# THETA FUNCTIONS, QUANTUM TORI AND HEISENBERG GROUPS

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**Abstract.** A linear algebraic group $G$ is represented by the linear space of its algebraic functions $F(G)$ endowed with multiplication and comultiplication which turn it into a Hopf algebra. Supplying $G$ with a Poisson structure, we get a quantized version $F_q(G)$ which has the same linear structure and comultiplication, but deformed multiplication. This paper develops a similar theory for abelian varieties. A description of abelian varieties $A$ in terms of linear algebra data was given by Mumford: $F(G)$ is replaced by the graded ring of theta functions with symmetric automorphy factors, and comultiplication is replaced by the Mumford morphism $M^*$ acting on pairs of points as $M(x,y) = M(x+y, x-y)$. After supplementing this by a Poisson structure and replacing the classical theta functions by the quantized ones, introduced by the author earlier, we obtain a structure which essentially coincides with the classical one so far as comultiplication is concerned, but has a deformed multiplication which moreover becomes only partial. The classical graded ring is thus replaced by a linear category. Another important difference from the linear case is that abelian varieties with different period groups (for multiplication) and different quantization parameters (for comultiplication) become interconnected after quantization.

## 0. Introduction and summary

**0.1. Classical theta functions.** Let $\mathbf{C}^d$ be a complex vector space, $\Lambda' \oplus \Lambda \subset \mathbf{C}^d$ a discrete sublattice of rank $2d$ split into the sum of two sublattices of rank $d$. Classical theta functions are entire functions $\Theta(z)$, $z \in \mathbf{C}^d$, such that

$$\Theta(z+\lambda') = \Theta(z) \quad \text{for all} \quad \lambda' \in \Lambda', \tag{0.1}$$

$$\Theta(z+\lambda) = c(\lambda)\, e^{q(\lambda,z)}\, \Theta(z) \quad \text{for all} \quad \lambda \in \Lambda, \tag{0.2}$$

where $c : \Lambda \to \mathbf{C}$ is a map and $q : \Lambda \times \mathbf{C} \to \mathbf{C}$ is a biadditive pairing linear in $z$.

$\Lambda' \oplus \Lambda$ is called *the period lattice* for $\Theta$, and $(c, q)$ *the automorphy factors.* These data are not arbitrary. It is convenient to break the classical restrictions, imposed on them, into two groups.

(i) *Group action and Riemann symmetry conditions.* For each $\lambda' + \lambda \in \Lambda' \oplus \Lambda$, consider the following map of the (total space of the) trivial line bundle $\mathbf{C}^d \times \mathbf{C}$ extending the shift by $\lambda' + \lambda$ on the base space:

$$t_{\lambda'+\lambda} : (z, u) \mapsto (z + \lambda' + \lambda, c(\lambda)\, e^{q(\lambda,z)}\, u). \tag{0.3}$$

Then we must have

$$t_{\lambda'_1+\lambda'_2+\lambda_1+\lambda_2} = t_{\lambda'_1+\lambda_1} \circ t_{\lambda'_2+\lambda_2}. \tag{0.4}$$

Solvability of (0.4) is essentially ensured by *Riemann's symmetry conditions* imposed upon the lattice and the automorphy factors.

If they hold, the quotient of $\mathbf{C}^d \times \mathbf{C}$ modulo the action of $\Lambda' \oplus \Lambda$ thus defined will be a line bundle $L$ over the complex torus $\mathcal{A} := \mathbf{C}^d/(\Lambda' \oplus \Lambda)$, and any theta function satisfying (0.1) and (0.2) will descend to a holomorphic section of $L$.

(ii) *Formal theta functions and Riemann's positivity conditions.* Equations (0.1) are satisfied for the basic exponents $e^{2\pi i\mu(z)}$ where $\mu$ runs over all linear functions $\mathbf{C}^d \to \mathbf{C}$ taking integral values on $\Lambda'$. Such $\mu$ form an abstract lattice which we will denote $H$. Usually one passes to a basis of $\mathbf{C}^d$ which is simultaneously a basis for $\Lambda'$; then the basic exponents are simply $e^{2\pi i\mathbf{n}^t\cdot z}$, $\mathbf{n} \in \mathbf{Z}^d$.

Any entire function satisfying (0.1) can be expanded into a Fourier series

$$\Theta(z) = \sum_{\mu\in H} a_\mu \, e^{2\pi i\mu(z)}, \tag{0.5}$$

whose coefficients decay swiftly enough.

Equations (0.2) written for (0.5) translate into some linear recurrent relations for coefficients $a_\mu$ which may have solutions satisfying only weaker decay conditions, or even defining everywhere divergent series. We will interpret all these solutions as defining generally a linear space of *formal* theta functions.

We can ask, when all formal solutions are in fact entire. The answer is given by *Riemann's positivity conditions.* (We will reproduce both conditions explicitly below in the form more appropriate for our purposes).

If both symmetry and positivity conditions are satisfied for the period lattice $\Lambda' \oplus \Lambda$ and the system of automorphy factors $(c, q)$, the complex torus $\mathcal{A} := \mathbf{C}^d/(\Lambda' \oplus \Lambda)$ has the canonical structure of *an abelian variety*, the line bundle $L$ on $\mathcal{A}$ defined above is ample, and the space of all theta functions satisfying (0.1), (0.2) can be canonically identified with $\Gamma(\mathcal{A}, L)$.

However, we will be definitely interested in more general thetas.

**0.2. Theta functions on an algebraic torus.** The total group of shifts by $\Lambda' \oplus \Lambda$ acting upon $\mathbf{C}^d$ directly and upon $\mathbf{C}^d \times \mathbf{C}$ via automorphy factors has the subgroup $\Lambda'$. Let us make an inventory of the data which we obtain after moding out only this subgroup.

(a) *Algebraic torus $T(H)$ and its characters.* This is an affine commutative algebraic group over $\mathbf{C}$ isomorphic to a product of $d$ multiplicative groups $\mathbf{G}_m$. More precisely, it is the spectrum of the group ring of $H$, the abstract lattice

Hom $(\Lambda', \mathbf{Z})$ introduced in (0.1) (ii). In the group ring, we write the elements of $H$ multiplicatively, and then denote them $e(h)$:

$$e(g+h) = e(g)\, e(h), \quad g, h \in H. \tag{0.6}$$

We have an analytic map, which is in fact an isomorphism:

$$p: \mathbf{C}^d \to T(H)(\mathbf{C}): \quad p^*(e(h)) := e^{2\pi i \mu(.)}, \tag{0.7}$$

where $\mu(.)$ is the linear function on $\mathbf{C}^d$ extending $h$ as a function on $\Lambda'$.

(b) *Multiplicative period lattice* $B \subset T(H)(\mathbf{C})$. By definition, this is the image of $\Lambda$ with respect to the map (0.7), $B = p(\Lambda)$, so that the complex torus $\mathcal{A}$ is now represented as

$$\mathcal{A} = T(H)/B \tag{0.8}$$

(c) *Automorphy factors and theta functions on* $T$. Any formal theta function (0.5) can be interpreted as a formal function on $T$, an infinite linear combination $\theta = \sum_{h \in H} a_h\, e(h)$. This takes care of equations (0.1).

Clearly, $p$ induces an isomorphism of $\Lambda$ with $B$.

After replacing $\lambda \in \Lambda$ by $b := p(\lambda) \in B$, the left hand side of (0.2) becomes the multiplicatively shifted series which we now denote

$$b^*(\theta) := \sum_{h \in H} a_h\, h(b)\, e(h) \tag{0.9}$$

where $h(b)$ is the value of the character $e(h)$ at the point $b \in T(H)(\mathbf{C})$.

To treat (0.2), we simply replace $c(\lambda)$ by $c_b^{-1}$, $b \in B$, and $e^{q(\lambda, z)}$ by $e(-h_b)$ in the definition of automorphy factors. Here $b \mapsto h_b$ is a certain map $B \to H$ (which is easily seen to be a group homomorphism, if we want (0.4) to hold). Extra inversions were introduced in order to rewrite the equations (0.2) in the following form:

$$\forall\, b \in B, \quad c_b\, e(h_b)\, b^*(\theta) = \theta. \tag{0.10}$$

Of course, in the case $d = 1$, $B = \{q^{\mathbf{Z}}\} \subset \mathbf{C}^*$, we get Jacobi's elliptic theta functions, for example, the basic theta

$$\theta_q(t) = \sum_{n \in Z} q^{n^2}\, t^n \tag{0.11}$$

where we write $t = e(h_0)$, $h_0$ being a generator of $H$. It converges everywhere on $\mathbf{C}^*$ for $|q| < 1$, defines a distribution on $|t| = 1$ for $|q| = 1$ and is a formal theta–function generally, satisfying the equations

$$q^{m^2}\, t^m\, \theta_q(q^{2m} t) = \theta_q(t) \tag{0.12}$$

for all $m \in \mathbf{Z}$.

This reformulation of basic definitions gives some immediate benefits, the most important of which is the possibility to extend the formal theory to arbitrary base fields $K$, and analytic theory to $p$–adic fields and, more generally, to arbitrary complete discretely normed fields instead of $\mathbf{C}$. In fact, over such fields reasonable functions with additive period lattices do not exist so that the equations (0.1) have no interesting solutions. Moding them out, we are left with (0.10), which do admit analytic solutions and, in particular, provide the theory of $p$–adic uniformization of abelian varieties with multiplicative reduction (Tate, Morikawa).

**0.3. Homogeneous coordinate ring of an abelian variety.** Briefly, this theory proceeds as follows. Consider an algebraic torus $T(H)$ over a complete normed field $K$, endowed with a period subgroup $B \subset T(H,1)(K)$ which is free of the same rank as $H$ and discrete. Consider moreover all theta functions on $T(K)$ with multiplicative periods $B$ and automorphy factors satisfying Riemann's symmetry and positivity conditions (the latter can be defined also in the $p$–adic case, cf. the main text below). Assume that this set (consisting of "ample" automorphic factors) is non–empty. Denote by $\Gamma(\mathcal{L})$ the space of theta functions with automorphy factors $\mathcal{L}$.

