REMARKS ON THE WEIERSTRASS POINTS ON CURVES

by

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MPI/90 – 14

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§ 1. Introduction.

Let C be a smooth curve over an algebraically closed field of arbitrary characteristic and L a line bundle on C . Laksov gave a complete treatment of the theory of Weierstrass points of L on C in [1], [2]. In this paper we try to give a different version of his treatment by introducing the i–th differential line bundles of L . The difference is inessential, but our treatment is simpler in my point of view. For example the discussion on the Wronskian. Besides, we define the (total) i–th weight at any point and give an algorithm for computing the gap sequence of L and the weight at a point.

Though we could give an independent treatment without referring to [1], [2] we prefer to use them since actually our main idea comes from them.

§ 2. Weierstrass points

We shall adopt most of the notation in [1], [2] and [3].

Let C be as above and D a positive divisor with $L = \mathcal{O}(D)$ of degree d and dimension r .

Denote by $P^m(D)$ the sheaf of m–th principal parts of D and by $a_m : V_c = H^0(C,L) \otimes_k \mathcal{O}_c \longrightarrow P^m(L)$ the m–th canonical homomorphism. Let $B^m = \operatorname{Im} a_m$ and $A^m = \operatorname{coker} a_m$ (see [1] § 1 for details).

Now we have the diagrams

$$
\begin{array}{ccccc}
V_c & \longrightarrow & B^m & \longrightarrow & 0 \\
\Vert & & \big\downarrow b^m & & \\
V_c & \longrightarrow & B^{m-1} & \longrightarrow & 0 \\
 & & \big\downarrow & & \\
 & & 0 & &
\end{array}
$$

and

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^m & \longrightarrow & \Omega_c^{\otimes m}(L) & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^m & \longrightarrow & P_c^m(L) & \longrightarrow & A^m \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & B^{m-1} & \longrightarrow & P_c^{m-1}(L) & \longrightarrow & A^{m-1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}
\qquad (*)
$$

where b^m is induced by $P^m \longrightarrow P^{m-1}$ and $G^m = \ker b^m$. Note that $G^0 = B^0$ and B^m hence the G^m are locally free.

<u>Proposition 2.1</u>. There are integers $0 = b_0 < b_1 < \dots < b_\gamma \leq d < \infty$ such that $\mathrm{rk}\, G^j = 0$ for $j \neq b_i$ and $\mathrm{rk}\, G^{b_i} = 1$ for all b_i.

<u>Proof</u>. Since there exists a sequence $0 = b_0 < \dots < b_r < b_{r+1} = \infty$ such that, if $b_m \leq j < b_{m+1}$ then $\mathrm{rk}\, B^j = m + 1$ by [1], so we have the conclusion.

We call $b_0, b_1, \dots, b_r$ the gap sequence of L which is the same as in [1].

<u>Remark</u>. If we replace V by a subspace $V' \subset V$ with $\dim V' = 1 + r'$. We shall get a corresponding $b_0, \dots, b_{r'}$ by the same way. We still call it the gap sequence for the linear system V'.

<u>Definition</u>. Let $G^{b_i} = G_i$. We call G_i the i–th differential line bundle of L (or of V').

For G_i we have a natural morphism

$$h_i : 0 \longrightarrow G_i \longrightarrow \Omega_C^{\otimes b_i}(D)$$

induced by a_{b_i}.

Definition. Let D_i be the degeneracy of h_i defined by the Fitting ideal F^0 (coker h_i). For any point $x \in C$ we call the multiplicity $w_i(x)$ of x in D_i the i–th weight of x with respect to L (or V') and $w(x) = \sum_{i=0}^{r} w_i(x)$ the weight of x and $W = \sum_{x \in C} w(x)$ the total weight.

A point x is a Weierstrass point for L (resp. a Wronski point for V') if for some i $w_i(x) \neq 0$. $I = \bigcup_{i=0}^{r} D_i$ as a scheme–theoretic union is called as the Weierstrass locus of L (resp. the Wronski locus of V') and the invertible sheaf corresponding to I as the Wronskian for I (resp. for V').

Proposition 2.2. The Weierstrass point, the Wronskian and the total weight defined above all of them are the same as that defined in [1], [2].

Proof. Recall that in [1] the Weierstrass point x is a point which satisfies the condition: there exists an integer i such that $\operatorname{rk} A^{b_i}(x) > b_i - i$. In other words, for such i a_{b_i} degenerates at x.

