then $|\lambda|=1$ and the group H reduces the orthogonal group module its centre. This means that the hypersurface admits locally a pseudo-Riemannian structure which is projectively invariant. The fundamental tensor g_{ij} (assume index $g_{ij} \geq 0$) and the connection form $\sigma_i{}^j$ are given by

$$(3.2) \qquad g_{ij} = |F| h_{ij} \; , \; \sigma_i{}^j = \pi_i{}^j - \pi^\circ \delta_i{}^j \; .$$

Usually we take ± 1 as F to simplify the formula. In this case

$$(3.3) \qquad \omega^0 = 0 \; , \; g_{ij} = h_{ij} \; , \; \sigma_i{}^j = \pi_i{}^j \; .$$

We shall consider this case hereafter. Then e_0 and e_{n+1} are uniquely determined up to sign ± 1. The vector e_{n+1} is called a *projective normal* vector of a hypersurface. Refer the book [2]. The formulas of a frame change are now reduced to

$$\tilde{\omega}^j = \omega^i A_i{}^j \; , \; \tilde{\omega}_i{}^{n+1} = a_i{}^k \omega_k{}^{n+1} \; ,$$

$$(3.4) \qquad \tilde{\omega}_i{}^k = da_i{}^j A_j{}^k + a_i{}^l \omega_l{}^j A_j{}^k \; , \; \tilde{\omega}_i{}^0 = a_i{}^j \omega_j{}^0 \; ,$$

$$\tilde{\omega}_{n+1}{}^i = \omega_{n+1}{}^j A_j{}^i \; , \; \tilde{\omega}_{n+1}{}^0 = \omega_{n+1}{}^0 \; .$$

So we may put

$$(3.5) \qquad \omega_j{}^0 = T_{jk} \omega^k \; , \; \omega_{n+1}{}^j = h^{jk} S_{kl} \omega^l \; ,$$

so that from (3.3),

$$(3.6) \qquad \tilde{T}_{ij} = a_i{}^k T_{kl} a_j{}^l \; , \; \tilde{S}_{ij} = h_i{}^k S_{kl} a_j{}^l \; , \; \tilde{\gamma}_i = a_i{}^j \gamma_j \; .$$

The invariant L_{ij} is given by

(3.7) $$L_{ij} = S_{ij} - T_{ij} .$$

We put

(3.8) $$U_{ij} = S_{ij} + T_{ij} .$$

The next proposition gives a relation connecting the normal vector e_{n+1} and the first vector e_0 ; this is a projective analogue of the relation in the affine geometry (see [9], [19]).

<u>Proposition</u> 3.1. Let Δ be the Laplacian of the metric g_{ij} . Then

$$\Delta e_0 = n e_{n+1} + \mathrm{Tr}(T) e_0 .$$

<u>Proof</u>. Since $de_0 = \omega^i e_i$, the covariant derivative of e_0 is e_i . The derivative of e_i is

$$de_i - e_j \sigma_i{}^j = \omega_i e_0 + \omega_i{}^{n+1} e_{n+1} - \tau_i{}^k e_k ,$$

so $e_{i,j} = T_{ij} e_0 - \frac{1}{2} h_{ij}{}^k e_k + g_{ij} e_{n+1}$. Then taking traces, we have the formula.

<u>Remark</u> 3.2. Let us consider an affinely homogeneous hyperbolic affine hypersphere which is not a quadric; f. ex. $M=\{x \in A^{n+1};\ x^1 \cdot \ldots \cdot x^{n+1}=1,\ x^i>0\}$ $(n \geq 3)$. Then the coframe (1.29) already satisfies $|F|=\mathrm{const} \neq 0$, so does the coframe (1.30). There-

fore (1.30) is the normalized coframe in the sense above. Then it is seen that $Tr(T) = const.1$, in particular $Tr(T) \neq 0$, i.e. $Tr(T)$ is a nontrivial absolute invariant.

The Codazzi-Minardi equation is summarized in the next

Proposition 3.3. (1) S_{ij}, T_{ij}, L_{ij} and U_{ij} are all symmetric tensors.

(2) $h_{ijk,l}-h_{ijl,k}=g_{ik}L_{jl}-g_{il}L_{jk}+g_{jk}L_{il}-g_{jl}L_{ik}$.