The set of automorphy factors is endowed with the natural commutative and associative multiplication. This holds even if we drop the positivity conditions. However, with positivity conditions, all relevant theta functions have swiftly decaying coefficients and therefore as well can be multiplied in the space of formal series in exponents. Moreover, a product of theta functions with the same periods $B$ is again a theta function with these periods.

This multiplication induces a mapping $\Gamma(\mathcal{L}') \otimes \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L}'\mathcal{L})$.

Construct the ring $\oplus\Gamma(\mathcal{L})$ graded by the semigroup of ample and symmetric $\mathcal{L}$ (symmetric means invariant with respect to the involution $x \mapsto x^{-1}$ of $T(H,1)$). Its projective spectrum is an abelian variety $A$ which is a model of $T(H,1)/B$ in algebraic geometry. Automorphy factors $\mathcal{L}$ can be interpreted then as invertible sheaves on $A$.

This beautiful theory still lacks some important structures present over $\mathbf{C}$. Namely, consider an abelian variety $\mathcal{A}$ over the complex field. Two theta functions on the *universal cover* $\varphi : \widetilde{\mathcal{A}} \to \mathcal{A}(\mathbf{C})$ (as opposed to the toric cover $T(H)(\mathbf{C})$), may look differently but correspond to the same section of the same line bundle $L$ on $\mathcal{A}$, because we can choose different splittings of the period lattice $\operatorname{Ker}\varphi$ and change the trivialization of $\varphi^*(L)$ appropriately.

In the analytical language, this translates into the functional equations for theta functions expressing their modular properties with respect to the matrix fractional linear transformation acting upon both $z$ and the half of the period matrix corresponding to $\Lambda$.

We cannot expect a simple–minded toric version of these equations to exist in general. For example, in the $p$–adic case an additive uniformization of abelian varieties does not exist because free abelian subgroups of $p$–adic vector spaces cannot be discrete.

Notice that this sort of difficulties in principle vanishes in non–commutative geometry, where "bad quotient spaces" like $\mathbf{Q}_p^d/\mathbf{Z}^d$ might have a perfectly well defined and rich theory. Moreover, in non–commutative geometry we can form another model of $T(H)/B$, independently of whether $B$ is discrete or not. Such a model is represented by the cross product of a ring of functions of $T(H)$ with the transformation group $B$, which is a special case of *quantum torus*.

Remarkably, over $\mathbf{C}$ a replacement of modularity for some quantum tori with maximally non–discrete (unitary) periods does exist and is expressed in the form of a beautiful Morita equivalence: see [RiSch] and the references therein.

We return now to the algebraic tori.

**0.4. Heisenberg groups.** In the toric context, over an arbitrary base field $K$, we can also succintly express the conditions (0.4). To this end let us introduce *the Heisenberg group* $\mathcal{G}(H)$ of the torus $T(H)$. By definition, this is the group of *linear automorphisms* of the space of (formal or algebraic) functions on $T(H)$ generated by (and consisting of) the following maps:

$$[c;\, x, h] : \ \Phi \mapsto c\, e(h)\ x^*(\Phi) \tag{0.13}$$

where $c \in K^*$, $x \in T(H)(K)$, $h \in H$, and $x^*(e(h)) := h(x)\, e(h)$, $h(x)$ being the value of $e(h)$ at $x$ as above.

In these terms, a system consisting of a period subgroup in $T(H)(K)$ and compatible automorphy factors becomes simply a homomorphism, which we will call *a multiplier*,

$$\mathcal{L} : \ B \mapsto \mathcal{G}(H),\ \mathcal{L}(b) = [c_b;\, x_b, h_b], \tag{0.14}$$

where $B$ is a free abelian group of the same rank as $H$ and $b \mapsto x_b$ is a bijection. (We will find convenient in the future to weaken these restrictions).

The equations for formal theta functions (0.10) simply say that $\theta$ is invariant with respect to the (image of) $B$.

We can also consider more general equations of the form

$$\left(\sum_{\Gamma\in\mathcal{G}(H)} c_\Gamma \Gamma\right)\Phi = 0, \quad c_\Gamma \in K. \tag{0.15}$$

Such equations of greater length than two (the number of non–zero $c_\Gamma$) are satisfied by many classical automorphic products, for example $e_q(t) := \prod_{n\ge 0}(1+q^{2n+1}t)$:

$$e_q(q^2t) - (1+qt)\, e_q(t) = 0. \tag{0.16}$$

**0.5. Quantizing abelian varieties.** Let us now introduce into the classical picture sketched above a new element, namely, *a holomorphic Poisson structure* $\alpha$ *on* $\mathcal{A}$ (and its covers) whose coefficients are constant in a basis of invariant vector fields.

In the theory of quantum linear groups $G$, such a structure defines a quantum deformation of all relevant algebraic data. In particular, the function ring of $G$ is replaced by a new non–commutative ring, with the same underlying linear space but deformed multiplication. Comultiplication, to the contrary, remains unchanged. In this way we get a Hopf algebra $A_q[G]$ considered as an algebra of functions on the appropriate quantum group.

We would like to play the same game with the homogeneous coordinate ring of a Poisson abelian variety $(A, \alpha)$. In this game, the peculiarities of multiplication and comultiplication look rather different from each other and from the linear case.

(i) *Multiplication and quantum tori.* Consistently with our approach to abelian varieties, we start with quantizing the respective covering torus $T(H)$. This is the standard procedure: we get a quantum torus which we denote $T(H, \alpha)$. Over $\mathbf{C}$, after exponentiation (0.7) the Poisson structure becomes an alternating bimultiplicative pairing on $H$. Generally, over a base field $K$, we consider a pairing $\alpha: H \times H \to K^*$ such that for all $h, g \in H$

$$\alpha(h,g) = \alpha(g,h)^{-1}, \quad \alpha(h_1 + h_2, g) = \alpha(h_1, g)\alpha(h_2, g). \tag{0.17}$$

The space of functions of any type (algebraic, analytic, smooth, formal ...) is topologically spanned by the set of invertible elements, *formal exponents* $e(h)$. They are linearly independent over $K$ and satisfy the deformed multiplication rule (0.6):

$$e(h)e(h') = \alpha(h, h')\, e(h + h'). \tag{0.18}$$

We may write $e_{H,\alpha}(h)$ for $e(h)$ if different $H$ and/or $\alpha$ are discussed simultaneously.

As in the theory of quantum groups, $\alpha$ plays the role of *quantization parameter.* Notice, however that for nontrivial $\alpha$, $T(H, \alpha)$ *is not a quantum group*: comultiplication is lost.

*The ring of algebraic functions* $Al\,(H, \alpha)$ of the torus $T(H, \alpha)$, by definition, is the linear space spanned over $K$ by $e(h)$, $h \in H$ with multiplication (0.18).

We will also consider the two–sided $Al\,(H, \alpha)$–module of *formal functions* $Af\,(H, \alpha)$ consisting of infinite linear combinations $\sum_h a_h e(h)$, $a_h \in K$, and, in the case of a complete normed field $K$ and *an unitary* quantization parameter $\alpha$ (that is, $|\alpha| = 1$) the ring of *analytic functions* $An\,(H, \alpha)$ consisting of those formal functions for which $|a_h|\,\|h\|^N \to 0$ for any $N$ as $\|h\| \to \infty$, $\|h\|$ being any Euclidean norm on $H$.

When $\alpha \equiv 1$, we get the usual notions of commutative geometry, so that $T(H, 1)$ is the algebraic torus previously denoted $T(H)$. A $K$–point $x \in T(H, 1)(K)$ is thus the same as a homomorphism $H \to K^*$ given by the values of all exponents $e_{H,1}(h)$ at $x$.

*A formal theta function* on $T(H,\alpha)$, by definition, is an element $\Phi \in Af\,(H,\alpha)$ invariant with respect to an abelian group $B$ consisting of transformations of the form

$$[c;\, x, h_l, h_r] : \, \Phi \mapsto c\, e_{H,\alpha}(h_l)\, x^*(\Phi)\, e_{H,\alpha}(h_r)^{-1}. \tag{0.19}$$

generalizing (0.10). Here $c \in K^*$, $g, h \in H$, $x \in T(H,1)(K)$, and $x^*$ is the shift automorphism of the function ring defined by

$$x^*(e_{H,\alpha}(h)) := h(x)\, e_{H,\alpha}(h), \;\; h(x) := e_{H,1}(x). \tag{0.20}$$

Notice the all–important presence of formal exponents at both sides of $x^*(\Phi)$ in (0.19): $l$ stands for left and $r$ for right in the notation $h_{l,r}$. Exponents $e_{H,\alpha}(h)$ are not central, and although we can transfer $e_{H,\alpha}(h_l)$ to the right or $e_{H,\alpha}(h_r)$ to the left, this will result in changing $x$.

In order to keep track of such changes, it is convenient to see $\alpha$ as a homomorphism $H \to T(H,1)(K) : \, h \mapsto A_h$, defined by

$$\forall g, h \in H, \;\; g(A_h) = \alpha(g,h). \tag{0.21}$$

Clearly, we have $g(A_h) = h(A_{-g})$, and any homomorphism satisfying these conditions comes from some $\alpha$.

¿From (0.18) and (0.21) one sees that inner automorphisms corresponding to formal exponents can be alternatively seen as special shift automorphisms:

$$e(h)\, \Phi\, e(h)^{-1} = (A_h^{-2})^*(\Phi). \tag{0.22}$$

We will call the subgroup of $T(H,1)(K)$ consisting of all $A_h^2$ *the hidden period group* of the quantum torus $T(H,\alpha)$.

Quantum identities for the classical modular functions invoked in 0.7 below clearly exhibit this double role of the parameter $q^2$: as the quantization parameter of the torus and as the multiplicative period of certain automorphic functions on this torus.

The quantum thetas studied in this paper may have either hidden, or more general shift period groups.

Namely, imitating the classical formalism, we can define quantum multipliers, develop the quantum versions of symmetry and positivity conditions and finally, in the case of unitary $\alpha$, to use the multiplication of analytic theta functions in order to introduce a quantized version of the homogeneous coordinate ring of an abelian variety.

