Volumes in Hyperbolic 5-Space

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0. Introduction

Volumes of hyperbolic polytopes are of considerable interest in various branches of mathematics. In function theory, the classical polylogarithms $\mathrm{Li}_k(z)$, defined by

$$\mathrm{Li}_1(z) = -\log(1-z) \quad \text{and} \quad \mathrm{Li}_k(z) = \int_0^z \frac{\mathrm{Li}_{k-1}(t)}{t}\,dt \quad \text{for} \quad k \geq 2 \quad , \quad z \in \mathbf{C} \quad ,$$

are involved in volume computations, at least in low dimensions.
By the Rigidity Theorem of Mostow and Prasad, the volume of a hyperbolic n-manifold, $n \geq 3$, is a topological invariant; the spectrum $\mathbf{Vol_n}$ of such values is therefore an important object to study and manifests different behaviour with respect to the dimension n. For $n = 3$, by work of Jørgensen and Thurston, $\mathbf{Vol_3}$ is a non-discrete subset of $\mathbf{R}_+$ whose order type and structure around limit points are well understood; much activity is going on in search of the minimum in $\mathbf{Vol_3}$ (cf. [K2, **14.4.1**]). For $n \neq 3$, the volume spectra $\mathbf{Vol_n}$ are discrete; for example, for $n = 2m \geq 2$, the Theorem of Gauss-Bonnet says that the volume of a hyperbolic manifold is, up to constant depending only on m, given by its Euler-Poincaré characteristic. A polytopal analogue is Schläfli's Reduction Principle expressing the volume of a $(2m)$-dimensional polytope in terms of the volumes of certain lower- and odd-dimensional ones (cf. [S] and [K2, **14.2.2**]).
Furthermore, there is also a strong number theoretical aspect. Volumes of hyperbolic 3-space forms defined arithmetically over an algebraic number field F are related to Dedekind's zeta function $\zeta_F(2)$. This in combination with Lobachevskij's result expressing volumes of three-dimensional hyperbolic simplices in terms of his (dilogarithmic) function $\text{Л}_2(\omega) = -\int_0^\omega \log|2\sin t|\,dt, \omega \in \mathbf{R}$, (cf. [Lo], and see **2.2**) was the initial motivation for Zagier's Conjecture about $\zeta_F(m)$, $m \geq 2$, and certain modified polylogarithms. Although Zagier's Conjecture holds for $m = 3$, as was shown by Goncharov (see [C] for a survey and an extensive list of references), it is unclear whether there is any connection between Dedekind zeta functions at $m = 3$ and volumes of arithmetic hyperbolic 5-space forms.
First results about volumes in hyperbolic 5-space were obtained by Müller [M] in 1954 and by Böhm [B] in 1960. Using different approaches, their main result consists in showing that the trilogarithm, as a function of a single variable, suffices to express volumes of

hyperbolic five-dimensional polytopes. Both contributions, however, show some disadvantages; Böhm's result is not expressed in closed form, while Müller's work does not allow a generalization to higher dimensions.

In this paper, we present a complete solution of the volume problem in hyperbolic 5-space. We derive an explicit volume formula for generic simplices, that is, for doubly asymptotic orthoschemes; the formula is expressed in terms of trilogarithmic functions in the dihedral angles which allows both, evaluation in concrete cases and generalization to higher dimensions.

An orthoscheme R in an n-dimensional space of constant curvature is a simplex bounded by hyperplanes $H_0, \ldots, H_n$ such that

$$H_i \perp H_j \quad \text{for} \quad |i-j|>1 \quad .$$

It is, up to isometry, uniquely determined by its (at most n) non-right dihedral angles, and it is very conveniently described by means of schemes or weighted graphs (see **1.1**). For hyperbolic orthoschemes R with vertices p_i opposite to H_i, $i=0,\ldots,n$, at most p_0 and/or p_n may be points at infinity. In such cases, R is called *simply* or *doubly asymptotic*. Doubly asymptotic orthoschemes are characterized by many nice properties (see **1.2**). For example, they are parametrizable by $n+3$ points on $\mathrm{P}_1(\mathbf{R})$, and each doubly asymptotic n-orthoscheme gives rise to a cycle of $n+1$ of such ones (see Proposition 1.3, **1.2**); by forming circular graphs out of them, they allow to construct new polytopes in higher dimensional spaces (see Proposition 1.4, **1.2**). For our purpose, one of the most important properties is reflected by the result, due to Sah [Sa] and Debrunner [D] (we reproduce the proof in **1.2**), that every hyperbolic polytope of *odd* dimension can be equidissected into doubly asymptotic orthoschemes; therefore, in hyperbolic 5-space, it suffices to solve the volume problem for doubly asymptotic orthoschemes.

A fundamental tool for volume computations is Schläfli's differential formula expressing the volume differential of a simplex in terms of small angle perturbations. This beautiful result (see Theorem 1, **2.1**), combined with Lobachevskij's volume formula for hyperbolic three-dimensional orthoschemes, allows to tackle the remaining single integration in the case of doubly asymptotic 5-orthoschemes. For this, we distinguish two cases. By expressing the cycle property of a doubly asymptotic 5-orthoscheme R in terms of its dihedral angles $\alpha_i = \angle(H_{i-1}, H_i)$, $1 \le i \le 5$, we obtain the relation

$$\lambda := \cot\alpha_0 \tan\alpha_3 = \tan\alpha_1 \cot\alpha_4 = \cot\alpha_2 \tan\alpha_5 \quad ,$$

wherein the additional angle α_0 can be seen as some angle in R.

The integration of the volume differential in the case $\lambda = 1$ was already performed in [K3]; there, we make use only of the so-called Trilobachevskij function $\text{Л}_3(\omega) = \frac{1}{4}\,\mathrm{Re}(\mathrm{Li}_3(e^{2i\omega}))$, $\omega \in \mathbf{R}$ (see Theorem 2, **2.3**). With the aid of that result, all covolumes of hyperbolic Coxeter groups with linear diagrams could be computed giving a first insight into $\mathbf{Vol}_5$.

The case $\lambda \neq 1$ is much more difficult; nevertheless, Theorem 3 of **2.3** provides a volume formula for an arbitrary doubly asymptotic 5-orthoscheme in terms of trilogarithm functions in complicated arguments related to its dihedral angles. It would be interesting to know whether the formula can be simplified in terms of Trilobachevskij functions. Moreover, by means of cutting and pasting, and after analytical continuation, Theorem 3 solves also the volume problem for arbitrary non-Euclidean 5-polytopes, at least in principle. As an application, Theorem 3 allows us to compute the volume of the totally asymptotic regular simplex in hyperbolic 5-space which maximizes all volumes of hyperbolic 5-simplices (see Theorem 4, **2.4**).

This work is organized as follows: In Chapter 1, we discuss the algebro-geometrical properties of doubly asymptotic orthoschemes making use of the language of weighted graphs. Chapter 2 contains the analytical part with all volume computations. We conclude the paper with two appendices; in Appendix A, we collect some useful determinant identities. The first half of Appendix B contains a summary on polylogarithms, in particular, for orders two and three. In the second half, we show how the characteristic volume integral can be represented in terms of trilogarithmic functions.

Finally, I would like to thank Herbert Gangl for helpful discussions about trilogarithm identities.

1. The geometry of polytopes in hyperbolic space

1.1. DESCRIPTION OF POLYTOPES IN SPACES OF CONSTANT CURVATURE

Let X^n be either the sphere S^n, Euclidean space E^n, or hyperbolic space $\overline{H^n} = H^n \cup \partial H^n$ extended by the set ∂H^n of points at infinity. Represent them by their natural embedding in Y^{n+1}, where Y^{n+1} is E^{n+1} or Minkowski space $E^{n,1}$ of signature $(n,1)$.
An n-dimensional convex polytope $P \subset X^n$ is the non-empty intersection of finitely many closed half-spaces bounded by hyperplanes H_i with outer unit normal vectors $e_i \in Y^{n+1}$, $i \in I$, say. We always assume that P is indecomposable (i.e., $\{e_i\}_{i \in I}$ does not split into two mutually orthogonal subsets) and of finite volume.
To a polytope $P \subset X^n \subset Y^{n+1}$ we can associate its Gram matrix $G(P) = (< e_i, e_j >_{Y^{n+1}})_{i,j \in I}$ of the vectors $\{e_i\}_{i \in I}$. We assume from now on that P is acute-angled, which means, by abuse of language, that all non-right dihedral angles $\alpha_{ij} = \angle(H_i, H_j)$ are strictly less than $\frac{\pi}{2}$. Then P is determined by $G(P)$ in the following way (cf. [V, §2]):

Proposition 1.1.
Let $G = (g_{ij})$ be an indecomposable symmetric $m \times m$-matrix of rank $n+1$ with $g_{ii} = 1$ and $g_{ij} \leq 0$ for $i \neq j$. Then G is the Gram $G(P)$ of an acute-angled polytope $P \subset X^n$ of finite volume defined uniquely up to isometry. In particular,

(1) *if G is positive definite (elliptic), then $m = n+1$, and P is a simplex on the sphere S^n;*
(2) *if G is positive semidefinite (parabolic), then $m = n+2$, and P is a simplex in E^{n+1};*
(3) *if G is of signature $(n,1)$ (hyperbolic), then P is a convex polytope in $\overline{H^n}$ with m facets.*

If P has many right dihedral angles, then P can be better visualized through its scheme $\Sigma(P)$: In general, a scheme Σ is a weighted graph whose nodes i, j are either joined by

an edge with positive weight ω_{ij}, or i,j are disjoint with weight $\omega_{ij}=0$. The number $|\Sigma|$ of nodes is called the order of Σ. To every (connected) scheme Σ of order m corresponds an (indecomposable) symmetric matrix $A(\Sigma)$ of order m with entries $a_{ii}=1$ and $a_{ij}=-\omega_{ij}\leq 0$ for $i\neq j$. Rank, determinant and character of definiteness of Σ are defined by the corresponding data of $A(\Sigma)$. In particular, Σ is elliptic, or parabolic, or hyperbolic if either all its components are elliptic, or – beside of elliptic ones – there is at least one parabolic component, or precisely one component is hyperbolic.
The scheme $\Sigma(P)$ of an acute-angled polytope $P\subset X^n$ is the scheme whose matrix coincides with the Gram matrix $G(P)$: The nodes i correspond to the bounding hyperplanes $H_i=e_i^{\perp}$ of P and the weights equal $-<e_i,e_j>_{Y^{n+1}}$, $i,j\in I$. $\Sigma(P)$ describes P uniquely up to isometry.
As for Coxeter polytopes in X^n (all the dihedral angles look like $\frac{\pi}{p}$, $p\in\mathbf{N}$, $p\geq 2$), we join two nodes related by the weight $\cos\alpha$ ($\alpha=\frac{p\pi}{q}$, $p,q\in\mathbf{N}$ coprime with $1\leq p<q$) by a $(q-2)$-fold line for $p=1$, $q=3,4$, and by a single line marked $\alpha=\frac{q}{p}$, otherwise. If two bounding hyperplanes of $P\subset X^n$, $X^n\neq S^n$, are parallel, their nodes are connected by an edge marked ∞; if they are divergent in hyperbolic space, we join them by a dotted line discarding the weight ≥ 1.

In the following we consider acute-angled polytopes in $\overline{H^n}$. Their geometry is particularly rich since their Gram matrices can be of arbitrarily high order as long as the index of inertia is one; moreover, depending on whether a vertex is an ordinary point, or a point at infinity, or ultrainfinite (i.e., lying outside the cone in $E^{n,1}$ defining hyperbolic space and forcing its truncation to reach finite volume), the scheme of its vertex polytope is elliptic, or parabolic, or hyperbolic, encoding therefore all three geometries of constant curvature (cf. [V, §3]). For the purpose of volume computations, we restrict to appropriate families of hyperbolic polytopes, to the ones represented by simplest schemes, which, as we shall see, are simultaneously the most important ones.

1.2. DOUBLY ASYMPTOTIC ORTHOSCHEMES

The most basic and important family of polytopes are n-orthoschemes $R\subset X^n$, that is, simplices bounded by hyperplanes $H_0,\dots,H_n$ such that $H_i\perp H_j$ for $|i-j|>1$; their schemes $\Sigma(R)$ are linear of order $n+1$ with weights $\alpha_i=\angle(H_{i-1},H_i)$, $1\leq i\leq n$:

$$\Sigma(R)\qquad:\qquad \circ\overset{\alpha_1}{\text{———}}\circ\text{—}\cdots\text{—}\circ\overset{\alpha_n}{\text{———}}\circ\quad .$$

Denote by p_i the vertex of R opposite to the facet $H_i\cap R$. Its vertex polytope in R is the simplex described by the subscheme of order n of $\Sigma(R)$ arising by discarding the node i together with the edges emanating from it. In hyperbolic space, among all vertices of R, at most p_0 and p_n may be points at infinity (the only parabolic subschemes of $\Sigma(R)$ may be

$$\circ\overset{\alpha_1}{\text{———}}\circ\text{—}\cdots\text{—}\circ\overset{\alpha_{n-1}}{\text{———}}\circ\quad\text{and}\quad\circ\overset{\alpha_2}{\text{———}}\circ\text{—}\cdots\text{—}\circ\overset{\alpha_n}{\text{———}}\circ\quad).$$

The vertices p_0,p_n are called principal vertices of R, and if p_0 or/and p_n are at infinity, R is said to be *simply* or *doubly asymptotic*. Notice that $R\subset\overline{H^n}$ is always acute-angled.