(3) $T_{ij,k}-T_{ik,j}=\frac{1}{2}(h_{ij}{}^{l}T_{lk}-h_{ik}{}^{l}T_{lj})+(g_{ij}\gamma_k-g_{ik}\gamma_j)$,

$S_{ij,k}-S_{ik,j}=-\frac{1}{2}(h_{ij}{}^{l}S_{lk}-h_{ik}{}^{l}S_{lj})-(g_{ij}\gamma_k-g_{ik}\gamma_j)$,

$L_{ij,k}-L_{ik,j}=-\frac{1}{2}(h_{ij}{}^{l}U_{lk}-h_{ik}{}^{l}U_{lj})-2(g_{ij}\gamma_k-g_{ik}\gamma_j)$,

$U_{ij,k}-U_{ij,k}=-\frac{1}{2}(h_{ij}{}^{l}L_{lk}-h_{ik}{}^{l}L_{lj})$

$\gamma_{i,j}-\gamma_{j,i}=T_{jk}L^{k}{}_{i}-T_{ik}L^{k}{}_{j}$.

(4) $h_{jik,}{}^{k} = -nL_{ij}$

$L_{ij,}{}^{j} = -\frac{1}{2}h_{i}{}^{jk}U_{jk}+2(n-1)\gamma_i$.

The proof is similar to that of Proposition 2.5 and Corollary 2.6. The calculation of the Riemann curvature tensor R_{ijkl} is also a routine. Namely one has

Proposition 3.4. (1) $R_{ijkl}=\frac{1}{2}(U_{ik}g_{jl}-U_{il}g_{jk}+U_{jl}g_{ik}-U_{jk}g_{il})$

$$+ \frac{1}{4}(h_{ik}{}^{m}h_{jlm} - h_{jk}{}^{m}h_{ilm}) \ .$$

(2) The Ricci tensor $R_{ij} = R_i{}^k{}_{kj}$ and the scalar curvature $R = R_i{}^i$ are given by

$$R_{ij} = -\frac{1}{2}(n-2)U_{ij} - \frac{1}{2}T_r(U)g_{ij} + \frac{1}{4}K_{ij} \ ,$$

$$R = -(n-1)T_r(U) + \frac{1}{4}F \ .$$

§ 4. Some remarks

In this section we shall make four remarks. The first one is to remark to § 3. The second remark is to interpret the reduction in § 1 in terms of coefficients of a defining equation of a hypersurface. Using this we list an explicit expression in the third remark. The final remark treats the case $n=1$.

1. Fubini frame and Wilczynski frame. The normalized frame in § 3 was first considered by Fubini. Let $x=x(u,v)$ be a parametric representation of a two-dimensional surface. Assume the surface is of hyperbolic type and the coordinate (u,v) is chosen so that $e^{2\theta}:=|\det(x, x_u, x_v, x_{uv})|\neq 0$, $\det(x, x_u, x_v, x_{uu}) = \det(x, x_u, x_v, x_{vv})=0$. Then the surface satisfies

$$(4.1)\qquad \begin{cases} x_{uu} = \theta_u x_u + \beta x_v + px \\ x_{vv} = \gamma x_u + \theta_v x_v + qx . \end{cases}$$

Fubini's normalization is, in this situation, $e^{\theta}=8|\beta\gamma|$ (see [10] Chapter IV, [15] p. 123). To be more precise, we choose a projective frame given by

$$e_0=x,\ e_1=\lambda x_u,\ e_2=\lambda x_v,\ e_3=\mu x+\lambda^2 x_{uv} ,$$

where

$$\lambda=e^{-\frac{1}{2}\theta},\ \mu=-\frac{1}{2}\lambda^2(\theta_{uv}+\beta\gamma) .$$

Then we have

$$\omega^1 = \frac{du}{\lambda},\ \omega^2 = \frac{dv}{\lambda},\ h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\phi_2 = e^{\phi} du dv,\ \phi_3 = -2e^{\theta}(\beta du^3 + \gamma dv^3)$$

$$\omega_0{}^0 = 0,\ |F| = 1 .$$

The invariants p and q are involved in U_{11} and U_{22}.

In some cases another choice of a frame is useful. Suppose also $n=2$ and $F \neq 0$, $h=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Note that the apolarity condition implies $h_{12*}=0$. Then the formula (1.25) shows that we may assume $L_{ij}=0$, because the equation $L_{11}-c^1h_{111}=L_{22}-c^2h_{222}=0$ is always solvable with respect to c^k. A frame with this property is called a Wilczynski frame (see(118,11) and (118,13) in [2]). It has played a fundamental role in Bols' book, especially in Chapter VIII. However, it is not possible to generalize this choice in higher dimension.