Now, however, we do not get a ring. We cannot say that the deformed product of two quantum theta functions with the same shift period groups is again such theta. In fact, what in the classical case was a period subgroup $B$ in $T(H,1)(K)$ splits now into *two* period subgroups, left and right, associated with each quantum multiplier and which differ from each other by shifts by hidden periods, encoded in $\alpha$. A product of two analytic thetas is a theta, only if the right periods of one of them coincide with the left periods of another.

It turns out that the quantum multiplication of analytic theta functions can be viewed as composition of morphisms in a linear category, and it is this category that should be considered as *the quantum deformation of the classical graded coordinate ring of an abelian variety.*

Objects of this category are *period maps*: homomorphisms of abelian groups $B \to T(H,1)(K)$. The case of embeddings of discrete lattices is the most important one, but considering arbitrary maps is imposed by our formalism.

It is interesting to remark that the mirror symmetry for abelian varieties can be formulated exactly in terms of these period maps. Namely, any diagram of the form:

$$(i, i^t):\ T(H,1)(K) \leftarrow B \to T(H^t,1)(K)$$

determines mirror dual pairs of quotients (in particular, abelian varieties) ($\mathcal{A} := T(H,1)/i(B), i^t$) and ($\mathcal{B} := T(H^t,1)/i^t(B), i$). Here $H^t = \mathrm{Hom}\,(H, \mathbf{Z})$, and $i^t$ should be considered as a version of physicists' $B$–field for $\mathcal{A}$, whereas $i$ as a $B$–field for $\mathcal{B}$. We refer to [Ma2], Introduction and §1, for further explanations.

It seems that our viewpoint is consistent with the remark made in [KaO]: in the presence of a $B$–field the category of coherent sheaves on an abelian variety must be modified in order to fit into the framework of homological mirror duality. We discuss only invertible sheaves in this paper.

(ii) *Comultiplication and Mumford's formalism.* An abelian variety is, of course, a group variety. This should provide an additional structure on its graded function ring. This additional structure cannot, however, be the standard Hopf comultiplication, because the pullback of an invertible sheaf $L$ with respect to the addition morphism $A \times A \to A : (x,y) \mapsto x+y$, generally cannot be expressed through $pr_1^*(L)$ and $pr_2^*(L)$. D. Mumford in [Mu] remarked that pullbacks behave nicely with respect to the morphism $M:\ A\times A \to A\times A:\ (x,y) \mapsto (x+y, x-y)$, namely, $M^*(L \boxtimes L) \cong L^2 \boxtimes L^2$, if $L$ is symmetric.

Moreover, $\Gamma(L)$ carries an irreducible representation of what we will call *the small Heisenberg group* $\mathcal{G}(L)$ which in our setup can be defined as the natural subquotient of $\mathcal{G}(H,1)$ transforming $\Gamma(L)$ into itself.

Mumford developed a rich and beautiful theory showing how multiplication of sections of invertible sheaves and $M^*$ interacts with the action of small Heisenberg groups and allowing him to produce explicit coordinates on the moduli space of abelian varieties (with some rigidity) and on the varieties themselves, and to write the comultiplication law (or rather $M$) explicitly in terms of these coordinates.

We have seen already that multiplication becomes crucially deformed in our picture.

Remarkably, Mumfords morphism $M$ not only survives quantization (recall that already $T(H,\alpha)$ is not a quantum group), but also remains undeformed in a very precise sense. We will now proceed to explaining this.

**0.6. Heisenberg groups of quantum tori and their rigidity.** By analogy with subsection 0.4, we will define the Heisenberg group $\mathcal{G}(H,\alpha)$ as the group generated by (an consisting of) all linear transformations (0.19). A quantum multiplier $\mathcal{L}$ is a homomorphism of an abelian group $B \to \mathcal{G}(H,\alpha)$. The respective space of quantum theta functions, invariant with respect to the image of $B$, will be denoted $\Gamma(\mathcal{L})$.

Now, it turns out that as any decent Heisenberg group, $\mathcal{G}(H,\alpha)$ together with its basic irreducible representation $Al\,(H,\alpha)$ is a rigid object, and in particular, does not depend on the quantization parameter. This is true in the following sense: there is a unique "untwisting" isomorphism of $\mathcal{G}(H,\alpha)$ with $\mathcal{G}(H,1)$ identical on the subgroups $K^*$ and compatible with representations on functions which are identified via $e_{H,\alpha}(h) \mapsto e_{H,1}(h)$ for all $h \in H$ (Proposition 1.2.1).

In particular, all our formal and analytic theta functions and, more generally, solutions to the equations (0.15), are obtained by lifting similar objects from the classical commutative world. They are even represented by Fourier series with the same coefficients. The small Heisenberg groups also are identified with their commutative counterparts. This is why Mumford's comultiplication formalism remains unchanged, as soon as we established the existence of the quantum version of $M$.

Of course, this does not mean that the theory of quantum thetas is void: the deformed multiplication makes all the difference, in the same way as in the theory of quantum linear groups.

**0.7. Geometry of non–commutative tori vs theory of special functions.** In the spirit of A. Connes [Co], studying topology (smooth geometry, analytic geometry ...) of $T(H,\alpha)$ is the same as studying modules over various function algebras of $T(H,\alpha)$, their $K$–theory, connections and Chern characters. One of the most interesting effects is the existence of non–trivial Morita equivalences between $T(H,\alpha)$ with various $\alpha$, and similar equivalences for modules with connections.

A recent and very intriguing motive is the appearance of non–commutative spaces, tori in particular, as "degenerations" or "extended deformations" of usual geometric objects: abelian and Calabi–Yau varieties, vector bundles etc. Non–commutative tori appear also in some contemporary versions of quantum strings and $M$–theory.

For a sample of these developments and further references cf. [Ri], [RiSch], [So], [CoDSch], [AsSch], [Wi], [KoSo].

This paper is focused on another aspect of the theory, that of special functions.

Physical applications of special functions are related to statistical physics, completely integrable lattice models etc. Roughly speaking, this happens because as we have seen a functional equation (0.15) satisfied by the function $F=\sum_h a_h e(h)$ can be translated into a recurrent relation between the values of the function $h\mapsto a_h$ or the operator function $h\mapsto a_h e_{H,\alpha}(h)$ on the lattice $H$. Even classical functions lifted in different ways to a noncommutative torus may exhibit unexpected properties and satisfy new functional equations.

One of the best known examples is the *$q$–exponential function* which in our notation is simply $e_q(t)$ from (0.16). To see why it deserves this name, consider the two–dimensional torus $T_q$ with the lattice $H=\mathbf{Z}h_1\oplus\mathbf{Z}h_2$ and the scalar product $\alpha(h_1,h_2)=q$, $\alpha(h_1,h_1)=\alpha(h_2,h_2)=1$. Put $u=e(h_1), v=e(h_2)$ so that $uv=q^2vu$. Then for $|q|<1$ in the ring $An\,(T_q)$ we have

$$e_q(u)\,e_q(v)=e_q(u+v). \tag{0.23}$$

Still more interesting is the formula

$$e_q(v)\,e_q(u)=e_q(u)\,e_q(qvu)\,e_q(v) \tag{0.24}$$

derived in [FK] as the quantum version of the Rogers pentagon identity for the classical dilogarithm.

Classical theta functions also made appearance in this context. For example, two lifts of $\theta_q(t)$ (see (0.11)) satisfy a non–commutative identity

$$\theta_q(u)\,\theta_q(v)\,\theta_q(u)=\theta_q(v)\,\theta_q(u)\,\theta_q(v). \tag{0.25}$$

Notice however, that that, unlike (0.23), (0.24), the products (0.25) are only formal functions even for $|q|<1$.

In the same vein, the function

$$r(z,t):=\frac{\theta_q(t)}{e_q(zt)\,e_q(zt^{-1})},\ \ z\in K^*,$$

satisfies the Yang–Baxter type equation

$$r(z,u)\,r(zz',v)\,r(z',u)=r(z',v)\,r(zz',u)\,r(z,v) \tag{0.26}$$

For a proof and discussion, see [FV]. The recent preprint [FKV] describes beautiful applications of these identities and their generalizations to some tori of higher dimension.

I hope that results of this paper can be developed in this direction and integrated with the more geometric theory due to Connes and others.

*Acknowledgement.* General quantum thetas were first introduced in [Ma1]. The approach presented here was developed after the talk given by the author at the Workshop "Trends in Non-commutative Algebra and Geometry" held at the Max Planck Institute for Mathematics in Bonn, Sept. 25–29, 2000. I am very grateful to the participants of this Workshop, especially to Alain Connes, Albert Schwarz and Yan Soibelman, discussions with whom allowed me to see more clearly the basic features of the theory.

## §1. Heisenberg groups of quantum tori

**1.1. Heisenberg groups and related objects.** In this subsection we fix $(H,\alpha)$ as in 0.5 (i). Denote by $\widetilde{\mathcal{G}}(H,\alpha)$ the set consisting of all quadruples $c\in K^*; x\in T(H,1)(K);\, g,h\in H$, with the multiplication law

$$[c';\, x',g',h']\cdot[c;\, x,g,h]=$$

$$[c'c\, g(x')\, h(x')^{-1}\,\alpha(g',g)\,\alpha(h',h)^{-1};\, xx',g+g',h+h']. \tag{1.1}$$

**1.1.1. Lemma–Definition.** *(a) $\widetilde{\mathcal{G}}(H,\alpha)$ is a group. It acts on each of the spaces $Al\,(H,\alpha)$, $An\,(H,\alpha)$, $Af\,(H,\alpha)$ by the rule*

$$[c;\, x,g,h]:\ \Phi\mapsto c\,e(g)\,x^*(\Phi)\,e(h)^{-1}. \tag{1.2}$$

*(b) Denote by $\widetilde{\mathcal{G}}_0(H,\alpha)$ the subset consisting of the elements $[1;\, A_h^2,h,h]$ for all $h\in H$ (see (0.21)). This is a central subgroup of $\widetilde{\mathcal{G}}(H,\alpha)$, kernel of the representation (1.2).*

*(c) Put $\mathcal{G}(H,\alpha)=\widetilde{\mathcal{G}}(H,\alpha)/\widetilde{\mathcal{G}}_0(H,\alpha)$ and call it the large Heisenberg group of $T(H,\alpha)$. Denote by $\mathcal{G}_l(H,\alpha)$ (resp. $\mathcal{G}_r(H,\alpha)$) the subgroup of $\widetilde{\mathcal{G}}(H,\alpha)$ consisting of all elements $[c;\, x,h,0]$ (resp. $[c;\, x,0,h]$). Then the natural projections $\mathcal{G}_{l,r}(H,\alpha)\to\mathcal{G}(H,\alpha)$ are isomorphisms.*

*(d) The map $[c;\, x,g,h]\mapsto g-h$ induces a well defined surjective morphism $h^-:\mathcal{G}(H,\alpha)\to H$ with kernel isomorphic to $K^*\times T(H,1)(K)$.*

**Proof.** A straightforward computation using (0.17), (0.18) shows that $\widetilde{\mathcal{G}}(H,\alpha)$ is a group and (1.2) is a representation.