The notion of orthoschemes was introduced and systematically studied by Schläfli (cf. [S]), however, in spherical space only. These simplices are generalizations of right-angled triangles and arise in a very natural manner out of general polytopes by successive dropping of perpendiculars to lower dimensional faces.
This amounts to say that the scissors congruence groups $\mathcal{P}(X^n)$ (that is, the abelian groups generated by $[P]$ for each polytope P in X^n equipped with the relations (i) $[P \sqcup Q] = [P] + [Q]$ ($\sqcup$ denotes disjoint interior union) and (ii) $[P] = [Q]$ for P isometric to Q) are generated by the classes of orthoschemes.
In the hyperbolic case, there is an isomorphism between the groups $\mathcal{P}(H^n)$ and $\mathcal{P}(\overline{H^n})$ for $n > 0$ (cf. [DSa, Theorem 2.1, p. 162]). Moreover, they are 2-divisible (cf. [Sa, p. 197]) which means that to each polytope P there exists a polytope Q such that $[P] = 2[Q]$. Another important property is the following (cf. [Sa, Prop. 3.7, p. 195], [D, Prop. 6.4, p. 142]):

Proposition 1.2.
For $n > 1$ odd, $\mathcal{P}(\overline{H^n})$ is generated by the classes of doubly asymptotic orthoschemes.

Proof:
Let $P \in \mathcal{P}(\overline{H^n})$. Then, by the 2-divisibility of $\mathcal{P}(\overline{H^n})$, there exists a polytope Q such that $[P] = 2[Q]$, and $[Q]$ can be written as algebraic sum of classes of orthoschemes.
We show that the class $[R]$ of each orthoscheme R is equal to the algebraic sum of classes of doubly asymptotic orthoschemes.
The first step is to represent $[R]$ by simply asymptotic orthoschemes. The following standard process supplies this representation (cf. [M, p. 9] and [BH, p. 191 ff]): Let $R = p_0 \cdots p_n$ be the convex hull of its vertices p_i opposite to $H_i \cap R$, $0 \leq i \leq n$. Denote by $p_i\vec{p}_0$ the ray through the edge p_ip_0 starting at p_i, by $q_0 \in \partial H^n$ the intersection point of $p_1\vec{p}_0$ with the boundary of H^n at infinity, and set $q_n := p_0$. Choose points q_{n+i} (indices modulo $n+1$) on $p_i\vec{p}_0$ such that the plane spanned by $q_0, \ldots, q_{n-1}$ is orthogonal to the line through $q_{n-1}q_n$ in q_{n-1}. Then, it is easy to check that the simplices $S_i := q_0 \cdots q_i p_{i+1} \cdots p_n$, $0 \leq i \leq n$, are simply asymptotic orthoschemes which, on the scissors congruence level, yield the equation

$$[R] = \sum_{i=0}^{n} (-1)^i \, [S_i] \quad . \tag{1}$$

Notice that $S_n = q_0 \cdots q_n$, by construction, is the simply asymptotic orthoscheme having the same (spherical) vertex figure at $q_n = p_0$ like R; therefore, S_n is given by

$$\Sigma(S_n) \quad : \quad \circ \overset{\alpha_2}{\text{---}} \circ \text{---} \cdots \text{---} \circ \overset{\alpha_n}{\text{---}} \circ \overset{\beta}{\text{---}} \circ \quad , \tag{2}$$

where the (elliptic) scheme $\circ \overset{\alpha_2}{\text{---}} \circ \text{---} \cdots \text{---} \circ \overset{\alpha_n}{\text{---}} \circ$ is also a subscheme of $\Sigma(R)$, and $0 < \beta < \frac{\pi}{2}$ is such that $\circ \overset{\alpha_3}{\text{---}} \circ \text{---} \cdots \text{---} \circ \overset{\alpha_n}{\text{---}} \circ \overset{\beta}{\text{---}} \circ$ is parabolic.
It remains to show that every simply asymptotic orthoscheme R can be written in terms of doubly asymptotic ones. For this, we dissect R into asymptotic orthoschemes in two

different ways. Let $R = p_0 \cdots p_n$ with $p_n \in \partial H^n$. Denote by q_0 the intersection point of $\vec{p_0 p_1}$ with ∂H^n. Again, choose points q_{n+i} (indices modulo $n+1$) on $\vec{p_0 p_i}$, $2 \le i \le n$, such that the hyperplane generated by $q_0, \ldots, q_{n-1}$ is orthogonal to $\vec{p_0 p_n}$, and put $q_n := p_0$. Then, the simplices $T_i := q_0 \cdots q_i p_{i+1} \cdots p_n$, $0 \le i \le n$, are asymptotic orthoschemes. In fact, apart from $T_n = q_0 \cdots q_n$ which is simply asymptotic, $T_0, \ldots, T_{n-1}$ are doubly asymptotic. Moreover, there is the following relation (cf. [D, Theorem, (i), p.127])

$$[R] = -\sum_{i=0}^{p} [T_i] + \sum_{i=p+1}^{n} [T_i] \quad , \tag{3}$$

where $p = p(R)$, $0 \le p \le n-1$, depends on the dihedral angles of R. For example, $p = 0$ if the double of the dihedral angle of R, visible as planar angle of R at p_2 in the triangle $p_0 p_1 p_2$, is still acute.
Notice that the simply asymptotic orthoschemes R and $T_n = q_0 \cdots q_n$ share the same (spherical) vertex figure at $p_0 = q_n$ implying $\Sigma(T_n) = \Sigma(S_n)$ (see (2)).
Therefore, combining the two cutting and pasting procedures (1) and (3), we obtain for a simply asymptotic orthoscheme R the relation

$$2[R] = \sum_{i=0}^{n} (-1)^i [S_i] - \sum_{i=0}^{p} [T_i] + \sum_{i=p+1}^{n} [T_i] \quad (0 \le p \le n-1) \quad . \tag{4}$$

On the right hand side, S_n and T_n are the only simply asymptotic orthoschemes; by the above remarks, they are isometric to each other. Hence, for n *odd*, we obtain the relation

$$2[R] = \sum_{i=0}^{n-1} (-1)^i [S_i] - \sum_{i=0}^{p} [T_i] + \sum_{i=p+1}^{n-1} [T_i] \quad (0 \le p \le n-1) \quad . \tag{5}$$

The application of the cutting and pasting procedures (1) and (5) to the orthoschemes dissecting the polytope $[Q]$ in $[P] = 2[Q]$ implies that $[P]$ can be represented as algebraic sum of classes of doubly asymptotic orthoschemes. Q.E.D.

Doubly asymptotic orthoschemes $R \subset \overline{H^n}$ are represented by schemes

$$\Sigma(R) \quad : \quad \circ\overset{\alpha_1}{—}\circ — \cdots — \circ\overset{\alpha_n}{—}\circ$$

with the parabolic subschemes

$$\circ\overset{\alpha_1}{—}\circ — \cdots — \circ\overset{\alpha_{n-1}}{—}\circ \quad \text{and} \quad \circ\overset{\alpha_2}{—}\circ — \cdots — \circ\overset{\alpha_n}{—}\circ \quad .$$

Define the angle $0 < \alpha_0 < \frac{\pi}{2}$ by the condition

$$\begin{aligned}\cos^2 \alpha_0 &= \det(\circ\overset{\alpha_1}{—}\circ — \cdots — \circ\overset{\alpha_{n-2}}{—}\circ) \,/\, \det(\circ\overset{\alpha_2}{—}\circ — \cdots — \circ\overset{\alpha_{n-2}}{—}\circ) \\ &= 1 - \frac{\cos^2 \alpha_1|}{|1} - \cdots - \frac{\cos^2 \alpha_{n-2}|}{|1} \quad ,\end{aligned}$$

using Pringsheim's notation for continued fractions. This condition is, by (A1) of Appendix A, equivalent to the parabolicity of $\circ\!\overset{\alpha_0}{\text{---}}\!\circ\!-\cdots-\!\circ\!\overset{\alpha_{n-2}}{\text{-----}}\!\circ$.

Proposition 1.3.

The schemes $\Sigma_i\ :\ \circ\!\overset{\alpha_i}{\text{---}}\!\circ\!-\cdots-\!\circ\!\overset{\alpha_{n+i-1}}{\text{-----}}\!\circ$, $i\in\mathbf{Z}$ *modulo* $n+1$, *form a cycle of* $n+1$ *doubly asymptotic orthoschemes in* $\overline{H^n}$ *wherein two neighbours* Σ_i, Σ_{i+1} *have a principal vertex in common. Moreover,* $\det\Sigma_i = \det\Sigma_j$ *for* $i,j\in\mathbf{Z}$ *modulo* $n+1$.

Proof:
We prove the periodicity of the weights, that means, $\alpha_{n+i+1}=\alpha_i$, and the determinant property for $i=j'-1=j''-2=0$, only.
Let $0<\alpha_{n+1}<\frac{\pi}{2}$ be such that $\circ\!\overset{\alpha_3}{\text{---}}\!\circ\!-\cdots-\!\circ\!\overset{\alpha_{n+1}}{\text{-----}}\!\circ$ is parabolic. By (A1), this means that

$$\cos^2\alpha_{n+1} = \det(\circ\!\overset{\alpha_3}{\text{---}}\!\circ\!-\cdots-\!\circ\!\overset{\alpha_n}{\text{---}}\!\circ)\,/\,\det(\circ\!\overset{\alpha_3}{\text{---}}\!\circ\!-\cdots-\!\circ\!\overset{\alpha_{n-1}}{\text{-----}}\!\circ) \quad .$$

We have to show that $\alpha_{n+1}=\alpha_0$. For this, look at the extended schemes

$$\begin{aligned}\Sigma' &: \circ\!\overset{\alpha_0}{\text{---}}\!\circ\!\overset{\alpha_1}{\text{---}}\!\circ\!-\cdots-\!\circ\!\overset{\alpha_{n-1}}{\text{-----}}\!\circ\!\overset{\alpha_n}{\text{---}}\!\circ \quad \text{and}\\ \Sigma'' &: \circ\!\overset{\alpha_1}{\text{---}}\!\circ\!\overset{\alpha_2}{\text{---}}\!\circ\!-\cdots-\!\circ\!\overset{\alpha_n}{\text{---}}\!\circ\!\overset{\alpha_{n+1}}{\text{-----}}\!\circ \quad ,\end{aligned}$$

whose determinants equal, by parabolicity,

$$\begin{aligned}\det\Sigma' &= \det\Sigma_1 = \det\Sigma_0 = \det(\circ\!\overset{\alpha_0}{\text{---}}\!\circ\!-\cdots-\!\circ\!\overset{\alpha_{n-1}}{\text{-----}}\!\circ) \quad ,\\ \det\Sigma'' &= \det\Sigma_1 = \det\Sigma_2 = \det(\circ\!\overset{\alpha_2}{\text{---}}\!\circ\!-\cdots-\!\circ\!\overset{\alpha_{n+1}}{\text{-----}}\!\circ) \quad .\end{aligned}$$

Therefore, $\det\Sigma' = \det\Sigma'' = \det\Sigma_0 = \det\Sigma_1 = \det\Sigma_2$ which implies that

$$\begin{aligned}\det\Sigma_1 &= -\cos^2\alpha_0\ \det(\circ\!\overset{\alpha_2}{\text{---}}\!\circ\!-\cdots-\!\circ\!\overset{\alpha_{n-1}}{\text{-----}}\!\circ)\\ &= -\cos^2\alpha_{n+1}\ \det(\circ\!\overset{\alpha_2}{\text{---}}\!\circ\!-\cdots-\!\circ\!\overset{\alpha_{n-1}}{\text{-----}}\!\circ) \quad .\end{aligned}$$

Hence, $\alpha_0=\alpha_{n+1}$. Since all Σ_i, $i\in\mathbf{Z}$ modulo $n+1$, have (equal) negative determinant, two parabolic subschemes of rank $n-1$ and elliptic subschemes of order less or equal to n, Σ_i are of signature $(n,1)$ and, being linear, describe therefore doubly asymptotic n-orthoschemes. Q.E.D.

Remarks.

(i) The hyperbolic orthoscheme cycle of Proposition 1.3 is a by-product of Schläfli's generalization of Napier's rule (embodied in the Pentagramma Mirificum of Gauss) for *spherical* triangles (cf. [S, p. 259-260], [C1], [C2], [IH]):
Start with a *spherical* $(n-2)$-orthoscheme $\circ\!\overset{\alpha_2}{\text{---}}\!\circ\!-\cdots-\!\circ\!\overset{\alpha_{n-1}}{\text{-----}}\!\circ$, bounded by hyperplanes $H_1,\dots,H_{n-1}$, say. Denote by $p_1,\dots,p_{n-1}$ its vertices and by H_0, H_n the

polar hyperplanes of the principal vertices p_1, p_{n-1}. By polarity, the additional positive weights are given by (use the linear dependence of $H_i, \ldots, H_{n+i-1}$, i modulo $n+1$)

$$
\begin{aligned}
\cos^2\alpha_n &= \cos^2(\angle(H_{n-1},H_n)) = \frac{\det(\circ\overset{\alpha_2}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n-1}}{\text{——}}\circ)}{\det(\circ\overset{\alpha_2}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n-2}}{\text{——}}\circ)} \quad , \\
\cos^2\alpha_0 &= \cos^2(\angle(H_n,H_0)) = \frac{\det(\circ\overset{\alpha_3}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n}}{\text{——}}\circ)}{\det(\circ\overset{\alpha_3}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n-1}}{\text{——}}\circ)} \quad , \\
\cos^2\alpha_1 &= \cos^2(\angle(H_0,H_1)) = \frac{\det(\circ\overset{\alpha_2}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n-1}}{\text{——}}\circ)}{\det(\circ\overset{\alpha_3}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n-2}}{\text{——}}\circ)} \quad .
\end{aligned}
\tag{6}
$$

Analogously to the two-dimensional case, and by induction, the angles α_n, α_0, α_1 can easily be seen as edge lengths in the spherical orthoscheme

$$
\circ\overset{\alpha_2}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n-1}}{\text{——}}\circ \quad .
$$

More precisely,

$$
\begin{aligned}
\alpha_n &= \angle(H_{n-1},H_n) = \frac{\pi}{2} - l(p_{n-2},p_{n-1}) \quad , \\
\alpha_0 &= \angle(H_n,H_0) = l(p_{n-1},p_1) \quad , \\
\alpha_1 &= \angle(H_0,H_1) = \frac{\pi}{2} - l(p_1,p_2) \quad ,
\end{aligned}
$$

where $l(p_i,p_j)$ denotes the length of the edge p_ip_j.
Finally, $H_0, \ldots, H_n$ form a cycle wherein non-consecutive hyperplanes, by polarity, are mutually perpendicular, and any consecutive set of $n-1$ hyperplanes bound a spherical $(n-2)$-orthoscheme (for $n=4$, this is Gauss' Pentagramma Mirificum).

