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Congruences between modular forms of genus 1 and of genus 2

According to the general conjectures of Birch, Swinnerton-Dyer, Bloch, Kato, Deligne, Beilinson the so-called critical values of the L-function $L(M,\nu)$ of a motive M have arithmetical consequences for the underlying motive M. I want to explain how we can use the data from van der Geer's talk to give some very strong evidence for these conjectures in the case that $M = M(f)$ is the Scholl motive attached to a modular cusp form.

Let f be a cuspidal modular form of weight k for $SL_2(\mathbb{Z})$. We assume that it is an eigenform for the Hecke algebra and for simplicity we assume that its Fourier coefficients are rational integers

$$f(q) = q + a_2 q^2 + a_3 q^3 + ...$$

This probably reduces us to the finitely many cases $k = 12, 14, 16, 18, 20, 22, 26$, but the conjectures below can be formulated also for arbitrary eigenforms f, then the formulation is more complicated. We can attach a motive $M(f)$ to this modular form. The L-function attached to this motive is of course the classical Hecke L-series of f (modified by a Γ-factor):

$$L(f,s) = L(M(f),s) = \frac{\Gamma(s)}{(2\pi)^s}\sum \frac{a_n}{n^s} = \frac{\Gamma(s)}{(2\pi)^s}\prod_p \frac{1}{1 - a_p p^{-s} + p^{2k-1-2s}}.$$

It is holomorphic in the entire s-plane and satisfies the functional equation $L(f,s) = L(k-s)$. By a theorem of Manin we know that we can define two periods $\Omega_-(f), \Omega_+(f)$ such that for $\nu \in [k-1, k-2, \dots, 1]$

$$L^*(f,\nu) = \frac{L(f,\nu)}{\Omega_{\epsilon(\nu)}(f)} \in \mathbb{Q}.$$

A priori the periods $\Omega_-(f), \Omega_+(f)$ are only defined up to an element in $\mathbb{Q}^\times$. But in [Ha-Ms, sect. 2] it is shown how we can use the theory of Whittaker models and integrality considerations to determine them up to an element in $\mathbb{Z}^\times = \{\pm 1\}$. With these normalized periods we can show that the numbers $L^*(f,\nu)$ have bounded denominators, actually they are even in $\mathbb{Z}$ except for the value $L^*(f,k-1) = L^*(f,1)$, whose denominator is the numerator of $\zeta(1-k)$.

To a pair k,ν with $\nu \in [k-1,\dots,k/2]$ we determine l,m by $k = l+m+4, \nu = l+3$. We consider the coefficient system $\mathbb{V}_{l,m}$ on $\mathcal{A}_2(\mathbb{C}) = \Gamma\backslash\mathbb{H}_2$ as in van der Geer's talk ($\Gamma = Sp(4,\mathbb{Z}) = Sp_2(\mathbb{Z})$). We think of this coefficient system as obtained from a rational representation of GSp_2 and we choose a Γ -invariant lattice $\mathcal{V}_{l,m}$ inside $\mathbb{V}_{l,m}$. Such a lattice is unique up to homothety if we tensorize by a ring $R = \mathbb{Z}[1/(2\cdot 3\cdot \dots \cdot r)]$ where we invert the "small" primes $< \max(l-m, m)$. The coefficient system $\mathcal{V}_{l,m} \otimes R$ is again called $\mathcal{V}_{l,m}$.

I want to formulate two conjectures. I say that a prime is large if it is $> k$. We consider the cohomology sequence (see [Ha-Ms, sect. 1],)

$$\to H^3_c(\Gamma\backslash\mathbb{H}_2, \mathcal{V}_{l,m}) \to H^3(\Gamma\backslash\mathbb{H}_2, \mathcal{V}_{l,m}) \to H^3(\partial(\Gamma\backslash\mathbb{H}_2), \mathcal{V}_{l,m}) \to .$$

This is a sequence of modules for the Hecke algebra. This Hecke algebra is essentially generated by Hecke operators

$$T_p^{(1)} \simeq \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, T_p^{(2)} \simeq \begin{pmatrix} p^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where p runs over all primes.

Under our assumption that $k \leq 26$ we know that the modular form f defines a direct summand $Rf \subset H^3(\partial(\Gamma\backslash\mathbb{H}_2), \mathcal{V}_{l,m})$, on which the two Hecke operators $T_p^{(1)}, T_p^{(2)}$ act by the eigenvalues $p^{m+1}+a_p+p^{l+2}$ and $a_p(p^{m+1}+p^{l+2})$, respectively. If we are in the regular case $l > m > 0$, the theory of Eisenstein series and the classical argument of Manin-Drinfeld give us a canonical lifting

$$\mathrm{Eis} : Rf \to H^3(\Gamma\backslash\mathbb{H}_2, \mathcal{V}_{l,m} \otimes \mathbb{Q}).$$

But in general $\mathrm{Eis}(Rf) \notin \mathrm{Im}(H^3(\partial(\Gamma\backslash\mathbb{H}_2), \mathcal{V}_{l,m}) \to H^3(\partial(\Gamma\backslash\mathbb{H}_2), \mathcal{V}_{l,m} \otimes \mathbb{Q})$, the denominator $\delta(f)$ is the smallest integer > 0 which multiplies $\mathrm{Eis}(f)$ into this image.