Clearly, $[c;\, x,g,h]$ produces the identical map, iff we have for all $k\in H$:

$$c\,e(g)\,x^*(e(k))\,e(h)^{-1}=e(k).$$

This means that $g=h$ and for all $k$, $c\,k(x)\,\alpha(h,k)^2=1$, in other words, $c=1$, $k(x)=\alpha(k,h)^2$. Comparing this to (0.21), one sees that $x=A_h^2$.

Using this description, one easily checks that $\widetilde{\mathcal{G}}(H,\alpha)$ is a semidirect product of either of the groups $\mathcal{G}_{l,r}(H,\alpha)$ and of the central subgroup $\widetilde{\mathcal{G}}_0(H,\alpha)$, which finishes the proof.

Many properties of $\mathcal{G}(H,\alpha)$ can be conveniently checked by using its identification with, say, $\mathcal{G}_l(H,\alpha)$.

However, in order to treat the multiplication of theta functions below, we will have to use both versions $\mathcal{G}_l$ and $\mathcal{G}_l$ on the equal footing. To this end we will collect several formulas for future use. Call the left (resp. the right) representative of $[c;x,g,h]$ its natural projection of the form $[c_l;x_l,h_l,0]$ (resp. $[c_r;x_r,0,h_r]$). We have then

$$[c_l;x_l,h_l,0]=[c\,\alpha(h,g)\varepsilon(h);xA_h^{-2},g-h,0], \tag{1.3}$$

$$[c_r;\,x_r,0,h_r]=[c\,\alpha(h,g)\varepsilon(g);xA_g^{-2},0,h-g]. \tag{1.4}$$

Here $\varepsilon(h):=\alpha(h,h)$; this is a character of $H$ with values in $\{\pm 1\}$ which in [Ma2] was called *the characteristic* of the quantization parameter $\alpha$.

To put this differently, two elements $[c_l;x_l,h_l,0]$, $[c_;x_r,0,h_r]$ have the same image in $\mathcal{G}(H,\alpha)$ iff

$$h_l=-h_r,\ x_r=x_lA_{h_l}^{-2}=x_lA_{h_r}^2,\ c_r=c_l\varepsilon(h_l)=c_l\varepsilon(h_r). \tag{1.5}$$

**1.2. Rigidity.** As we mentioned in the Introduction, our Heisenberg group together with its basic irreducible representation is a rigid object independent of $\alpha$. More precisely, let us refer to the quantization parameter explicitly by adding an appropriate subscript to our notation. Then we have:

**1.2.1. Proposition.** *(a) The map written in terms of left representatives*

$$u_{1,\alpha}:\,\mathcal{G}(H,1)\to\mathcal{G}(H,\alpha):\ [c;\,x,h,0]_1\mapsto[c;\,xA_h,h,0]_\alpha \tag{1.6}$$

*is a group isomorphism.*

*(b) The linear map (denoted by the same symbol)*

$$u_{1,\alpha}:\,Al\,(H,1)\to Al\,(H,\alpha):\ e_{H,1}(h)\mapsto e_{H,\alpha}(h) \tag{1.7}$$

*is an isomorphism of the representations extending (1.6), and similarly for other spaces of functions.*

*(c) $u_{1,\alpha}$ is the unique isomorphism satisfying (b).*

**Proof.** In fact, one easily checks that the map (1.6) is bijective and compatible with products.

Moreover,

$$[c;\,x,h,0]_1\,(e_{H,1}(g))=c\,e_{H,1}(h)\,x^*(e_{H,1}(g))=c\,g(x)\,e_{H,1}(h+g),$$

$$[c;\,xA_h,h,0]_\alpha\,(e_{H,\alpha}(g))=c\,e_{H,\alpha}(h)\,(xA_h)^*(e_{H,\alpha}(g))=c\,g(xA_h)\,\alpha(h,g)\,e_{H,\alpha}(h+g),$$

and the coefficients at the right hand sides coincide in view of (0.21). Finally, unicity follows from the fact that our representation is faithful.

This proves the Proposition.

More generally, we can define a coherent system of twisting isomorphisms

$$u_{\alpha,\beta} := u_{1,\beta} \circ u_{1,\alpha}^{-1} : \mathcal{G}_l(H,\alpha) \to \mathcal{G}_l(H,\beta) : [c;\, x, h, 0]_\alpha \mapsto [c;\, xA_h^{-1}B_h, h, 0]_\beta, \quad (1.8)$$

where the points $B_h$ are determined by the relations (0.21) with $\beta$ replacing $\alpha$.

Of course, twisting isomorphisms generally are not compatible with multiplication. We will now introduce and study the structure on the Heisenberg group which reflects multiplication of functions and which actually depends on $\alpha$.

**1.3. Partial composition in $\mathcal{G}(H,\alpha)$.** Let $\Gamma'', \Gamma' \in \mathcal{G}(H,\alpha)$. Call these elements *composable* (in this order) iff $x''_r = x'_l$ where $x''_r$ (resp. $x'_l$) is taken from the right (resp. left) representative of $\Gamma''$ (resp. $\Gamma'$):

$$\Gamma'' \equiv [c''_r;\, x''_r, 0, h''_r] \bmod \widetilde{\mathcal{G}}_0(H,\alpha),$$

$$\Gamma' \equiv [c'_l;\, x'_l, h'_l, 0] \bmod \widetilde{\mathcal{G}}_0(H,\alpha).$$

If this condition is satisfied, define the composition

$$\Gamma'' \circ \Gamma' := [c''_r;\, 1, 0, h''_r] \cdot [c'_l;\, x'_l, h'_l, 0] \bmod \widetilde{\mathcal{G}}_0(H,\alpha). \quad (1.9)$$

Calculating with the help of (1.1), (1.3), (1.4), we get

$$\Gamma'' \circ \Gamma' \equiv [c''_r c'_l\, \alpha(h'_l, h''_l)\, \varepsilon(h''_l);\, x'_l A^2_{h''_l}, h'_l + h''_l, 0] \equiv$$

$$[c''_r c'_l\, \alpha(h'_l, h''_l)\, \varepsilon(h'_l);\, x''_r A^2_{h'_r}, 0, h'_r + h''_r] \bmod \widetilde{\mathcal{G}}_0(H,\alpha). \quad (1.10)$$

**1.3.1. Proposition.** *(a) Assume that the pairs $(\Gamma''', \Gamma'')$ and $(\Gamma'', \Gamma')$ are composable. Then the pairs $(\Gamma''', \Gamma'' \circ \Gamma')$ and $(\Gamma''' \circ \Gamma'', \Gamma')$ are composable as well, and the triple composition is associative.*

*(b) We have for any $\Phi, \Psi \in Al\,(H,\alpha)$ (or $An\,(H,\alpha)$)*

$$\Gamma'(\Phi)\, \Gamma''(\Psi) = (\Gamma'' \circ \Gamma')\, (\Phi\Psi). \quad (1.11)$$

**Proof.** We start with checking compatibilities needed to form two triple products. Given equalities $x'''_r = x''_l$ and $x''_r = x'_l$, we obtain, using (1.10):

$$x_l^{\prime\prime\circ\prime} = x'_l A^2_{h''_l} = x''_r A^2_{h''_l}$$

which in view of (1.5) coincides with $x_l''$ and therefore with $x_r'''$. Here $x_l''^{\circ\prime}$ denotes the $x$–component of the left representative of $\Gamma''\circ\Gamma'$.

Similarly, $x_r'''^{\circ\prime\prime} = x_l'$ because both sides are equal to $x_l'' A_{h_l''}^{-2} = x_r''$.

Now we verify (1.11):

$$\Gamma'(\Phi)\,\Gamma''(\Psi) = c_l' e(h_l')\,x_l'^*(\Phi)\,c_r'' x_r''^*(\Psi)\,e(h_r'')^{-1} =$$

$$c_l' c_r'' e(h_l')\,x_l'^*(\Phi\Psi) e(h_r'')^{-1} =$$

$$[c_r'';\,1,0,h_r'']\cdot[c_l';\,x_l',h_l',0]\,(\Phi\Psi) = (\Gamma''\circ\Gamma')\,(\Phi\Psi)$$

in view of (1.9).

Finally, applying (1.11) several times, we see that

$$(\Gamma'''\circ(\Gamma''\circ\Gamma'))\,(\Phi\Psi\Xi) = \Gamma'(\Phi)\,\Gamma''(\Psi)\,\Gamma'''(\Xi) = ((\Gamma'''\circ\Gamma'')\circ\Gamma')\,(\Phi\Psi\Xi)$$

for any $\Phi,\Psi,\Xi$, which proves associativity.

We will show now that $\mathcal{G}(H,\alpha)$ with its partial multiplication can be treated as the set of morphisms of an amusing category $\mathcal{C}(H,\alpha)$. By definition, its objects are points $\xi\in T(H,1)(K)$ and morphisms are

$$\mathrm{Hom}\,(\xi,\eta) := \{\Gamma\in\mathcal{G}(H,\alpha)\,|\,x_r(\Gamma)=\xi,\,x_l(\Gamma)=\eta\}. \qquad (1.12)$$

For composition we take $\circ$. Proposition 1.3.1 shows that this category is well defined. Identity morphisms are $[1;\,\xi,0,0]$.