(ii) Let $\Sigma(R) \; : \; \circ\overset{\alpha_1}{\text{——}}\circ-\cdots-\circ\overset{\alpha_n}{\text{——}}\circ$ be a doubly asymptotic n-orthoscheme. Then, $\Sigma(R)$ is (up to congruence) uniquely determined by $n-2$ of its n dihedral angles and can be parametrized through $\mathrm{P}_1(\mathbf{R})$ in the following way (cf. [S, Nr. 27, p. 256ff], [C2, §3]): Let $c_i := \cos^2\alpha_i$, $1 \le i \le n-2$. Set, for $k = 0, 1, \ldots$,

$$
\begin{aligned}
c_{n-1+k} := \cos^2\alpha_{n-1+k} &= \frac{\det(\circ\overset{\alpha_{k+1}}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n+k-2}}{\text{——}}\circ)}{\det(\circ\overset{\alpha_{k+1}}{\text{——}}\circ-\cdots-\circ\overset{\alpha_{n+k-3}}{\text{——}}\circ)} \\
&= 1 - \frac{c_{n-2+k}|}{|1} - \cdots - \frac{c_{k+1}|}{|1} \quad .
\end{aligned}
\tag{7}
$$

By Proposition 1.3, the sequence $\{c_l\}_{l\ge1}$ has period $n+1$, that is, $c_{n+2} = c_1$. So, let x_0, x_1, x_2 be three distinct points on $\mathrm{P}_1(\mathbf{R})$ and choose $n-2$ further points $x_3, \ldots, x_{n+1}$ such that their cross-ratios give

$$
\{x_{l-1}, x_{l+2}; x_l, x_{l+1}\} = \frac{x_{l-1}-x_l}{x_{l+2}-x_l} : \frac{x_{l-1}-x_{l+1}}{x_{l+2}-x_{l+1}} = c_l \quad , \quad l = 1, \ldots, n-2 \quad . \tag{8}
$$

Combining some properties of cross-ratios one can check that

$$\{x_{n-2}, x_{n+1}; x_{n-1}, x_n\} = 1 - \frac{c_{n-2}|}{1|} - \cdots - \frac{c_1|}{1|} \quad .$$

Hence, $\{x_{n-2}, x_{n+1}; x_{n-1}, x_n\} = c_{n-1}$ (see (7)). Therefore, by identifying $x_{l+n+2} = x_l$, (8) holds for all $l = 1, \dots, n$ and we have a cycle of $n+1$ points on $P_1(\mathbf{R})$ parametrizing the Napier cycle of Proposition 1.3.

Starting with a Napier cycle of doubly asymptotic orthoschemes in $\overline{H^m}$, we can construct new families of polytopes in $\overline{H^{lm+k}}$. Look at full periods of doubly asymptotic orthoschemes in $\overline{H^{2n+1}}$, $n \geq 1$, which split into identical halves (cf. [S, No. 28, p. 261 ff]); that is, their (extended) schemes (of order $2n+3$) are of the form

$$\begin{array}{lll} \Sigma_0^{2n+3} & : & \circ\overset{\alpha_0}{—}\circ—\cdots—\circ\overset{\alpha_n}{—}\circ\overset{\alpha_0}{—}\circ\overset{\alpha_1}{—}\circ—\cdots—\circ\overset{\alpha_n}{—}\circ \quad \text{such that} \\ \Sigma_i^{2n+1} & : & \circ\overset{\alpha_i}{—}\circ—\cdots—\circ\overset{\alpha_n}{—}\circ\overset{\alpha_0}{—}\circ\overset{\alpha_1}{—}\circ—\cdots—\circ\overset{\alpha_{n-2+i}}{—}\circ \quad , \end{array} \tag{9}$$

for $i = 0, 1, 2$, are parabolic subschemes. By (A2), this means that

$$\begin{aligned} &\det\Sigma_0^n \cdot \{\det\Sigma_0^{n+1} - \cos^2\alpha_n \det\Sigma_1^{n-1}\} = \\ &\quad \det(\circ\overset{\alpha_0}{—}\circ—\cdots—\circ\overset{\alpha_{n-2}}{—}\circ)\cdot\{\det(\circ\overset{\alpha_0}{—}\circ—\cdots—\circ\overset{\alpha_{n-1}}{—}\circ) - \\ &\quad -\cos^2\alpha_n \det(\circ\overset{\alpha_1}{—}\circ—\cdots—\circ\overset{\alpha_{n-2}}{—}\circ)\} = 0 \quad . \end{aligned} \tag{10}$$

Since Σ_0^n is elliptic, we obtain:

$$\begin{aligned} \cos^2\alpha_n &= \frac{\det\Sigma_0^{n+1}}{\det\Sigma_1^{n-1}} = \frac{\det\Sigma_0^n \cdot \det\Sigma_0^{n+1}}{\det\Sigma_1^{n-1}\cdot\det\Sigma_0^n} \\ &= \{1 - \frac{\cos^2\alpha_0|}{|1} - \cdots - \frac{\cos^2\alpha_{n-1}|}{|1}\}\cdot\{1 - \frac{\cos^2\alpha_{n-1}|}{|1} - \cdots - \frac{\cos^2\alpha_0|}{|1}\} \end{aligned}$$

and cyclic permutations of it. Apart from the $n+1$ (usually) different doubly asymptotic $(2n+1)$-orthoschemes Σ_i^{2n+2}, $0 \leq i \leq n$, in the Napier cycle, we can construct the following hyperbolic polytopes:

Proposition 1.4.
Let $m, n \in \mathrm{N}$ such that $(m,n) \neq (1,1)$. Suppose Σ_0^{2n+3} to be as in (9), and denote by Ω_n^m the cyclic scheme of m repetitions of Σ_0^{n+2} : $\circ\overset{\alpha_0}{—}\circ—\cdots—\circ\overset{\alpha_n}{—}\circ$. Then, Ω_n^m is hyperbolic and of finite volume for $m = 1, 2, 4$ and arbitrary $n \geq 1$. In particular,
(a) *Ω_n^1 describes a compact simplex in H^n;*
(b) *Ω_n^2 describes a totally asymptotic simplex in $\overline{H^{2n+1}}$;*
(c) *Ω_n^4 describes a doubly truncated orthoscheme in $\overline{H^{4n+1}}$.*

Proof:
First, we compute the determinants $\det\Omega_n^m$ for $m, n \geq 1$, $(m,n) \neq (1,1)$. By Lemma A, Appendix A, we obtain

$$\det\Omega_n^m = \begin{cases} 0 & \text{for } m \equiv 0(4) \quad ; \\ -2\prod_{i=0}^{n} \cos^m \alpha_i & \text{for } m \equiv 1,3(4) \quad ; \\ -4\prod_{i=0}^{n} \cos^m \alpha_i & \text{for } m \equiv 2(4) \quad . \end{cases}$$

Ad (a): Suppose that $n > 1$. Since Ω_n^1 is of order $n+1$ and contains elliptic subschemes of order n, $\det\Omega_n^1 < 0$ implies that $\text{sign}\Omega_n^1 = (n,1)$. Hence, by **1.1**, Ω_n^1 is a compact hyperbolic n-simplex.
Ad (b): Ω_n^2 is of order $2(n+1)$ with elliptic subschemes of order $2n$. Again, $\det\Omega_n^2 < 0$ guarantees that $\text{sign}\Omega_n^2 = (2n+1,1)$. Therefore, Ω_n^2 yields a non-compact $(2n+1)$-simplex of finite volume *all* of whose vertices are at infinity.
Ad (c): Here, $\det\Omega_n^4 = 0$ for $n \geq 1$. Ω_n^4 is of order $4n+4$ and contains, by discarding any two antipodal nodes, two (disjoint) parabolic subschemes of rank $2n$, each; by discarding two non-adjacent non-antipodal nodes in Ω_n^4, we are left with one hyperbolic and one elliptic subscheme whose signatures add up to $(4n+1,1)$. Together with $\det\Omega_n^4 = 0$ this implies that $\text{sign}\Omega_n^4 = (4n+1,1)$. It is easy to see that Ω_n^4 is an orthoscheme in $\overline{H^{4n+1}}$ whose ultrainfinite principal vertices are cut off by means of their polar hyperplanes (cf. [IH, 3., p. 530]; see also Remark (i)).

It remains to show that, for $m = 3$ and $m \geq 5$, $n \geq 1$, Ω_n^m cannot describe a hyperbolic polytope of finite volume.
Let $m = 3$. Since Ω_n^3 is of order $3(n+1)$, has negative determinant and contains elliptic subschemes of order $3n+1$, we obtain $\text{sign}\Omega_n^3 = (3n+2,1)$. Therefore, Ω_n^3 is a $(3n+2)$-simplex containing an open subset of $\overline{H^{3n+2}}$; but it is of infinite volume since all its vertices are ultrainfinite (described by hyperbolic subschemes of order $3n+2$).
For $m \geq 5$, Ω_n^m is superhyperbolic, that means, of index of inertia bigger than one. To prove this, we look first at the case $m = 5$. It is easy to see that Ω_n^5 contains the two disjoint subschemes

$$\circ\overset{\alpha_0}{\text{---}}\circ\text{---}\cdots\text{---}\circ\overset{\alpha_n}{\text{---}}\circ\overset{\alpha_0}{\text{---}}\circ\text{---}\cdots\text{---}\circ\overset{\alpha_{n-1}}{\text{------}}\circ \quad \text{and}$$

$$\circ\overset{\alpha_0}{\text{---}}\circ\text{---}\cdots\text{---}\circ\overset{\alpha_n}{\text{---}}\circ\overset{\alpha_0}{\text{---}}\circ\text{---}\cdots\text{---}\circ\overset{\alpha_n}{\text{---}}\circ \quad ,$$

which are both hyperbolic of signature $(2n+1,1)$. Therefore, Ω_n^5 contains a subscheme of signature $(4n+2,2)$. The same reasoning works also for $m > 5$. Q.E.D.

1.3. THE TOTALLY ASYMPTOTIC REGULAR SIMPLEX

A regular simplex $S_{reg}(2\alpha) \subset \overline{H^n}$, $n \geq 2$, with dihedral angles 2α satisfying $\frac{1}{n} < \cos(2\alpha) \leq \frac{1}{n-1}$ can be dissected into orthoschemes; by drawing perpendiculars starting from its center or from a vertex, $S_{reg}(2\alpha)$ admits the subdivisions

$$[\,S_{reg}(2\alpha)\,] = (n+1)!\,[\,\sigma_{n+1}\,] = n!\,[\,\nu_{n+1}\,] \quad ,$$

where the simplices σ_{n+1} and ν_{n+1} are defined by the schemes

$$\sigma_{n+1} \quad : \quad \circ\overset{\alpha}{\text{———}}\circ-\circ-\cdots-\circ-\circ-\circ \quad ;$$
$$\nu_{n+1} \quad : \quad \circ\overset{2\alpha}{\text{———}}\circ\overset{\alpha}{\text{———}}\circ-\circ-\cdots-\circ-\circ-\circ \quad .$$

If $S^{\infty}_{reg}(2\alpha)$ is the totally asymptotic regular n-simplex, that is, $\cos(2\alpha)=\frac{1}{n-1}$, then σ_{n+1} is simply asymptotic, and ν_{n+1} is doubly asymptotic. Put $\nu^0_{n+1} := \nu_{n+1}$ and define, for $0 < i \leq [\frac{n-1}{2}]$,

$$\nu^i_{n+1} \quad : \quad \underset{0}{\circ}-\circ-\cdots-\circ-\circ\overset{\alpha}{\text{———}}\underset{i}{\circ}\overset{2\alpha}{\text{———}}\circ\overset{\alpha}{\text{———}}\circ-\circ-\cdots-\circ-\circ-\underset{n}{\circ} \quad .$$

Then, $\nu^i_{n+1} = \nu^{n-1-i}_{n+1}$, and there are the following identities:

$$[\,\nu^i_{n+1}\,] = \binom{n+1}{i+1}[\,\sigma_{n+1}\,] \qquad \text{for } i = 0,1,\ldots,\left[\frac{n-1}{2}\right] \quad .$$

These relations are consequences of different dissections of $S_{reg}(2\alpha)$ as Schläfli observed for their spherical analogues (cf. [S, I., p. 271] and [D, (7.4), p. 147]). On the other hand, in the hyperbolic totally asymptotic case, the dissecting doubly asymptotic orthoschemes ν^i_{n+1}, $i=0,\ldots,n-1$, belong to a Napier cycle (see Proposition 1.3). For the remaining Napier neighbor

$$\nu^n_{n+1} \quad : \quad \circ\overset{\alpha}{\text{———}}\circ-\circ-\cdots-\circ-\circ\overset{\alpha}{\text{———}}\circ \quad ,$$

however, it is an open problem whether there is a scissors congruence relation connecting $[\nu^n_{n+1}]$ to $[\nu^i_{n+1}], i=0,\ldots,n-1$.