Now from the diagram (∗) we have a standard form of the matrix representing a_{b_i} at x

$$\left[\begin{array}{c:c} \text{matrix of } a_{b_i - 1} & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hdashline 0 \cdots\cdots 0 & h_i \end{array}\right].$$

Therefore, the degeneracy of a_{b_i} is defined by that of h_i and the i–th minors of the matrix for a_{b_i-1}. Hence for every i the degenerateness of h_i implies the degenerateness of a_{b_i}.

For the other direction, if a_{b_i} degenerate at x and $h_i(x) \neq 0$ then a_{b_i-1} degenerate at x ; if now $a_{b_i-1} \neq a_{b_{i-1}}$ the matrix of a_{b_i-1} at x has the following shape

$$\begin{bmatrix} & \cdot & * \\ & \cdot & \cdot \\ \text{matr. of } a_{b_i-2} & \cdot & \cdot \\ & \cdot & \cdot \\ & \cdot & * \end{bmatrix} .$$

Then in this case a_{b_i-1} degenerates at x implies that a_{b_i-2} degenerates at x . We proceed in this way until $a_{b_i-k} = a_{b_{i-1}}$. If now the degeneracy of $a_{b_{i-1}}$ at x is equivalent the degeneracy of h_{i-1} we are through: if not, we proceed further as above until we arrive at $B^0 = G^0$. So the first assertion is proved.

As for the second assertion we note that the invertible sheaf corresponding to the degeneracy of h_i is

$$\Omega_c^{\otimes b_i}(L) \otimes G_i^{-1}$$

and $G_0 = B^0$, $B^{b_r} = \mathcal{O}_c^{\oplus(r+1)}$, then the Wronskian

$$\mathcal{O}(I) = \Omega_c^{\otimes \sum_{i=0}^{r} b_i} \otimes L^{(r+1)} \otimes (G_0^{-1} \otimes G_1^{-1} \otimes \dots \otimes G_r^{-1}) .$$

But from (*)

$$G_i = (\Lambda^{i+1}B^{b^i})^{-1} \otimes \Lambda^i B^{b_i-1} \simeq (\Lambda^{i+1}B^{b_i})^{-1} \otimes \Lambda^i B^{b_{i-1}} ,$$

therefore

$$\mathcal{O}(I) \simeq \Omega^{\otimes \sum_{i=0}^{r} b_i} \otimes L^{(r+1)} ,$$

the same expression as in [1], [2].

The third assertion comes from the second.

§ 3. i–th part of Weierstrass point

By introducing i–th differential line bundle we can define the i–th part of Weierstrass points I_i by the degeneracy of h_i. Intuitively these points are that Weierstrass points which existness depends on the appearence of the new term of higher differentials.

In classical case i.e. the gap sequence being 0,1,2, ... ,r , the line bundle corresponding to I_i is $G_i^{-1} \otimes \Omega^{\otimes b_i}(L) \simeq (\Lambda^{i+1}B^i)^{-1} \otimes \Lambda^{i+1}p^i \otimes ((\Lambda^i B^{i-1}) \otimes \Lambda^i p^{i-1})^{-1}$, which corresponds the divisor of the degeneracy of a_i excluded by the degeneracy of a_{i-1} .

In general case it is not so simple. From the matrix of a_{b_i} namely

$$\left[\begin{array}{c:c} \text{matrix of } a_{b_i-1} & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hdashline 0 \cdots\cdots 0 & h_i \end{array}\right]$$

we see the degeneracy of a_{b_i} is defined by the ideal $h_i\mathfrak{A}$, where $\mathfrak{A}$ is generated by the (i–1)–th minors of a_{b_i-1} . Therefore the i–th part I_i in this case is the difference of the degeneracy of a_{b_i} and that of a_{b_i-1} and generally the degeneracy of a_{b_i-1} is different from that of $a_{b_{i-1}}$.

In fact, from the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^m & \longrightarrow & \Omega^{\otimes m}(L) & \longrightarrow & C^m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^m & \longrightarrow & P^m(L) & \longrightarrow & A^m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^{m-1} & \longrightarrow & P_c^{m-1}(L) & \longrightarrow & A^{m-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} ,$$

if $m = b_i$ we have $F^{b_i-i}(A^{b_i}) = F^0(C^{b_i}) \cdot F^{b_i-i}(A^{b_i-1})$, where $F^k(M)$ denotes the k–th Fitting ideal of M. Therefore I_i is the difference of divisor determined by $F^{b_i-i}(A^{b_i})$ and that by $F^{b_i-i}(A^{b_i-1})$, but generally $F^{b_i-i}(A^{b_i-1}) \neq F^{b_{i-1}-(i-1)}(A^{b_{i-1}})$. There is an example in [1] which explains this situation, so we only know that I_i includes the difference of the degeneracy of a_{b_i} and that of $a_{b_{i-1}}$.