**1.3.2. Proposition.** *$\mathcal{C}(H,\alpha)$ is a groupoid with the set of isomorphism classes of objects $T(H,1)(K)/P_\alpha$, where $P_\alpha := \{A_h^2\,|\,h\in H\}$ is the group of hidden periods. The automorphism group of any object is isomorphic to $K^*\times K_0$ where $K_0$ is the kernel of the form $\alpha^2$.*

**Proof.** If there is a morphism $\Gamma:\ \xi\to\eta$, represented by $[c_r;\,x_r,0,h_r]$ and $[c_l;\,x_l,h_l,0]$, we must have $x_r=\xi, x_l=\eta$. In view of (1.5) this is possible precisely when $\xi\equiv\eta \bmod P_\alpha$. Moreover, we then have $\xi=\eta A_{h_l}^{-2}=\eta A_{h_r}^2$.

In this case we van solve the equation $\Gamma'\circ\Gamma=[1;\xi,0,0]$ for the left inverse to $\Gamma$: namely $\Gamma'=[c_l^{-1};\,\eta,-h_l,0]$, and similarly for the right inverse.

Finally, the (left representatives of the) automorphism group of $\xi$ are $[c;\,\xi,h,0]$, $h\in\mathrm{Ker}\,\alpha^2$, with the multiplication law

$$[c';\,\xi,h',0]\circ[c;\,\xi,h,0] = [c'c;\,\xi,h'+h,0].$$

**1.3.3. Double–sided representatives.** Elements of the form $[c;\,1,g,h]$ form a subgroup $\mathcal{G}_d(H,\alpha)$ in $\widetilde{\mathcal{G}}(H,\alpha)$. Assume that $\alpha^2$ is non–degenerate. Then the natural morphism $\mathcal{G}_d(H,\alpha) \to \mathcal{G}(H,\alpha)$ is an embedding (see Lemma 1.1.1 (b)). We will identify in this case $\mathcal{G}_d(H,\alpha)$ with its image, and call its elements double–sided representatives.

Two such representatives $[c'', 1, h''_l, h''_r]$, $[c', 1, h'_l, h'_r]$ are composable, iff $h'_r = h''_l$, and in this case their composition is double–sided as well:

$$[c'', 1, h''_l, h''_r] \circ [c', 1, h'_l, h'_r] = [c''c';\, h'_l, h''_r].$$

We leave the straightforward check as an exercise.

**1.4. Functorial properties of the Heisenberg group.** In this paper as in [Ma2] the category of quantum tori is defined as opposite to the category of their rings of *algebraic* functions. This means that a morphism $F : T(H,\alpha) \to T(H',\alpha')$, is given by the contravariant $K$–algebra homomorphism $F^* : Al\,(H',\alpha') \to Al\,(H,\alpha)$. The following (easy) result is proved in [Ma2]:

**1.4.1. Proposition.** *a) The set of invertible elements of* $Al\,(H,\alpha)$ *is* $\{a\, e(h)\,|\, a \in K^*, h \in H\}$. *If* $F : T(H_2,\alpha_2) \to T(H_1,\alpha_1)$ *is a morphism of non–commutative tori, then the induced map* $f = [F] : H_1 \to H_2$ *determined by* $F^*(e_{H_1,\alpha_1}(h)) = a_h e_{H_2,\alpha_2}(f(h))$, $a_h \in K^*$, *is additive and compatible with the squares of the quantization forms:*

$$\alpha_2^2(f(h), f(g)) = \alpha_1^2(h,g). \tag{1.13}$$

*b) The set of all morphisms* $F$ *with a fixed* $f = [F]$ *is either empty, or has a natural structure of the principal homogeneous space over the group* $T(H_1, 1)(K) =$ Hom$\,(H_1, K^*)$. *In particular, if the characteristic of* $f$ *is 1, then* $F^* : e(h) \mapsto e(f(h))$ *extends to a uniquely defined morphism of rings of algebraic functions.*

*c) Any morphism* $F^*$ *extends to* $Af$ *by* $F^*(\sum a_h e(h)) = \sum a_h F^*(e(h))$. *If* $K$ *is normed and* $\alpha$ *unitary, then this extension maps analytic functions to analytic.*

The bilinear form

$$\varepsilon_f(h,g) := \alpha_1(h,g)\alpha_2^{-1}(f(h), f(g))$$

with values in $\{\pm 1\}$ is called the characteristic of $f$ and of $F$. This form is trivial iff $a_{g+h} = a_g a_h$ for all $g, h$.

Shifts (0.20) of $T(H,\alpha)$ by points of $T(H,1)(K)$ are endomorphisms of $T(H,\alpha)$. Besides them, we will use the *multiplication by* $n$

$$[n] : T(H,\alpha) \to T(H,\alpha^{n^2}), \quad [n]^*(e_{H,\alpha^{n^2}}(h)) = e_{H,\alpha}(nh). \tag{1.14}$$

and *the Mumford morphism*

$$M : T(H\oplus H, \alpha\oplus\alpha) \to T(H\oplus H, \alpha^2\oplus\alpha^2),$$

$$M^*(e_{H\oplus H,\alpha^2\oplus\alpha^2}(h,g)) = e_{H\oplus H,\alpha\oplus\alpha}(h+g,h-g). \tag{1.15}$$

It is well defined, because

$$(\alpha\oplus\alpha)[(h+g,h-g),(h'+g',h'-g')] =$$

$$\alpha(h+g,h'+g')\alpha(h-g,h'-g') = \alpha^2(h,h')\alpha^2(g,g').$$

Consider now a morphism of tori $F : T(H_2,\alpha_2) \to T(H_1,\alpha_1)$ as in Proposition 1.4.1 such that $F^*(e(h)) = a_h e(f(h))$, $a_h \in K^*$, for a homomorphism of character groups $f = [F] : H_1 \to H_2$. Assume that $\operatorname{Ker} f = \{0\}$ so that $F^*$ is an embedding. Denote by $\mathcal{G}(F)$ the following subquotient of $\mathcal{G}(H_2,\alpha_2)$: take the full subgroup stabilizing the subspace $F^*(Al\,(H_1,\alpha_1))$ and mode out the kernel of this representation.

**1.4.2. Proposition.** *Assume that $F^*$ satisfies the following condition: $h\mapsto a_h$ is a group homomorphism. Then there exists a well defined isomorphism $\mathcal{G}(F)\to\mathcal{G}(H_1,\alpha_1)$ compatible with the natural representations of these groups on $Al\,(H_1,\alpha_1)$.*

**Proof.** Let us work with left representatives.

Clearly $[c;\,x,h,0]_{\alpha_2}$ stabilizes the subspace $F^*(Al\,(H_1,\alpha_1))$ iff $h\in\operatorname{Im} f$. Such an element acts identically on this subspace iff $c=1$, $h=0$, and $f^*(g)(x)=1$ for all $g\in H_1$, in other words, $x$ belongs to the kernel $\mathcal{K}$ of the homomorphism $\varphi : T(H_2,1)(K)\to T(H_1,1)(K)$ induced by $f$.

For any $g,h\in H_1$ we then have

$$[\,ca_g;\,x,f(g),0]_{\alpha_2}\,F^*(e(h)) = F^*([\,c;\,\varphi(x),g,0]_{\alpha_1}\,e(h))$$

(here the multiplicativity of $a_h$ is used). One now easily sees that the map

$$\mathcal{G}(F)\to\mathcal{G}(H_1,\alpha_1)\ :\ [\,ca_g;\,x \bmod \mathcal{K},f(g),0\,]_{\alpha_2}\mapsto[\,c;\,\varphi(x),g,0\,]_{\alpha_1}$$

is a well defined isomorphism which we looked for.

**1.4.3. Homomorphisms $\psi_{n,d}$.** Let $d,n$ be two integers, $d/n$. Consider the map

$$\psi_{d,n} : \widetilde{\mathcal{G}}(H,\alpha^d)\to\widetilde{\mathcal{G}}(H,\alpha) :\ [c;\,x,h_l,h_r]_{\alpha^d}\mapsto[c^{n^2/d};\,x^{n/d},nh_l,nh_r]_\alpha. \tag{1.16}$$

A straightforward check shows that it is a homomorphism mapping $\widetilde{\mathcal{G}}_0(H,\alpha^d)$ into $\widetilde{\mathcal{G}}_0(H,\alpha)$. Hence it induces a homomorphism of the large Heisenberg groups which we denote by the same symbol:

$$\psi_{d,n} : \mathcal{G}(H,\alpha^d)\to\mathcal{G}(H,\alpha).$$

## §2. Multipliers and theta functions

**2.1. Definition.** *(i) A (formal) theta multiplier for $T(H,\alpha)$ is a homomorphism $\mathcal{L}: B \to \mathcal{G}(H,\alpha)$ where $B$ is an abelian group.*

*(ii) A theta function with multiplier $\mathcal{L}$ is a formal function on $T(H,\alpha)$ invariant with respect to the action of (the image of) $B$.*

*(iii) $\Gamma(\mathcal{L})$ is the linear space of theta functions with multiplier $\mathcal{L}$.*

This notation is supposed to remind the case of usual abelian varieties where we deal with invertible sheaves and their sections.

Only the image of $B$ determines the space of theta–functions, but using arbitrary homomorphisms gives more flexibility.

