Max-Planck-Institut für Mathematik
Bonn

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Christian Le Merdy
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Preprint Series 2020 (43)

Date of submission: August 8, 2020

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Christian Le Merdy
Safoura Zadeh

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Laboratoire de Mathématiques de Besançon
Université Bourgogne Franche-Comté
France

MPIM 20-43

ON FACTORIZATION OF SEPARATING MAPS ON NONCOMMUTATIVE L^p-SPACES

CHRISTIAN LE MERDY[1] AND SAFOURA ZADEH

Abstract. For any semifinite von Neumann algebra $\mathcal{M}$ and any $1 \leq p < \infty$, we introduce a natutal S^1-valued noncommutative L^p-space $L^p(\mathcal{M}; S^1)$. We say that a bounded map $T: L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is S^1-bounded (resp. S^1-contractive) if $T \otimes I_{S^1}$ extends to a bounded (resp. contractive) map $T\overline{\otimes} I_{S^1}$ from $L^p(\mathcal{M}; S^1)$ into $L^p(\mathcal{N}; S^1)$. We show that any completely positive map is S^1-bounded, with $\|T\overline{\otimes} I_{S^1}\| = \|T\|$. We use the above as a tool to investigate the separating maps $T: L^p(\mathcal{M}) \to L^p(\mathcal{N})$ which admit a direct Yeadon type factorization, that is, maps for which there exist a w^*-continuous $*$-homomorphism $J: \mathcal{M} \to \mathcal{N}$, a partial isometry $w \in \mathcal{N}$ and a positive operator B affiliated with $\mathcal{N}$ such that $w^*w = J(1) = s(B)$, B commutes with the range of J, and $T(x) = wBJ(x)$ for any $x \in \mathcal{M} \cap L^p(\mathcal{M})$. Given a separating isometry $T: L^p(\mathcal{M}) \to L^p(\mathcal{N})$, we show that T is S^1-contractive if and only if it admits a direct Yeadon type factorization. We further show that if $p \neq 2$, the above holds true if and only if T is completely contractive.

1. Introduction

Let $\mathcal{M}, \mathcal{N}$ be two semifinite von Neumann algebras. For any $1 \leq p < \infty$, let $L^p(\mathcal{M})$ and $L^p(\mathcal{N})$ denote their associated noncommutative L^p-spaces. A bounded map $T: L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is called separating if for any $x, y \in L^p(\mathcal{M})$ such that $x^*y = xy^* = 0$, we have $T(x)^*T(y) = T(x)T(y)^* = 0$. Separating maps are a noncommutative analog of Lamperti operators, that is, operators on classical (=commutative) L^p-spaces preserving disjoint supports. We refer to [4,18,19,23] for information and deep results on Lamperti operators.

In the noncommutative setting, pairs (x, y) such that $x^*y = xy^* = 0$ were first considered on Schatten classes S^p in [1], as a tool to describe onto sujective isometries on S^p for $1 \leq p \neq 2 < \infty$. Later on, separating maps were used either implicitly or explicitly, and with different names, in [2, 3] (see also [22]) and in Yeadon's paper [34] providing a full description of isometries $L^p(\mathcal{M}) \to L^p(\mathcal{N})$, for $1 \leq p \neq 2 < \infty$.

Recently the two authors [21] and, independently, G. Hong, S. K. Ray and S. Wang [11] established the following characterization property. A bounded map $T: L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is separating if and only if there exist a w^*-continuous Jordan homomorphism $J: \mathcal{M} \to \mathcal{N}$, a partial isometry $w \in \mathcal{N}$ and a positive operator B affiliated with $\mathcal{N}$ such that $w^*w = J(1) = s(B)$, the support of B, B commutes with the range of J, and

$$T(x) = wBJ(x), \qquad x \in \mathcal{M} \cap L^p(\mathcal{M}). \tag{1}$$

This remarkable factorization property was discovered by Yeadon in the above mentioned paper. Indeed he showed in [34] that for $p \neq 2$, any linear isometry $T: L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is separating and further admits a factorization of the type (1). In reference to this seminal

Date: August 8, 2020.

Key words and phrases. Noncommutative L^p-spaces, isometries, tensor products, completely positive maps.

work, we call (1) a Yeadon type factorization of T. It turns out that if T is separating, the triple (w, B, J) in its Yeadon type factorization is unique.

We note that analogs of separating maps are currently investigated in other settings. On the one hand, they are used on general noncommutative functions spaces, in order to obtain a Yeadon type description of isometries on a large class of such spaces [12]. On the other hand, they are investigated in operator algebras (the case $p = \infty$) and play a fundamental role in the classification of nuclear C^*-algebras, see [33] and the references therein.

The present paper looks at separating maps $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ for which the Jordan homomorphism J in the Yeadon type factorization is actually a $*$-homomorphism (equivalently, is multiplicative). We say that T has a direct Yeadon type factorization in this case. The first motivation for considering this notion is a result by M. Junge, D. Sherman and Z.-J. Ruan [15, Proposition 3.2] which asserts that for $p \neq 2$, a linear isometry $L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is a complete isometry if and only if it has a direct Yeadon type factorization. The second motivation is the L^2-case. In [21, Theorem 4.2], we proved that an isometry $T\colon L^2(\mathcal{M}) \to L^2(\mathcal{N})$ is separating (equivalently, has a Yeadon type factorization) if and only if $T \otimes I_{\ell^1}$ extends to a contractive map $L^2(\mathcal{M}; \ell^1) \to L^2(\mathcal{N}; \ell^1)$. Here $L^2(\mathcal{M}; \ell^1)$ and $L^2(\mathcal{N}; \ell^1)$ denote Junge's ℓ^1-valued non commutative L^2-spaces from [13].

We introduce S^1-valued noncommutative L^p-spaces $L^p(\mathcal{M}; S^1)$, which naturally extend previous constructions from [13, 26]. We say that a bounded map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is S^1-bounded (resp. S^1-contractive) if $T \otimes I_{S^1}$ extends to a bounded (resp. contractive) map

$$T\overline{\otimes} I_{S^1}\colon L^p(\mathcal{M}; S^1) \longrightarrow L^p(\mathcal{N}; S^1).$$

When $\mathcal{M}, \mathcal{N}$ are hyperfinite, S^1-boundedness coincides with complete regularity in the sense of [27] (see also [5, 14]) and $\|T\overline{\otimes} I_{S^1}\| = \|T\|_{reg}$. We prove that any map with a direct Yeadon type factorization is S^1-bounded, with $\|T\overline{\otimes} I_{S^1}\| = \|T\|$ (see Proposition 4.5). Our main result is that conversely, any S^1-contractive separating isometry admits a direct Yeadon type factorization (see Theorem 5.4). The resulting statement (see Corollary 5.9) that an isometry $T\colon L^2(\mathcal{M}) \to L^2(\mathcal{N})$ is S^1-contractive if and only if it admits a direct Yeadon type factorization is both an L^2-version of [15, Proposition 3.2] and a matricial version of [21, Theorem 4.2].

The spaces $L^p(\mathcal{M}; S^1)$ and S^1-boundedness are investigated in Section 3. We prove in passing that any completely positive map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is S^1-bounded, with $\|T\overline{\otimes} I_{S^1}\| = \|T\|$ (see Theorem 3.13).

We also establish comparisons between direct Yeadon type factorizations and complete boundedness. After observing that any separating map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ with a direct Yeadon type factorization is completely bounded, with $\|T\|_{cb} = \|T\|$ (see Proposition 4.4), we show that conversely if $p \neq 2$, any completely contractive isometry $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ admits a direct Yeadon type factorization (see Theorem 5.6). This result strengthens [15, Proposition 3.2].

2. Noncommutative L^p-spaces and representations of matrix spaces

In this section, we give some background and preliminary facts on noncommutative L^p-spaces built over semifinite von Neumann algebras.

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful (n.s.f.) trace [31, Definition V.2.1]. Except otherwise stated, this trace will be denoted by $\tau_{\mathcal{M}}$. Assume that $\mathcal{M} \subset B(\mathcal{H})$ acts on some Hilbert space $\mathcal{H}$. Let $L^0(\mathcal{M})$ denote the $*$-algebra of all closed densely defined (possibly unbounded) operators on $\mathcal{H}$, which are $\tau_{\mathcal{M}}$-measurable. Then for any $0 < p < \infty$, the noncommutative L^p-space $L^p(\mathcal{M})$, associated with $(\mathcal{M}, \tau_{\mathcal{M}})$, can be defined as

$$L^p(\mathcal{M}) := \big\{x \in L^0(\mathcal{M}) \,:\, \tau_{\mathcal{M}}(|x|^p) < \infty\big\}.$$

We set $\|x\|_p := \tau_{\mathcal{M}}(|x|^p)^{\frac{1}{p}}$ for any $x \in L^p(\mathcal{M})$. If $p \geq 1$, $L^p(\mathcal{M})$ equipped with $\|\cdot\|_p$ is a Banach space. The reader is referred to [16, 29, 32] and the references therein for details on the algebraic operations on $L^0(\mathcal{M})$ and the construction of $L^p(\mathcal{M})$, and for further properties.

We let $L^\infty(\mathcal{M}) = \mathcal{M}$ for convenience and for any $x \in \mathcal{M}$, we let $\|x\|_\infty$ denote its operator norm. We recall that if $0 < p, q, r \leq \infty$ are such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then for any $x \in L^p(\mathcal{M})$ and $y \in L^q(\mathcal{M})$, the product xy belongs to $L^r(\mathcal{M})$, with $\|xy\|_r \leq \|x\|_p \|y\|_q$. In particular, for any $1 \leq p < \infty$, let $p' = \frac{p}{p-1}$ be the conjugate number of p. Then xy belongs to $L^1(\mathcal{M})$ for any $x \in L^p(\mathcal{M})$ and $y \in L^{p'}(\mathcal{M})$. Further the duality pairing

$$\langle x, y\rangle = \tau_{\mathcal{M}}(xy), \qquad x \in L^p(\mathcal{M}),\ y \in L^{p'}(\mathcal{M}),$$

yields an isometric isomorphism $L^p(\mathcal{M})^* = L^{p'}(\mathcal{M})$. In particular, we may identify $L^1(\mathcal{M})$ with the (unique) predual of $\mathcal{M}$. These duality results will be used without further reference in the paper.

For any $0 < p \leq \infty$, we let $L^p(\mathcal{M})^+$ denote the cone of positive elements of $L^p(\mathcal{M})$.

If $\mathcal{A}$ is a w^*-closed $*$-subalgebra of $\mathcal{M}$ such that the restriction of $\tau_{\mathcal{M}}$ to $\mathcal{A}^+$ is semifinite, then for any $0 < p < \infty$, we may define $L^p(\mathcal{A})$ using this restriction and $L^p(\mathcal{A})$ isometrically embeds in $L^p(\mathcal{M})$. In particular, for any projection e in $\mathcal{M}$, the restriction of $\tau_{\mathcal{M}}$ to the corner algebra $e\mathcal{M}e$ is semifinite, and therefore we have a natural embedding

$$L^p(e\mathcal{M}e) \subset L^p(\mathcal{M}).$$

For any two von Neumann algebras $\mathcal{M}_1, \mathcal{M}_2$, we let $\mathcal{M}_1\overline{\otimes}\mathcal{M}_2$ denote their von Neumann tensor product [31, Section IV.5]. If $\tau_{\mathcal{M}_1}$ and $\tau_{\mathcal{M}_2}$ are n.s.f. traces on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, then $\tau_{\mathcal{M}_1} \otimes \tau_{\mathcal{M}_2}$ uniquely extends to a n.s.f. trace on $\mathcal{M}_1\overline{\otimes}\mathcal{M}_2$. Then for any any $0 < p < \infty$, we have a natural embedding $L^p(\mathcal{M}_1) \otimes L^p(\mathcal{M}_2) \subset L^p(\mathcal{M}_1\overline{\otimes}\mathcal{M}_2)$, and

$$\|x \otimes y\|_p = \|x\|_p \|y\|_p, \qquad x \in L^p(\mathcal{M}_1),\, y \in L^p(\mathcal{M}_2). \tag{2}$$

We further recall that $x \otimes y \in L^p(\mathcal{M}_1\overline{\otimes}\mathcal{M}_2)^+$ if $x \in L^p(\mathcal{M}_1)^+$ and $y \in L^p(\mathcal{M}_2)^+$.

We also note that the direct sum $\mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ satisfies

$$L^p\big(\mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2\big) = L^p(\mathcal{M}_1) \overset{p}{\oplus} L^p(\mathcal{M}_2) \tag{3}$$

for any $0 < p < \infty$.

We now fix some notations regarding matrix spaces. Let $\mathcal{H}$ be a Hilbert space and let tr be the usual trace on $B(\mathcal{H})$. For any $0 < p < \infty$, we let $S^p(\mathcal{H})$ denote the Schatten p-class of operators on $\mathcal{H}$; this is the noncommutative L^p-space associated with $(B(\mathcal{H}), \mathrm{tr})$. If $\mathcal{H} = \ell^2$, we simply denote these spaces by S^p. For any $n \geq 1$, we let tr_n denote the usual trace on M_n and we let S^p_n denote the Schatten p-class of $n \times n$ matrices. We let E_{ij}, $1 \leq i, j \leq n$, denote the usual matrix units on M_n and we let $I_n \in M_n$ be the identity matrix. Finally whenever $\mathcal{M}$ is a semifinite von Neumann algebra equipped with a n.s.f. trace $\tau_{\mathcal{M}}$, we let $\tau_{\mathcal{M},n} = \mathrm{tr}_n \otimes \tau_{\mathcal{M}}$ denote the natural trace on $M_n\overline{\otimes}\mathcal{M}$. We note that $L^p(M_n\overline{\otimes}\mathcal{M})$ can be naturally regarded as a space of $n \times n$ matrices with values in $L^p(\mathcal{M})$. This brings us to the algebraic identification

$$L^p(M_n\overline{\otimes}\mathcal{M}) \simeq S_n^p \otimes L^p(\mathcal{M}). \tag{4}$$

Let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a bounded operator between two noncommutative L^p-spaces. Following usual terminology we say, using (4), that T is completely bounded if there exists a constant $C \geq 0$ such that

$$\big\| I_{S_n^p} \otimes T\colon L^p(M_n\overline{\otimes}\mathcal{M}) \to L^p(M_n\overline{\otimes}\mathcal{N})\big\| \leq C$$

for any $n \geq 1$. In this case we let $\|T\|_{cb}$ denote the smallest $C \geq 0$ satisfying this uniform estimate; it is called the completely bounded norm of T. We say that T is completely contractive if $\|T\|_{cb} \leq 1$. Further we say that T is positive if it maps $L^p(\mathcal{M})^+$ into $L^p(\mathcal{N})^+$ and we say that T is completely positive maps if $I_{S_n^p} \otimes T$ is positive for any $n \geq 1$. We recall that in the case $p = 2$, we have that any bounded $T\colon L^2(\mathcal{M}) \to L^2(\mathcal{N})$ is automatically completely bounded, with $\|T\|_{cb} = \|T\|$. This follows from the fact that $L^2(M_n\overline{\otimes}\mathcal{M})$ (resp. $L^2(M_n\overline{\otimes}\mathcal{N})$) coincides with the Hilbertian tensor product of S_n^2 and $L^2(\mathcal{M})$ (resp. $L^2(\mathcal{N})$).