The regular simplex and its volume play an important role in many branches of mathematics. Its importance stems also from the extremality property that the volume of a hyperbolic simplex is maximal if and only if it is regular and totally asymptotic (see **2.4**).

2. Volumes of doubly asymptotic 5-orthoschemes

2.1. SCHLÄFLI'S VOLUME DIFFERENTIAL FORMULA

Our aim is to derive an explicit formula for the volume of a generic hyperbolic 5-simplex. For this, by means of Proposition 1.2, it is sufficient to consider doubly asymptotic orthoschemes. We shall make use of Schläfli's formula expressing the volume differential of a polytope in terms of infinitesimal angle perturbations. In its hyperbolic form, it says (cf. [K1, §2, p. 549]):

Theorem 1.

Let $n \geq 2$, and denote by Π^n_κ the set of all acute-angled hyperbolic n-polytopes P of combinatorial type κ, with dihedral angles α_j, $j \in J$, attached at the codimension two faces F_j of P. Then, the differential of vol_n on Π^n_κ can be represented by

$$d\,\mathrm{vol}_n(P) = \frac{1}{n-1} \sum_{j\in J} \mathrm{vol}_{n-2}(F_j)\, d\,\alpha_j \quad , \quad \mathrm{vol}_0(F_j) := 1 \quad . \tag{11}$$

That the congruence class of an acute-angled hyperbolic polytope is uniquely determined by its dihedral angles was shown by Andreev (cf. [K1, §2, p. 549]) and follows from Proposition 1.1. Hence, volume is expressible in terms of dihedral angles, and to achieve this, by Theorem 1, we have to perform a single integration. Notice that, for $n = 3$, (11) makes sense for families Π^n_κ of compact type, only.

2.2. THE THREE DIMENSIONAL CASE

In order to integrate Schläfli's differential (11) on Π^5_κ, we have to express the volume coefficients in terms of dihedral angles. This can be achieved by means of Lobachevskij's formula for hyperbolic 3-orthoschemes (cf. [Lo]): Let R denote a hyperbolic 3-orthoscheme,

$$\Sigma(R) \quad : \quad \circ\overset{\alpha_1}{\text{—}}\circ\overset{\alpha_2}{\text{—}}\circ\overset{\alpha_3}{\text{—}}\circ \quad .$$

Since $\det\Sigma(R) < 0$, we obtain the realization condition $\cos\alpha_2 > \sin\alpha_1 \sin\alpha_3$ for R. It is very convenient to introduce an additional angle $0 \le \theta \le \frac{\pi}{2}$ defined by

$$\tan^2\theta = \frac{|\det\Sigma(R)|}{\cos^2\alpha_1 \cos^2\alpha_3} = \frac{\cos^2\alpha_2 - \sin^2\alpha_1 \sin^2\alpha_3}{\cos^2\alpha_1\cos^2\alpha_3} \quad ,$$

the so-called principal parameter of R. In terms of the imaginary part of Euler's Dilogarithm $\mathrm{Li}_2(z) = \sum\limits_{r=1}^{\infty} \frac{z^r}{r^2}$, $|z| \le 1$,(see Appendix B1)

$$Л_2(\omega) = \frac{1}{2}\,\mathrm{Im}\,(\mathrm{Li}_2(e^{2i\omega})) = \frac{1}{2}\sum_{r=1}^{\infty} \frac{\sin(2r\omega)}{r^2} \quad ,$$

Lobachevskij derived the formula

$$\begin{aligned} \mathrm{vol}_3(R) = \frac{1}{4}\,\{&Л_2(\alpha_1+\theta) - Л_2(\alpha_1-\theta) + Л_2(\frac{\pi}{2}+\alpha_2-\theta) + Л_2(\frac{\pi}{2}-\alpha_2-\theta) + \\ &+ Л_2(\alpha_3+\theta) - Л_2(\alpha_3-\theta) + 2\,Л_2(\frac{\pi}{2}-\theta)\,\} \quad . \end{aligned} \tag{12}$$

This volume formula for a hyperbolic 3-orthoscheme $R = p_0p_1p_2p_3$ is invariant with respect to polar truncation of an ultrainfinite principal vertex p_0 or p_3 of R as long as the line through its (longest) hypotenuse p_0p_3 is hyperbolic; otherwise, (12) has to be slightly modified (cf. [K1]).

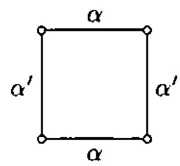
By means of (12) and obvious dissections, we obtain further results. For example:

For a doubly asymptotic orthoscheme

$$\Sigma(R) \quad : \quad \circ\overset{\alpha}{\text{—}}\circ\overset{\alpha'}{\text{—}}\circ\overset{\alpha}{\text{—}}\circ \quad , \quad \alpha' := \frac{\pi}{2} - \alpha \quad : \quad \mathrm{vol}_3(R) = \frac{1}{2}\,Л_2(\alpha) \quad . \tag{13}$$

Therefore, since $\mathcal{P}(\overline{H^3})$ is generated by doubly asymptotic orthoschemes, the Lobachevskij function $\text{Л}_2(\omega)$, $\omega \in \mathbf{R}$, is characterizable as hyperbolic 3-volume.
For a totally asymptotic simplex Q with scheme $\Sigma(Q) = \Omega_1^2$ (see Proposition 1.4):

$\Sigma(Q)$:



$$: \quad \mathrm{vol}_3(Q) = \text{Л}_2(\alpha) + \text{Л}_2(\alpha') \quad . \tag{14}$$

For a general totally asymptotic simplex $S^\infty(\alpha,\beta,\gamma)$, $\alpha+\beta+\gamma=\pi$:

$$\mathrm{vol}_3(S^\infty(\alpha,\beta,\gamma)) = \text{Л}_2(\alpha) + \text{Л}_2(\beta) + \text{Л}_2(\gamma) \quad . \tag{15}$$

In particular, for the regular simplex $S^\infty_{reg}(\frac{\pi}{3})$, we get $\mathrm{vol}_3(S^\infty_{reg}(\frac{\pi}{3})) = 3\,\text{Л}_2(\frac{\pi}{3}) \simeq 1.0149$.

2.3. THE VOLUME FORMULA

Let $R \subset \overline{H^5}$ denote a doubly asymptotic orthoscheme represented by

$$\Sigma(R) \quad : \quad \circ \overset{\alpha_1}{—} \circ \overset{\alpha_2}{—} \circ \overset{\alpha_3}{—} \circ \overset{\alpha_4}{—} \circ \overset{\alpha_5}{—} \circ \quad .$$

By Proposition 1.3, $R =: R_1$ is part of a 6-cycle of doubly asymptotic orthoschemes R_i, $i = 0,\dots,5$,

$$\Sigma(R_i) \quad : \quad \circ \overset{\alpha_i}{—} \circ \overset{\alpha_{i+1}}{—} \circ \overset{\alpha_{i+2}}{—} \circ \overset{\alpha_{i+3}}{—} \circ \overset{\alpha_{i+4}}{—} \circ \quad , \quad i \in \mathbf{Z} \text{ modulo } 6 \quad .$$

This cycle property can be put into the following analytical form (cf. also [K3, Lemma, p. 652]):

Proposition 2.1.

The Napier cycle of doubly asymptotic 5-orthoschemes R_i with graphs

$$\Sigma(R_i) \quad : \quad \circ \overset{\alpha_i}{—} \circ \overset{\alpha_{i+1}}{—} \circ \overset{\alpha_{i+2}}{—} \circ \overset{\alpha_{i+3}}{—} \circ \overset{\alpha_{i+4}}{—} \circ \quad , \quad i \in \mathbf{Z} \text{ modulo } 6 \quad ,$$

associated to $R = R_1$ satisfies the relation

$$\cot\alpha_0 \tan\alpha_3 = \tan\alpha_1 \cot\alpha_4 = \cot\alpha_2 \tan\alpha_5 = \tan\Theta \quad , \tag{16}$$

where $0 \le \Theta \le \frac{\pi}{2}$ is the angle given by

$$\tan^2\Theta = \frac{|\det\Sigma(R)|}{\cos^2\alpha_1 \cos^2\alpha_3 \cos^2\alpha_5} \quad .$$

Proof:
First, we remark that the parabolicity condition for R_i yields

$$\det\Sigma(R_i) = -\cos^2\alpha_{i-2}\det(\circ\overset{\alpha_i}{\text{——}}\circ\overset{\alpha_{i+1}}{\text{——}}\circ\overset{\alpha_{i+2}}{\text{——}}\circ) \quad , \quad i\in\mathbf{Z} \text{ modulo } 6 \quad .$$

Moreover, by Proposition 1.3, $\det\Sigma(R_i) = \det\Sigma(R_j)$ for $i,j\in\mathbf{Z}$ modulo 6, $i\neq j$. Therefore,

$$\Theta = \theta_{2i+1} = \frac{\pi}{2} - \theta_{2i} \quad , \tag{17}$$

where $0\leq\theta_i\leq\frac{\pi}{2}$ is defined by

$$\tan^2\theta_i = \frac{\det(\circ\overset{\alpha_i}{\text{——}}\circ\overset{\alpha_{i+1}}{\text{——}}\circ\overset{\alpha_{i+2}}{\text{——}}\circ)}{\cos^2\alpha_i\cos^2\alpha_{i+2}} \quad , \quad i,j\in\mathbf{Z} \text{ modulo } 6 \quad .$$

The relations (16) follow now from properties of the Napier cycle for spherical 3-orthoschemes (see **1.2**, Remark (i)) and (17): The length l_i of the edge where the dihedral angle α_i sits equals $\frac{\pi}{2}-\alpha_{i+3}$, and, by **1.2**, (6), there is the correspondence

$$\tan l_i = \cot\alpha_{i+3} = \tan\theta_i\cot\alpha_i \quad , \quad i,j\in\mathbf{Z} \text{ modulo } 6 \quad . \tag{18}$$

Q.E.D.

Notice that for $\Theta = 0$, that is, $\det\Sigma(R) = 0$, R is degenerated in dimension implying $\text{vol}_5(R) = 0$.

Let R be the convex hull of p_i (see Figure 1), which are opposite to the bounding hyperplanes H_i, $0\leq i\leq 5$, as usually; denote by F_i the apex face of R associated to α_i, $1\leq i\leq 5$. Notice that the angle α_0 (see (18) and **1.2**, Remark (i)) can be seen as dihedral angle of R; more precisely, $\alpha_0' = \frac{\pi}{2}-\alpha_0 = p_0p_4p_1 = p_4p_1p_5$ (see Figure 1).

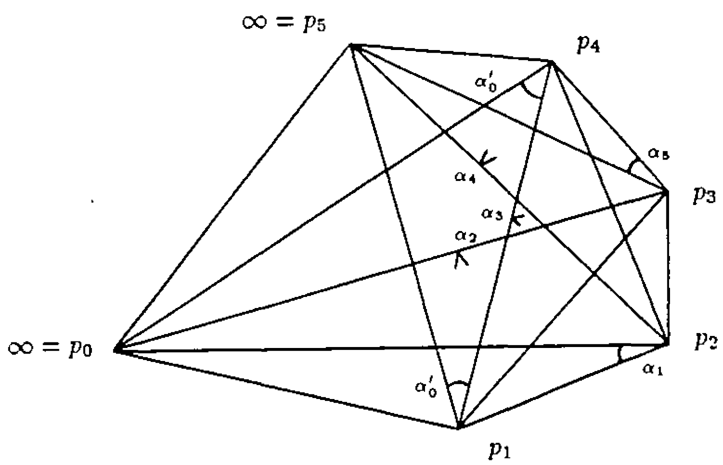


Fig. 1

By Theorem 1, the volume differential $d\mathrm{vol}_5(R)$ of R takes the form

$$(-4)\, d\mathrm{vol}_5(R) = \sum_{j=1}^{5} \mathrm{vol}_3(F_j)\, d\alpha_j \quad .$$

Notice that $\alpha_1, \ldots, \alpha_5$ are not independent parameters of R, wherefore the coefficients $\mathrm{vol}_3(F_j)$ are not all partial derivatives $\dfrac{\partial \mathrm{vol}_5(R)}{\partial \alpha_j}$. Since p_0, p_5 are vertices of R at infinity, the faces

$$F_j = R \cap H_{j-1} \cap H_j = p_0 \cdots p_{j-2} \widehat{p_{j-1}} \widehat{p}_j p_{j+1} \cdots p_5 \quad , \quad 1 \le j \le 5 \quad ,$$

are asymptotic 3-orthoschemes with the schemes

$$\begin{aligned}
\Sigma(F_1) \quad &: \quad \circ \overset{\alpha_4'}{\text{---}} \circ \overset{\alpha_4}{\text{---}} \circ \overset{\alpha_5}{\text{---}} \circ \quad , \\
\Sigma(F_2) \quad &: \quad \circ \overset{\alpha_5}{\text{---}} \circ \overset{\alpha_5'}{\text{---}} \circ \overset{\alpha_5}{\text{---}} \circ \quad , \\
\Sigma(F_3) \quad &: \quad \circ \overset{\alpha_0'}{\text{---}} \circ \overset{\alpha_0}{\text{---}} \circ \overset{\alpha_0'}{\text{---}} \circ \quad , \\
\Sigma(F_4) \quad &: \quad \circ \overset{\alpha_1}{\text{---}} \circ \overset{\alpha_1'}{\text{---}} \circ \overset{\alpha_1}{\text{---}} \circ \quad , \\
\Sigma(F_5) \quad &: \quad \circ \overset{\alpha_1}{\text{---}} \circ \overset{\alpha_2}{\text{---}} \circ \overset{\alpha_2'}{\text{---}} \circ \quad .
\end{aligned}$$