Image of any element $b \in B$ has left and right representatives in $\mathcal{G}_{l,r}$ which we will denote respectively

$$[\,c_{l,b};\, x_{l,b}, h_{l,b}, 0],\ \ [\,c_{r,b};\, x_{r,b}, 0, h_{r,b}].$$

As in (1.5), their coincidence modulo $\mathcal{G}_0(H,\alpha)$ means that for all $b \in B$

$$-h_{r,b} = h_{l,b} = h_b^-,\ \ x_{r,b} = x_{l,b}\, A_{h_{l,b}}^{-2},\ \ c_{r,b} = \varepsilon(h_b^-)\, c_{l,b}. \tag{2.1}$$

**2.2. Lemma.** *Let $B$ be an abstract abelian group. Consider two families of elements $[\,c_{l,b};\, x_{l,b}, h_{l,b}, 0]$ and $[\,c_{r,b};\, x_{r,b}, 0, h_{r,b}]$ in $\mathcal{G}_{l,r}(H,\alpha)$ respectively, indexed by $B$ and satisfying (2.1). Then these families constitute a theta multiplier as above, iff the following conditions are satisfied:*

*(i) The maps $B \to H: b \mapsto h_{l,b}$ and and $B \to T(H,1)(K): b \mapsto x_{l,b}$ are group homomorphisms, and the same for $h_r$, $x_r$.*

*(ii) The map*

$$B \times B \to K^*:\ (b_1, b_2) \mapsto \langle b_1, b_2\rangle := h_{b_2}^-(x_{l,b_1})\, \alpha(h_{b_1}^-, h_{b_2}^-) \tag{2.2}$$

*is a symmetric bimultiplicative pairing.*

*(iii) We have identically*

$$\frac{c_{l,b_1+b_2}}{c_{l,b_1} c_{l,b_2}} = \langle b_1, b_2\rangle \tag{2.3}$$

*and the same for $c_{r,b}$.*

The proof is straightforward.

If a symmetric bimultiplicative pairing $(b_1, b_2)$ is chosen such that $(b_1, b_2)^2 = \langle b_1, b_2\rangle$, then any solution of (2.3) can be uniquely represented in the form

$$c_{l,b} = \psi_l(b)\,(b,b), \tag{2.4}$$

where $\psi_l : B \to K^*$ is a character. We have $\psi_r(b) = \varepsilon(h_b^-)\psi_l(b)$.

Such choice of a square root becomes always possible after a finite extension of $K$. We will also use the explicit representation of coefficients in the form (2.4), because it matches the classical notation.

**2.3. Definition.** *(i) Quadruples $(\psi_l, (\,,\,), x_l, h_l)$ (resp. $(\psi_r, (\,,\,), x_r, h_r)$) are called the left (resp. right) automorphy factors of the multiplier $\mathcal{L}$.*

*(ii) The maps $x_{l,r} : B \to T(H,1)(K)$ and their images are called the left (resp. the right) period groups of $\mathcal{L}$.*

The abstract definition of multipliers 2.1 makes transparent their functorial properties, whereas the description 2.2 provides the link with the classical constructions in the theory of theta functions. In particular, (2.2) for $\alpha \equiv 1$ becomes the Riemann symmetry condition so that generally (2.2) represents its quantum deformation.

Twisting isomorphisms (1.8) identify the sets of theta multipliers for various $\alpha$'s and their actions on the spaces of functions. What remains specific for each $\alpha$, is the interaction of a multiplier with the deformed multiplication. The period groups also change under twisting isomorphisms.

**2.4. Analytic theta functions and Riemann's symmetry conditions.** By definition, a theta function with multiplier $\mathcal{L}$ must satisfy the functional equations

$$\psi_l(b)\,(b,b)\,e(h_{l,b})\,(x_{l,b})^*(\theta) = \theta = \psi_r(b)\,(b,b)\,(x_{r,b})^*(\theta)\,e(h_{r,b})^{-1} \tag{2.5}$$

for all $b \in B$.

**2.4.1. Theorem.** *(a) We have*

$$\dim \Gamma(\mathcal{L}) \le [H : h^-(B)]. \tag{2.6}$$

*Equality holds if $x_{l,b}, c_{l,b}$ depend only on $h_b^-$, for example, if $h^- : B \to H$ is an embedding.*

*(b) Let the last condition be satisfied. Assume that $K$ is a normed field. Then all theta functions with multiplier $\mathcal{L}$ are analytic if $[H : h^-(B)] < \infty$ and $\log|\langle b,b\rangle|$ is a negatively defined quadratic form on $B$.*

*In this case $\operatorname{rk} B = \operatorname{rk} H$, and $B$ is discrete in $T(H,1)(K)$.*

*Such multipliers will be called ample.*

**Proof.** Let $\theta = \sum_{h\in H} a_h e(h)$, $a_h \in K$, $b \in B$. We have

$$(x_{l,b})^*(\theta) = \sum_{h\in H} a_h h(x_{l,b}) e(h) = \sum_{h\in H} a_{h-h_b^-}\, h_b^-(x_{l,b})^{-1}\, h(x_{l,b})\, e(h-h_b^-),$$

so that the left hand side of (2.5) becomes

$$c_{l,b}\, e(h_b^-) \sum_{h\in H} a_{h-h_b^-}\, h_b^-(x_{l,b})^{-1}\, h(x_{l,b})\, e(h-h_b^-) =$$

$$\sum_{h\in H} c_{l,b}\, a_{h-h_b^-}\, h_b^-(x_{l,b})^{-1}\, h(x_{l,b})\, \alpha(h_b^-, h)\, \varepsilon(h_b^-)\, e(h).$$

Replacing here $h_b^-(x_{l,b})^{-1}$ by $\langle b,b\rangle\,\varepsilon(h_b^-)$ (see (2.2)), we see that (2.5) is equivalent to the following equations for coefficients:

$$a_{h-h_b^-} = a_h\,c_{l,b}^{-1}\,\langle b,b\rangle\,h(x_{l,b})^{-1}\,\alpha(h,h_b^-) \tag{2.7}$$

for all $h\in H$ and $b\in B$. Therefore if values $a_h$ for some system of representatives of $H/h^-(B)$ are fixed, $\theta$ is defined uniquely, if it exists at all.

Such a choice can be done arbitrarily, if $h^-$ is injective; otherwise (2.7) might be overdetermined. This proves the first statement of the theorem. It also shows that if $\Gamma(\mathcal{L})$ is not finite dimensional, it necessarily contains non–analytic functions.

Assume now that $\Gamma(\mathcal{L})$ is finite dimensional. Then on each coset $h-h^-(B)$ we have, in the notation (2.4),

$$\log|a_{h-h_b^-}| = \log|a_h| + \log|(b,b)| - \log|\psi_l(b)\,h(x_{l,b})\,\alpha(h_b^+,h)|.$$

The second summand in the right hand side is quadratic in $b$ whereas the third is linear. Hence analyticity follows from the negative definiteness of $(b,b)$ or $\langle b,b\rangle$ which is the same.

**2.5. Operations on multipliers.** *(i) Powers.* We can compose any multiplier $\mathcal{L}:\,B\to\mathcal{G}(H,\alpha^d)$ with the homomorphism $\psi_{d,n}:\,\mathcal{G}(H,\alpha^d)\to\mathcal{G}(H,\alpha)$ defined in 1.4.3. The resulting composition will be a multiplier as well.

Especially important is the case $d=n$. We will denote $\psi_{n,n}\circ\mathcal{L}$ by $\mathcal{L}^n$. From (1.16) we see that

$$\mathcal{L}^n(b) = [c_{l,b}^n;\,x_{l,b},nh_{l,b},0]. \tag{2.8}$$

In particular, the left and the right period groups for $\mathcal{L}^n$ are the same as for $\mathcal{L}$.

If $\alpha\equiv 1$, $\mathcal{L}\mapsto\mathcal{L}^n$ simply corresponds to the $n$–th power of invertible sheaves, in particular, product of theta functions induces a map $\Gamma(\mathcal{L})^{\otimes n}\to\Gamma(\mathcal{L}^n)$. This cannot be true in general, because $\mathcal{L}$ and $\mathcal{L}^n$ correspond to different quantization parameters. The only polylinear map of this kind I can think of can be obtained by first untwisting $\alpha^n$, multiplying in the commutative ring, and then again twisting to $\alpha$.

*(ii) External tensor products.* Consider two multipliers $\mathcal{L}':\,B'\to T(H',\alpha')$, and $\mathcal{L}'':\,B''\to T(H'',\alpha'')$. We will use their left representatives denoted as in 2.1.

Their external tensor product $\mathcal{L}=\mathcal{L}'\boxtimes\mathcal{L}''$ is a multiplier

$$\mathcal{L}'\boxtimes\mathcal{L}'':\;B'\oplus B'' :\to \mathcal{G}(H'\oplus H'',\alpha'\oplus\alpha'')$$

which is defined by the following left representatives:

$$\mathcal{L}(b_1,b_2) = [c'_{l,b_1}c''_{l,b_2};\,(x'_{l,b_1},x''_{l,b_2}),(h'_{l,b_1},h''_{l,b_2}),0]. \tag{2.9}$$

*(iii) Pullbacks.* Consider a morphism of non–commutative tori $F: T(H_2,\alpha_2) \to T(H_1,\alpha_1)$ such that $F^*$ is an embedding and the map $h \mapsto a_h$ is a homomorphism, as in Proposition 1.4.2. Let $\mathcal{K}$ be the group defined in the proof of that Proposition. Denote by $\widetilde{\mathcal{G}}(F)$ the subgroup of $\mathcal{G}(H_2,\alpha_2)$ stabilizing $F^*(Al\,(H_1,\alpha_1))$ so that we have the exact sequence

$$1 \to \mathcal{K} \to \widetilde{\mathcal{G}}(F) \to \mathcal{G}(F) \to 1.$$

A multiplier $\mathcal{L}: B \to \mathcal{G}(H_1,\alpha_1)$ can be identified with a homomorphism $\mathcal{L}: B \to \mathcal{G}(F)$. Any lift of this homomorphism to a homomorphism $F^*(\mathcal{L}): B \to \widetilde{\mathcal{G}}(F)$ is a multiplier for $T(H_2,\alpha_2)$. Such lifts certainly exist if $\mathcal{K}$ is finite and the subgroup of shifts $\{x_{l,b}\}$ of $\mathcal{L}$ is free.

For any lift, any function $\Phi$ on $T(H_1,\alpha_1)$ and any $b \in B$ we will have

$$F^*(\mathcal{L})(b)(F^*(\Phi)) = F^*(\mathcal{L}(b)\Phi). \tag{2.10}$$

As a corollary, we get

$$F^*(\Gamma(\mathcal{L})) \subset \Gamma(F^*(\mathcal{L})). \tag{2.11}$$

This construction is applicable, in particular, to shifts, multiplications $[n]$ and Mumford's morphisms (see (1.14) and (1.15)).