<u>2. A normal form of a hypersurface.</u> Consider a hypersurface given by an equation

$$y = \psi(x) \quad x = (x^1, \ldots, x^n)$$

in the affine space A^{n+1}. The problem here is to derive a certain normalization of ψ Which corresponds to the reduction in § 1. Assume the hypersurface is non-degenerate and passes through the origin $(x,y)=(0,0)$. Then the above expression has a formal development like as

$$(4.2) \qquad y = \langle x, x \rangle + \sum_{d\geq 3} \psi_d ,$$

where $\langle\ ,\ \rangle$ is a certain non-degenerate bilinear form and ψ_d is a homogeneous polynominal of degree d . (Make an affine change of coordinates if necessary). Define coefficients of ψ_d by

$$\psi_d = \frac{1}{d!} \sum a_{i_1 \ldots i_d} x^{i_1} \ldots x^{i_d} ,$$

which are symmetric with respect to indices. Then, putting $\langle x,x\rangle = \frac{1}{2} xh{}^t x$ for a non-degenerate matrix $h=(h_{ij})$, we define

$$\mathrm{Tr}\psi_3 = \sum h^{ij} a_{ijk} ,$$

$$\mathrm{Tr}^2\psi_4 = \sum h^{ij} h^{kl} a_{ijkl} .$$

<u>Proposition</u> 4.1. (1) For every point of a non-degenerate hypersurface, there exists a projective change of coordinates such that the hypersurface has an expression at that point

$$\mathrm{Tr}\psi_3 = \mathrm{Tr}^2\psi_4 = 0 .$$

(2) Every projective transformation which keeps this expression invariant belongs to the isotropy group at the origin of the

quadric $y = < x,x >$.

We shall sketch how to calculate to show this proposition. A projective transformation that keeps the origin has a form

$$Y = \frac{\nu \tilde{y}}{\lambda + b\tilde{x} + \mu\tilde{y}}, \qquad X = \frac{a\tilde{x} + c\tilde{y}}{\lambda + b\tilde{x} + \mu\tilde{y}},$$

where $a=(a_i^{\,j})$, $b=(b_i)$ and $c=(c^j)$ as before. Assume the expression (4.2) is transformed into

$$\tilde{y} = < \tilde{x}, \tilde{x} > + \sum \tilde{\psi}_d .$$

Then the comparison of the second-degree terms gives

$$ah^t a = \lambda \nu h . \tag{4.3}$$

Since we are concerned with a projective transformation, we may assume

$$|\lambda\nu| = 1 . \tag{4.4}$$

The comparison of the third-degree terms then gives

$$\lambda\nu\tilde{a}_{ijk} = \nu a_{pqr}a_i^{\,p}a_j^{\,q}a_k^{\,r} + 2\,\mathrm{Sym}(D_i h_{jk}) , \tag{4.5}$$

$$D_i = a_i^{\,k}h_{kj}c^j - \nu b_i ,$$

where $\tilde{}$ is used to denote quantities associated to $\tilde{\psi}_d$. The notation Sym denotes a symmetrization. Take a trace of (4.5), then one sees that a suitable choice of c makes $\mathrm{Tr}\tilde{\psi}_3=0$. Assuming next $\mathrm{Tr}\psi_3=0$ also, compute the fourth-degree terms. Then

$$(4.6)\qquad 6(\mu\nu-\langle c,c\rangle)\,\mathrm{Sym}(h_{ij}h_{kl})+\lambda\nu\tilde{a}_{ijkl}$$
$$=\lambda^{-2}a_{pqrs}a_i{}^p a_j{}^q a_k{}^r a_l{}^s-4\lambda^{-2}\mathrm{Sym}(b_i a_{pqr}a_j{}^p a_k{}^q a_l{}^r)\ .$$

Take traces twice times :

$$(4.7)\qquad 2n(n+2)(\mu\nu-\langle c,c\rangle)+\lambda\nu\mathrm{Tr}^2\tilde{\psi}_4=\lambda^{-2}\mathrm{Tr}^2\psi_4\ .$$

Then a suitable choice of μ proves the first part of Proposition. The second part can be seen from (4.5) and (4.7) together with (4.3) and (4.4).

<u>3. Explicit expression of invariants.</u> Let $y=\psi(x)$ be an equation of a hypersurface, which is normalized as in the previous remark. We use a following normalized frame:

$$(4.8)\qquad \begin{aligned} e_0 &= (1,\ x^1,\ \ldots.,\ x^n,\ \psi(x)) \\ e_i &= \alpha(0,\ \ldots.,\ 0,\ 1,\ 0,\ \ldots,\ 0,\ \psi_i(x)) \\ e_{n+1} &= -\tfrac{1}{2}\,e_0+c^i e_i+\alpha^{-n}(0,\ \ldots,\ 0,\ 1)\ , \end{aligned}$$

where

$$\alpha = (\det \psi_{ij})^{-1/n(n+2)} \ , \ d\alpha = \alpha_j dx^j$$

$$c^i = n\alpha^{-n-2}\alpha_j\psi^{ji} \ , \ \psi^{ji} = (\psi_{ij})^{-1} \ ,$$

$$l = \alpha^{-n-2}(\alpha\alpha_{ij} - (n+1)\alpha_i\alpha_j - \alpha\alpha_k\psi^{kl}\psi_{lij})\psi^{ij} \ .$$