It is clear that $\sum_{x \in C} w_i(x) = \deg I_i = b_i(2g-2) + d - \deg G_i$ and we call it the <u>i–th total weight</u>.

§ 4. Local computation

By our description for Weierstrass points, at an arbitrary point $x \in C$ the b_r– truncated Taylor series of all the sections of $H^0(C,D)$ in the local ring $\mathcal{O}_x$ can be reduced to a "standard form"

$$\begin{bmatrix} t^{\nu_0}, * & * & * & \ldots\ldots\ldots & * \\ 0 \,..\, 0 & t^{\nu_1} & * & * \;\ldots\ldots & * \\ 0 \;\ldots.\; 0 & 0 & t^{\nu_2} & * \,..\, & * \\ & \cdots & \cdots & \cdots & \\ 0 \;\ldots\ldots & & & 0 & t^{\nu_r} \end{bmatrix},$$

and which can be pasted as the morphism of sheaves, where the row means $t^{\nu_i} dt^{b_i} + * \, dt^{(b_i+1)} + \ldots$ and $1, dt, \ldots, dt^{b_r}$ is a basis for $P^{b_r}(D)$ at x. Therefore, $\nu_i = w_i(x)$ and a Weierstrass point implies that $\nu_i(x) > 0$ for some i.

In other words, one could start from a k–linearly independent basis $f_0, \dots, f_r$ for $H^0(C,D)$ and a basis for $P^{b_r}(D)$ at x then using the elementary transformation in $\mathcal{O}_x$ we get this standard form.

So we may derive from this an algorithm for computing the gap sequence of D and the i–th weight at any point. Perhaps it will be never used for actual computation except for using a computer.

At the beginning we choose a basis $f_0, \dots, f_r$ for $H^0(C,D)$ such that the formal series of them in the completion of $\mathcal{O}_x$ have the form:

$$f_0 = t^{\mu_{00}} + ct^{\mu_{01}} + \dots$$

$$\cdot \quad \cdot \quad \cdot$$

$$f_r = t^{\mu_{r0}} + \dots$$

and $\mu_{00} < \mu_{10} < \dots < \mu_{r0} \le d$.

We assume

$$T_{b_r}(f_i) = \sum_{j=0}^{b_r} (c_{ij} t^{\nu_{ij}} + \dots)(dt)^j , \quad c_{ij} \neq 0 .$$

(1) $\nu_0 = \mu_{00}$.

Since $\nu_{i0} = \mu_{i0}$, the first column of the matrix can be reduced to $\begin{bmatrix} t^{\mu_{00}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

(2) Now we consider $f_1, \dots, f_r$.

We select an integer ℓ (if exists) such that $\mu_{\ell 0} < p$ but $\mu_{(\ell+1)0} \geq p$. Then $\nu_i = \mu_{i0} - i$ for $i = 1, \dots, \ell$, where we take $p = \infty$ if $\mathrm{char} k = 0$. Note that $\ell < p$.

Under this assumption

$$T_{b_r}(f_i) = (t + dt)^{\mu_{i0}} + c(t + dt)^{\mu_{i1}} + \dots = f_i + \dots + \left[\begin{bmatrix} \mu_{i0} \\ i \end{bmatrix} t^{\mu_{i0}-i} + \dots \right] dt^i + \dots$$

where C is a constannt and $\begin{bmatrix} \mu_{i0} \\ i \end{bmatrix} \neq 0$. Therefore the submatrix corresponding to $f_0, \dots, f_\ell$ is

$$\begin{bmatrix} t^{\mu_{00}} & & \\ & t^{\mu_{10}-1} & * \\ 0 & \ddots & t^{\mu_{\ell 0}-\ell} \end{bmatrix}$$

and $b_0 = 0$, $b_1 = 1, \dots, b_\ell = \ell$.

In particular, if $\mathrm{char} k = 0$ or $p > d$ we have the classical gap sequence and the i–th weight $w_i(x) = \mu_i - i$.

(3) Assume $\ell < p-1$. We consider $f_{\ell+1}, \dots, f_r$ and the set $S_{\ell+1} = \{\mu_{ij} | i \geq \ell + 1$ and $\mu_{ij} (\text{mod } p) \geq \ell + 1\}$.