A positive map $T\colon (\mathcal{M}, \tau_{\mathcal{M}}) \to (\mathcal{N}, \tau_{\mathcal{N}})$ is called trace preserving if $\tau_{\mathcal{N}} \circ T = \tau_{\mathcal{M}}$ on $\mathcal{M}^+$.

Lemma 2.1. *Let $T\colon (\mathcal{M}, \tau_{\mathcal{M}}) \to (\mathcal{N}, \tau_{\mathcal{N}})$ be a trace preserving $*$-homomorphism. Then for any $1 \leq p < \infty$, the restriction of T to $\mathcal{M} \cap L^1(\mathcal{M})$ extends to a complete isometry $L^p(\mathcal{M}) \to L^p(\mathcal{N})$.*

Proof. Since T is a $*$-homomorphism, $|(I_{M_n} \otimes T)(x)|^p = (I_{M_n} \otimes T)(|x|^p)$ for any $x \in M_n\overline{\otimes}\mathcal{M}$. The result follows at once. $\square$

We now give two elementary results on the representation of matrix spaces into semifinite von Neumann algebras.

Lemma 2.2. *Suppose that $\mathcal{M}$ is a semifinite von Neumann algebra, let $n \geq 1$ and let $\theta\colon M_n \to \mathcal{M}$ be a unital $*$-homomorphism. Then there exist a projection $e \in \mathcal{M}$ and a bijective $*$-homomorphism $\rho : \mathcal{M} \to M_n\overline{\otimes}(e\mathcal{M}e)$ such that*

$$(\rho \circ \theta)(a) = a \otimes e, \qquad a \in M_n,$$

and ρ is trace preserving.

Proof. Let $e = \theta(E_{11})$, this is a projection. Since θ is a unital $*$-homomorphism, the family $\{\theta(E_{ij}) \,:\, 1 \leq i, j \leq n\}$ is a system of matrix units on $\mathcal{M}$. Hence as is well-known (see e.g. the proof of [31, Proposition IV.1.8]), $x_{ij} := \theta(E_{1i})x\theta(E_{j1})$ belongs to $e\mathcal{M}e$ for any $x \in \mathcal{M}$ and any $1 \leq i, j \leq n$, and the mapping

$$\rho\colon \mathcal{M} \to M_n\overline{\otimes}(e\mathcal{M}e), \qquad \rho(x) = \sum_{i,j=1}^n E_{ij} \otimes x_{ij},$$

is a bijective $*$-homomorphism. It is clear that $(\rho \circ \theta)(a) = a \otimes e$ for every a in M_n.

To check that ρ is trace preserving, let $u_i = \theta(E_{i1})$ and $e_i = \theta(E_{ii})$ for all $1 \leq i \leq n$. Then $u_i u_i^* = e_i$ and $e_1 + \cdots + e_n = 1$. Hence for any $x \in \mathcal{M}^+$, x_{ii} belongs to $(e\mathcal{M}e)^+$ for any $1 \leq i \leq n$ and we have

$$\sum_{i=1}^{n} \tau_{\mathcal{M}}(x_{ii}) = \sum_{i=1}^{n} \tau_{\mathcal{M}}(u_i^* x u_i) = \sum_{i=1}^{n} \tau_{\mathcal{M}}(e_i x) = \tau_{\mathcal{M}}(x).$$

Therefore, $(tr_n \otimes \tau_{e\mathcal{M}e}) \circ \rho = \tau_{\mathcal{M}}$ on $\mathcal{M}^+$. □

It is a classical fact that any non abelian von Neumann algebra contains a copy of M_2. Here is a more precise statement in the semifinite case.

Lemma 2.3. *Let $\mathcal{M}$ be a non abelian semifinite von Neumann algebra. There exists a non zero $*$-homomorphism $\gamma\colon M_2 \to \mathcal{M}$ valued in $\mathcal{M} \cap L^1(\mathcal{M})$.*

In the above statement, the condition that γ is valued in $\mathcal{M} \cap L^1(\mathcal{M})$ does not come for free. Consider for example an infinite dimensional Hilbert space H and let $\mathcal{M} = B(H \overset{2}{\oplus} H) \simeq M_2 \overline{\otimes} B(H)$. Then the mapping $a \mapsto a \otimes I_H$ is a $*$-homomorphism from M_2 into $\mathcal{M}$ and for any $a \in {M_2}^+$, $a \neq 0$, the trace of $a \otimes I_H$ is infinite.

Proof of Lemma 2.3. Let $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ be the direct sum decomposition of $\mathcal{M}$ into a type I summand $\mathcal{M}_1$ and a type II summand $\mathcal{M}_2$ (see e.g. [31, Section V]).

Assume that $\mathcal{M}_2 \neq \{0\}$. According to [17, Lemma 6.5.6], there exist 2 equivalent mutually orthogonal projections e, f in $\mathcal{M}_2$ such that $e + f = 1$. Then by [31, Proposition V.1.22] and its proof, $\mathcal{M}_2 \simeq M_2 \overline{\otimes} (e\mathcal{M}_2 e)$. Let $\varepsilon \in e\mathcal{M}_2 e$ be a non zero projection with finite trace. Then $\tau_{\mathcal{M}_2}(a \otimes \varepsilon) = \mathrm{tr}_2(a)\tau_{e\mathcal{M}_2 e}(\varepsilon) < \infty$ for any $a \in {M_2}^+$. Hence the mapping $\gamma\colon M_2 \to \mathcal{M}_2 \subset \mathcal{M}$ defined by $\gamma(a) = a \otimes \varepsilon$ is a non zero $*$-homomorphism taking values in $L^1(\mathcal{M})$.

If $\mathcal{M}_2 = \{0\}$, then $\mathcal{M} = \mathcal{M}_1$ is type I. Since $\mathcal{M}$ is non abelian, it follows from [31, Theorem V.1.27] that there exists a Hilbert space H with $\dim(H) \geq 2$ and an abelian von Neumann algebra W such that $\mathcal{M}$ contains $B(H)\overline{\otimes} W$ as a summand. Let $e \in B(H)$ be a rank one projection and define $\tau_W \colon W^+ \to [0, \infty]$ by $\tau_W(z) = \tau_{\mathcal{M}}(e \otimes z)$. Then τ_W is a n.s.f. trace and $\tau_{\mathcal{M}}$ coincides with $\mathrm{tr} \otimes \tau_W$ on $B(H)^+ \otimes W^+$. Let $\varepsilon \in W$ be a non zero projection with finite trace. Then it follows from above that $\tau_{\mathcal{M}}(a \otimes \varepsilon) < \infty$ for any finite rank $a \in B(H)^+$. Now let (e_1, e_2) be an orthonormal family in H. Then the mapping $\gamma\colon M_2 \to \mathcal{M}_2$ taking any $[a_{ij}]_{1 \leq i,j \leq 2}$ to $\sum_{i,j} a_{ij}\, \overline{e_j} \otimes e_i \otimes \varepsilon$ is a non zero $*$-homomorphism and the restriction of $\tau_{\mathcal{M}}$ to the positive part of its range is finite. Hence γ is valued in $L^1(\mathcal{M})$. □

3. S^1-boundedness

In this section we introduce S^1-valued noncommutative L^p-spaces, in a way which extends the definition provided by [26, Chapter 3] in the hyperfinite case. Then we introduce the notions of S^1-boundedness and S^1-contractivity for bounded maps between noncommutative L^p-spaces, and we discuss the connection between S^1-boundedness and complete positivity.

We fix a semifinite von Neumann algebra $\mathcal{M}$. We recall the definitions and basic properties of column/row valued $L^p(\mathcal{M})$-spaces for which we refer to [28] (see also [13, 21, 29]). Let Λ be an index set, and consider the Hilbert space ℓ^2_Λ. For any $1 \leq p \leq \infty$, let

$L^p(\mathcal{M};\{\ell^2_\Lambda\}_c)$ denote the space of all families $(b_\lambda)_{\lambda\in\Lambda}$ of elements in $L^p(\mathcal{M})$ such that the sums $\sum_{\lambda\in F} b_\lambda^* b_\lambda$, for finite $F\subset\Lambda$, are uniformly bounded in $L^{\frac{p}{2}}(\mathcal{M})$. Then for any such family, set

$$\big\|(b_\lambda)_\lambda\big\|_{L^p(\mathcal{M};\{\ell^2_\Lambda\}_c)} \,=\, \sup\Bigl\{\Bigl\|\sum_{\lambda\in F} b_\lambda^* b_\lambda\Bigr\|_{\frac{p}{2}}^{\frac{1}{2}}\Bigr\},$$

where the supremum runs over all finite $F\subset\Lambda$. This defines a norm on $L^p(\mathcal{M};\{\ell^2_\Lambda\}_c)$ and $L^p(\mathcal{M};\{\ell^2_\Lambda\}_c)$ is complete.

Likewise for any $1\leq p\leq\infty$, we let $L^p(\mathcal{M};\{\ell^2_\Lambda\}_r)$ denote the space of all families $(a_\lambda)_{\lambda\in\Lambda}$ of elements in $L^p(\mathcal{M})$ such that the sums $\sum_{\lambda\in F} a_\lambda a_\lambda^*$, for finite $F\subset\Lambda$, are uniformly bounded in $L^{\frac{p}{2}}(\mathcal{M})$. This is a Banach space for the norm

$$\big\|(a_\lambda)_\lambda\big\|_{L^p(M;\{\ell^2_\Lambda\}_r)} \,=\, \sup\Bigl\{\Bigl\|\sum_{\lambda\in F} a_\lambda a_\lambda^*\Bigr\|_{\frac{p}{2}}^{\frac{1}{2}}\Bigr\},$$

where the supremum runs over all finite $F\subset\Lambda$. It is plain that $(a_\lambda)_{\lambda\in\Lambda}$ belongs to $L^p(\mathcal{M};\{\ell^2_\Lambda\}_r)$ if and only if $(a_\lambda^*)_{\lambda\in\Lambda}$ belongs to $L^p(\mathcal{M};\{\ell^2_\Lambda\}_c)$.

Let $(E_{\lambda,\mu})_{\lambda,\mu\in\Lambda}$ be the matrix units in $B(\ell^2_\Lambda)$ corresponding to the standard basis of ℓ^2_Λ. We may regard any $z\in L^p(B(\ell^2_\Lambda)\overline{\otimes}\mathcal{M})$ as a matrix $(z_{\lambda,\mu})_{\lambda,\mu\in\Lambda}$ of elements in $L^p(\mathcal{M})$, with $E_{\lambda,\mu}\otimes z_{\lambda,\mu} = (E_{\lambda,\lambda}\otimes 1)z(E_{\mu,\mu}\otimes 1)$. Then $L^p(\mathcal{M};\{\ell^2_\Lambda\}_c)$ can be identified with any column subspace of $L^p(B(\ell^2_\Lambda)\overline{\otimes}\mathcal{M})$. More precisely fix any $\mu_0\in\Lambda$. If $z\in L^p(B(\ell^2_\Lambda)\overline{\otimes}\mathcal{M})$ is such that $z_{\lambda,\mu}=0$ for any $\mu\neq\mu_0$ and any λ, then $(z_{\lambda,\mu_0})_{\lambda\in\Lambda}$ belongs to $L^p(\mathcal{M};\{\ell^2_\Lambda\}_c)$ and its norm in the latter space is equal to the norm of z in $L^p(B(\ell^2_\Lambda)\overline{\otimes}\mathcal{M})$. Conversely for any $(b_\lambda)_{\lambda\in\Lambda}$ in $L^p(\mathcal{M};\{\ell^2_\Lambda\}_c)$, the matrix $(z_{\lambda,\mu})_{\lambda,\mu\in\Lambda}$ defined, for any $\lambda\in\Lambda$, by $z_{\lambda,\mu_0}=b_\lambda$ and $z_{\lambda,\mu}=0$ if $\mu\neq\mu_0$, represents an element z of $L^p(B(\ell^2_\Lambda)\overline{\otimes}\mathcal{M})$.

Likewise $L^p(\mathcal{M};\{\ell^2_\Lambda\}_r)$ can be identified with any row subspace of $L^p(B(\ell^2_\Lambda)\overline{\otimes}\mathcal{M})$.

We will use the fact that if $p\geq 1$ is finite, then for any $(a_\lambda)_{\lambda\in\Lambda}$ in $L^p(\mathcal{M};\{\ell^2_\Lambda\}_r)$ and for any $(b_\lambda)_{\lambda\in\Lambda}$ in $L^p(\mathcal{M};\{\ell^2_\Lambda\}_c)$, the family $(a_\lambda b_\lambda)_{\lambda\in\Lambda}$ is summable in $L^{\frac{p}{2}}(\mathcal{M})$ for the usual topology. This allows to define the sums

$$\sum_\lambda a_\lambda b_\lambda,\qquad \sum_\lambda a_\lambda a_\lambda^*\qquad\text{and}\qquad \sum_\lambda b_\lambda^* b_\lambda \tag{5}$$

as elements of $L^{\frac{p}{2}}(\mathcal{M})$.

In the case when $p=\infty$, the spaces $L^\infty(\mathcal{M};\{\ell^2_\Lambda\}_r)$ and $L^\infty(\mathcal{M};\{\ell^2_\Lambda\}_c)$ coincide with the row space $R^\omega_\Lambda(M)$ and the column space $C^\omega_\Lambda(M)$ from [6, 1.2.26-1.2.29], respectively. For any $(a_\lambda)_{\lambda\in\Lambda}$ in $L^\infty(\mathcal{M};\{\ell^2_\Lambda\}_r)$ and for any $(b_\lambda)_{\lambda\in\Lambda}$ in $L^\infty(\mathcal{M};\{\ell^2_\Lambda\}_c)$, the family $(a_\lambda b_\lambda)_{\lambda\in\Lambda}$ is summable in the w^*-topology of $\mathcal{M}$ and the sums in (5) are defined in $\mathcal{M}$ according to this topology.

The next lemma is a polar decomposition principle which will be used several times in our arguments. We state it for column valued $L^p(\mathcal{M})$-spaces; a similar statement holds for row valued $L^p(\mathcal{M})$-spaces.