Their volumes can be computed by Lobachevskij's formula (12) in the following way:

$$\begin{aligned}
&\mathrm{vol}_3(F_1) = \frac{1}{4}\,\{\, Л_2(\frac{\pi}{2} - \alpha_4 + \alpha_5) - Л_2(\frac{\pi}{2} + \alpha_4 + \alpha_5) + 2Л_2(\alpha_4) \,\} \quad ; \\
&\mathrm{vol}_3(F_2) = \frac{1}{2} Л_2(\alpha_5) \quad ; \quad \mathrm{vol}_3(F_3) = \frac{1}{2} Л_2(\alpha_0') \quad ; \quad \mathrm{vol}_3(F_4) = \frac{1}{2} Л_2(\alpha_1) \quad ; \qquad (19) \\
&\mathrm{vol}_3(F_5) = \frac{1}{4}\,\{\, Л_2(\frac{\pi}{2} - \alpha_2 + \alpha_1) - Л_2(\frac{\pi}{2} + \alpha_2 + \alpha_1) + 2Л_2(\alpha_2) \,\} \quad ,
\end{aligned}$$

with the dependences (see Proposition 2.1)

$$\lambda := \tan \Theta = \cot \alpha_0 \tan \alpha_3 = \tan \alpha_1 \cot \alpha_4 = \cot \alpha_2 \tan \alpha_5 \quad . \qquad (20)$$

Hence, we need to integrate the differential

$$\begin{aligned}
(-8)\, d\mathrm{vol}_5(R) = &- \frac{1}{2}\,\{\, Л_2(\frac{\pi}{2} - \alpha_5 + \alpha_4) + Л_2(\frac{\pi}{2} + \alpha_5 + \alpha_4) \,\}\, d\alpha_1 + Л_2(\alpha_4)\, d\alpha_1 + \\
&+ Л_2(\alpha_5)\, d\alpha_2 + Л_2(\alpha_0')\, d\alpha_3 + Л_2(\alpha_1)\, d\alpha_4 - \qquad (21) \\
&- \frac{1}{2}\,\{\, Л_2(\frac{\pi}{2} - \alpha_1 + \alpha_2) + Л_2(\frac{\pi}{2} + \alpha_1 + \alpha_2) \,\}\, d\alpha_5 + Л_2(\alpha_2)\, d\alpha_5
\end{aligned}$$

subject to the relation (20), that is,

$$\begin{aligned}
\tan \alpha_2 &= \tan \theta_2 \tan \alpha_5 \quad , \\
\tan \alpha_0 &= \tan \theta_2 \tan \alpha_3 \quad , \qquad (20') \\
\tan \alpha_4 &= \tan \theta_2 \tan \alpha_1 \quad ,
\end{aligned}$$

with

$$\tan^2\theta_2 = \frac{\sin^2\alpha_2\sin^2\alpha_4 - \cos^2\alpha_3}{\cos^2\alpha_2\cos^2\alpha_4} = \cot^2\Theta \quad .$$

Observe that we can choose a path of integration along which the parameter Θ is constant: R is characterized by three independent parameters. If we fix α_1, α_4 and let α_2 vary, for example, then, by (20′), Θ is constant. Hence, in the sequel, we may assume $\lambda = \tan\Theta$ to be constant.

Suppose $\alpha_1, \alpha_3, \alpha_5$ to be the free parameters among the five dihedral angles $\alpha_1, \ldots, \alpha_5$ with the mutual dependences (20), and integrate the (complete) volume differential beginning from the (collapsed) orthoscheme R_{deg} of volume zero with dihedral angles $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = \frac{\pi}{2}$, $\alpha_0 = \alpha_3 = 0$. This yields, together with a symmetrization argument,

$$\begin{aligned}
(-8)\,\mathrm{vol}_5(R) = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (22)\\
& I(\lambda^{-1},0;\alpha_1) + \frac{1}{2}I(\lambda,0;\alpha_2) - I(\lambda^{-1},0;\alpha_0') + \frac{1}{2}I(\lambda,0;\alpha_4) + I(\lambda^{-1},0;\alpha_5) - \\
& - \frac{1}{4}\{\, I(\lambda,-(\frac{\pi}{2}+\alpha_1);\frac{\pi}{2}+\alpha_1+\alpha_2) + I(\lambda,-(\frac{\pi}{2}-\alpha_1);\frac{\pi}{2}-\alpha_1+\alpha_2) - \\
& \quad - I(\lambda,-(\frac{\pi}{2}+\alpha_1);\pi+\alpha_1) - I(\lambda,-(\frac{\pi}{2}-\alpha_1);\pi-\alpha_1) - \\
& \quad - I(\lambda,-(\frac{\pi}{2}+\alpha_5);\pi+\alpha_5) - I(\lambda,-(\frac{\pi}{2}-\alpha_5);\pi-\alpha_5) + \\
& \quad + I(\lambda,-(\frac{\pi}{2}+\alpha_5);\frac{\pi}{2}+\alpha_5+\alpha_4) + I(\lambda,-(\frac{\pi}{2}-\alpha_5);\frac{\pi}{2}-\alpha_5+\alpha_4)\,\} \quad .
\end{aligned}$$

Here, $\lambda = \tan\Theta$ is as in (20), and $I(a,b;x)$ is the function in the variable x defined by

$$I(a,b;x) = \int_{\frac{\pi}{2}}^{x} Л_2(y)\, d\arctan(a\tan(b+y)) \quad , \quad a,b \in \mathbf{R} \text{ fixed} \quad , \qquad (23)$$

with $I(1,b;x) = -Л_3(x) - \frac{3}{16}\zeta(3)$.

The volume problem consists now in expressing the integrals (23) in terms of polylogarithms of orders less or equal to three, related or even simpler functions (see also **Introduction**). There are special cases which can be treated easier. In fact, there is a basic difference between $a = 1$ and $a \neq 1$; for the volume problem, the case $\lambda = 1$, that means, $\alpha_1 = \alpha_4$, $\alpha_2 = \alpha_5$, $\alpha_3 = \alpha_0$, was already solved in [K3, Theorem, p. 659]. The result in terms of the Trilobachevskij function $Л_3(\omega)$, $\omega \in \mathbf{R}$, (see Appendix B1) is as follows:

Theorem 2.

Let R denote a doubly asymptotic 5-orthoscheme with $\lambda = \tan\Theta = 1$, that is,

$$\Sigma(R) \quad : \quad \circ\overset{\alpha_1}{\text{---}}\circ\overset{\alpha_2}{\text{---}}\circ\overset{\alpha_3}{\text{---}}\circ\overset{\alpha_1}{\text{---}}\circ\overset{\alpha_2}{\text{---}}\circ \quad , \quad \cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1 \quad .$$

Then,

$$\text{vol}_5(R) = \frac{1}{4}\{\text{Л}_3(\alpha_1) + \text{Л}_3(\alpha_2) - \frac{1}{2}\text{Л}_3(\frac{\pi}{2} - \alpha_3)\} - \frac{1}{16}\{\text{Л}_3(\frac{\pi}{2} + \alpha_1 + \alpha_2) + \\ + \text{Л}_3(\frac{\pi}{2} - \alpha_1 + \alpha_2)\} + \frac{3}{64}\zeta(3) \quad . \tag{24}$$

Remarks.

The orthoscheme

$$\Sigma(R) \quad : \quad \circ\overset{\alpha_1}{\text{——}}\circ\overset{\alpha_2}{\text{——}}\circ\overset{\alpha_3}{\text{——}}\circ\overset{\alpha_1}{\text{——}}\circ\overset{\alpha_2}{\text{——}}\circ \quad , \quad \cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1 \quad ,$$

of Theorem 2 belongs to a Napier period splitting into identical halves and generating further hyperbolic polytopes (see Proposition 1.4). For example, we obtain the totally asymptotic 5-simplex Q with diagram

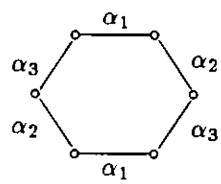


Fig. 2

Its volume can be computed in the following way (cf. [K4, 3.4, Theorem 3]):

$$\text{vol}_5(Q) = \frac{1}{2}\{\,\text{Л}_3(\alpha_1) + \text{Л}_3(\alpha_2) + \text{Л}_3(\alpha_3) - \\ - \text{Л}_3(\frac{\pi}{2} - \alpha_1) - \text{Л}_3(\frac{\pi}{2} - \alpha_2) - \text{Л}_2(\frac{\pi}{3} - \alpha_1)\,\} + \frac{7}{32}\zeta(3) \quad . \tag{25}$$

By a similar construction (see Proposition 1.4), we obtain another asymptotic polytope $R_2 \subset \overline{H^5}$ described by

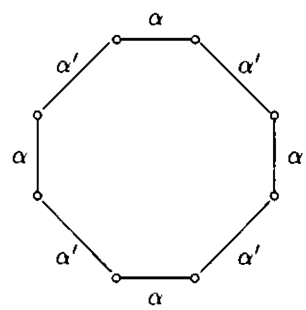


Fig. 3

R_2 is a doubly truncated 5-orthoscheme whose dihedral angles α resp. α' are attached at the (doubly asymptotic) orthoscheme faces

$$\circ\overset{\alpha'}{\text{——}}\circ\overset{\alpha}{\text{——}}\circ\overset{\alpha'}{\text{——}}\circ \quad \text{resp.} \quad \circ\overset{\alpha}{\text{——}}\circ\overset{\alpha'}{\text{——}}\circ\overset{\alpha}{\text{——}}\circ \quad .$$

For its volume, we obtain

$$\mathrm{vol}_5(R_2) = -\frac{1}{2}\left\{ Л_3(\alpha) + Л_3(\alpha') \right\} + \frac{1}{32}\zeta(3) \quad . \tag{26}$$

In comparison with the result for $\lambda = 1$ as presented in Theorem 2, the case $\lambda \neq 1$ is much more difficult; one reason for this is hidden behind the function theoretical behavior of the inverse tangent function

$$\arctan(x) = \frac{1}{2i}\log\frac{1+ix}{1-ix} \quad ,$$

which does not allow to transform the integral $I(a,b;x)$ into $Л_3(\omega), \omega \in \mathbf{R}$. In order to derive a formula for $\mathrm{vol}_5(R)$ in the case $\lambda \neq 1$ in terms of trilogarithmic functions with arguments connected to the dihedral angles, we relate the integrals

$$I(a,b;x) = \int_{\frac{\pi}{2}}^{x} Л_2(y)\, d\arctan(a\tan(b+y)) \quad , \quad a,b \in \mathbf{R} \text{ fixed} \quad ,$$

to functions of the form (see Appendix B1)

$$J(a,b,c;z) := \int_0^z \log(1+at)\log(1+bt)d\log(1+ct) \quad , \quad a,b,c \in \mathbf{C} \quad .$$

Then, according to (B24), $I(a,b;x)$ can be expressed in terms of polylogarithms of orders less or equal to three as follows:

$$\begin{aligned} I(a,b;x) = {} & (x-\frac{\pi}{2})\arctan(a\cot b)\log 2 + Л_2(x)\left\{\arctan(a\tan(b+x)) + \arctan(a\cot b)\right\} + \\ & + \frac{1}{2}\log 2\left\{\mathrm{Li}_2\big((1+a)\sin(b+x), \frac{\pi}{2}-(b+x)\big) - \mathrm{Li}_2\big((1-a)\sin(b+x), \frac{\pi}{2}-(b+x)\big)\right\} - \\ & - \frac{1}{2}\log 2\left\{\mathrm{Li}_2\big((1+a)\sin(b+\frac{\pi}{2}), b\big) - \mathrm{Li}_2\big((1-a)\sin(b+\frac{\pi}{2}), b\big)\right\} + \\ & + H(\omega, b; \cot x) \quad , \end{aligned} \tag{27}$$

where $\mathrm{Li}_2(r,\phi)$ denotes $\mathrm{Re}(\mathrm{Li}_2(re^{i\phi}))$, $\tan\omega = ia$; moreover, for $c = c(\omega,b) = \cot(b+\omega)$ and $u = u(x) = \cot x$, $H(\omega,b;\cot x)$ is defined by (see (B20))

$$\begin{aligned} & H(\omega,b;\cot x) = 2\mathrm{Re}\,(h(\omega,b;x)) = \qquad (28) \\ & \frac{-1}{4}\mathrm{Re}\Big[F(\frac{-ic}{1-ic};1+iu) - F(\frac{ic}{1+ic};1-iu) + F(\frac{1}{1+ic};1-cu) - F(\frac{1}{1-ic};1-cu) + \\ & \quad + F(\frac{1}{ic};\frac{1-cu}{1+iu}) - F(-\frac{1}{ic};\frac{1-cu}{1-iu}) + F(\frac{2}{1-ic};\frac{1-cu}{1-iu}) - F(\frac{2}{1+ic};\frac{1-cu}{1+iu}) + \\ & \quad + \log(1-cu)\left\{\log(1+iu)\log\frac{ic(1-iu)}{1+ic} - \log(1-iu)\log\frac{ic(1+iu)}{1-ic}\right\} + \\ & \quad + \frac{1}{2}\log^2(1+iu)\log\frac{1+ic}{2c(c+i)} - \frac{1}{2}\log^2(1-iu)\log\frac{1-ic}{2c(c-i)}\Big] \quad , \end{aligned}$$

where, for $s, z \in \mathbf{C}$, $z \notin \mathbf{R}_{\leq 0}$,

$$F(s;z) = \mathrm{Li}_3(s) - \mathrm{Li}_3(sz) + \log z \cdot \mathrm{Li}_2(sz) \quad . \tag{29}$$

Summarizing, we obtain

Theorem 3.

