An algebraic geometry view of a model quantum field theory on a curve

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MPI/95-59

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Abstract

We survey an algebraic geometry approach that we have developed to a certain model quantum field theory on a compact Riemann surface, which first arose in string theory. Our aim has been to use algebraic geometry to formulate it in a way which is not only mathematically rigorous, but which also brings out the physics as well as the geometry of the model. This is achieved through an algebro-geometric formulation of the *analyticity constraints* implicit in the physicist's description of the model by the so-called *operator product expansion* (OPE). Algebraic geometry serves not only to formulate the model, but also provides powerful computational tools. As a consequence we are able to show that these analyticity axioms are sufficient to determine all the *field correlation functions* (needed by the physicist to describe the system), which we show to be sections (in general meromorphic) of line bundles on cartesian products of the Riemann surface. In the process we obtain a rigorous, unified proof of celebrated identities of Cauchy, Frobenius and Fay, which are hereby seen to express the *fermionic* nature of the quantum system. The *current correlation functions*, formally defined by coalescing arguments in the field correlation functions, are obtained using *ringed spaces with nilpotent elements*. This provides an example of a solution using *global geometry* to a problem of *normal-ordering* a product of fields. The technique of using ringed spaces can be extended to give a global geometric formulation of the *Sugawara* construction of the energy-momentum tensor as a normal-ordered product of currents. We thereby arrive at a new algebro-geometric construction of a *projective connection*. Algebraic geometry proofs of classical formulas in the theory of functions on a Riemann surface are also obtained in the course of this study.

1 Algebraic geometry formulation

The procedure that we shall follow here repeatedly will be to first introduce the heuristic expressions from quantum field theory that we wish to interpret in terms of algebraic geometry. The mathematician may regard them as symbolic expressions yet to be defined. We shall then formulate those that we shall work with in terms of algebraic geometry. Thus the sytem that we shall consider consists of a pair of "quantum fields" b and c on a compact, connected Riemann surface M of arbitrary genus $g \geq 0$. The quotes indicate that we do not attempt to define a quantum field, which is a very singular object and extremely difficult to deal with mathematically with any rigour. We shall always deal, instead, with certain well-behaved functionals of the fields, called the *correlation functions* of the system. These are, in fact, what the physicist would like to compute.

We denote the general correlation function $C(m, n)$ by the *symbolic* expression

$$C(m,n) \quad = \quad < b(Q_1)\dots b(Q_m)c(P_1)\dots c(P_n) > \tag{1.1}$$

where $Q_1, \dots, Q_m, P_1, \dots, P_n$ are points on M. In physics, such a correlation function should give the expectation value of finding m particles of the field b and n of the field c at $Q_1, \dots, Q_m$ and $P_1, \dots, P_n$ respectively, in their ground state. It is difficult, however, to sustain a serious physical interpretation of the correlation functions in view of the physical artificiality of the model, although these fields do arise naturally in string theory as Faddeev-Popov ghost fields. Their rather pedestrian origin, however, does not give any idea of the remarkable interest of this model from the viewpoint of mathematical physics. Our aim here is to develop a single coherent viewpoint of this system and accordingly we make only occasional mention of some of the many interesting papers that have been written about the system.

We shall regard the correlation functions $C(m, n)$ as functionals of the fields, complete knowledge of which specifies the model. More precisely, the $C(m, n)$ will be seen to be - not necessarily holomorphic- *sections of holomorphic line bundles*. The analyticity properties of these sections are constrained by the *operator product expansion* (OPE) of the b and c fields, which is given by the heuristic relation [1]

$$b(z)c(w) = \frac{I}{z-w} + \text{holomorphic terms}, \tag{1.2}$$

where I is the identity operator and this relation is supposed to make sense only inside one of the $C(m, n)$.