Lemma 3.1. *Let $1\leq p<\infty$, let Λ be an index set and consider a family $(b_\lambda)_{\lambda\in\Lambda}$ of $L^p(\mathcal{M})$. The following assertions are equivalent.*

(i) *The family $(b_\lambda)_{\lambda\in\Lambda}$ belongs to $L^p(\mathcal{M};\{\ell^2_\Lambda\}_c)$ and $\|(b_\lambda)_\lambda\|_{L^p(\mathcal{M};\{\ell^2_\Lambda\}_c)}\leq 1$.*

(ii) *There exist a family* $(w_\lambda)_{\lambda\in\Lambda}$ *in* $L^\infty(\mathcal{M};\{\ell^2_\Lambda\}_c)$ *and* b *in* $L^p(\mathcal{M})$ *with*

$$\|(w_\lambda)_\lambda\|_{L^\infty(\mathcal{M};\{\ell^2_\Lambda\}_c)} \leq 1 \qquad \textit{and} \qquad \|b\|_p \leq 1,$$

such that for all $\lambda \in \Lambda$, $b_\lambda = w_\lambda b$.

Proof. Assume (i). Following the above discussion we fix $\mu_0 \in \Lambda$ and consider the element $z \in L^p(B(\ell^2_\Lambda)\overline{\otimes}\mathcal{M})$ such that $z_{\lambda,\mu_0} = b_\lambda$ and $z_{\lambda,\mu} = 0$ for any $\mu \neq \mu_0$. Then we have

$$z = \sum_\lambda E_{\lambda,\mu_0} \otimes b_\lambda,$$

with norm convergence in $L^p(B(\ell^2_\Lambda)\overline{\otimes}\mathcal{M})$. Consider the polar decomposition $z = w|z|$ of z, with $w \in B(\ell^2_\Lambda)\overline{\otimes}\mathcal{M}$ and $|z| \in L^p(B(\ell^2_\Lambda)\overline{\otimes}\mathcal{M})$. Then we have

$$|z| = E_{\mu_0,\mu_0} \otimes b, \qquad \text{with} \qquad b = \Big(\sum_\lambda b_\lambda^* b_\lambda\Big)^{\frac{1}{2}}.$$

We note that $\|b\|_p = \|(b_\lambda)_\lambda\|_{L^p(\mathcal{M};\{\ell^2_\Lambda\}_c)} \leq 1$.

Now if $(w_{\lambda,\mu})_{\lambda,\mu\in\Lambda}$ is the family of $\mathcal{M}$ representing w, then for any $\lambda \in \Lambda$, we have $b_\lambda = w_{\lambda,\mu_0} b$ and $w_{\lambda,\mu} = 0$ if $\mu \neq \mu_0$. Hence the family $(w_{\lambda,\mu_0})_{\lambda\in\Lambda}$ belongs to $L^\infty(\mathcal{M};\{\ell^2_\Lambda\}_c)$ and its norm in the latter space is $\|w\| \leq 1$. This yields (ii).

The converse implication "(ii) $\Rightarrow$ (i)" folows from the fact that for any finite $F \subset \Lambda$, we have

$$\sum_{\lambda\in F} (w_\lambda b)^*(w_\lambda b) = b^*\Big(\sum_{\lambda\in F} w_\lambda^* w_\lambda\Big) b.$$

□

Definition 3.2. Let $1 \leq p < \infty$. We let $L^p(\mathcal{M}; S^1)$ denote the space of all infinite matrices $[x_{ij}]_{i,j\geq 1}$ in $L^p(\mathcal{M})$ for which there exist families

$$(a_{ik})_{i,k\geq 1} \in L^{2p}(\mathcal{M};\{\ell^2_{\mathbb{N}^2}\}_r) \qquad \text{and} \qquad (b_{kj})_{k,j\geq 1} \in L^{2p}(\mathcal{M};\{\ell^2_{\mathbb{N}^2}\}_c)$$

such that for all $i, j \geq 1$,

$$x_{ij} = \sum_{k=1}^\infty a_{ik} b_{kj}.$$

We equip $L^p(\mathcal{M}; S^1)$ with the following norm,

$$\|[x_{ij}]\|_{L^p(\mathcal{M};S^1)} = \inf\Big\{\|(a_{ik})_{i,k}\|_{L^{2p}(\mathcal{M};\{\ell^2_{\mathbb{N}^2}\}_r)}\,\|(b_{kj})_{k,j}\|_{L^{2p}(\mathcal{M};\{\ell^2_{\mathbb{N}^2}\}_c)}\Big\}, \tag{6}$$

where the infimum is taken over all families $(a_{ik})_{i,k\geq 1}$ and $(b_{kj})_{k,j\geq 1}$ as above.

When applying (6), we will use the fact that we both have

$$\|(a_{ik})_{i,k}\|_{L^{2p}(\mathcal{M};\{\ell^2_{\mathbb{N}^2}\}_r)} = \Big\|\sum_{i,k} a_{ik}a_{ik}^*\Big\|_p^{\frac{1}{2}} \quad \text{and} \quad \|(b_{kj})_{k,j}\|_{L^{2p}(\mathcal{M};\{\ell^2_{\mathbb{N}^2}\}_c)} = \Big\|\sum_{j,k} b_{kj}^* b_{kj}\Big\|_p^{\frac{1}{2}}.$$

The above definition is a natural extension of Junge's spaces $L^p(\mathcal{M};\ell^1)$ introduced in [13]. A similar argument as in the proof of [13, Lemma 3.5] shows that $L^p(\mathcal{M}; S^1)$ is a vector space and that (6) is indeed a norm. Moreover $L^p(\mathcal{M}; S^1)$ endowed with this norm is a Banach space.

For any integer $n \geq 1$, let $\mathbb{N}_n = \{1, \ldots, n\}$. We let $L^p(\mathcal{M}; S^1_n)$ be the subspace of $L^p(\mathcal{M}; S^1)$ of matrices $[x_{ij}]_{i,j\geq 1}$ with support in $\mathbb{N}_n \times \mathbb{N}_n$. We note that $\bigcup_n L^p(\mathcal{M}; S^1_n)$ is dense in $L^p(\mathcal{M}; S^1)$.

Remark 3.3. Identifying a finite matrix $[x_{ij}]_{1\leq i,j\leq n}$ of elements in $L^p(\mathcal{M})$ with the sum $\sum_{i,j=1}^n x_{ij} \otimes E_{ij}$, we see that at the algebraic level, $L^p(\mathcal{M}; S^1_n) = L^p(\mathcal{M}) \otimes S^1_n$. More generally we have a natural embedding

$$L^p(\mathcal{M}) \otimes S^1 \subset L^p(\mathcal{M}; S^1). \tag{7}$$

More precisely, consider a matrix $c = [c_{ij}]_{i,j\geq 1}$ in S^1 and $x \in L^p(\mathcal{M})$. Let $c' = [c'_{ik}]_{i,k\geq 1}$ and $c'' = [c''_{kj}]_{k,j\geq 1}$ in S^2 such that $c'c'' = c$ and let $x', x'' \in L^{2p}(\mathcal{M})$ such that $x'x'' = x$. Then $(c'_{ik}x')_{i,k\geq 1}$ and $(c''_{kj}x'')_{k,j\geq 1}$ belong to $L^{2p}(\mathcal{M}; \{\ell^2_{\mathbb{N}^2}\}_r)$ and $L^{2p}(\mathcal{M}; \{\ell^2_{\mathbb{N}^2}\}_c)$, respectively, and $c_{ij}x = \sum_k (c'_{ik}x')(c''_{kj}x'')$ for all $i, j \geq 1$. Thus $[c_{ij}x]_{i,j\geq 1}$ belongs to $L^p(\mathcal{M}; S^1)$. Identifying this matrix with $x \otimes c$, this yields (7). It is clear that with this convention, $L^p(\mathcal{M}) \otimes S^1$ is a dense subspace of $L^p(\mathcal{M}; S^1)$.

Lemma 3.4 below shows that for elements of $L^p(\mathcal{M}; S^1_n)$, the infimum in (6) can be taken over finite families only. This will turn out to be very convenient in future arguments. To obtain this property we will use a natural connection between the definition of the norm on $L^p(\mathcal{M}; S^1)$ and decomposable operators.

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. A linear map $\theta\colon \mathcal{A} \to \mathcal{B}$ is said to be decomposable if θ is a linear combination of completely positive maps from $\mathcal{A}$ into $\mathcal{B}$. In this case, θ may be written as $\theta = (\theta_1 - \theta_2) + i(\theta_3 - \theta_4)$, for four completely positive maps $\theta_j\colon \mathcal{A} \to \mathcal{B}$. Note, for example, that any finite rank operator between C^*-algebras is decomposable. In [9], Haagerup introduced a norm $\|\cdot\|_{dec}$ on the space of all decomposable maps from $\mathcal{A}$ into $\mathcal{B}$. We refer to the latter paper and also to [25, Chap. 11 & 14] for basic properties of this norm. (This norm is given in Remark 3.14, however we will not need it explicitly here.)

Let $n \geq 1$ and let $\theta\colon M_n \to \mathcal{M}$ be a linear map. According to [20, Prop. 4.5],

$$\|\theta\|_{dec} = \inf\Big\{ \|(v_{ik})_{i,k}\|_{L^\infty(\mathcal{M}; \{\ell^2_{\mathbb{N}_n\times\mathbb{N}}\}_r)}\, \|(w_{kj})_{k,j}\|_{L^\infty(\mathcal{M}; \{\ell^2_{\mathbb{N}\times\mathbb{N}_n}\}_c)} \Big\},$$

where the infimum runs over all families $(v_{ik})_{i,k}$ and $(w_{kj})_{k,j}$ in $\mathcal{M}$ such that $\theta(E_{ij}) = \sum_{k=1}^\infty v_{ik}w_{kj}$ for any $1 \leq i, j \leq n$. Applying Lemma 3.1 and its row counterpart, we deduce that for any linear map $u\colon M_n \to L^p(\mathcal{M})$,

$$\big\|\big[u(E_{ij})\big]\big\|_{L^p(\mathcal{M}; S^1_n)} = \inf\left\{\|a\|_{2p}\|\theta\|_{dec}\|b\|_{2p}\right\}, \tag{8}$$

where the infimum runs over all $a, b \in L^{2p}(\mathcal{M})$ and all linear maps $\theta\colon M_n \to \mathcal{M}$ such that

$$u(s) = a\theta(s)b, \qquad s \in M_n.$$

We will use Pisier's delta norm δ on $\mathcal{M} \otimes S^1_n$ introduced in [25, Chapter 12] (see also [6, Sections 6.4-6.5]). Given a matrix $[y_{ij}]_{1\leq i,j\leq n}$ of elements in $\mathcal{M}$, consider the associated operator $\theta\colon M_n \to \mathcal{M}$ defined by $\theta(E_{ij}) = y_{ij}$ for any $1 \leq i, j \leq n$. By [25, Corollary 12.4], we have $\|\theta\|_{dec} = \|[y_{ij}]\|_\delta$. Combining with (8), we deduce that for any matrix $[x_{ij}]_{1\leq i,j\leq n}$ of elements in $L^p(\mathcal{M})$, we have

$$\|[x_{ij}]\|_{L^p(\mathcal{M}; S^1_n)} = \inf\big\{\|a\|_{2p}\|[y_{ij}]\|_\delta\|b\|_{2p}\big\}, \tag{9}$$

where the infimum is taken over all factorizations of $[x_{ij}]$ of the form

$$x_{ij} = ay_{ij}b, \qquad 1 \leq i, j \leq n,$$

with a, b in $L^{2p}(\mathcal{M})$ and y_{ij} in $\mathcal{M}$.

Lemma 3.4. *Let $1 \leq p < \infty$ and let $n \geq 1$. For any $[x_{ij}]_{1\leq i,j\leq n}$ in $L^p(\mathcal{M}; S^1_n)$, the following assertions are equivalent.*

(i) $\|[x_{ij}]\|_{L^p(\mathcal{M};S^1_n)} < 1$.

(ii) *There exist an integer $m \geq 1$ and families $(a_{ik})_{1\leq i\leq n, 1\leq k\leq m}$ and $(b_{kj})_{1\leq k\leq m, 1\leq j\leq n}$ in $L^{2p}(\mathcal{M})$ such that $x_{ij} = \sum_{k=1}^m a_{ik}b_{kj}$, for all $1 \leq i,j \leq n$, and*

$$\Big\|\sum_{i=1}^n \sum_{k=1}^m a_{ik}a_{ik}^*\Big\|_p < 1 \qquad \text{and} \qquad \Big\|\sum_{j=1}^n \sum_{k=1}^m b_{kj}^* b_{kj}\Big\|_p < 1.$$

Proof. Assume (i), that is, $\|[x_{ij}]\|_{L^p(\mathcal{M};S^1_n)} < 1$. By (9), there exist a matrix $[y_{ij}]_{1\leq i,j\leq n}$ of elements in $\mathcal{M}$ and $a, b \in L^{2p}(\mathcal{M})$ such that

$$\|a\|_{2p} < 1, \qquad \|b\|_{2p} < 1, \qquad \|[y_{ij}]\|_{\delta} < 1,$$

and $x_{ij} = ay_{ij}b$ for all $1 \leq i,j \leq n$. According to [6, Proposition 6.5.2] there exist $m \geq 1$, and families $(v_{ik})_{1\leq i\leq n,1\leq k\leq m}$ and $(w_{kj})_{1\leq k\leq m,1\leq j\leq n}$ in $\mathcal{M}$ such that $y_{ij} = \sum_{k=1}^m v_{ik}w_{kj}$ for any $1 \leq i,j \leq n$, and

$$\Big\|\sum_{i=1}^n \sum_{k=1}^m v_{ik}v_{ik}^*\Big\|_\infty < 1, \qquad \Big\|\sum_{j=1}^n \sum_{k=1}^m w_{kj}^* w_{kj}\Big\|_\infty < 1.$$

For any $1 \leq i,j \leq n$ and any $1 \leq k \leq m$, set $a_{ik} = av_{ik}$ and $b_{kj} = w_{kj}b$. Then they satisfy the assertion (ii).

The converse implication "(ii) $\Rightarrow$ (i)" is obvious. $\square$

Remark 3.5. We may naturally identify $L^p(\mathcal{M}; S^1_n)$ with $L^p(M_n\overline{\otimes}\mathcal{M})$ as vector spaces (the norms on these two spaces are however different). Let $L^p(\mathcal{M}; S^1_n)^+$ be the set of all the $[x_{ij}]_{1\leq i,j\leq n} \in L^p(\mathcal{M}; S^1_n)$ which belong (under this identification) to the positive cone $L^p(M_n\overline{\otimes}\mathcal{M})^+$. For such a matrix, we have

$$\|[x_{ij}]\|_{L^p(\mathcal{M};S^1_n)} = \Big\|\sum_{i=1}^n x_{ii}\Big\|_p. \tag{10}$$

Indeed since $[x_{ij}]_{1\leq i,j\leq n}$ belongs to $L^p(\mathcal{M}; S^1_n)^+$, there exist a matrix $B = [b_{kj}]_{1\leq k,j\leq n}$ of elements in $L^{2p}(\mathcal{M})$ such that $[x_{ij}] = B^*B$, which reads

$$x_{ij} = \sum_{k=1}^n b_{ki}^* b_{kj}, \qquad 1 \leq i,j \leq n.$$

Then with $a_{ik} = b_{ki}^*$, we have

$$\Big\|\sum_{i,k=1}^n a_{ik}a_{ik}^*\Big\|_p = \Big\|\sum_{k,j=1}^n b_{kj}^* b_{kj}\Big\|_p = \Big\|\sum_{i=1}^n x_{ii}\Big\|_p.$$

This implies the inequality $\leq$ in (10).