Concretely, if as usual we denote by $[c_{l,b};\, x_{l,b}, h_{l,b}, 0]$ the left representative of $\mathcal{L}(b)$, then

$$F^*(\mathcal{L})(b) = [c_{l,b}\, a_{h_{l,b}};\, x'_{l,b}, f(h_{l,b}), \tag{2.12}$$

where as above, $F^*(e(h)) = a_h e(f(h))$, $\varphi$ is the morphism induced on commutative tori, and $x'_{l,b} \in \varphi^{-1}(x_{l,b})$.

Applying this to shift morphisms and $[n]$, we get

$$y^*(\mathcal{L})(b) = [c_{l,b}h_{l,b}(y);\, x_{l,b}y^{-1}, h_{l,b}, 0], \tag{2.13}$$

$$[n](\mathcal{L})(b) = [c_{l,b};\, x_{l,b}^{1/n}, nh_{l,b}, 0]_{\alpha^{n^2}}. \tag{2.14}$$

(Note the change of the quantization parameter in the last formula).

The multiplier $\mathcal{L}$ is called *symmetric*, if $[-1]\mathcal{L}(-b) \equiv \mathcal{L}(b)$. In view of (2.14), this is equivalent to $c_{l,b} \equiv c_{l,-b}$, or, in terms of the automorphy factors (2.4), $\psi(b) \in \{\pm 1\}$.

**2.6. Twisting isomorphisms.** The existence and unicity of the twisting maps (1.6), (1.7), (1.8) shows that most of the structures described in this section for different $\alpha$ are connected by the canonical isomorphisms. More precisely, if $\mathcal{L}$ is a (formal) theta multiplier for $T(H,\alpha)$, then $u_{\alpha,\beta}\circ\mathcal{L}$ is a theta multiplier for $T(H,\beta)$. Moreover, $\Gamma(\mathcal{L})$ turns into $\Gamma(u_{\alpha,\beta}\circ\mathcal{L})$ by simply replacing $e_{H,\alpha}(h)$ with $e_{H,\beta}(h)$.

The behaviour of multipliers with respect to morphisms of quantum tori becomes slightly more complex, if shifts are involved. However, any commutative diagram involving only morphisms which transform formal exponents to formal exponents makes sense and remains commutative after untwisting all quantization parameters to 1. Introducing theta functions into this picture, we can coherently untwist them as well.

The most important application of this remark concerns Mumford's morphisms (1.15), to which I will return in §4. It follows that the "comultiplication" of quantum theta functions remains essentially the same as in the classical case.

We now pass to the more interesting effects related to multiplication.

## §3. Composition of multipliers and multiplication of thetas

**3.1. Composition of multipliers.** Consider two multipliers $\mathcal{L}'', \mathcal{L}' : B \to \mathcal{G}(H,\alpha)$. Call them *composable*, if for every $b \in B$, $\mathcal{L}''(b)$ and $\mathcal{L}'(b)$ are composable in the sense of 1.3.

When $\mathcal{L}''$ and $\mathcal{L}'$ are composable, we can define their pointwise composition $\mathcal{L} : B \to \mathcal{G}(H,\alpha)$, $\mathcal{L}(b) := \mathcal{L}''(b) \circ \mathcal{L}'(b)$. A priori this is only a map of sets. The following theorem shows that we, in fact, get a multiplier.

We will combine the notation of 1.3 and of 2.1, 2.2, so that, for example, the left representative of $\mathcal{L}'(b)$ is $[c'_{l,b}; x'_{l,b}, h'_{l,b}, 0]$, the structure form (2.2) of $\mathcal{L}''$ is $\langle\,,\rangle''$ etc.

**3.1.1. Theorem.** *(a) $\mathcal{L}$ is a multiplier. Its structure form is*

$$\langle b_1, b_2\rangle = \langle b_1, b_2\rangle'' \, \langle b_1, b_2\rangle' \, \alpha(h'_{l,b_2}, h''_{l,b_1}) \, \alpha(h'_{l,b_1}, h''_{l,b_2}). \tag{3.1}$$

*(b) If $\theta' \in \Gamma(\mathcal{L}')$, $\theta'' \in \Gamma(\mathcal{L}'')$, and $\theta := \theta'\theta''$ exists as a formal series, then $\theta \in \Gamma(\mathcal{L})$.*

*(c) If $\alpha$ is unitary and $\mathcal{L}''$, $\mathcal{L}'$ are ample, then $\mathcal{L}$ is also ample, and all such pairwise products of theta functions are well defined.*

**Proof.** (a) From (1.10) we get the left representative of $\mathcal{L}(b)$ in the form:

$$[c''_{l,b} c'_{l,b} \, \alpha(h'_{l,b}, h''_{l,b}); \, x'_{l,b} A^2_{h''_{l,b}}, h'_{l,b} + h''_{l,b}, 0]. \tag{3.2}$$

We have to check for these representatives the conditions (2.2) and (2.3) of Lemma 2.2. With (3.2) as input, the right hand side of (2.2) becomes

$$h'_{l,b_2}(x'_{l,b_1} A^2_{h''_{l,b_1}}) \, h''_{l,b_2}(x'_{l,b_1} A^2_{h''_{l,b_1}}) \times$$

$$\alpha(h'_{l,b_1}, h'_{l,b_2}) \, \alpha(h''_{l,b_1}, h''_{l,b_2}) \, \alpha(h'_{l,b_1}, h''_{l,b_2}) \, \alpha(h''_{l,b_1}, h'_{l,b_2}). \tag{3.3}$$

Using the composability condition $x''_{r,b} = x'_{l,b}$ and (2.1) in the form $x''_{r,b} = x''_{l,b} A^{-2}_{h''_{l,b}}$, we can replace the second factor in (3.3) by $h''_{l,b_2}(x''_{l,b_1})$. Taking into account (2.2) for the two multipliers, we obtain the symmetric form (3.1).

It remains to check (2.3). Again from (3.2) we obtain

$$\frac{c_{l,b_1+b_2}}{c_{l,b_1} c_{l,b_2}} = \frac{c''_{l,b_1+b_2}}{c''_{l,b_1} c''_{l,b_2}} \cdot \frac{c'_{l,b_1+b_2}}{c'_{l,b_1} c'_{l,b_2}} \cdot \alpha(h''_{r,b_1}, h'_{l,b_2})\, \alpha(h''_{r,b_2}, h''_{l,b_1}).$$

Using (2.2) and replacing the right components $h_r$ by the left ones, we finally get (3.1).

(b) This immediately follows from (1.11).

(c) If $\alpha$ is unitary, then the logarithm of the modulus of form (3.1) is negative defined, when this holds for the factors.

**3.2. Categorical interpretation.** Extending (1.12), we can define two categories in which composition of multipliers and multiplication of theta functions become the composition of morphisms. They have common objects: "period homomorphisms" $\xi : B \to T(H,1)(K)$ ($B$ is assumed to be fixed; the most important case is that of a lattice of the same rank as $H$).

In the category PIC $(H, \alpha)$ morphisms are multipliers:

$$\mathrm{Hom}_{\mathrm{PIC}}(\xi, \eta) := \{\, \mathcal{L} \mid x_r(\mathcal{L}) = \xi,\, x_l(\mathcal{L}) = \eta\} \tag{3.4}$$

Here, say, $x_r(\mathcal{L})$ is the period map $b \mapsto x_{r,b}$ associated with $\mathcal{L}$.

In the category Pic $(H, \alpha)$ morphisms are

$$\mathrm{Hom}_{\mathrm{Pic}}(\xi, \eta) := \{\, \oplus' \,\Gamma(\mathcal{L}) \mid x_r(\mathcal{L}) = \xi,\, x_l(\mathcal{L}) = \eta\} \tag{3.5}$$

where the summation $\oplus'$ is taken only over ample multipliers. The resulting morphism spaces are of course graded by ample multipliers, and the composition is compatible with this grading.

**3.3. Multipliers with hidden periods.** We will say that $\mathcal{L}$ and its thetas have *hidden periods*, if the image of $B$ lies in the group of double–sided representatives $\mathcal{G}_d(H, \alpha)$ described in 1.3.3. We continue to assume that $\alpha^2$ is non–degenerate.

Put $\mathcal{L}(b) = [c_b;\, 1, h_{l,b}, h_{r,b}]$. From the condition $\mathcal{L}(b_1)\mathcal{L}(b_2) = \mathcal{L}(b_1 + b_2)$ we see that such a family defines a multiplier iff the alternate form on $B$

$$\{b_1, b_2\} := \alpha(h_{l,b_1}, h_{l,b_2})\, \alpha(h_{r,b_1}, h_{r,b_2})^{-1} \tag{3.6}$$

is symmetric, that is, takes values $\{\pm 1\}$, and moreover

$$c_{b_1+b_2} = c_{b_1} c_{b_2}\, \{b_1, b_2\} \tag{3.7}$$

for all $b_1, b_2 \in B$.

A large class of such multipliers corresponds to morphisms between quantum tori which are quotients of $T(H,\alpha)$, in particular, to the automorphisms of $T(H,\alpha)$.

More precisely, assume that $\mathcal{L}$ has the following property: *for any $b$, $h_{r,b}$ is uniquely defined by $h_{l,b}$.* This holds, for example, if $B \to H : b \mapsto h_{l,b}$ has trivial kernel. Put $T_{l,r} := T(h_{l,r}(B), \alpha)$. There is a group morphism $f : h_l(B) \to h_r(B)$ such that $h_{r,b} = f(h_{l,b})$ for all $b$. If the form (3.6) takes only values $\pm 1$, $f$ is compatible with $\alpha^2$ and therefore determines a morphism of quantum tori $F : T_r \to T_l : F^*(e(h)) = e(f(h))$. Conversely, any such morphism of quantum tori associated with subgroups of $H$ produces a family of multip[liers with hidden periods

$$\mathcal{L}(b) = [\chi(b);\, 1, h_{l,b}, f(h_{l,b})] \tag{3.8}$$

where $\chi : H \to K^*$ is an arbitrary character.