Denote by $R \subset \overline{H^5}$ *a doubly asymptotic 5-orthoscheme represented by*

$$\Sigma(R) \quad : \quad \circ \overset{\alpha_1}{\text{---}} \circ \overset{\alpha_2}{\text{---}} \circ \overset{\alpha_3}{\text{---}} \circ \overset{\alpha_4}{\text{---}} \circ \overset{\alpha_5}{\text{---}} \circ \quad \text{with}$$

$$\lambda = \tan\Theta = \frac{|\det \Sigma(R)|^{1/2}}{\cos\alpha_1 \cos\alpha_3 \cos\alpha_5} \quad , \quad 0 \leq \Theta \leq \frac{\pi}{2} \quad .$$

Let $0 \leq \alpha_0 \leq \frac{\pi}{2}$ *such that* $\tan\alpha_0 = \cot\Theta \tan\alpha_3$. *Then,*

$$\begin{aligned}
\mathrm{vol}_5(R) = & \\
& -\frac{1}{8}\{I(\lambda^{-1},0;\alpha_1) + \frac{1}{2}I(\lambda,0;\alpha_2) - I(\lambda^{-1},0;\alpha_0') + \frac{1}{2}I(\lambda,0;\alpha_4) + I(\lambda^{-1},0;\alpha_5)\}+ \\
& + \frac{1}{32}\{I(\lambda,-(\frac{\pi}{2}+\alpha_1);\frac{\pi}{2}+\alpha_1+\alpha_2) + I(\lambda,-(\frac{\pi}{2}-\alpha_1);\frac{\pi}{2}-\alpha_1+\alpha_2) - \\
& \quad - I(\lambda,-(\frac{\pi}{2}+\alpha_1);\pi+\alpha_1) - I(\lambda,-(\frac{\pi}{2}-\alpha_1);\pi-\alpha_1) - \\
& \quad - I(\lambda,-(\frac{\pi}{2}+\alpha_5);\pi+\alpha_5) - I(\lambda,-(\frac{\pi}{2}-\alpha_5);\pi-\alpha_5) + \\
& \quad + I(\lambda,-(\frac{\pi}{2}+\alpha_5);\frac{\pi}{2}+\alpha_5+\alpha_4) + I(\lambda,-(\frac{\pi}{2}-\alpha_5);\frac{\pi}{2}-\alpha_5+\alpha_4)\} \quad ,
\end{aligned} \tag{30}$$

where $I(a,b;x)$ *is the trilogarithmic function according to* (27) – (29) *with the property* $I(1,b;x) = -\mathrm{JI}_3(x) - \frac{3}{16}\zeta(3)$.

Remark.

Theorem 3 contains the complete solution of the volume problem for five-dimensional non-Euclidean polytopes. Namely, by Proposition 1.2 and its constructive proof, the volume of a *compact* hyperbolic 5-orthoscheme is expressible as sum and difference of volumes of doubly asymptotic 5-orthoschemes. According to the trigonometric principle (that is, hyperbolic k-volume is i^k times spherical k-volume (cf. [BH, p. 20-21, p. 210])), the formula in the compact case can be dualized by means of analytical continuation to yield a volume formula for spherical 5-orthoschemes. Finally, any non-Euclidean polytope is equidissectable to orthoschemes (see **1.2**).

2.4. APPLICATIONS

Based on Theorem 3 of **2.3**, there are various applications and further directions to study. Computations of covolumes of hyperbolic Coxeter groups and consequences for the volume spectrum of hyperbolic space forms of dimension five were already presented in [K3] and [K4]; there, we determined the volumes of all arithmetic hyperbolic Coxeter polytopes with linear and cyclic schemes of order six. Those results, combined with certain dissections of the sort (3), **1.2**, allowed to compute the contents of the quasicrystallographic simplices

$$\nu_1 \quad : \quad \circ \overset{5}{\text{---}} \circ \overset{\frac{5}{2}}{\text{---}} \circ \overset{5}{\text{---}} \circ \overset{\frac{5}{2}}{\text{---}} \circ \overset{5}{\text{---}} \circ \quad ,$$

$$\nu_2 \quad : \quad \circ \overset{\frac{5}{2}}{\text{---}} \circ \overset{5}{\text{---}} \circ \overset{\frac{5}{2}}{\text{---}} \circ \overset{5}{\text{---}} \circ \overset{\frac{5}{2}}{\text{---}} \circ \quad ,$$

which do not satisfy the conditions of Theorem 2 and serve therefore as test objects for (30):

$$\begin{aligned} \mathrm{vol}_5(\nu_1) &= \frac{1}{96}\,\text{Л}_3(\frac{\pi}{5}) + \frac{\zeta(3)}{800} \simeq 0.0020 \quad , \\ \mathrm{vol}_5(\nu_2) &= \frac{1}{96}\,\text{Л}_3(\frac{\pi}{5}) \simeq 0.0005 \quad . \end{aligned}$$

Here, we study implications for the totally asymptotic regular 5-simplex $S^{\infty}_{reg}(2\alpha)$ with $\cos(2\alpha) = \frac{1}{4}$. Its importance is, among other things, expressed by the following

Theorem 4.

In $\overline{H^n}$, $n \geq 2$, a simplex is of maximal volume if and only if it is totally asymptotic and regular.

This result was proved by Haagerup and Munkholm [HM] by purely functional analytical methods; they also showed that (cf. [HM, Proposition 2, p. 4])

$$\frac{n-1}{n^2} \leq \frac{\mathrm{vol}_{n+1}(S^{\infty}_{reg})}{\mathrm{vol}_n(S^{\infty}_{reg})} \leq \frac{1}{n} \quad .$$

For $n = 5$, this leads to the estimate (cf. [K2, **14.3.2**, (14.60)])

$$0.0510 \leq \mathrm{vol}_5(S^{\infty}_{reg}) \leq 0.0638 \quad .$$

On the other hand, by **1.3**, we know that

$$\mathrm{vol}_5(S^{\infty}_{reg}) = 5!\,\mathrm{vol}_5(\nu_6) \quad ,$$

where ν_6 is the doubly asymptotic orthoscheme

$$\nu_6 \quad : \quad \circ \overset{2\alpha}{\text{---}} \circ \overset{\alpha}{\text{---}} \circ \text{---} \circ \text{---} \circ \text{---} \circ \quad \text{with} \quad \lambda = \frac{\sqrt{\det(\circ \text{---} \circ \text{---} \circ \text{---} \circ)}}{\cos^2 \frac{\pi}{3}} = \sqrt{5} \quad .$$

Therefore, by Theorem 3 of **2.3**, we obtain

$$\mathrm{vol}_5(S^{\infty}_{reg}) \simeq 0.0578 \quad .$$

Appendix A. Some useful determinant identities

Let $\Sigma_n \ : \ \circ\overset{\alpha_1}{\text{---}}\circ - \cdots - \circ\overset{\alpha_n}{\text{---}}\circ$ be a scheme of order $n+1$ with weights $\cos\alpha_i$, $1 \le i \le n$, $\det\Sigma_0 := 1$ and $\det\Sigma_1 = \sin^2\alpha_1$. Put

$$\sigma_i^j \ : \ \circ\overset{\alpha_i}{\text{---}}\circ - \cdots - \circ\overset{\alpha_j}{\text{---}}\circ \qquad \text{for } 1 \le i \le j \le n \ .$$

Apart from $\sigma_1^n = \Sigma_n$, σ_i^j is a proper subscheme of Σ_n. There are the following recursion formulae (cf. [S, (1), p. 258, and (1), p. 261]):

$$\det\Sigma_n = \det\Sigma_{n-1} - \cos^2\alpha_n \det\Sigma_{n-2} \quad , \quad n \ge 2 \ : \tag{A1}$$

$$\det\Sigma_n = \det\sigma_1^{k-1}\det\sigma_{k+1}^n - \cos^2\alpha_k \det\sigma_1^{k-2}\det\sigma_{k+2}^n \quad \text{for } 2 < k \le n-2 \, . \tag{A2}$$

Moreover, by [S, (2), p. 259],

$$\det\sigma_1^{n-1}\det\sigma_2^n - \det\sigma_1^n\det\sigma_2^{n-2} = \prod_{i=1}^n \cos^2\alpha_i \quad . \tag{A3}$$

For a cyclic scheme Δ_n of order n with weights $\cos\alpha_i$, $1 \le i \le n$,

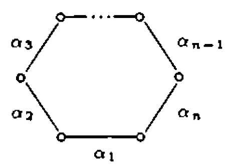


Fig. 4

one can show (for one choice of indices) that (cf. [S, p. 262])

$$\det\Delta_n = -2\prod_{i=1}^n \cos\alpha_i + \det\sigma_1^{n-1} - \cos^2\alpha_n \det\sigma_2^{n-2} \quad . \tag{A4}$$

Lemma A.

Let $m, n \in \mathbb{N}$, $(m,n) \ne (1,1)$. Suppose Σ_0^{2n+3} to be as in (9), and denote by Ω_n^m the cyclic scheme of m repetitions of $\Sigma_0^{n+2} \ : \ \circ\overset{\alpha_0}{\text{---}}\circ - \cdots - \circ\overset{\alpha_n}{\text{---}}\circ$. Then,

$$\det\Omega_n^m = \begin{cases} 0 & \text{for } m \equiv 0(4) \quad ; \\ -2\prod_{i=0}^n \cos^m\alpha_i & \text{for } m \equiv 1,3(4) \quad ; \\ -4\prod_{i=0}^n \cos^m\alpha_i & \text{for } m \equiv 2(4) \quad . \end{cases} \tag{A5}$$

Proof:
Denote by

$$\begin{array}{lll} \omega^1_m & : & \circ\overset{\alpha_0}{\rule{2em}{0.4pt}}\circ - \cdot \mid \cdot - \circ\overset{\alpha_{n-1}}{\rule{2em}{0.4pt}}\circ\overset{\alpha_n}{\rule{2em}{0.4pt}}\circ\overset{\alpha_0}{\rule{2em}{0.4pt}}\circ - \cdot \mid \cdot - \circ\overset{\alpha_{n-1}}{\rule{2em}{0.4pt}}\circ \quad \text{resp.} \\ \omega^2_m & : & \circ\overset{\alpha_1}{\rule{2em}{0.4pt}}\circ - \cdot \mid \cdot - \circ\overset{\alpha_{n-1}}{\rule{2em}{0.4pt}}\circ\overset{\alpha_n}{\rule{2em}{0.4pt}}\circ\overset{\alpha_0}{\rule{2em}{0.4pt}}\circ - \cdot \mid \cdot - \circ\overset{\alpha_{n-2}}{\rule{2em}{0.4pt}}\circ \end{array}$$

the schemes of order $m(n+1)$ resp. $m(n+1)-2$ arising from Ω^m_n by discarding an edge resp. two nodes together with the edges emanating from them (the verticals indicate repetitions of Σ^{n+2}_0); set $\delta^1_m := \det\omega^1_m$ and $\delta^2_m := \cos^2\alpha_n \det\omega^2_m$. Then, by (A4),

$$\det\Omega^m_n = -2\prod_{i=0}^{n}\cos^m\alpha_i + \delta^1_m - \delta^2_m \quad .$$

In order to prove (A5), we show for the ingredients δ^1_m and δ^2_m, by induction on m, that

$$\delta^1_m = \begin{cases} (-1)^{\frac{m}{2}}\prod\limits_{i=0}^{n}\cos^m\alpha_i & \text{for } m \text{ even;} \\ (-1)^{\frac{m-1}{2}}\prod\limits_{i=0}^{n}\cos^{m-1}\alpha_i \det(\circ\overset{\alpha_0}{\rule{2em}{0.4pt}}\circ - \cdots - \circ\overset{\alpha_{n-1}}{\rule{2em}{0.4pt}}\circ) & \text{for } m \text{ odd,} \end{cases}$$

and

$$\delta^2_m = \begin{cases} (-1)^{\frac{m}{2}+1}\prod\limits_{i=0}^{n}\cos^m\alpha_i & \text{for } m \text{ even;} \\ (-1)^{\frac{m-1}{2}}\prod\limits_{i=0}^{n}\cos^{m-1}\alpha_i \det(\circ\overset{\alpha_0}{\rule{2em}{0.4pt}}\circ - \cdots - \circ\overset{\alpha_{n-1}}{\rule{2em}{0.4pt}}\circ) & \text{for } m \text{ odd.} \end{cases}$$