The converse inequality (which is true without any positivity assumption) follows from the fact that if $x_{ij} = \sum_k a_{ik}b_{kj}$ for any $1 \leq i,j \leq n$ and some $a_{ik}, b_{kj} \in L^{2p}(\mathcal{M})$, then

$$\Big\|\sum_{i=1}^n x_{ii}\Big\|_p = \Big\|\sum_{i,k} a_{ik}b_{ki}\Big\|_p \leq \Big\|\sum_{i,k=1}^n a_{ik}a_{ik}^*\Big\|_p^{\frac12} \Big\|\sum_{i,k=1}^n b_{ki}^* b_{ki}\Big\|_p^{\frac12},$$

by Hölder's inequality.

We now establish an injectivity property of the $L^p(\mathcal{M}; S^1_n)$-norms.

Lemma 3.6. *Assume that $e \in \mathcal{M}$ is a projection with finite trace. Let $n \geq 1$ be an integer. For any matrix $[x_{ij}]_{1\leq i,j\leq n}$ of elements in $L^p(e\mathcal{M}e)$, we have*

$$\|[x_{ij}]\|_{L^p(e\mathcal{M}e;S^1_n)} = \|[x_{ij}]\|_{L^p(\mathcal{M};S^1_n)}. \tag{11}$$

Proof. The inequality $\geq$ is obvious. To prove the converse, it suffices, by density of $e\mathcal{M}e$ in $L^p(e\mathcal{M}e)$, to verify the inequality $\leq$ in (11) when each x_{ij} belongs to $e\mathcal{M}e$. Assume this property, along with $\|[x_{ij}]\|_{L^p(\mathcal{M};S^1_n)} < 1$.

By (9), there exist a and b in $L^{2p}(\mathcal{M})$ and a matrix $[y_{ij}]_{1\leq i,j\leq n}$ of elements in $\mathcal{M}$ such that $x_{ij} = ay_{ij}b$ for any $1 \leq i,j \leq n$, $\|a\|_{2p} < 1$, $\|b\|_{2p} < 1$ and $\|[y_{ij}]\|_{\delta} < 1$. By assumption, $ex_{ij}e = x_{ij}$ hence we actually have $x_{ij} = eay_{ij}be$ for any $1 \leq i,j \leq n$. Using polar decompositions we can write $be = wb'$ and $ea = a'v$, with $b' = |be|$, $a' = |a^*e|$ and $v, w \in \mathcal{M}$ such that $\|v\| \leq 1$ and $\|w\| \leq 1$. Note that $a', b' \in L^{2p}(e\mathcal{M}e)^+$ and that $\|a'\|_{2p} < 1$ and $\|b'\|_{2p} < 1$. It follows from these factorizations that

$$x_{ij} = a'vy_{ij}wb', \qquad 1 \leq i,j \leq n. \tag{12}$$

Since e has a finite trace, it belongs to $L^{2p}(\mathcal{M})$ hence we can choose $\varepsilon > 0$ such that

$$\|a' + \varepsilon e\|_{2p} < 1 \qquad \text{and} \qquad \|b' + \varepsilon e\|_{2p} < 1. \tag{13}$$

Both $a' + \varepsilon e$ and $b' + \varepsilon e$ have an inverse in $e\mathcal{M}e$. Then we can define

$$z_{ij} = (a' + \varepsilon e)^{-1} x_{ij} (b' + \varepsilon e)^{-1}, \qquad 1 \leq i,j \leq n. \tag{14}$$

Since each x_{ij} belongs to $e\mathcal{M}e$, each z_{ij} belongs to $e\mathcal{M}e$ as well. Further we have

$$x_{ij} = (a' + \varepsilon e) z_{ij} (b' + \varepsilon e), \qquad 1 \leq i,j \leq n. \tag{15}$$

Let us now show that

$$\|[z_{ij}]\|_{\delta} \leq \|[y_{ij}]\|_{\delta}\,. \tag{16}$$

Here the delta norm on the left-hand side is computed in $e\mathcal{M}e \otimes S^1_n$ whereas the delta norm on the right-hand side is computed in $\mathcal{M} \otimes S^1_n$. We observe that since $a' \in L^{2p}(e\mathcal{M}e)^+$, $(a' + \varepsilon e)^{-1} a'$ belongs to $e\mathcal{M}e$ and we have $\|(a' + \varepsilon e)^{-1} a'\|_\infty \leq 1$. Likewise, we have $\|b'(b' + \varepsilon e)^{-1}\|_\infty \leq 1$. This implies that

$$\|(a' + \varepsilon e)^{-1} a' v\|_\infty \leq 1 \qquad \text{and} \qquad \|wb'(b' + \varepsilon e)^{-1}\|_\infty \leq 1. \tag{17}$$

Let $\theta \colon M_n \to \mathcal{M}$ be the linear map associated with $[y_{ij}]$ and let $\varphi \colon M_n \to e\mathcal{M}e$ be associated with $[z_{ij}]$. By (12) and (14), we have $z_{ij} = (a' + \varepsilon e)^{-1} a' v y_{ij} w b' (b' + \varepsilon e)^{-1}$ for any $1 \leq i,j \leq n$. Hence

$$\varphi(s) = \big[(a' + \varepsilon e)^{-1} a' v\big] \theta(s) \big[wb'(b' + \varepsilon e)^{-1}\big], \quad s \in M_n.$$

It therefore follows from e.g. [25, (11.4)] and (17) that $\|\varphi\|_{dec} \leq \|\theta\|_{dec}$.

Since $\|\theta\|_{dec} = \|[y_{ij}]\|_\delta$ and $\|\varphi\|_{dec} = \|[z_{ij}]\|_\delta$, by [25, Corollary 12.4], this yields (16).

Now combining (15), (13) and (16), and using (9) in $L^p(e\mathcal{M}e; S^1_n)$, we obtain that $\big\|[x_{ij}]\big\|_{L^p(e\mathcal{M}e;S^1_n)} < 1$. This proves the result. $\square$

For any semifinite and hyperfinite von Neumann algebra $\mathcal{M}$, and for any operator space E, Pisier [26, Chapter 3] introduced a vector valued noncommutative L^p-space, denoted by $L^p(\mathcal{M})[E]$. The next statement shows that Definition 3.2 is consistent with [26].

Proposition 3.7. *Let $\mathcal{M}$ be a semifinite and hyperfinite von Neumann algebra, and let $1 \leq p < \infty$. Equip the spaces S^1 and S^1_n with their natural operator space structures (see e.g. [6, 1.14.5]). Then*

$$L^p(\mathcal{M}; S^1) = L^p(\mathcal{M})[S^1] \qquad \text{and} \qquad L^p(\mathcal{M}; S^1_n) = L^p(\mathcal{M})[S^1_n]$$

isometrically, for all $n \geq 1$.

Proof. We assume that the semifinite von Neumann algebra $\mathcal{M}$ is hyperfinite. By density it suffices to prove that for any $n \geq 1$ and for any matrix $[x_{ij}]_{1\leq i,j\leq n}$ of elements in $L^p(\mathcal{M})$, we have

$$\|[x_{ij}]\|_{L^p(\mathcal{M};S^1_n)} = \|[x_{ij}]\|_{L^p(\mathcal{M})[S^1_n]}. \tag{18}$$

Assume first that $\mathcal{M}$ is finite. For any matrix $[y_{ij}]_{1\leq i,j\leq n}$ of elements in $\mathcal{M}$, let $\|[y_{ij}]\|_{\min}$ denote its norm in the minimal tensor product $\mathcal{M} \otimes_{\min} S^1_n$. It follows from the definition of $\Lambda_p(E)$ in [26, p.41] and from [26, Theorem 3.8] that for any matrix $[x_{ij}]_{1\leq i,j\leq n}$ of elements in $L^p(\mathcal{M})$, we have

$$\|[x_{ij}]\|_{L^p(\mathcal{M})[S^1_n]} = \inf\bigl\{\|a\|_{2p}\|[y_{ij}]\|_{\min}\|b\|_{2p}\bigr\},$$

where the infimum runs over all $a, b \in L^{2p}(\mathcal{M})$ and all matrices $[y_{ij}]$ of elements in $\mathcal{M}$ such that $x_{ij} = a y_{ij} b$ for any $1 \leq i, j \leq n$. Since $\mathcal{M}$ is hyperfinite, hence injective, we have

$$\|[y_{ij}]\|_{\delta} = \|[y_{ij}]\|_{\min}$$

for any such $[y_{ij}]$. This follows from the fact that if $\theta\colon M_n \to \mathcal{M}$ is the linear map associated with $[y_{ij}]$, then $\|[y_{ij}]\|_{\min} = \|\theta\|_{cb}$, $\|[y_{ij}]\|_{\delta} = \|\theta\|_{dec}$, as mentioned above, and $\|\theta\|_{cb} = \|\theta\|_{dec}$ (see [9]). Applying (9), we deduce the equality (18) in that case.

For a possibly non finite $\mathcal{M}$, consider $V = \cup e\mathcal{M}e$, where the union runs over all projections e in $\mathcal{M}$ with finite trace. The finite case considered above shows that

$$L^p(e\mathcal{M}e; S^1_n) = L^p(e\mathcal{M}e)[S^1_n]$$

isometrically, for any such e. Applying Lemma 3.6 and [25, Theorem 3.4], this implies that (18) holds true whenever $x_{ij} \in V$ for all $1 \leq i, j \leq n$. Since V is dense in $L^p(\mathcal{M})$, this yields (18) for any $x_{ij} \in L^p(\mathcal{M})$. □

In the sequel we consider a second semifinite von Neumann algebra $\mathcal{N}$. Recall the embedding (7) from Remark 3.3.

Definition 3.8. Let $1 \leq p < \infty$ and let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a bounded map. We say that T is

(i) S^1-bounded if $T \otimes I_{S^1}$ extends to a bounded map
$$T\overline{\otimes} I_{S^1}\colon L^p(\mathcal{M}; S^1) \longrightarrow L^p(\mathcal{N}; S^1).$$
In this case, the norm of $T\overline{\otimes} I_{S^1}$ is called the S^1-bounded norm of T and is denoted by $\|T\|_{S^1}$;

(ii) S^1-contractive if it is S^1-bounded and $\|T\|_{S^1} \leq 1$.

Remark 3.9. It is plain that $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is S^1-bounded if and only if there exists a constant $K \geq 0$ such that

$$\bigl\|T \otimes I_{S^1_n}\colon L^p(\mathcal{M}; S^1_n) \longrightarrow L^p(\mathcal{N}; S^1_n)\bigr\| \leq K$$

for any $n \geq 1$. In this case, $\|T\|_{S^1}$ is the smallest $K \geq 0$ satisfying this property.

Remark 3.10. We have natural isometric identifications

$$L^2(\mathcal{M};\{\ell^2_{\mathbb{N}^2}\}_r) = L^2(\mathcal{M};\{\ell^2_{\mathbb{N}^2}\}_c) = L^2(B(\ell^2)\overline{\otimes}\mathcal{M}).$$

They imply that

$$L^1(\mathcal{M};S^1) = L^1(B(\ell^2)\overline{\otimes}\mathcal{M}) \qquad \text{isometrically.}$$

Consequently, a bounded map $T\colon L^1(\mathcal{M}) \to L^1(\mathcal{N})$ is S^1-bounded if and only if T is completely bounded and $\|T\|_{S^1} = \|T\|_{cb}$ is this case.

Assume that $\mathcal{M},\mathcal{N}$ are two semifinite and hyperfinite von Neumann algebras, and let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a bounded map. We say that T is completely regular if there exists a constant $K \geq 0$ such that for any $n \geq 1$,

$$\big\|T\otimes I_{M_n}\colon L^p(\mathcal{M})[M_n] \longrightarrow L^p(\mathcal{N})[M_n]\big\| \leq K.$$

In this case, the completely regular norm $\|T\|_{reg}$ is defined as the least possible K satisfying this property. This concept was introduced in [27]. It is shown in the latter paper that if T is completely regular, then for any operator space E, $T\otimes I_E$ extends to a bounded operator $T\overline{\otimes}I_E$ from $L^p(\mathcal{M})[E]$ into $L^p(\mathcal{N})[E]$, with

$$\big\|T\overline{\otimes}I_E\colon L^p(\mathcal{M})[E] \longrightarrow L^p(\mathcal{N})[E]\big\| \leq \|T\|_{reg}. \tag{19}$$

We refer to [14] and [5] for developments and further results.

Proposition 3.11. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are semifinite and hyperfinite von Neumann algebras, let $1\leq p<\infty$ and let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a bounded operator. Then T is S^1-bounded if and only if T is completely regular and in this case, we have $\|T\|_{S^1} = \|T\|_{reg}$.*

Proof. Suppose that T is S^1-contractive. By Proposition 3.7, we have

$$\|T\otimes I_{S^1_n}\colon L^p(\mathcal{M})[S^1_n] \longrightarrow L^p(\mathcal{N})[S^1_n]\| \leq \|T\|_{S^1}. \tag{20}$$

for every $n\geq 1$. Assume that $p>1$ and let $p'=p/(p-1)$ be the conjugate number of p. By [26, Theorem 4.1], we both have

$$\big(L^p(\mathcal{M})[S^1_n]\big)^* \cong L^{p'}(\mathcal{M})[M_n] \qquad \text{and} \qquad \big(L^p(\mathcal{N})[S^1_n]\big)^* \cong L^{p'}(\mathcal{N})[M_n].$$

isometrically. Passing to the adjoint in (20), we obtain that

$$\|T^*\otimes I_{M_n}\colon L^{p'}(\mathcal{N})[M_n] \longrightarrow L^{p'}(\mathcal{M})[M_n]\| \leq \|T\|_{S^1}$$

for every $n\geq 1$. Thus T^* is completely regular, with $\|T^*\|_{reg}\leq \|T\|_{S^1}$. It now follows from [27, Lemma 2.3] that T is completely regular as well, with $\|T\|_{reg}\leq\|T\|_{S^1}$. The case $p=1$ is proved similarly, using Remark 3.10.