Subindices mean derivatives with respect to $(x^i): \psi_i = \frac{\partial\psi}{\partial x^i}$, $\alpha_j = \frac{\partial\alpha}{\partial x^j}$, ... The dual frame ω is given by

$$(4.9) \qquad \omega = \begin{pmatrix} 0 & \omega^i & 0 \\ \frac{1}{2}\omega_j{}^{n+1} & d\log\alpha\,\delta_j{}^i - c^i\omega_j{}^{n+1} & \omega_j{}^{n+1} \\ \frac{1}{2}dl & -\frac{1}{2}\omega^i + dc^i + c^i d\log\alpha & 0 \end{pmatrix}$$

where

$$\omega^i = \alpha^{-1}dx^i \ , \ \omega_j{}^{n+1} = \alpha^{n+2}\psi_{jk}\omega^k \ .$$

Then we have

$$(4.10) \qquad h_{ij} = \alpha^{n+2}\psi_{ij}$$

$$h_{ijk} = \alpha^{n+2}(\alpha\psi_{ijk} + n(\alpha_k\psi_{ij} + \alpha_i\psi_{jk} + \alpha_j\psi_{ki})) \ .$$

Assume, for simplicity, $\psi_{ij}(0) = \delta_{ij}$. The apolarity condition at the origin is $\sum \psi_{iij} = 0$, hence $\alpha_j(0) = 0$. This shows, at the origin

(4.11) $\quad h_{ij}(0) = \delta_{ij}\ ,\ h_{ijk}(0)=a_{ijk}\ ,\ F(0)=a_{ijk}a_{ijk}\ .$

A further calculation shows

$$L_{ij}(0) = \frac{1}{n+2}(a_{pqi}a_{pqj}-a_{ppij}) - \frac{1}{n(n+2)} F\delta_{ij}$$

$$\gamma_1(0) = -\frac{1}{2n(n+2)}(a_{iijjl}-2a_{ijk}a_{ijkl}-2a_{ijl}a_{ijkk}+3a_{ijk}a_{jkm}a_{mil})$$

(Take summations for repeated indices)

<u>4. The case n=1 .</u> The reduction in § 1 shows $h_{111}=0$ as well as $L=L_{11}=0$ if $n=1$. So the treatments in § 2 and § 3 cannot apply for a curve. We will reproduce a part of [5] in our notation. The connection form ω has a form

$$\begin{pmatrix} \omega^0 & \omega^1 & 0 \\ \omega_1 & 0 & \omega^1 \\ \omega_2 & \omega_1 & -\omega^0 \end{pmatrix}$$

and by a frame change it varies as

$$\tilde{\omega}^0 = d\log\lambda + \omega - b\omega$$

$$\tilde{\omega}_2 = \lambda^{-2}\omega_2\ .$$

A point where $\omega_2=0$ is called a *sextactic* point. If we write a curve locally as

$$y = \frac{1}{2} x^2 + \frac{e}{120} x^5 + 0(x^6)$$

(see Proposition 4.1), then (4.11) shows that (0,0) is a sextactic point if and only if $e=0$. It is known that if $\omega_2=0$ everywhere, then a curve is a conic ([5], [15]) .

Proposition 4.2. Assume the curve has no sextactic points. Then there is a unique frame such that ω takes a form

$$\begin{pmatrix} 0 & \omega^1 & 0 \\ \omega_1 & 0 & \omega^1 \\ -\omega^1 & \omega_1 & 0 \end{pmatrix} .$$

Proof. Since $\omega_2 \neq 0$ by assumption, we can choose λ so that $\omega_2 = -\omega^1$. Choose next b so that $\omega^0=0$. Uniqueness can be seen readily from (1.10).

When we choose a frame as above, ω^1 is called the *projective line element*. When we write $\omega_1=k\omega^1$, the coefficient k is called the *projective curvature* of a curve. The above reduction corresponds to the expression of a curve as

$$y = \frac{1}{2} x^2 - \frac{1}{20} x^5 + \frac{g}{7!} x^7 + 0(x^8) .$$

Then, at the origin, the projective line element is dx and the projective curvature is equal to $g/18$. Refer [5] for a variational problem concerning the projective curvature.

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