For $m=1$, the assertions for δ^1_1 and δ^2_1 follow from **1.2**, (10).
Let $m=2$. Since $\det\Sigma^{2n+1}_i = 0$ for $i=0,1$ (see **1.2**, (9)),

$$\delta^1_2 = \det(\circ\overset{\alpha_0}{\rule{2em}{0.4pt}}\circ - \cdots - \circ\overset{\alpha_{n-1}}{\rule{2em}{0.4pt}}\circ\overset{\alpha_n}{\rule{2em}{0.4pt}}\circ\overset{\alpha_0}{\rule{2em}{0.4pt}}\circ - \cdots - \circ\overset{\alpha_{n-1}}{\rule{2em}{0.4pt}}\circ)$$

equals the determinant of the extended scheme

$$\circ\overset{\alpha_0}{\rule{2em}{0.4pt}}\circ - \cdots - \circ\overset{\alpha_{n-1}}{\rule{2em}{0.4pt}}\circ\overset{\alpha_n}{\rule{2em}{0.4pt}}\circ\overset{\alpha_0}{\rule{2em}{0.4pt}}\circ - \cdots - \circ\overset{\alpha_n}{\rule{2em}{0.4pt}}\circ \quad ,$$

and also the determinant of

$$\circ\overset{\alpha_1}{\rule{2em}{0.4pt}}\circ - \cdots - \circ\overset{\alpha_{n-1}}{\rule{2em}{0.4pt}}\circ\overset{\alpha_n}{\rule{2em}{0.4pt}}\circ\overset{\alpha_0}{\rule{2em}{0.4pt}}\circ - \cdots - \circ\overset{\alpha_n}{\rule{2em}{0.4pt}}\circ \quad .$$

By (A3), it follows that $(\delta^1_2)^2 = \prod\limits_{i=0}^{n}\cos^4\alpha_i$. Hence, $\delta^1_2 = -\prod\limits_{i=0}^{n}\cos^2\alpha_i$, and $\delta^2_2 = -\delta^1_2$.
The general case is shown for δ^1_m, only. Write

$$\omega^1_m \;:\; \circ\overset{\alpha_0}{\rule{2em}{0.4pt}}\circ - \cdot \mid \cdots \mid \cdot - \circ\overset{\alpha_n}{\rule{2em}{0.4pt}}\circ\overset{\alpha_0}{\rule{2em}{0.4pt}}\circ - \cdots - \circ\overset{\alpha_{n-1}}{\rule{2em}{0.4pt}}\circ\overset{\alpha_n}{\rule{2em}{0.4pt}}\circ\overset{\alpha_0}{\rule{2em}{0.4pt}}\circ - \cdots - \circ\overset{\alpha_{n-1}}{\rule{2em}{0.4pt}}\circ \; .$$

Since $\det\Sigma_1^{2n+1} = 0$, (A3) yields

$$\det(\omega_m^1) = \delta_m^1 = \delta_{m-2}^1 \cdot \delta_2^1 \quad .$$

Now, use the induction hypothesis to finish the proof. Q.E.D.

Appendix B. The Trilogarithm

B1. BASIC DEFINITIONS AND PROPERTIES

For the material of this paragraph, we refer to [L].
The Trilogarithm $\mathrm{Li}_3(z)$, as every polylogarithm function $\mathrm{Li}_k(z)$, arises as iterated logarithm in the following way: Let

$$\mathrm{Li}_1(z) := -\log(1-z), \quad \text{with} \quad \mathrm{Li}_1(z) = \sum_{r=1}^{\infty} \frac{z^r}{r} \quad \text{for} \quad |z| < 1 \quad ,$$

and define the k-logarithm or polylogarithm of order $k \geq 2$ by

$$\mathrm{Li}_k(z) := \int_0^z \frac{\mathrm{Li}_{k-1}(t)}{t}\, dt, \quad \text{with} \quad \mathrm{Li}_k(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^k} \quad \text{for} \quad |z| \leq 1 \quad . \qquad (B1)$$

Then, $\mathrm{Li}_k(1) = \zeta(k)$, $k \geq 2$, and there is the identity

$$\frac{1}{m^{k-1}} \mathrm{Li}_k(z^m) = \mathrm{Li}_k(z) + \mathrm{Li}_k(\omega z) + \cdots + \mathrm{Li}_k(\omega^{m-1} z) \quad , \qquad (B2)$$

where $\omega = e^{2\pi i/m}$, $m \geq 1$. In particular,

$$\frac{1}{2^{k-1}} \mathrm{Li}_k(z^2) = \mathrm{Li}_k(z) + \mathrm{Li}_k(-z) \quad . \qquad (B3)$$

Moreover,

$$\overline{\mathrm{Li}_k(z)} = \mathrm{Li}_k(\overline{z}) \quad , \quad k \geq 1 \quad . \qquad (B4)$$

There are only a few special polylogarithmic values known; for example,

$$\begin{aligned} &\mathrm{Li}_2(-1) = -\frac{\pi^2}{12} \quad ; \quad \mathrm{Li}_2(\tfrac{1}{2}) = \frac{\pi^2}{12} - \frac{\log^2 2}{2} \quad ; \quad \mathrm{Li}_2(1) = \frac{\pi^2}{6} \quad ; \\ &\mathrm{Li}_2(\frac{\sqrt{5}-1}{2}) = \frac{\pi^2}{10} - \log^2(\frac{\sqrt{5}-1}{2}) \quad ; \quad \mathrm{Li}_2(\frac{3-\sqrt{5}}{2}) = \frac{\pi^2}{15} - \frac{1}{4}\log^2(\frac{3-\sqrt{5}}{2}) \quad . \end{aligned} \qquad (B5)$$

$$\begin{aligned} &\mathrm{Li}_3(-1) = -\frac{3}{4}\zeta(3) \quad ; \quad \mathrm{Li}_3(\tfrac{1}{2}) = \frac{7}{8}\zeta(3) - \frac{\pi^2}{12}\log 2 + \frac{1}{6}\log^3(2) \quad ; \\ &\mathrm{Li}_3(\frac{3-\sqrt{5}}{2}) = \frac{4}{5}\zeta(3) + \frac{\pi^2}{15}\log(\frac{3-\sqrt{5}}{2}) - \frac{1}{12}\log^3(\frac{3-\sqrt{5}}{2}) \quad . \end{aligned} \qquad (B6)$$

These values can be computed by means of functional equations for $\mathrm{Li}_2(z)$ and $\mathrm{Li}_3(z)$; we mention here the functional equations of Kummer type in two variables of the form (cf. [L, (18), p. 284, and (11), p. 297]):

$$\mathrm{Li}_2\left(\frac{x(1-y)}{y(1-x)}\right) = \mathrm{Li}_2(y) - \mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{x}{y}\right) + \mathrm{Li}_2\left(\frac{1-y}{1-x}\right) - \frac{\pi^2}{6} + \log y \log \frac{1-y}{1-x} \quad ; \qquad (B7)$$

$$\begin{aligned} &\mathrm{Li}_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right) + \mathrm{Li}_3(xy) + \mathrm{Li}_3\left(\frac{x}{y}\right) - 2\,\mathrm{Li}_3\left(\frac{x(1-y)}{y(1-x)}\right) - 2\,\mathrm{Li}_3\left(\frac{x(1-y)}{x-1}\right) - \\ &\quad - 2\,\mathrm{Li}_3\left(\frac{1-y}{1-x}\right) - 2\,\mathrm{Li}_3\left(\frac{1-y}{y(x-1)}\right) - 2\,\mathrm{Li}_3(x) - 2\,\mathrm{Li}_3(y) + 2\zeta(3) = \qquad (B8) \\ &\quad = \log^2 y \log\left(\frac{1-y}{1-x}\right) - \frac{\pi^2}{3}\log y - \frac{1}{3}\log^3 y \quad . \end{aligned}$$

Write $z = re^{i\phi}$, $r > 0$, $0 \le \phi < 2\pi$, and put $\mathrm{Li}_k(r,\phi) := \mathrm{Re}(\mathrm{Li}_k(re^{i\phi}))$ in the standard way. Then, $\mathrm{Li}_k(r,\phi) = \mathrm{Li}_k(r,-\phi)$, and

$$\mathrm{Li}_2(re^{i\phi}) = \mathrm{Li}_2(r,\phi) + i\,[\omega \log r + \text{Л}_2(\omega) + \text{Л}_2(\phi) - \text{Л}_2(\omega+\phi)] \quad , \qquad (B9)$$

where $\tan\omega = \dfrac{r\sin\phi}{1 - r\cos\phi}$. For $\mathrm{Li}_3(re^{i\phi})$, however, there is no equivalent to (B9). But there are the following identities:

$$\begin{aligned} \mathrm{Li}_2(r,\phi) + \mathrm{Li}_2\left(\frac{1}{r},\phi\right) &= -\frac{1}{2}\log^2 r + \frac{1}{2}(\pi-\phi)^2 - \frac{\pi^2}{6} \quad ; \\ \mathrm{Li}_3(r,\phi) - \mathrm{Li}_3\left(\frac{1}{r},\phi\right) &= -\frac{1}{6}\log^3 r + \frac{1}{6}(3(\pi-\phi)^2 - \pi^2)\log r \quad , \end{aligned} \qquad (B10)$$

and, by (B2)–(B4),

$$\begin{aligned} &\mathrm{Li}_2(r,\pi) = \mathrm{Li}_2(-r) \quad ; \quad \mathrm{Li}_3(r,\pi) = \frac{1}{4}\mathrm{Li}_3(r^2) - \mathrm{Li}_3(r) \quad ; \\ &\mathrm{Li}_2\left(r,\frac{\pi}{2}\right) = \frac{1}{4}\mathrm{Li}_2(-r^2) \quad ; \quad \mathrm{Li}_3\left(r,\frac{\pi}{2}\right) = \frac{1}{8}\mathrm{Li}_3(-r^2) \quad ; \qquad (B11) \\ &\mathrm{Li}_2\left(r,\frac{\pi}{3}\right) = \frac{1}{6}\mathrm{Li}_2(-r^3) - \frac{1}{2}\mathrm{Li}_2(-r) \quad ; \quad \mathrm{Li}_3\left(r,\frac{\pi}{3}\right) = \frac{1}{18}\mathrm{Li}_3(-r^3) - \frac{1}{2}\mathrm{Li}_3(-r) \quad . \end{aligned}$$

For arguments $z = e^{2i\alpha}$, $\alpha \in \mathbf{R}$, on the unit circle, real and imaginary part of $\mathrm{Li}_k(z)$ play a particular role. Define the higher Lobachevskij functions by

$$\begin{aligned} \text{Л}_{2k}(\alpha) &= \frac{1}{2^{2k-1}}\,\mathrm{Im}(\mathrm{Li}_{2k}(e^{2i\alpha})) = \frac{1}{2^{2k-1}}\sum_{r=1}^{\infty}\frac{\sin(2r\alpha)}{r^{2k}} \quad , \\ \text{Л}_{2k+1}(\alpha) &= \frac{1}{2^{2k}}\,\mathrm{Re}(\mathrm{Li}_{2k+1}(e^{2i\alpha})) = \frac{1}{2^{2k}}\sum_{r=1}^{\infty}\frac{\cos(2r\alpha)}{r^{2k+1}} \quad , \end{aligned} \qquad (B12)$$

generalizing Lobachevskij's function (see **2.2**)

$$\text{Л}_2(\alpha) = \frac{1}{2}\,\mathrm{Im}(\mathrm{Li}_2(e^{2i\alpha})) = -\int_0^\alpha \log|2\sin t|\,dt \quad .$$

There are the relations

$$\text{Л}_{2k}(\alpha) = \int_0^\alpha \text{Л}_{2k-1}(t)dt \quad , \quad \text{Л}_{2k+1}(\alpha) = \frac{1}{2^{2k}}\,\zeta(2k+1) - \int_0^\alpha \text{Л}_{2k}(t)dt \quad . \qquad (B13)$$

Moreover, $\text{Л}_k(\alpha)$ is π-periodic, even (odd) for k odd (even) and distributes according to

$$\frac{1}{m^{k-1}}\,\text{Л}_k(m\alpha) = \sum_{r=0}^{m-1} \text{Л}_k(\alpha + \frac{r\pi}{m}) \quad .$$

Now, $\mathrm{Li}_k(e^{2i\alpha})$ splits in a part consisting of the Lobachevskij function $\text{Л}_k(\alpha)$, while the other part is always elementary (cf. [L, (16, (17), (22), p. 300]); for example, for $0 \leq \alpha \leq 2\pi$,

$$\mathrm{Li}_2(e^{2i\alpha}) = (\frac{\pi}{2} - \alpha)^2 - \frac{\pi^2}{12} + 2i\text{Л}_2(\alpha) \quad ;$$
$$\mathrm{Li}_3(e^{2i\alpha}) = 4\text{Л}_3(\alpha) + \frac{\alpha(\pi-\alpha)(\pi-2\alpha)}{3}i \quad .$$

Finally, by using the integral representations

$$\mathrm{Li}_2(z) = \int_1^\infty \log t\,\{\frac{1}{t-z} - \frac{1}{t}\}\,dt \quad , \quad \mathrm{Li}_3(z) = \frac{1}{2}\int_1^\infty \log^2 t\,\{\frac{1}{t-z} - \frac{1}{t}\}\,dt \quad ,$$

one can deduce the identities

$$\frac{1}{2}\int_1^\infty \log^2(t-a)\,\{\frac{1}{t-b} - \frac{1}{t-c}\}\,dt = \mathrm{Li}_3(\frac{b-a}{1-a}) - \mathrm{Li}_3(\frac{c-a}{1-a}) +$$
$$+ \log(1-a)\,\{\,\mathrm{Li}_2(\frac{b-a}{1-a}) - \mathrm{Li}_2(\frac{c-a}{1-a})\,\} - \frac{1}{2}\log^2(1-a)\log\,(\frac{1-b}{1-c}) \quad ;$$

$$\int_1^\infty \log t \log(t-a)\,\{\frac{1}{t-b} - \frac{1}{t}\}\,dt = \mathrm{Li}_3(b) + \mathrm{Li}_3(\frac{b-a}{b}) + \mathrm{Li}_3(\frac{b-a}{1-a}) - \mathrm{Li}_3(\frac{b-a}{b(1-a)}) -$$
$$- \mathrm{Li}_3(\frac{a}{a-1}) + \log(1-a)\,\{\,\mathrm{Li}_2(\frac{b-a}{1-a}) - \mathrm{Li}_2(\frac{b-a}{b(1-a)}) - \mathrm{Li}_2(\frac{a}{(a-1)})\,\} +$$
$$+ \frac{1}{2}\log^2(1-a)\log\frac{a}{b} - \frac{1}{3}\log^3(1-a) \quad .$$