The converse is clear, using Proposition 3.7 again. □

Remark 3.12. Junge's space $L^p(\mathcal{M};\ell^1)$ from [13] coincides with the subspace of $L^p(\mathcal{M};S^1)$ of matrices $[x_{ij}]_{i,j\geq 1}$ such that $x_{ij}=0$ for any $i\neq j$. In [21, Definition 2.5], we introduced ℓ^1-boundedness by saying that a bounded map $T\colon L^p(\mathcal{M})\to L^p(\mathcal{N})$ is ℓ^1-bounded if $T\otimes I_{\ell^1}$ extends to a bounded map from $L^p(\mathcal{M};\ell^1)$ into $L^p(\mathcal{N};\ell^1)$. It is plain that any S^1-bounded map T is ℓ^1-bounded, with $\|T\|_{\ell^1}\leq\|T\|_{S^1}$. However [21, Example 2.7] shows that the converse is not true.

We now state the main result of this section.

Theorem 3.13. *Suppose that $T\colon L^p(\mathcal{M})\to L^p(\mathcal{N})$ is a completely positive operator. Then T is S^1-bounded and $\|T\|_{S^1}=\|T\|$.*

Proof. Let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a completely positive operator. Fix some $n \geq 1$. Let $x = [x_{ij}]_{1\leq i,j\leq n}$ be a matrix of elements in $L^p(\mathcal{M})$ with $\|x\|_{L^p(\mathcal{M};S^1_n)} < 1$. According to Lemma 3.4, there exist an integer $m \geq 1$ and families $(a_{ik})_{1\leq i\leq n,1\leq k\leq m}$ and $(b_{kj})_{1\leq k\leq m,1\leq j\leq n}$ in $L^{2p}(\mathcal{M})$ such that

$$\Big\|\sum_{i,k} a_{ik}a_{ik}^*\Big\|_p < 1, \qquad \Big\|\sum_{k,j=1} b_{kj}^*b_{kj}\Big\|_p < 1 \qquad \text{and} \qquad x_{ij} = \sum_{k=1}^m a_{ik}b_{kj}$$

for any $1 \leq i,j \leq n$. We introduce

$$r_{ij} = \sum_{k=1}^m a_{ik}a_{jk}^*, \qquad s_{ij} = \sum_{k=1}^m b_{ki}^*b_{kj} \qquad \text{and} \qquad z_{ij} = \begin{pmatrix} r_{ij} & x_{ij} \\ x_{ji}^* & s_{ij}\end{pmatrix}$$

for any $1 \leq i,j \leq n$. Then we set

$$r = [r_{ij}], \qquad s = [s_{ij}] \qquad \text{and} \qquad z = [z_{ij}].$$

With $x^* = [x_{ji}^*]$, we may write

$$z = \begin{pmatrix} r & x \\ x^* & s\end{pmatrix}.$$

Following Remark 3.5 and (4) we regard x, x^*, r, s as elements of $S^p_n \otimes L^p(\mathcal{M}) = L^p(M_n\overline{\otimes}\mathcal{M})$ and we regard z as an element of $S^p_{2n} \otimes L^p(\mathcal{M}) = L^p(M_{2n}\overline{\otimes}\mathcal{M})$.

Now consider $a = [a_{ik}]_{1\leq i\leq n,1\leq k\leq m}$ and $b = [b_{kj}]_{1\leq k\leq m,1\leq j\leq n}$, regarded as elements of $S^p_{n,m} \otimes L^p(\mathcal{M})$ and $S^p_{m,n} \otimes L^p(\mathcal{M})$, respectively, and let $c = \begin{pmatrix} a \\ b^*\end{pmatrix} \in S^p_{2n,m} \otimes L^p(\mathcal{M})$. It follows from the above definitions that

$$z = \begin{pmatrix} a \\ b^*\end{pmatrix}\begin{pmatrix} a^* & b\end{pmatrix} = cc^*,$$

hence $z \in L^p(M_{2n}\overline{\otimes}\mathcal{M})^+$.

Let us write $T_n = I_{S^p_n} \otimes T$ for simplicity. By assumption, T_{2n} is positive hence

$$T_{2n}(z) = \begin{pmatrix} T_n(r) & T_n(x) \\ T_n(x^*) & T_n(s)\end{pmatrix} \in L^p(M_{2n}\overline{\otimes}\mathcal{M})^+.$$

Consider the positive square root $(T_{2n}(z))^{1/2}$, which belongs to $L^{2p}(M_{2n}\overline{\otimes}\mathcal{M})^+$. We may write it as

$$(T_{2n}(z))^{1/2} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \delta\end{pmatrix},$$

with α, β, δ in $L^{2p}(M_n\overline{\otimes}\mathcal{M})$, and $\alpha \geq 0$, $\delta \geq 0$. Then,

$$T_n(r) = \alpha^2 + \beta\beta^*; \tag{21}$$

$$T_n(s) = \beta^*\beta + \delta^2; \tag{22}$$

$$T_n(x) = \alpha\beta + \beta\delta. \tag{23}$$

Write $\alpha = [\alpha_{ij}]$, $\beta = [\beta_{ij}]$ and $\delta = [\delta_{ij}]$. Using (23), we have

$$T(x_{ij}) = \sum_{k=1}^n \alpha_{ik}\beta_{kj} + \sum_{k=1}^n \beta_{ik}\delta_{kj}, \quad 1 \leq i,j \leq n.$$

Let

$$a = \Bigl(\sum_{i,k} \alpha_{ik}\alpha_{ik}^* + \sum_{i,k} \beta_{ik}\beta_{ik}^*\Bigr)^{1/2} \qquad \text{and} \qquad b = \Bigl(\sum_{j,k} \beta_{kj}^*\beta_{kj} + \sum_{k,j} \delta_{kj}^*\delta_{kj}\Bigr)^{1/2},$$

then the above factorization implies that

$$\|[T(x_{ij})]\|_{L^p(\mathcal{N};S_n^1)} \leq \|a\|_{2p}\|b\|_{2p}.$$

Now observe that by (21) and (22), and the fact that $\alpha^* = \alpha$ and $\delta^* = \delta$, we have

$$T(r_{ii}) = \sum_k \alpha_{ik}\alpha_{ik}^* + \beta_{ik}\beta_{ik}^* \qquad \text{and} \qquad T(s_{jj}) = \sum_k \beta_{kj}^*\beta_{kj} + \delta_{kj}^*\delta_{kj}$$

for any $1 \leq i, j \leq n$. Consequently,

$$\begin{aligned}
\|a\|_{2p}^{2p} &= \Bigl\|\sum_{i,k} \alpha_{ik}\alpha_{ik}^* + \sum_{i,k} \beta_{ik}\beta_{ik}^*\Bigr\|_p^p \\
&= \Bigl\|\sum_i T(r_{ii})\Bigr\|_p^p \\
&\leq \|T\|^p \Bigl\|\sum_i r_{ii}\Bigr\|_p^p = \|T\|^p \Bigl\|\sum_{i,k} a_{ik}a_{ik}^*\Bigr\|_p^p \leq \|T\|^p.
\end{aligned}$$

Similarly, we can show that $\|b\|_{2p}^{2p} \leq \|T\|^p$, and therefore

$$\|[T(x_{ij})]\|_{L^p(\mathcal{N};S_n^1)} \leq \|T\|.$$

The result follows at once. $\square$

Remark 3.14. Let $1 \leq p \leq \infty$. Following [5,14] we say that a bounded map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is decomposable if there exist two bounded maps $S_1, S_2\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ such that the mapping

$$\Gamma_{S_1,S_2} := \begin{pmatrix} S_1 & T \\ T_* & S_2 \end{pmatrix} : L^p(M_2\overline{\otimes}\mathcal{M}) \longrightarrow L^p(M_2\overline{\otimes}\mathcal{N})$$

taking any $\begin{pmatrix} r & x \\ y & s \end{pmatrix}$ to $\begin{pmatrix} S_1(r) & T(x) \\ T(y^*)^* & S_2(s) \end{pmatrix}$, with $x, y, r, s \in L^p(\mathcal{M})$, is completely posivive. This is equivalent to T being a linear combination of completely positive maps $L^p(\mathcal{M}) \to L^p(\mathcal{N})$. In this case, the decomposable norm of T is defined by

$$\|T\|_{dec} = \inf\bigl\{\max\{\|S_1\|, \|S_2\|\}\bigr\}, \tag{24}$$

where the infimum is taken over all possible pairs (S_1, S_2) such that Γ_{S_1,S_2} is completely positive . When T is completely positive, $\Gamma_{T,T}$ is completely positive and we have $\|T\|_{dec} = \|T\|$ in this case.

With these definitions in mind, it is clear from the proof of Theorem 3.13 that the latter generalizes as follows, for any $1 \leq p < \infty$:

(25) Any decomposable map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is S^1-bounded, with $\|T\|_{S^1} \leq \|T\|_{dec}$.

In the special case when $\mathcal{M}, \mathcal{N}$ are hyperfinite, the converse is true, that is, any S^1-bounded map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is decomposable, with $\|T\|_{S^1} \leq \|T\|_{dec}$. This follows from Proposition 3.11 and [5, Theorem 3.23]. We do not know if this property is true for general semifinite von Neumann algebras.

We finally mention that Proposition 3.11 together with [27, Proposition 2.2] show that when $\mathcal{M}$, $\mathcal{N}$ are semifinite and hyperfinite von Neumann algebras, every S^1-bounded operator $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is completely bounded. We do not know whether this property is true for general semifinite von Neumann algebras, except in the trivial cases $p = 2$ and $p = 1$ (see Remark 3.10).

4. Separating maps with a direct Yeadon type factorization

The notion of Yeadon type factorization was introduced in [21], in reference to Yeadon's characterization of isometries on noncommutative L^p-spaces for $1 \leq p \neq 2 < \infty$ [34]. In this section, we introduce the notion of direct Yeadon type factorization and we discuss the relationship between the norm, the completely bounded norm and the S^1-bounded norm of operators which admit such a factorization.

First we recall some prerequisite concepts and results. A Jordan homomorphism between von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ is a linear map $J\colon \mathcal{M} \to \mathcal{N}$ that preserves involution and the Jordan product $(x, y) \mapsto \frac{1}{2}(xy + yx)$. The interested reader is referred to [10, Chapter 7], [30] and [17, Exercises 10.5.21-10.5.31] for information on these maps. We note for further use that any Jordan homomorphism is positive.

We assume that $(\mathcal{M}, \tau_{\mathcal{M}})$ and $(\mathcal{N}, \tau_{\mathcal{N}})$ are semifinite and we let $1 \leq p < \infty$. Following [21], we say that an operator $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ has a Yeadon type factorization if there exist a w^*-continuous Jordan homomorphism $J\colon \mathcal{M} \to \mathcal{N}$, a partial isometry $w \in \mathcal{N}$, and a positive operator B affiliated with $\mathcal{N}$, which satisfy the following conditions:

(a) $w^*w = J(1) = s(B)$, the support projection of B;
(b) every spectral projection of B commutes with $J(x)$, for all $x \in \mathcal{M}$;
(c) $T(x) = wBJ(x)$ for all $x \in \mathcal{M} \bigcap L^p(\mathcal{M})$.

In this case, w, B and J are uniquely determined by T and we call (w, B, J) the Yeadon triple associated with T.

Yeadon's Theorem [34] asserts that if $p \neq 2$, any isometry $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ admits a Yeadon type factorization.

Following [21], we say that an operator $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is separating if it preserves disjointness of elements; that is, if for $x, y \in L^p(\mathcal{M})$ such that $x^*y = xy^* = 0$, then we have $T(x)^*T(y) = T(x)T(y)^* = 0$. It is shown in [11, 21] that T admits a Yeadon type factorization if and only if it is separating.

Let $J : \mathcal{M} \to \mathcal{N}$ be a Jordan homomorphism and let $\mathcal{D} \subset \mathcal{N}$ be the von Neumann algebra generated by $J(\mathcal{M})$. Then $J(1)$ is the unit of $\mathcal{D}$. By e.g. [30, Theorem 3.3], there exist projections e and f in the center of $\mathcal{D}$ such that

(i) $e + f = J(1)$.
(ii) $x \mapsto J(x)e$ is a $*$-homomorphism.
(iii) $x \mapsto J(x)f$ is an anti-$*$-homomorphism.

Let $\mathcal{N}_1 = e\mathcal{N}e$ and $\mathcal{N}_2 = f\mathcal{N}f$. We let $\pi\colon \mathcal{M} \to \mathcal{N}_1$ and $\sigma\colon \mathcal{M} \to \mathcal{N}_2$ be defined by $\pi(x) = J(x)e$ and $\sigma(x) = J(x)f$, for all $x \in \mathcal{M}$. Then J is valued in $\mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2$ and $J(x) = \pi(x) + \sigma(x)$, for all $x \in \mathcal{M}$. As in [21] we use the notations

$$J = \begin{pmatrix} \pi & 0 \\ 0 & \sigma \end{pmatrix} \quad \text{and} \quad J(x) = \begin{pmatrix} \pi(x) & 0 \\ 0 & \sigma(x) \end{pmatrix} \tag{26}$$

to refer to such a decomposition.

Definition 4.1. We say that a separating map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ admits a direct (resp. anti-direct) Yeadon type factorization if the Jordan homomorphism of its Yeadon triple is a $*$-homomorphism (resp. an anti-$*$-homomorphism).

The above definition is partly motivated by a result due to Junge-Ruan-Sherman [15, Proposition 3.2] which asserts that if $p \neq 2$, an isometry $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ admits a direct Yeadon type factorization if and only if T is a 2-isometry, if and only if T is a complete isometry.

Remark 4.2. Let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a separating map and let (w, B, J) be its Yeadon triple.

(a) The mapping $w^*T(\,\cdot\,)$, which maps any $x \in \mathcal{M} \cap L^p(\mathcal{M})$ to $BJ(x)$, is also a separating map. Its Yeadon triple is $(J(1), B, J)$. Since J is positive, B is positive and B commutes with the range of J, the mapping $w^*T(\,\cdot\,)$ is positive.

(b) Assume that $J = \pi$ is a $*$-homomorphism, so that T has a direct Yeadon type factorization. For any $n \geq 1$, $I_{M_n} \otimes \pi$ is a $*$-homomorphism from $M_n \overline{\otimes} \mathcal{M}$ into $M_n \overline{\otimes} \mathcal{N}$. Hence $I_{S^p_n} \otimes T\colon L^p(M_n \overline{\otimes} \mathcal{M}) \to L^p(M_n \overline{\otimes} \mathcal{N})$ admits a Yeadon type factorization. Indeed the Yeadon triple of $I_{S^p_n} \otimes T$ is equal to $(I_n \otimes w, I_n \otimes B, I_{M_n} \otimes \pi)$. It follows from (a) that in this case, $w^*T(\,\cdot\,)$ is completely positive.