Now I can formulate :

Conjecture A: *Let f be as above, let $\nu \in [k-1, \ldots, k/2]$. If a large prime ℓ divides $L^*(f,\nu)$ and if in addition ℓ does not divide $L^*(f,\nu-1)$ and $\zeta_\ell(l-m+1)$ then $\ell|\delta(f)$.*

This conjecture implies by a standard argument

Conjecture B: *Under the same assumptions on f,ν and ℓ there exists a Siegel modular form $F \in S_{l-m,m+3}$ (see v.d. Geer) which is an eigenform for the Hecke operators such that for all primes p*

$$\lambda^{(1)}(F,p) \equiv p^{m+1} + a_p + p^{l+2} \bmod \ell, \quad \lambda^{(2)}(F,p) \equiv a_p(p^{m+1} + p^{l+2}) \bmod \ell.$$

Conjecture B can now be checked numerically using the work of C. Faber and G. v.d. Geer.

The interesting modular forms are given by the forms f_{22} and f_{26} of weight 22 and 26, respectively. We find the following divisibilities by large primes

$$41|L^*(f_{22},14), \quad 43|L^*(f_{26},23), \quad 97|L^*(f_{26},21), \quad 29|L^*(f_{26},19).$$

Hence we must find Siegel modular eigenforms in the spaces $S_{4,10}, S_{20,5}, S_{16,7}, S_{12,9}$. It turns out that all these spaces are of dimension 1, and the computer program of C. Faber and v.d.Geer produces the eigenvalues for the Hecke operators for all primes $p \leq 37$. And indeed the congruences are true in all these cases. In fact, we also find that there are no other congruences modulo large primes ($\ell > k$) if we choose other values of ν and $k = 22$ and 26.

If the Siegel modular form F is the generator of one of the above spaces we can again attach a motive $M(F)$ and this gives us for all ℓ a 4-dimensional Galois representation

$$\rho(F) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(M(F)_{et,\ell}).$$

The cup product defines an alternating pairing $<\ ,\ >$ on $M(F)_{et,\ell}$ and hence the image of the Galois group is contained in a symplectic group. Then Conjecture A implies that after reduction mod ℓ the Galois module $M(F)_{et,\ell} \otimes \mathbb{F}_\ell$ contains an invariant submodule $\mathbb{F}_\ell(-\ell-2)$. Its orthogonal complement Y is of dimension 3 over $\mathbb{F}_\ell$ and the module $Y/\mathbb{F}_\ell(-\ell-2)$ is $M(f)_{et,\ell}$. The quotient $M(F)_{et,\ell} \otimes \mathbb{F}_\ell/Y \xrightarrow{\sim} \mathbb{F}_\ell(-m-1)$. The semi-simplification of $M(F)_{et,\ell} \otimes \mathbb{F}_\ell \xrightarrow{\sim} \mathbb{F}_\ell(-\ell-2) \oplus M(f)_{et,\ell} \oplus \mathbb{F}_\ell(-m-1)$. The semi-simplification generates the field extension of $\ell-$ division points $\mathbb{Q}(M(f)_{et,\ell}[\ell])$ and the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on Y produces an unramified extension of this last field. It is visible in the matrix representation

$$\rho(F)|Y(\sigma) = \begin{pmatrix} \alpha^{-l-2} & u(\sigma) & v(\sigma) \\ 0 & a(\sigma) & b(\sigma) \\ 0 & c(\sigma) & d(\sigma) \end{pmatrix}.$$

Here a, b, c, d are the matrix coefficients of $\rho(f) \otimes \mathbb{F}_\ell$. One has to prove that we really get an extension, i.e. there are elements σ in the Galois group with $\rho(f) \otimes \mathbb{F}_\ell(\sigma) = Id$ but $(u(\sigma), v(\sigma)) \neq 0$. Then we get an extension of $\mathbb{Q}(M(f)_{et,\ell}[\ell])$ of the form predicted by the Birch, Swinnerton-Dyer, Bloch, Kato, Beilinson, Deligne conjectures.

For more details I refer to the very provisorial preprints on my home-page, which can be found under www.math.uni-bonn.de. These preprints are in the folder eisenstein in my ftp directory.

[Ha-koll] A congruence between a Siegel and an elliptic modular form

[Ha-Ms] Modular symbols and special values of automorphic L-functions

[Ha-MM] Modular construction of mixed motives II