If $S_{\ell+1} \neq \phi$ let $\mu_{i_1 j_1} = \min S_{\ell+1}$, then $\nu_{\ell+1} = \mu_{i_1 j_1} - (\ell + 1)$. Now if still $\ell + 1 < p - 1$ and $S_{\ell+2} \neq \phi$ where $S_{\ell+2} = \{\mu_{ij} | i \geq \ell + 1, i \neq i_1$ and $\mu_{ij} (\text{mod } p) \geq \ell + 2\}$ then $\nu_{\ell+2} = \mu_{i_2 j_2} - (\ell + 2)$, where $\mu_{i_2 j_2} = \min S_{\ell+2}$. We continue this process until $\ell' \geq p - 1$ or $S_{\ell'} = \phi$. In this case $b_{\ell+1} = \ell + 1, \dots, b_{\ell'-1} = \ell' - 1$.

The reason for doing so is obvious if we note the non–zero terms of all the Taylor expansions.

Of course we may include Step (1) and (2) in Step (3) if we like.

We rearrange the indices for f_i by $f'_{\ell+1} = f_{i_1}$, $f'_{\ell+2} = f_{i_2}, \dots .\ f'_{\ell'} = f_{i_{\ell'-\ell}}$.

(4) We assume that after Step (3) there leaves $f_{\ell'+1}, \dots, f_r$ unmoved. Suppose $\ell' + 1 < p^2 - 1$.

Let $S_{\ell'+1} = \{\mu_{ij} | i \geq \ell' + 1$ and $\mu_{ij} (\text{mod } p^2) \geq \ell' + 1\}$. If $S_{\ell'+1} \neq \phi$ and $\mu_{i'j'} = \min S_{\ell' + 1}$ then

$$\nu_{\ell'+1} = \begin{cases} \mu_{i'j'} - (\ell' + 1) & \text{if } \ell' + 1 \geq p \\ \mu_{i'j} - (\ell' + 1 + p) & \text{if } \ell' + 1 < p \end{cases} .$$

We continue like this until $\ell'' \geq p^2 - 1$ or $S_{\ell''} = \phi$. In this case $b_{\ell'+1} = \ell' + 1$

or $\ell' + 1 + p$ depending on $\ell' + 1 \geq p$ or not.

The reason for doing so is the same as in Step (3).

(5) Now suppose $\ell'' + 1 < p^3 - 1$, and repeat the same process as in (4).

In this way we shall exhaust all the sections of D (resp. V′) and finally arrive the standard form.

But we have to determine how many terms of Taylor series of f_i's we need in this algorithm.

Since the total weight $(\sum_1^r b_i)(2g-2) + (r+1)d \leq (2d-r)r(g-1) + (r+1)d$ and since we worked in P^{b_r} where $b_r \leq d$, then every μ_{ij} which appears in ν_ℓ is less than $(2d-r)r(g-1) + (r+2)d$. This means we only need to expanse every f_i to $(2d-r)r(g-1) + (r+2)d$ order.

References

[1] D. Laksov: Weierstrass points on curves, Astér. 87–88 (1981) 221–247.

[2] ——— : Wronskian and Plücker formulas for linear system on curves, Ann. scient. Ec. Norm. sup. 4⁰ series t. 17 (1984) 45–66.

[3] R. Piene: Numerical characters of a curve in projective n–space, in Real and Complex singularities, Oslo (1976) 475–495.

Voraussichtliche Aufenthaltsdauer am Institut: von .... bis ....
(Anticipated duration of stay) (from) (to)

III. BERUFLICHE ANGABEN (VOCATIONAL DATA)

Fachgebiet: ....
(Field of research)

Zeitpunkt der Promotion: ....
(When did the scientist take his/her degree (doctorate ?)

Tätigkeiten seit der Promotion
oder dem Hochschulabschluß: ....
(ggf. auf separatem Blatt
angeben)
....

(Employments since taking the degree:
(please use separate sheet, if necessary))..

Gegenwärtige berufliche Stellung: ....
(Present professional position)

a) Bezeichnung:
(Title)

b) Seit wann in dieser Stellung tätig: ....
(Since when has this position been held?)

c) Institut (Anschrift): ....
(Institution, address)
....

IV. ANGABEN ÜBER DAS GEPLANTE GEMEINSAME FORSCHUNGSVORHABEN: (ggf. auf separatem Blatt erläutern)

(IN WHICH JOINT RESEARCH PROJECT WILL THE SCIENTIST BE INVOLVED DURING HIS STAY AT THE INSTITUTE ?
please use separate sheet, if necessary)

....

....

....

Name des Wissenschaftlers, mit dem der Stipendien- oder Honorarempfänger in erster Linie zusammenarbeiten soll:

(Name of the German scientist with whom the visiting scientist will work together in the first line)

....