Remark 4.3. Let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a separating operator, with Yeadon triple (w, B, J). Assume that $w = J(1)$, so that

$$T(x) = BJ(x), \qquad x \in \mathcal{M} \cap L^p(\mathcal{M}).$$

Consider a decomposition of J as in (26). This induces a direct/anti-direct decomposition of T, as follows.

Recall $\mathcal{N}_1 = e\mathcal{N}e$ and $\mathcal{N}_2 = f\mathcal{N}f$. Then $\mathcal{N}_1, \mathcal{N}_2$ are semifinite and we have

$$L^p(\mathcal{N}_1) \overset{p}{\oplus} L^p(\mathcal{N}_2) \;=\; L^p(\mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2) \subset L^p(\mathcal{N}).$$

Set $B_1 = Be$ and $B_2 = Bf$. Since $B = BJ(1)$, we have $B = B_1 + B_2$. Moreover B commutes with the range of J, that is, B is affiliated with $J(\mathcal{M})'$. This implies that B commutes with e and f. Consequently, B_1 is affiliated with $\mathcal{N}_1$ and B_2 is affiliated with $\mathcal{N}_2$. Now define

$$T_1\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{N}_1) \qquad \text{and} \qquad T_2\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{N}_2)$$

by setting

$$T_1(x) = T(x)e \qquad \text{and} \qquad T_2(x) = T(x)f, \quad x \in L^p(\mathcal{M}).$$

Then

$$T = T_1 + T_2.$$

Further T_1 is a separating operator and its Yeadon triple is equal to $(1_{\mathcal{N}_1}, B_1, \pi)$. Likewise T_2 is a separating operator and its Yeadon triple is equal to $(1_{\mathcal{N}_2}, B_2, \sigma)$. In particular, T_1 has a direct Yeadon type factorization whereas T_2 has an anti-direct Yeadon type factorization.

In the case when $w \neq J(1)$, one can apply the following decomposition principle to the mapping $w^*T(\,\cdot\,)$ from Remark 4.2 (a).

Proposition 4.4. *Let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a bounded operator with a direct Yeadon type factorization. Then T is completely bounded and $\|T\|_{cb} = \|T\|$.*

Proof. Suppose that T has a direct Yeadon type factorization, with Yeadon triple (w, B, π) and fix some integer $n \geq 1$. Set $\pi_n = I_{M_n} \otimes \pi$, $w_n = I_n \otimes w$ and $B_n = I_n \otimes B$. By Remark 4.2 (b), $I_{S_n^p} \otimes T$ is separating with Yeadon triple equal to (w_n, B_n, π_n).

We note that for any $x \in \mathcal{M} \cap L^p(\mathcal{M})$, we have $|T(x)|^p = B^p \pi(|x|^p)$, hence

$$\|T(x)\|_p^p = \tau_{\mathcal{N}}\big(B^p \pi(|x|^p)\big). \tag{27}$$

Let $y \in (M_n \overline{\otimes} \mathcal{M}) \cap L^p(M_n \overline{\otimes} \mathcal{M})$. Then similarly we have

$$\|(I_{S_n^p} \otimes T)(y)\|_p^p = \tau_{\mathcal{N},n}\big(B_n^p \pi_n(|y|^p)\big).$$

Write $x = |y|^p$ and decompose it as $x = [x_{ij}]_{1 \leq i,j \leq n}$. Then

$$\tau_{\mathcal{N},n}\big(B_n^p \pi_n(|y|^p)\big) = \sum_{i=1}^{n} \tau_{\mathcal{N}}\big(B^p \pi(x_{ii})\big).$$

For any $1 \leq i \leq n$, we have

$$\tau_{\mathcal{N}}\big(B^p \pi(x_{ii})\big) = \|T(x_{ii}^{\frac{1}{p}})\|_p^p \leq \|T\|^p \|x_{ii}^{\frac{1}{p}}\|_p^p = \|T\|^p \tau_{\mathcal{M}}(x_{ii}),$$

by (27). We infer that

$$\|(I_{S_n^p} \otimes T)(y)\|_p^p \leq \|T\|^p \sum_{i=1}^{n} \tau_{\mathcal{M}}(x_{ii}) = \|T\|^p \, \tau_{\mathcal{M},n}(x).$$

This yields $\|(I_{S_n^p} \otimes T)(y)\|_p \leq \|T\| \|y\|_p$, which proves that T is completely bounded, with $\|T\|_{cb} = \|T\|$. ☐

Proposition 4.5. *Let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a bounded operator with a direct Yeadon type factorization. Then T is S^1-bounded and $\|T\|_{S^1} = \|T\|$.*

Proof. Suppose that T has a direct Yeadon type factorization, with Yeadon triple (w, B, π). By Remark 4.2 (b), $U := w^* T(\,\cdot\,)$ is completely positive. Hence by Theorem 3.13, U is S^1-bounded, with $\|U\|_{S^1} = \|U\|$. Since $wU(x) = T(x)$ for any $x \in L^p(\mathcal{M})$, this immediately implies that T is also S^1-bounded, with $\|T\|_{S^1} = \|U\|_{S^1}$. Further we have $\|T\| = \|U\|$, which yields the result. ☐

In the case when $\mathcal{M}, \mathcal{N}$ are hyperfinite, it follows from [5, 27] that any completely positive map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is automatically completely bounded, with $\|T\|_{cb} = \|T\|$. We do not know if this holds true in general. If this were true, Proposition 4.4 would be a direct consequence of Remark 4.2 (b).

5. Direct Yeadon type factorization and isometries

We proved in the previous section (Propositions 4.4 and 4.5) that if a contraction $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ admits a direct Yeadon type factorization, then it is both completely contractive and S^1-contractive. The purpose of this section is to establish converse statements for isometries. Namely we will show that an isometry $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ admits a direct Yeadon type factorization provided that either T is completely contractive and $p \neq 2$, or T is S^1-contractive.

We need three preparatory lemmas.

Lemma 5.1. *Let $1 \leq p < \infty$. Let $\mathcal{M}$ and $\mathcal{N}$ be semifinite von Neumann algebras and let $b \in L^p(\mathcal{N})$. Consider a matrix $[x_{ij}]_{1\leq i,j\leq n}$ of elements in $L^p(\mathcal{M})$. We have*

$$\|[x_{ij} \otimes b]\|_{L^p(\mathcal{M}\overline{\otimes}\mathcal{N};S^1_n)} = \|b\|_p \, \|[x_{ij}]\|_{L^p(\mathcal{M};S^1_n)}. \tag{28}$$

Proof. The case $p = 1$ follows from Remark 3.10, so we may assume that $p \neq 1$. Let $p' = \frac{p}{p-1}$ be the conjugate number of p. Let $b \in L^p(\mathcal{N})$ and let $c \in L^{p'}(\mathcal{N})$ such that $\|c\|_{p'} = 1$ and $\tau_{\mathcal{N}}(bc) = \|b\|_p$. Define

$$T\colon L^{p'}(\mathcal{M}) \to L^{p'}(\mathcal{M}\overline{\otimes}\mathcal{N}), \quad T(z) = z \otimes c.$$

We claim that T is decomposable, with $\|T\|_{dec} \leq 1$, see (24) for the definition. To check this, consider the polar decomposition $c = u|c|$ of c. Then $|c^*| = u|c|u^*$. In the space $L^{p'}(M_2\overline{\otimes}\mathcal{N})$, the matrix $\begin{pmatrix} |c| & |c| \\ |c| & |c| \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes |c|$ is positive, hence

$$C := \begin{pmatrix} |c^*| & c \\ c^* & |c| \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |c| & |c| \\ |c| & |c| \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u^* \end{pmatrix} \geq 0.$$

Consequently the operator

$$L^{p'}(M_2\overline{\otimes}\mathcal{M}) \longrightarrow L^{p'}(M_2\overline{\otimes}\mathcal{M}) \otimes L^{p'}(M_2\overline{\otimes}\mathcal{N}) \subset L^{p'}(M_4\overline{\otimes}\mathcal{M}\overline{\otimes}\mathcal{N})$$

taking X to $X \otimes C$ for any $X \in L^{p'}(M_2\overline{\otimes}\mathcal{M})$ is completely positive. For any r, x, y, s in $L^{p'}(\mathcal{M})$, and $X = \begin{pmatrix} r & x \\ y & s \end{pmatrix}$, the matrix $\begin{pmatrix} r \otimes |c^*| & x \otimes c \\ y \otimes c^* & s \otimes |c| \end{pmatrix}$ is an extracted square matrix of $X \otimes C$. We deduce that the mapping $\Gamma\colon L^{p'}(M_2\overline{\otimes}\mathcal{M}) \to L^{p'}(M_2\overline{\otimes}\mathcal{M}\overline{\otimes}\mathcal{N})$ defined by

$$\Gamma \begin{pmatrix} r & x \\ y & s \end{pmatrix} = \begin{pmatrix} r \otimes |c^*| & x \otimes c \\ y \otimes c^* & s \otimes |c| \end{pmatrix}, \qquad r, x, y, s \in L^{p'}(\mathcal{M}),$$

is completely positive. Since $r \mapsto r \otimes |c^*|$ and $s \mapsto s \otimes |c|$ are contractive from $L^{p'}(\mathcal{M})$ into $L^{p'}(\mathcal{M}\overline{\otimes}\mathcal{N})$, this proves the claim.

Next the adjoint $T^*\colon L^p(\mathcal{M}\overline{\otimes}\mathcal{N}) \to L^p(\mathcal{M})$ is also decomposable, with $\|T^*\|_{dec} \leq 1$. By (25), this implies that T^* is S^1-contractive. The inequality $\geq$ in (28) follows since for any $x \in L^p(\mathcal{M})$, we have $T^*(x \otimes b) = \|b\|_p \, x$. The reverse inequality $\leq$ in (28) is immediate from the definitions. $\square$

The next result extends (3) to S^1-valued spaces.

Lemma 5.2. *Let $1 \leq p < \infty$ and let $\mathcal{N}_1$ and $\mathcal{N}_2$ be semifinite von Neumann algebras. For any $n \geq 1$, for any $[x^1_{ij}]_{1\leq i,j\leq n}$ in $L^p(\mathcal{N}_1; S^1_n)$ and for any $[x^2_{ij}]_{1\leq i,j\leq n}$ in $L^p(\mathcal{N}_2; S^1_n)$, we have*

$$\|[x^1_{ij}, x^2_{ij}]\|_{L^p(\mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2; S^1_n)} = \Bigl(\|[x^1_{ij}]\|^p_{L^p(\mathcal{N}_1;S^1_n)} + \|[x^2_{ij}]\|^p_{L^p(\mathcal{N}_2;S^1_n)}\Bigr)^{\frac{1}{p}}. \tag{29}$$

Proof. Let $\varepsilon > 0$. By Lemma 3.4, there exist an integer $m \geq 1$, families $[a^1_{ik}]_{1\leq i\leq n, 1\leq k\leq m}$ and $[b^1_{kj}]_{1\leq k\leq m, 1\leq j\leq n}$ in $L^{2p}(\mathcal{N}_1)$, and families $[a^2_{ik}]_{1\leq i\leq n, 1\leq k\leq m}$ and $[b^2_{kj}]_{1\leq k\leq m, 1\leq j\leq n}$ in $L^{2p}(\mathcal{N}_2)$ such that we have $x^1_{ij} = \sum_k a^1_{ik} b^1_{kj}$ and $x^2_{ij} = \sum_k a^2_{ik} b^2_{kj}$ for all $1 \leq i, j \leq n$, as well as norm estimates

$$\|(a^1_{ik})_{i,k}\|_{L^{2p}(\mathcal{N}_1;\{\ell^2_{nm}\}_r)} = \|(b^1_{kj})_{kj}\|_{L^{2p}(\mathcal{N}_1;\{\ell^2_{mn}\}_c)} \leq \bigl(\|[x^1_{ij}]\|_{L^p(\mathcal{N}_1;S^1_n)} + \varepsilon\bigr)^{\frac{1}{2}}$$

and

$$\|(a^2_{ik})_{i,k}\|_{L^{2p}(\mathcal{N}_2;\{\ell^2_{nm}\}_r)} = \|(b^2_{kj})_{kj}\|_{L^{2p}(\mathcal{N}_2;\{\ell^2_{mn}\}_c)} \leq \big(\|[x^2_{ij}]\|_{L^p(\mathcal{N}_2;S^1_n)} + \varepsilon\big)^{\frac{1}{2}}$$

Let $\mathcal{N} = \mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2$. Using (3), we have

$$\begin{aligned}
\|(a^1_{ik}, a^2_{ik})_{i,k}\|_{L^{2p}(\mathcal{N};\{\ell^2_{nm}\}_r)} &= \Big\|\Big(\sum_k a^1_{ik}a^{1*}_{ik}\,, \sum_k a^2_{ik}a^{2*}_{ik}\Big)\Big\|^{\frac{1}{2}}_{L^p(\mathcal{N})} \\
&= \Bigg(\Big\|\sum_k a^1_{ik}a^{1*}_{ik}\Big\|^p_{L^p(\mathcal{N}_1)} + \Big\|\sum_k a^2_{ik}a^{2*}_{ik}\Big\|^p_{L^p(\mathcal{N}_2)}\Bigg)^{\frac{1}{2}} \\
&= \Big(\|(a^1_{ik})_{i,k}\|^{2p}_{L^{2p}(\mathcal{N}_1;\{\ell^2_{nm}\}_r)} + \|(a^2_{ik})_{i,k}\|^{2p}_{L^{2p}(\mathcal{N}_2;\{\ell^2_{nm}\}_r)}\Big)^{\frac{1}{2}} \\
&\leq \Big(\big(\|[x^1_{ij}]\|_{L^p(\mathcal{N}_1;S^1_n)} + \varepsilon\big)^p + \big(\|[x^2_{ij}]\|_{L^p(\mathcal{N}_2;S^1_n)} + \varepsilon\big)^p\Big)^{\frac{1}{2}}.
\end{aligned}$$

Likewise,

$$\|(b^1_{kj}, b^2_{kj})_{i,k}\|_{L^{2p}(\mathcal{N};\{\ell^2_{mn}\}_c)} \leq \Big(\big(\|[x^1_{ij}]\|_{L^p(\mathcal{N}_1;S^1_n)} + \varepsilon\big)^p + \big(\|[x^2_{ij}]\|_{L^p(\mathcal{N}_2;S^1_n)} + \varepsilon\big)^p\Big)^{\frac{1}{2}}.$$

Since $(x^1_{ij}, x^2_{ij}) = \sum_k (a^1_{ik}, a^2_{ik})(b^1_{kj}, b^2_{kj})$ for all $1 \leq i,j \leq n$ and $\varepsilon > 0$ is arbitrary, the above two estimates imply the inequality $\leq$ in (29). The proof of the reverse inequality is similar. $\square$

The next result may be known to operator space specialists. We include a proof for the sake of completeness.