In the combination, they give rise to the following very useful integral expression, for $c \neq a, b$ (notice that [M, (A33)] is incomplete and has a misprint):

$$J(a,b,c;z) = \int_0^z \log(1+at)\,\log(1+bt)\,d\log(1+ct) = \tag{B14}$$

$$= \mathrm{Li}_3\left(\frac{b}{a}\right) - \mathrm{Li}_3\left(\frac{c-b}{c-a}\right) + \mathrm{Li}_3\left(\frac{c}{c-a}\right) + \mathrm{Li}_3\left(\frac{c}{c-b}\right) + \mathrm{Li}_3\left(\frac{(c-b)(1+az)}{(c-a)(1+bz)}\right) -$$

$$- \mathrm{Li}_3\left(\frac{b(1+az)}{a(1+bz)}\right) - \mathrm{Li}_3\left(\frac{c(1+az)}{c-a}\right) - \mathrm{Li}_3\left(\frac{c(1+bz)}{c-b}\right) +$$

$$+ \log\left(\frac{1+az}{1+bz}\right)\left\{ \mathrm{Li}_2\left(\frac{b(1+az)}{a(1+bz)}\right) - \mathrm{Li}_2\left(\frac{(c-b)(1+az)}{(c-a)(1+bz)}\right) \right\} +$$

$$+ \log(1+az)\mathrm{Li}_2\left(\frac{c(1+az)}{c-a}\right) + \log(1+bz)\mathrm{Li}_2\left(\frac{c(1+bz)}{c-b}\right) +$$

$$+ \log(1+az)\,\log(1+bz)\,\log\frac{a(1+cz)}{a-c} - \frac{1}{2}\log^2(1+bz)\log\frac{a(c-b)}{b(c-a)}$$

$$= F\left(\frac{b}{a};\frac{1+az}{1+bz}\right) - F\left(\frac{c-b}{c-a};\frac{1+az}{1+bz}\right) + F\left(\frac{c}{c-a};1+az\right) + F\left(\frac{c}{c-b};1+bz\right) +$$

$$+ \log(1+az)\,\log(1+bz)\,\log\frac{a(1+cz)}{a-c} - \frac{1}{2}\log^2(1+bz)\log\frac{a(c-b)}{b(c-a)} \quad ,$$

where

$$F(s;z) = \mathrm{Li}_3(s) - \mathrm{Li}_3(sz) + \log z \cdot \mathrm{Li}_2(sz) \quad \text{with} \quad F(s;1) = F(0;z) = 0 \quad .$$

B2. THE INTEGRAL $I(a,b;x)$

Denote by $I(a,b;x)$, $a, b \in \mathbf{R}$ fixed, the integral

$$I(a,b;x) = \int_{\frac{\pi}{2}}^{x} Л_2(y)\,d\arctan(a\tan(b+y)) \quad , \tag{B15}$$

with $I(1,b;x) = -Л_3(x) - \frac{3}{16}\zeta(3)$ (see **B1**). Therefore, suppose that $a \neq 1$. Then, integration by parts yields

$$I(a,b;x) = Л_2(x)\arctan(a\tan(b+x)) + i_1(a,b;x) \quad \text{with} \tag{B16}$$

$$i_1(a,b;x) = \int_{\frac{\pi}{2}}^{x} \arctan(a\tan(b+y))\,\log 2\sin y\,dy \quad .$$

By [L, (28), p. 307], we obtain

$$\begin{aligned} i_1(a,b;x) &= \int_{\frac{\pi}{2}}^{x} \arctan(a\tan(b+y))\,\{\log 2 + \log\sin y\,\}\,dy \\ &= \frac{\log 2}{2}\,\{\,\mathrm{Li}_2\big((1+a)\sin(b+x),\frac{\pi}{2}-(b+x)\big) - \mathrm{Li}_2\big((1-a)\sin(b+x),\frac{\pi}{2}-(b+x)\big)\,\} - \\ &- \frac{\log 2}{2}\,\{\,\mathrm{Li}_2\big((1+a)\sin(b+\frac{\pi}{2}),b)\big) - \mathrm{Li}_2\big((1-a)\sin(b+\frac{\pi}{2}),b\big)\,\} + \\ &+ \int_{\frac{\pi}{2}}^{x} \arctan(a\tan(b+y))\,\log\sin y\,dy \quad , \end{aligned} \tag{B17}$$

where $\mathrm{Li}_2(r,\phi) = \mathrm{Re}(\mathrm{Li}_2(re^{i\phi}))$ according to (B9). Define

$$i_2(a,b;x) := \int_{\frac{\pi}{2}}^{x} \arctan(a\tan(b+y))\,\log\sin y\,dy \quad .$$

Then, we can write

$$i_2(a,b;x) = \frac{1}{2i}\int_{\frac{\pi}{2}}^{x} \log\,\frac{1+ia\tan(b+y)}{1-ia\tan(b+y)}\,\log\sin y\,dy \tag{B18}$$

$$= \arctan(a\cot b)\,\{(x-\frac{\pi}{2})\log 2 \,+\, Л_2(x)\} \,+\, \frac{1}{2i}\int_{\frac{\pi}{2}}^{x} \log\,\frac{1+\cot(\omega-b)\cot y}{1-\cot(\omega+b)\cot y}\,\log\sin y\,dy \quad ,$$

where $\omega \in \mathbf{C}$ is such that $\tan\omega = ia$ (that is, if $a =: \tanh A < 1$, we have $\omega = iA$, while for $a =: \coth\tilde{A} > 1$, we obtain $\omega = i(\tilde{A} + i\frac{\pi}{2})$). Next, we set

$$i_3(\omega,b;x) := \frac{1}{2i}\int_{\frac{\pi}{2}}^{x} \log\,\frac{1+\cot(\omega-b)\cot y}{1-\cot(\omega+b)\cot y}\,\log\sin y\,dy \quad . \tag{B19}$$

By means of the variable change $t := \cot y$, $i_3(\omega,b;x)$ transforms into

$$i_3(\omega,b;x) = \frac{1}{4i}\int_{0}^{\cot x} \log\,\frac{1+\cot(\omega-b)t}{1-\cot(\omega+b)t}\,\log(1+t^2)\,\frac{dt}{1+t^2} \quad ,$$

which can be written as

$$\begin{aligned} i_3(\omega,b;x) &= h(\omega,b;x) - h(-\omega,b;x) \quad \text{with} \\ h(\omega,b;x) &:= -\,\frac{1}{4i}\int_{0}^{\cot x} \log(1-\cot(\omega+b)t)\,\log(1+t^2)\,\frac{dt}{1+t^2} \quad . \end{aligned} \tag{B20}$$

Let $c = c(\omega, b) := \cot(\omega + b)$. Then, $c(\omega, b) = \overline{c(-\omega, \overline{b})}$ implying that $h(\omega, b; x) = -h(-\omega, b; x)$. Therefore,

$$i_3(\omega, b; x) = 2\mathrm{Re}(h(\omega, b; x)) \quad .$$

Now, decomposition into partial fractions yields

$$-8i\, h(\omega, b; x) = H_1^+ + H_1^- + H_2^+ + H_2^- \quad , \tag{$B21$}$$

wherein $H_i^{\pm} = H_i^{\pm}(\omega, b; x)$, $i = 1, 2$, stand for

$$H_1^{\pm} := \int_0^{\cot x} \log(1 - ct)\, \log(1 \pm it)\, \frac{dt}{1 \pm it} \quad ;$$

$$H_2^{\pm} := \int_0^{\cot x} \log(1 - ct)\, \log(1 \pm it)\, \frac{dt}{1 \mp it} \quad .$$

By (B14), we obtain, for $u = u(x) := \cot x$,

$$\begin{aligned} H_1^{\pm} &= \mp i \int_0^{\cot x} \log(1 - ct)\, \log(1 \pm it)\, \frac{\pm i\, dt}{1 \pm it} \\ &= \mp \frac{i}{2} \{ \log^2(1 \pm iu)\, \log(1 - cu) - \int_0^{\cot x} \log^2(1 \pm it)\, \frac{-c}{1 - ct}\, dt \} \\ &= \mp \frac{i}{2} \log^2(1 \pm iu)\, \log(1 - cu) \pm i \Big[\mathrm{Li}_3(\frac{\mp ic}{1 \mp ic}) - \mathrm{Li}_3(\frac{\mp ic}{1 \mp ic}(1 \pm iu)) + \\ &\quad + \log(1 \pm iu)\, \mathrm{Li}_2(\frac{\mp ic}{1 \mp ic}(1 \pm iu)) + \frac{1}{2} \log^2(1 \pm iu)\, \log \frac{1 - cu}{1 \mp ic} \Big] \quad ; \end{aligned} \tag{$B22$}$$

$$\begin{aligned} H_2^{\pm} &= \pm i \int_0^{\cot x} \log(1 - ct)\, \log(1 \pm it)\, \frac{\mp i\, dt}{1 \mp it} \\ &= \pm i \Big[\mathrm{Li}_3(\pm \frac{1}{ic}) - \mathrm{Li}_3(\frac{2}{1 \pm ic}) + \mathrm{Li}_3(\frac{1}{1 \pm ic}) + \mathrm{Li}_3(\frac{1}{2}) + \\ &\quad + \mathrm{Li}_3(\frac{2}{1 \pm ic}\, \frac{1 - cu}{1 \pm iu}) - \mathrm{Li}_3(\pm \frac{1}{ic}\, \frac{1 - cu}{1 \pm iu}) - \mathrm{Li}_3(\frac{1 - cu}{1 \pm ic}) - \\ &\quad - \mathrm{Li}_3(\frac{1 \pm iu}{2}) + \log \frac{1 - cu}{1 \pm iu} \{ \mathrm{Li}_2(\pm \frac{1}{ic}\, \frac{1 - cu}{1 \pm iu}) - \mathrm{Li}_2(\frac{2}{1 \pm ic}\, \frac{1 - cu}{1 \pm iu}) \} + \\ &\quad + \log(1 - cu)\, \mathrm{Li}_2(\frac{1 - cu}{1 \pm ic}) + \log(1 \pm iu)\, \mathrm{Li}_2(\frac{1 \pm iu}{2}) + \\ &\quad + \log(1 - cu)\, \log(1 \pm iu)\, \log \frac{ic(1 \mp iu)}{1 \pm ic} - \frac{1}{2} \log^2(1 \pm iu) \log \frac{\pm 2ic}{1 \pm ic} \Big] \quad . \end{aligned} \tag{$B23$}$$

Combining equations (B15) up to (B23) and respecting (B4), we finally obtain

$$I(a,b;x) = (x-\frac{\pi}{2})\log 2\arctan(a\cot b) + \text{Л}_2(x)\,\{\arctan(a\tan(b+x)) + \arctan(a\cot b)\,\}+$$
$$+\frac{1}{2}\log 2\,\{\mathrm{Li}_2\big((1+a)\sin(b+x),\frac{\pi}{2}-(b+x)\big) - \mathrm{Li}_2\big((1-a)\sin(b+x),\frac{\pi}{2}-(b+x)\big)\,\}-$$
$$-\frac{1}{2}\log 2\,\{\mathrm{Li}_2\big((1+a)\sin(b+\frac{\pi}{2}),b\big) - \mathrm{Li}_2\big((1-a)\sin(b+\frac{\pi}{2}),b\big)\,\}+$$
$$+2\,\mathrm{Re}(h(\omega,b;x)) \quad , \qquad (B24)$$

wherein, for $u = u(x) := \cot x$,

$$(-8)\,\mathrm{Re}\,(h(\omega,b;x)) = \qquad (B25)$$
$$\mathrm{Re}\Big[\,F(\frac{-ic}{1-ic};1+iu) \;-\; F(\frac{ic}{1+ic};1-iu) \;+\; F(\frac{1}{1+ic};1-cu) \;-F(\frac{1}{1-ic};1-cu)+$$
$$+\;F(\frac{1}{ic};\frac{1-cu}{1+iu}) \;-F(-\frac{1}{ic};\frac{1-cu}{1-iu}) \;+\; F(\frac{2}{1-ic};\frac{1-cu}{1-iu}) \;-F(\frac{2}{1+ic};\frac{1-cu}{1+iu})+$$
$$+\;\log(1-cu)\,\{\,\log(1+iu)\log\frac{ic(1-iu)}{1+ic} \;-\; \log(1-iu)\log\frac{ic(1+iu)}{1-ic}\,\}+$$
$$+\;\frac{1}{2}\,\log^2(1+iu)\,\log\frac{1+ic}{2c(c+i)} \;-\; \frac{1}{2}\,\log^2(1-iu)\,\log\frac{1-ic}{2c(c-i)}\,\Big] \quad .$$

Here, $\tan\omega = ia$, $c = c(\omega,b) = \cot(b+\omega)$, $u = \cot x$, and, for $s,z\in\mathbf{C}$, $z\notin\mathbf{R}_{\leq 0}$,

$$F(s;z) = \mathrm{Li}_3(s) \;-\; \mathrm{Li}_3(sz) \;+\; \log z\cdot\mathrm{Li}_2(sz) \quad \text{with} \quad F(0;z) = F(s;1) = 0 \quad .$$

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