On an appropriate subset of these data we can also exhibit a nice functorial description of $\circ$–multiplication:

**3.3.1. Proposition.** *Consider the following category $\mathcal{T}$: objects are subgroups of maximal rank in $H$, morphisms are isomorphisms compatible with $\alpha^2$.*

*For each morphism, construct a multiplier as in (3.8). Then the product of morphisms corresponds to the $\circ$–product of multipliers.*

**3.3.2. Example.** Consider the two–dimensional torus $T_q$ described in 0.7: $H = \mathbf{Z}^2 = \mathbf{Z}h_1 \oplus \mathbf{Z}h_2$, $\alpha(h_1,h_2) = q$, $\alpha(h_1,h_1) = \alpha(h_2,h_2) = 1$. Put $u = e(h_1), v = e(h_2)$. The functional equation for $\theta_q(t)$, $qu\theta_q(q^2t) = \theta_q(t)$, gives for two liftings of $\theta_q$ to $T_q$

$$quv^{-1}\theta_q(u)v = \theta_q(u),\ u\theta_q(u)u^{-1} = \theta_q(u)$$

and

$$qvu\theta_q(v)u^{-1} = \theta_q(v),\ v^{-1}\theta_q(v)v = \theta_q(v).$$

Hence they are thetas on $T_q$ with hidden periods corresponding to the the automorphisms $f : h_1 \mapsto h_1, h_2 \mapsto h_1 + h_2$, $g : h_1 \mapsto h_1 - h_2, h_2 \mapsto h_2$.

Cross–multiplying, we get similar functional equations for $\theta_q(u)\theta_q(v)$:

$$quv^{-1}\theta_q(u)\theta_q(v)v = \theta_q(u)\theta_q(v),\ q^{-1}u\theta_q(u)\theta_q(v)vu^{-1} = \theta_q(u)\theta_q(v).$$

**3.3.3. Example: Weinstein's theta distribution.** In [We], A. Weinstein considers a real quantum torus $T(\mathbf{Z}^d, \alpha)$ where $\alpha(e(m), e(n)) = e^{-\pi i\hbar P(m,n)}$, $P$ being a real skew–symmetric pairing. He represents the quantum multiplication by an integral operator with the distribution kernel $K(x,y,z) = L(y-x, z-x)$ where in his notation

$$L(y,z) = \sum_{m,n\in\mathbf{Z}^d} e^{-\pi i\hbar P(m,n) - 2\pi i(my+nz)}. \tag{3.9}$$

In our general setup, this means encoding the multiplication on $T(H,\alpha)$ in terms of the formal theta function on $T(H\oplus H,1)$

$$\theta_W := \sum_{(g,h)\in H\times H} \alpha(g,h)\, e(g,h). \tag{3.10}$$

We can rewrite (3.10) in the most natural way passing to the torus $T(H\oplus H,\beta)$ where $\beta((g,0),(0,h)) := \alpha(g,h)$ and $H\oplus\{0\}, \{0\}\oplus H$ are $\beta$–isotropic. Then we have

$$u_{1,\beta}(\theta_W) = \sum_{g\in H} e(g,0) \,*_\beta \sum_{h\in H} e(0,h) \tag{3.11}$$

where $*_\beta$ denotes the multiplication in $T(H\oplus H,\beta)$. Each factor in (3.11) is of course the delta distribution at zero on $T(H,1)$ lifted to $T(H\oplus H,\beta)$. Clearly, $u_{1,\beta}(\theta_W)$ has the hidden period group

$$H\oplus H \to \mathcal{G}(H\oplus H,\beta):\ (g,h)\mapsto [1;1,g,-h]_\beta. \tag{3.12}$$

Weinstein treats $\theta_W$ as a theta–distribution on the commutative torus. In order to calculate its automorphy factors, we can introduce *a symmetric pairing* $\langle\,,\rangle$ on $H\oplus H$ such that $\langle (g,h),(g,h)\rangle = \alpha(g,h)$. Then $\theta_W = \sum_{k\in H\oplus H}\langle k,k\rangle e(k)$ and moreover, for all $k\in H$

$$\langle k,k\rangle\, e(k)\, x_k^*(\theta_W) = \theta_W, \tag{3.13}$$

where $x_k^*(e(j)) = \langle k,j\rangle^2$.

## §4. Small Heisenberg groups

**4.1. Small Heisenberg groups $\mathcal{G}(\mathcal{L})$.** Consider a multiplier $\mathcal{L}:\ B\to\mathcal{G}(H,\alpha)$. Denote by $\widetilde{\mathcal{G}}(\mathcal{L})$ the normalizer of $\mathcal{L}(B)$ in $\mathcal{G}(H,\alpha)$.

We will call $\mathcal{G}(\mathcal{L}) := \widetilde{\mathcal{G}}(\mathcal{L})/\mathcal{L}(B)$ *the small Heisenberg group* of $\mathcal{L}$. Clearly, it acts upon $\Gamma(\mathcal{L})$.

In the following theorem we use the same notation for $\mathcal{L}(b)$ as in 2.1.

**4.1.1. Theorem.** *(a) Assume that the natural map $\mathcal{L}(B)\to T(H,1)(K)\times H$ is injective. Then*

$$\widetilde{\mathcal{G}}_l(\mathcal{L}) = \{\,[c;\xi,\gamma,0]\,|\,\forall b\in B,\ h_b^-(\xi) = \gamma(x_{l,b})\,\alpha^2(h_b^-,\gamma)\,\}. \tag{4.1}$$

*(b) Assume that $K$ is large enough in the following sense: the natural homomorphism $T(H,1)(K)\to T(h^-(B),1)(K)$ is surjective and all torsion points of its kernel are rational over $K$. Assume moreover that $\dim\Gamma(\mathcal{L}) = [H:h^-(B)]<\infty$. In this case the representation of $\mathcal{G}(\mathcal{L})$ in $\Gamma(\mathcal{L})$ is irreducible.*

**Proof.** (a) A straightforward calculation gives:

$$[c;\xi,\gamma,0]^{-1}[c_{l,b};x_{l,b},h_b^-,0]\,[c;\xi,\gamma,0] =$$

$$[c_{l,b}\,\gamma(x_{l,b})\,\alpha^2(h_b^-,\gamma)h_b^-(\xi);\; x_{l,b}, h_b^-, 0]. \tag{4.2}$$

If $[c;\,\xi,\gamma,0]$ normalizes $\mathcal{L}(B)$ and $\mathcal{L}(B) \to T(H,1)(K) \times H$ is injective, the right hand side of (4.2) must coincide with $[c_{l,b};\, x_{l,b}, h_b^-, 0]$. This proves (a).

(b) Let us fix an arbitrary $\gamma \in H$ and consider the conditions in (4.1) as equations for $\xi$:

$$\forall b \in B,\;\; h_b^-(\xi) = \gamma(b)\,\alpha^2(h_b^-,\gamma) \tag{4.3}$$

These equations uniquely determine the image of this point in $T(h^-(B),1)(K)$, and we can take for $\xi$ any lift of this image to $T(H,1)(K)$. If the assumptions of (b) hold, the set of such lifts is non–empty and is a torsor over the kernel $\mathcal{K}$ of $T(H,1)(K) \to T(h^-(B),1)(K)$, so that we obtain an isomorphism $\mathcal{G}(\mathcal{L})/K^* = \mathcal{K} \times H/h^-(B)$ induced by $[c;\,\xi,\gamma,0] \mapsto (\xi,\gamma)$. The pairing $(\xi,\gamma) \mapsto \gamma(\xi)$ induces a duality between the finite groups $\mathcal{K}$ and $H/h^-(B)$ with values in the group of roots of unity.

Let us now show that $\mathcal{G}(\mathcal{L})$ and its representation in $\Gamma(\mathcal{L})$ have exactly the same structure as their counterparts described in §1 of [Mu].

The easiest way to see this is to use the twisting isomorphism $u_{\alpha,1} = u_{1,\alpha}^{-1}$ (see (1.6), (1.7)) in order to reduce the situation to the case $\alpha \equiv 1$ which we will assume in the rest of the proof. Since we assumed that $\dim \Gamma(\mathcal{L}) = [H : h^-(B)] < \infty$, the space $\Gamma(\mathcal{L})$ splits into the direct sum of one–dimensional subspaces $\Gamma_\chi(\mathcal{L})$ labeled by all $\chi \in H/h^-(B)$: each $\Gamma_\chi(\mathcal{L})$ consists of thetas whose coefficients $a_h$ vanish outside the coset class $\chi$ (cf. the proof of Theorem 2.4.1). Clearly, this splitting coincides with the splitting into irreducible representations of $\mathcal{K}$ lifted to $\mathcal{G}(\mathcal{L})$ by $\xi \mapsto \xi^*$.

On the other hand, the set of these subspaces forms a torsor over any lifting of the group $H/h^-(B)$ to $\mathcal{G}(\mathcal{L})$. More precisely, from (4.1) with $\alpha \equiv 1$ one sees that the map $[c;\xi,\gamma] \mapsto \gamma$ descends to a well defined map $\mathcal{G}(\mathcal{L}) \to H/h^-(B)$ which we want to lift.

In order to see that such a lifting exists one can imitate Mumford's reasoning. We omit the details.

The most important consequence of this formalism is the existence of almost canonical bases in all spaces of theta functions $\Gamma(\mathcal{L})$ corresponding to (formally) ample multip[liers. More precisely, the basis depends on the choice of a lifting of $H/h^-(B) \to \mathcal{G}(\mathcal{L})$, it is indexed by the characters of this group, and defined up to a common constant factor.

**4.2. Rigidity of comultiplication.** Let $\mathcal{A}$ be an abelian variety together with its parametrization by a torus. Mumford's study of the morphisms $M^*$ : $\Gamma(\mathcal{A}\times\mathcal{A}, L\boxtimes L) \to \Gamma(\mathcal{A}\times\mathcal{A}, L^2\boxtimes L^2)$ for symmetric ample $L$ admits a straightforward rewriting in terms of our realization of theta functions. We can then apply twistings $u_{1,\alpha}$ (resp. $u_{1,\alpha^2}$) to the source (resp. target) of the morphism $M^*$ (see (1.15)). All formulas not involving quantum multiplications of thetas will remain the same.

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