Lemma 5.3. *Let $1 \leq p \leq \infty$, let $n \geq 2$ and let $t\colon S^p_n \to S^p_n$ denote the transposition operator. We have*

(i) $\|t : S^p_n \to S^p_n\|_{cb} = \|I_{S^p_n} \otimes t\colon S^p_n[S^p_n] \to S^p_n[S^p_n]\| = n^{2|\frac{1}{2}-\frac{1}{p}|}$;
(ii) $\|t : S^p_n \to S^p_n\|_{reg} = \|t \otimes I_{S^1_n}\colon S^p_n[S^1_n] \to S^p_n[S^1_n]\| = n$.

Proof. We will use the the Haagerup tensor product $\overset{h}{\otimes}$, the row and column operator spaces R_n and C_n, the interpolation spaces $R_n(\theta) = (C_n, R_n)_\theta$ for $\theta \in [0,1]$, introduced in [24], and the construction of operator space valued S^p-spaces from [26, Chapter 1]. We will also use the crucial fact that the Haagerup tensor product commutes with interpolation (see [24, Theorem 2.3] for a precise statement). We refer to the above references and to [6, 25] for some background.

Let $(e_1, \ldots, e_n)$ be the standard basis of ℓ^2_n. It follows from [26, Theorem 1.1] that for any operator space E, the mapping $E_{ij} \otimes x \mapsto e_i \otimes x \otimes e_j$, $1 \leq i,j \leq n$ and $x \in E$, uniquely extends to a completely isometric isomorphism

$$S^p_n[E] \simeq R_n\big(\tfrac{1}{p}\big) \overset{h}{\otimes} E \overset{h}{\otimes} R_n\big(1 - \tfrac{1}{p}\big). \tag{30}$$

(i) : First we note that $\|t\colon M_n \to M_n\|_{cb} = n$, see e.g. [8, Proposition 2.2.7]. Since we have $\|t\colon S^2_n \to S^2_n\|_{cb} = 1$, we obtain by interpolation that

$$\|t\colon S^p_n \to S^p_n\|_{cb} \leq n^{2|\frac{1}{2}-\frac{1}{p}|}.$$

We now turn to lower estimates. Consider the matrix $[E_{ij}]$ in $S^p_n[S^p_n]$ and note that $I_{S^p_n}\otimes t$ maps $[E_{ij}]$ to $[E_{ji}]$ and $[E_{ji}]$ to $[E_{ij}]$. Applying (30) with $E = S^p_n$ equipped with its canonical operator space structure, we have isometric identifications

$$\begin{aligned} S^p_n[S^p_n] &\simeq R_n\bigl(\tfrac{1}{p}\bigr)\overset{h}{\otimes} R_n\bigl(\tfrac{1}{p}\bigr)\overset{h}{\otimes} R_n\bigl(1-\tfrac{1}{p}\bigr)\overset{h}{\otimes} R_n\bigl(1-\tfrac{1}{p}\bigr)\\ &\simeq R_{n^2}\bigl(\tfrac{1}{p}\bigr)\overset{h}{\otimes} R_{n^2}\bigl(1-\tfrac{1}{p}\bigr)\\ &\simeq S^p_{n^2}. \end{aligned}$$

In the first of these identifications, $[E_{ij}]$ corresponds to $\sum_{i,j} e_i\otimes e_i\otimes e_j\otimes e_j$, which may be written as $\bigl(\sum_i e_i\otimes e_i\bigr)\otimes\bigl(\sum_j e_j\otimes e_j\bigr)$. Since the $e_i\otimes e_i$ are pairwise orthogonal in $\ell^2_{n^2}$, we deduce that

$$\|[E_{ij}]\|_{S^p_n[S^p_n]} \;=\; \Bigl\|\sum_{i=1}^n e_i\otimes e_i\Bigr\|_{R_{n^2}(\frac{1}{p})}\Bigl\|\sum_{j=1}^n e_j\otimes e_j\Bigr\|_{R_{n^2}(1-\frac{1}{p})} = n^{\frac{1}{2}}n^{\frac{1}{2}} = n.$$

Similarly, $[E_{ji}]$ corresponds to $\sum_{i,j} e_i\otimes e_j\otimes e_i\otimes e_j$. Further $\{e_i\otimes e_j\,:\,1\leq i,j\leq n\}$ is an orthonormal basis of $\ell^2_{n^2}$. Hence through the identification of $S^p_n[S^p_n]$ with $S^p_{n^2}$, $[E_{ji}]$ corresponds to the identity map on $\ell^2_{n^2}$. Its S^p-norm is equal to $n^{\frac{2}{p}}$, hence

$$\|[E_{ji}]\|_{S^p_n[S^p_n]} = n^{\frac{2}{p}}.$$

These computations show that $\|I_{S^p_n}\otimes t\colon S^p_n[S^p_n]\to S^p_n[S^p_n]\|\geq n^{2|1/2-1/p|}$. Since the cb-norm of t is greater than or equal to $\|I_{S^p_n}\otimes t\colon S^p_n[S^p_n]\to S^p_n[S^p_n]\|$, this proves the double equality in (i).

(ii) : Note that

$$\|t\colon M_n\to M_n\|_{reg} = \|t\colon M_n\to M_n\|_{cb} = n$$

and that $\|t : S^1_n\to S^1_n\|_{reg} = \|t\colon M_n\to M_n\|_{reg}$ by duality. Hence by interpolation,

$$\|t\colon S^p_n\to S^p_n\|_{reg}\leq n.$$

We now turn to lower estimates. We have $S^1_n\simeq R_n\overset{h}{\otimes} C_n$ completely isometrically hence applying (30) with $E=S^1_n$, we have an isometric identification

$$S^p_n[S^1_n]\simeq R_n\bigl(\tfrac{1}{p}\bigr)\overset{h}{\otimes} R_n\overset{h}{\otimes} C_n\overset{h}{\otimes} R_n\bigl(1-\tfrac{1}{p}\bigr).$$

According to e.g. [8, Proposition 1.5.14 (6) & (8)], we have

$$R_n\bigl(\tfrac{1}{p}\bigr)\overset{h}{\otimes} R_n\simeq \bigl(C_n\overset{h}{\otimes} R_n, R_n\overset{h}{\otimes} R_n\bigr)_{\frac{1}{p}}\simeq (M_n, S^2_n)_{\frac{1}{p}} = S^{2p}_n.$$

Likewise,

$$C_n\overset{h}{\otimes} R_n\bigl(1-\tfrac{1}{p}\bigr)\simeq\bigl(C_n\overset{h}{\otimes} C_n, C_n\overset{h}{\otimes} R_n\bigr)_{1-\frac{1}{p}}\simeq (S^2_n, M_n)_{1-\frac{1}{p}} = S^{2p}_n.$$

Hence arguing as in the proof of (i), we have

$$\begin{aligned}\|[E_{ij}]\|_{S^p_n[S^1_n]} \;&=\; \Bigl\|\sum_{i=1}^n e_i\otimes e_i\Bigr\|_{R_n(\frac{1}{p})\overset{h}{\otimes} R_n}\Bigl\|\sum_{j=1}^n e_j\otimes e_j\Bigr\|_{C_n\overset{h}{\otimes} R_n(1-\frac{1}{p})}\\ &=\; \bigl\|I_n\colon \ell^2_n\to\ell^2_n\bigr\|^2_{S^{2p}_n} = n^{\frac{1}{p}}.\end{aligned}$$

Next using as in (i) the correspondance between $[E_{ji}]$ and $\sum_{i,j} e_i \otimes e_j \otimes e_i \otimes e_j$, as well as the functorial property of the Haagerup tensor product (see e.g. [6, 1.5.5]), we have

$$\Big\|\sum_{i,j=1}^{n} e_i \otimes e_j \otimes e_i \otimes e_j\Big\|_{R_n \overset{h}{\otimes} R_n \overset{h}{\otimes} C_n \overset{h}{\otimes} C_n} \\ \leq \big\|I_n \colon R_n\big(\tfrac{1}{p}\big) \to R_n\big\|_{cb} \big\|I_n \colon C_n \to R_n\big(1-\tfrac{1}{p}\big)\big\|_{cb} \, \|[E_{ji}]\|_{S^p_n[S^1_n]}.$$

Using the facts that $CB(C_n, R_n) \simeq S^2_n$ and $CB(C_n, C_n) \simeq M_n$ (see e.g. [7, Section 4]), we both have $\big\|I_n \colon C_n \to R_n\big\|_{cb} = n^{\frac{1}{2}}$ and $\big\|I_n \colon C_n \to C_n\big\|_{cb} = 1$. Hence

$$\big\|I_n \colon R_n\big(\tfrac{1}{p}\big) \to R_n\big\|_{cb} \leq n^{\frac{1}{2}(1-\frac{1}{p})},$$

by interpolation. Likewise,

$$\big\|I_n \colon C_n \to R_n\big(1-\tfrac{1}{p}\big)\big\|_{cb} \leq n^{\frac{1}{2}(1-\frac{1}{p})}$$

Further $R_n \overset{h}{\otimes} R_n \overset{h}{\otimes} C_n \overset{h}{\otimes} C_n \simeq R_{n^2} \overset{h}{\otimes} C_{n^2} \simeq S^1_{n^2}$ hence

$$\Big\|\sum_{i,j=1}^{n} e_i \otimes e_j \otimes e_i \otimes e_j\Big\|_{R_n \overset{h}{\otimes} R_n \overset{h}{\otimes} C_n \overset{h}{\otimes} C_n} = \big\|I_{n^2} \colon \ell^2_{n^2} \to \ell^2_{n^2}\big\|_1 = n^2.$$

These estimate yield

$$\|[E_{ji}]\|_{S^p_n[S^1_n]} \geq n^{1+\frac{1}{p}}.$$

Hence we obtain that

$$\|t \otimes I_{S^1_n} \colon S^p_n[S^1_n] \to S^p_n[S^1_n]\| \geq \frac{n^{1+\frac{1}{p}}}{n^{\frac{1}{p}}} = n.$$

Since $\|t \colon S^p_n \to S^p_n\|_{reg} \geq \|t \otimes I_{S^1_n} \colon S^p_n[S^1_n] \to S^p_n[S^1_n]\|$, (ii) follows at once. $\square$

Theorem 5.4. *Let $T \colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be an isometry. The following statements are equivalent.*

(i) *T admits a direct Yeadon type factorization.*
(ii) *T is S^1-contractive.*

Proof. The implication "$(i) \Rightarrow (ii)$" follows from Proposition 4.5 so we only need to prove "$(ii) \Rightarrow (i)$".

We first show this implication in the case when $\mathcal{M} = M_n$, with $n \geq 2$. Let $T \colon S^p_n \to L^p(\mathcal{N})$ be an isometry and assume that T is S^1-contractive. By Remark 3.12 and [21, Theorem 4.2], T admits a Yeadon type factorisation. Let (w, B, J) be its Yeadon triple. Changing T into $w^*T(\,\cdot\,)$, see Remark 4.2 (a), we can assume that $w = J(1)$. Consider a decomposition $J = \begin{pmatrix} \pi & 0 \\ 0 & \sigma \end{pmatrix}$ as in (26). We aim at showing that $\sigma = 0$.

Let us apply Remark 4.3 to T. In the sequel we use the elements $\mathcal{N}_1, \mathcal{N}_2$, B_1, B_2 and

$$T_1 \colon S^p_n \longrightarrow L^p(\mathcal{N}_1), \quad T_2 \colon S^p_n \longrightarrow L^p(\mathcal{N}_2)$$

from this remark. By construction we have $T_1(x) = B_1\pi(x)$ and $T_2(x) = B_2\sigma(x)$ for any $x \in S^p_n$.

Applying Lemma 2.2 to the unital $*$-homomorphism $\pi\colon M_n\to\mathcal{N}_1$, we obtain a projection ε_1 in $\mathcal{N}_1$ and a bijective $*$-homomorphism $\rho_\pi:\mathcal{N}_1\to M_n\overline{\otimes}(\varepsilon_1\mathcal{N}_1\varepsilon_1)$ such that

$$(\rho_\pi\circ\pi)(x)=x\otimes\varepsilon_1,\qquad x\in M_n,$$

and ρ_π is trace preserving. By Lemma 2.1, ρ_π induces an isometry (still denoted by)

$$\rho_\pi\colon L^p(\mathcal{N}_1)\longrightarrow L^p(M_n\overline{\otimes}(\varepsilon_1\mathcal{N}_1\varepsilon_1))\,\simeq\,S^p_n\otimes L^p(\varepsilon_1\mathcal{N}_1\varepsilon_1).$$

We have $B_1=T_1(I_n)$, hence $B_1\in L^p(\mathcal{N}_1)$. Further for any $x\in S^p_n$, we have

$$\begin{aligned}(\rho_\pi\circ T_1)(x)&=\rho_\pi\left(B_1\pi(x)\right)\\&=\rho_\pi(B_1)\rho_\pi(\pi(x))\\&=\rho_\pi(B_1)(x\otimes\varepsilon_1).\end{aligned}$$

Since $B_1\pi(x)=\pi(x)B_1$, a similar computation shows that we also have $(\rho_\pi\circ T_1)(x)=(x\otimes\varepsilon_1)\rho_\pi(B_1)$. This shows that $\rho_\pi(B_1)$ commutes with $x\otimes\varepsilon_1$ for any $x\in S^p_n$. Consequently there exists b_1 in $L^p(\varepsilon_1\mathcal{N}_1\varepsilon_1)$ such that $\rho_\pi(B_1)=I_n\otimes b_1$. Then the above computation shows that

$$(\rho_\pi\circ T_1)(x)=x\otimes b_1,\qquad x\in S^p_n. \tag{31}$$

Recall that we let $t\colon M_n\to M_n$ denote the transposition map. The mapping $\sigma\circ t\colon M_n\to\mathcal{N}_2$ is a unital $*$-homomorphism. Hence arguing as above, we obtain a projection ε_2 in $\mathcal{N}_2$, a trace preserving bijective $*$-homomorphism $\rho_\sigma\colon\mathcal{N}_2\to M_n\overline{\otimes}(\varepsilon_2\mathcal{N}_2\varepsilon_2)$, inducing an isometry

$$\rho_\sigma\colon L^p(\mathcal{N}_2)\longrightarrow L^p(M_n\overline{\otimes}(\varepsilon_2\mathcal{N}_2\varepsilon_2))\simeq S^p_n\otimes L^p(\varepsilon_2\mathcal{N}_2\varepsilon_2),$$

and some b_2 in $L^p(\varepsilon_2\mathcal{N}_2\varepsilon_2)$, such that

$$(\rho_\sigma\circ T_2)(x)=t(x)\otimes b_2,\qquad x\in S^p_n. \tag{32}$$

Observe that $\rho_\pi\colon L^p(\mathcal{N}_1)\to L^p(M_n\overline{\otimes}(\varepsilon_1\mathcal{N}_1\varepsilon_1))$ and $\rho_\sigma\colon L^p(\mathcal{N}_2)\to L^p(M_n\overline{\otimes}(\varepsilon_2\mathcal{N}_2\varepsilon_2))$ are completely positive. Hence by Theorem 3.13, they are S^1-contractive.

Let $m\geq 1$ and let $[x_{ij}]_{1\leq i,j\leq m}$ in $S^p_n[S^1_m]$. Since ρ_π is S^1-contractive, we have

$$\|[\rho_\pi\circ T_1(x_{ij})]\|_{L^p(M_n\overline{\otimes}(\varepsilon_1\mathcal{N}_1\varepsilon_1);S^1_m)}\leq\|[T_1(x_{ij})]\|_{L^p(\mathcal{N}_1;S^1_m)}.$$

On the other hand, using (31), (32) and Lemma 5.1, we have

$$\|[\rho_\pi\circ T_1(x_{ij})]\|_{L^p(M_n\overline{\otimes}(\varepsilon_1\mathcal{N}_1\varepsilon_1);S^1_m)}=\|[x_{ij}\otimes b_1]\|_{L^p(M_n\overline{\otimes}(\varepsilon_1\mathcal{N}_1\varepsilon_1);S^1_m)}=\|[x_{ij}]\|_{S^p_n[S^1_m]}\|b_1\|_p.$$

Hence we obtain that

$$\|b_1\|_p\|[x_{ij}]\|_{S^p_n[S^1_m]}\leq\|[T_1(x_{ij})]\|_{L^p(\mathcal{N}_1;S^1_m)}.$$

Similarly, we have

$$\|b_2\|_p\|[t(x_{ij})]\|_{S^p_n[S^1_m]}\leq\|[T_2(x_{ij})]\|_{L^p(\mathcal{N}_2;S^1_m)}.$$

Taking the p-th powers and summing the above inequalities, we obtain that

$$\begin{aligned}\|b_1\|_p^p\|[x_{ij}]\|^p_{S^p_n[S^1_m]}&+\|b_2\|_p^p\|[t(x_{ij})]\|^p_{S^p_n[S^1_m]}\\&\leq\,\|[T_1(x_{ij})]\|^p_{L^p(\mathcal{N}_1;S^1_m)}+\|[T_2(x_{ij})]\|^p_{L^p(\mathcal{N}_2;S^1_m)}.\end{aligned}$$

According to Lemma 5.2, the right-hand side in the above inequality coincides with $\|[T(x_{ij})]\|^p_{L^p(\mathcal{N};S^1_m)}$. Since T is assumed S^1-contractive, we infer that

$$\|b_1\|_p^p\|[x_{ij}]\|^p_{S^p_n[S^1_m]}+\|b_2\|_p^p\|[t(x_{ij})]\|^p_{S^p_n[S^1_m]}\leq\|[x_{ij}]\|^p_{S^p_n[S^1_m]}. \tag{33}$$

Using (31) and (32) again, we note that for any $x \in S_n^p$,

$$\begin{aligned}\|T(x)\|_p^p &= \|T_1(x)\|_p^p + \|T_2(x)\|_p^p \\ &= \|x \otimes b_1\|_p^p + \|t(x) \otimes b_2\|_p^p,\end{aligned}$$

and hence

$$\|T(x)\| = \|x\|_p^p \big(\|b_1\|_p^p + \|b_2\|_p^p\big). \tag{34}$$

Since T is an isometry, this implies that

$$\|b_1\|_p^p + \|b_2\|_p^p = 1.$$

Replacing $\|b_1\|_p^p$ by $(1 - \|b_2\|_p^p)$ in (33), we obtain that

$$\|b_2\|_p \|[t(x_{ij})]\|_{S_n^p[S_m^1]} \leq \|b_2\|_p \|[x_{ij}]\|_{S_n^p[S_m^1]}$$

for any $m \geq 1$ and any $[x_{ij}]_{1 \leq i,j \leq m}$ in $S_n^p[S_m^1]$. By Lemma 5.3 (ii), the above inequality holds only if $b_2 = 0$. In this case, we have $\sigma = 0$, and hence J is a $*$-homomorphism.

We now consider the general case. We let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be an isometry and assume that T is S^1-contractive. As in the first part of the proof, this implies that T has a Yeadon type factorisation. Let $J\colon \mathcal{M} \to \mathcal{N}$ be the Jordan homomorphism in the Yeadon triple of T and let $J = \begin{pmatrix} \pi & 0 \\ 0 & \sigma \end{pmatrix}$ be a decomposition of J as in (26). Let $\mathcal{M}_1 = \mathrm{Ker}(\sigma)$. Since σ is w^*-continuous, $\mathcal{M}_1$ is a w^*-closed ideal of $\mathcal{M}$. Hence we have a direct sum decomposition

$$\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2.$$

Moreover $\sigma_{|\mathcal{M}_2}$ is one-to-one. To prove that J is a $*$-homomorphism, it suffices to show that $\mathcal{M}_2$ is abelian.

If not, then by Lemma 2.3, there exists a non zero $*$-homomorphism $\gamma\colon M_2 \to \mathcal{M}_2$ taking values in $\mathcal{M}_2 \cap L^1(\mathcal{M}_2)$. Let $\tau' = \tau_{\mathcal{M}} \circ \gamma\colon M_2 \to \mathbb{C}$. Then τ' is a non zero trace on M_2 hence there exists $\delta > 0$ such that $\tau' = \delta tr_2$. This readliy implies that

$$\delta^{-\frac{1}{p}}\gamma\colon S_2^p \longrightarrow L^p(\mathcal{M}_2)$$

is an isometry. Further $\delta^{-\frac{1}{p}}\gamma$ is completely positive. Hence by Theorem 3.13, $\delta^{-\frac{1}{p}}\gamma$ is S^1-contractive. By composition, we obtain that $\delta^{-\frac{1}{p}}T \circ \gamma$ is an S^1-contractive isometry from S_2^p into $L^p(\mathcal{N})$. According to the first part of this proof, $\delta^{-\frac{1}{p}}T \circ \gamma$ has therefore a direct Yeadon type factorization. We observe that the Jordan homomorphism of its Yeadon triple is equal to $J \circ \gamma$. The latter is therefore multiplicative, hence $\sigma \circ \gamma$ is multiplicative. Since $\sigma \circ \gamma$ also is anti-multiplicative, we actually have

$$\sigma \circ \gamma(ab) = [\sigma \circ \gamma(b)][\sigma \circ \gamma(a)] = \sigma \circ \gamma(ba)$$

for any $a, b \in M_2$. However $\sigma \circ \gamma$ is one-to-one, hence the above property implies that $ab = ba$ for any $a, b \in M_2$, a contradiction. Hence $\mathcal{M}_2$ is abelian as expected, which concludes the proof. ☐

Remark 5.5. Let $1 \leq p < \infty$ and let $\mathcal{N}$ be a semifinite von Neumann algebra. The argument in the first part of the proof of Theorem 5.4 shows that for any $n \geq 1$ and for any non zero separating map $T\colon S_n^p \to L^p(\mathcal{N})$, the operator $\|T\|^{-1}T$ is an isometry. Indeed this follows from (34).

Likewise for any Hilbert space $\mathcal{H}$ and for any non zero separating map $T\colon S^p(\mathcal{H}) \to L^p(\mathcal{N})$, the operator $\|T\|^{-1}T$ is an isometry.

Theorem 5.6. *Let $1 \leq p \neq 2 < \infty$ and let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be an isometry. The following statements are equivalent.*

(i) *T admits a direct Yeadon type factorization.*
(ii) *T is completely contractive.*

Proof. The implication "$(i) \Rightarrow (ii)$" follows from Proposition 4.4 so we only need to prove "$(ii) \Rightarrow (i)$". It turns out that the proof of the similar implication in Theorem 5.4 applies for this case, up to a few changes that we now explain.

Assume first that $\mathcal{M} = M_n$, with $n \geq 2$, and consider $T_1, T_2, \rho_\pi, \rho_\sigma, b_1, b_2$ given by the proof of Theorem 5.4. By Lemma 2.1, $\rho_\pi\colon L^p(\mathcal{N}_1) \to L^p(M_n \overline{\otimes}(\varepsilon_1\mathcal{N}_1\varepsilon_1))$ and $\rho_\sigma\colon L^p(\mathcal{N}_2) \to L^p(M_n \overline{\otimes}(\varepsilon_2\mathcal{N}_2\varepsilon_2))$ are complete isometries. Further for any $m \geq 1$ and any $[x_{ij}]_{1\leq i,j\leq m}$ in $S^p_m[S^p_n]$, we have

$$\|[x_{ij} \otimes b_1\|_{L^p(M_m \overline{\otimes} M_n \overline{\otimes}(\varepsilon_1 N_1 \varepsilon_1))} = \|[x_{ij}\|_{S^p_m[S^p_n]}\|b_1\|_p,$$

by (2). Hence

$$\|b_1\|_p\|[x_{ij}]\|_{S^p_m[S^p_n]} \leq \|[T_1(x_{ij})]\|_{L^p(M_m \overline{\otimes}\mathcal{N}_1)}.$$

Similarly

$$\|b_2\|_p\|[x_{ij}]\|_{S^p_m[S^p_n]} \leq \|[T_2(x_{ij})]\|_{L^p(M_m \overline{\otimes}\mathcal{N}_2)}.$$

Moreover by (3),

$$\|[T(x_{ij})]\|^p_{L^p(M_m \overline{\otimes}\mathcal{N})} = \|[T_1(x_{ij})]\|^p_{L^p(M_m \overline{\otimes}\mathcal{N}_1)} + \|[T_2(x_{ij}]\|^p_{L^p(M_m \overline{\otimes}\mathcal{N}_2)}.$$

Then using Lemma 5.3 (i), the argument in the proof Theorem 5.4 shows that $b_2 = 0$ and hence that T has a direct Yeadon type factorization.

In the general case, the proof of Theorem 5.4 applies almost verbatim, using the simple fact that $\delta^{-\frac{1}{p}}\gamma$ is a complete isometry. $\square$

Remark 5.7. Let $n \geq 2$ and consider $T\colon S^p_n \overset{p}{\oplus} S^p_n \to S^p_n \overset{p}{\oplus} S^p_n$ defined by

$$T(x, y) = \big(x, n^{-\frac{1}{p}}t(x)\big), \qquad x, y \in S^p_n.$$

Then T is a separating map and by Lemma 5.3, we have $\|T\| = \|T\|_{S^1} = \|T\|_{cb}$. However T does not have a direct Yeadon type factorization. This shows that Theorems 5.4 and 5.6 cannot hold true if we remove the isometric assumption on T.

Remark 5.8. Let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be an isometry. The proof of Theorem 5.4 actually shows that T admits a direct Yeadon type factorization if and only if T is S^1_2-contractive, that is,

$$\big\|T \otimes I_{S^1_2}\colon L^p(\mathcal{M}; S^1_2) \longrightarrow L^p(\mathcal{N}; S^1_2)\big\| \leq 1.$$

Likewise if $p \neq 2$, the proof of Theorem 5.6 shows that T admits a direct Yeadon type factorization if and only if T is 2-contractive.

Note that Theorem 5.6 and the above remark extend [15, Proposition 3.2]. Theorem 5.4 can be regarded as a variant of the latter. Its main feature is that it also applies to $p = 2$. We emphasize this in the next statements.

Corollary 5.9. *An isometry $T\colon L^2(\mathcal{M}) \to L^2(\mathcal{N})$ admits a direct Yeadon type factorization if and only if it is S^1-contractive.*

Corollary 5.10. *Any completely positive isometry $T\colon L^2(\mathcal{M}) \to L^2(\mathcal{N})$ admits a direct Yeadon type factorization.*

Proof. This follows from Theorem 3.13 and Theorem 5.4. □

Remark 5.11. Assume here that $\mathcal{M},\mathcal{N}$ are semifinite and hyperfinite von Neumann algebras. In the case when $p\neq 2$, Theorem 5.4 follows from Theorem 5.6, by Proposition 3.11 and [27, Proposition 2.2]. Moreover the L^2-case of Theorem 5.4, and hence Corollaries 5.9 and 5.10, have a much simpler proof. Indeed under the hyperfinite assumption, suppose that $T\colon L^2(\mathcal{M})\to L^2(\mathcal{N})$ is an S^1-contractive isometry. By Proposition 3.11, T is completely regular with $\|T\|_{reg}\leq 1$. Applying (19) with the specific operator space $E=S_2^2[\mathrm{Max}(\ell_2^1)]$ we obtain that

$$\big\|T\otimes I_{S_2^2}\otimes I_{\ell_2^1}\colon L^2(\mathcal{M})\big[S_2^2[\mathrm{Max}(\ell_2^1)]\big]\longrightarrow L^2(\mathcal{N})\big[S_2^2[\mathrm{Max}(\ell_2^1)]\big]\big\|\leq 1. \tag{35}$$

According to [26, Theorem 1.9], we have a Fubini type isometric identification between $L^2(\mathcal{M})\big[S_2^2[\mathrm{Max}(\ell_2^1)]\big]$ and $L^2(M_2\overline{\otimes}\mathcal{M})[\mathrm{Max}(\ell_2^1)]$. Combining with [21, (7)], we then have

$$L^2(\mathcal{M})\big[S_2^2[\mathrm{Max}(\ell_2^1)]\big]\simeq L^2(M_2\overline{\otimes}\mathcal{M};\ell_2^1).$$

We have a similar result for $\mathcal{N}$. Consequently (35) implies that

$$I_{S_2^2}\otimes T\colon L^2(M_2\overline{\otimes}\mathcal{M})\longrightarrow L^2(M_2\overline{\otimes}\mathcal{N})$$

is ℓ_2^1-contractive. Further $L^2(M_2\overline{\otimes}\mathcal{M})$ (resp. $L^2(M_2\overline{\otimes}\mathcal{N})$) coincides with the Hilbertian tensor product of S_2^2 and $L^2(\mathcal{M})$ (resp. $L^2(\mathcal{N})$). Hence $I_{S_2^p}\otimes T$ is an isometry. It therefore follows from [21, Theorem 4.2] that $I_{S_2^2}\otimes T$ admits a Yeadon type factorization. By [11, Theorem 3.6], this implies that T admits a direct Yeadon type factorization.

Acknowledgement. The work leading to this paper started whilst the second author was visiting "Laboratoire de Mathématiques de Besançon" (LmB). She greatly acknowledges LmB for hospitality and excellent working condition. The first named author is supported by the French "Investissement d'Avenir" program, project ISITE-BFC (contract ANR-15-IDEX-03).

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Laboratoire de Mathématiques de Besançon, Universite Bourgogne Franche-Comté, France

Email address: `christian.lemerdy@univ-fcomte.fr`

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111, Bonn, Germany &
Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Germany.

Email address: `jsafoora@gmail.com`