We begin the development of our algebraic geometry formulation of the model by associating a holomorphic line bundle α on M to the "quantum field" c and a holomorphic line bundle β on M to b. Thus $C(m, n)$ is a meromorphic section of the line bundle

$$p_1^*(\beta) \otimes \cdots \otimes p_m^*(\beta) \otimes p_{m+1}^*(\alpha) \otimes \cdots \otimes p_{m+n}^*(\alpha) \tag{1.3}$$

on $M^{m+n} \equiv M_1 \times \ldots \times M_{m+n}$, the cartesian product of $m+n$ copies of M, where $p_i : M^{m+n} \to M_i$ is the i-th canonical projection. The OPE (1.2) will now be interpreted as saying that the only singularities of $C(m,n)$ are simple poles when a Q and a P argument coincide. *We must emphasise that this global principle in no way restricts the possible singularities that can arise in practice.* All it means is that we require that *any singularity, other than the ones of physical origin coming from the OPE (1.2), must be forced on us by rigorous mathematical analysis.* (See the discussion of the spin $(1-J)-J$ system below). While our principle has a certain resemblance to the "principle of maximal analyticity" in vogue in physics many years ago, it is more compelling here, since our model belongs to a class of models essentially defined through their OPE's. It should be noted, however, that our principle extends the meaning of the OPE, a property of the model on the *complex plane* [1], to the *compact Riemann surface M.*

Our first conclusion from this is that the "one point functions" $C(1,0) = < b >$, $C(0,1) = < c >$ are simply holomorphic sections of β and α respectively. A physicist who was studying this system starting from a lagrangian would call these one point functions the *zero modes* of the system and would want to eliminate them. Starting from the OPE (1.2) as we are doing, however, there is no reason *up to now* to introduce constraints on the line bundles α and β so as to eliminate such sections.

The first nontrivial case is that of the "two point function" $C(1,1) = < b(Q)c(P) >$. In view of the OPE (1.2) we are led to the following definition :

Definition 1.1 *$< bc >$ is a meromorphic section of the line bundle $p_1^*(\beta) \otimes p_2^*(\alpha)$ whose only singularity is a simple pole along the diagonal Δ of $M \times M$.*

This means that a non-zero two point function exists if and only if the map i in the exact sequence

$$O \to H^0(M\times M, p_1^*(\beta)\otimes p_2^*(\alpha)) \xrightarrow{i} H^0(M\times M, p_1^*(\beta)\otimes p_2^*(\alpha)\otimes \mathcal{O}(\Delta)) \quad (1.4)$$

is not an isomorphism.

So far the line bundles α and β have been completely arbitrary. We cannot, of course, expect to get anything interesting without some restriction, but it is important to make these restrictions as weak as possible. The optimum condition turns out to be to simply bound the sum of the *degrees* of the two line bundles by $2g-2$. We then have the following elegant characterisation of the $b-c$ system (see [2] for the proof):

Theorem 1.2 *Let $deg(\alpha) + deg(\beta) \leq 2g-2$. Then a non-zero two point function $< bc >$ exists if and only if :*
(i) $\beta \otimes \alpha = K \equiv$ the holomorphic cotangent bundle of M
(ii) $deg(\alpha) = g-1 = deg(\beta)$

(iii) neither α nor β have any holomorphic sections.
If these conditions hold then the two point function not only exists, but it is also unique (after normalisation).

Thus we see that simply requiring the existence of a non-zero two point function imposes very stringent conditions on the line bundles α and β, even though the degree condition we imposed is very much weaker than the conditions one would be led to impose in a field-theoretic approach. In fact *we can use Theorem 1.2 to understand the conventional formulation of the model.* Thus condition (i) is precisely the condition that the integrand of the standard action for the $b-c$ system, viz. $S \sim \int_M b\bar{\partial}c$, is indeed a volume form, as it must be for the integration over M to make sense. A field associated with the line bundle $K^{\otimes\lambda}$, the λ-th tensor power of the holomorphic cotangent bundle K, will be said to have "conformal spin λ" and so condition (ii) means that we are in the case when b and c are fields of conformal spin $1/2$, or rather a "twisted" version of it since we do not require $\beta = \alpha = \sqrt{K}$. Finally, condition (iii) says that zero modes must be absent. However, we now see that this is necessary in order to have a two point function whose singularity structure is determined by the OPE (1.2), rather than because some undefined functional integral will otherwise give trouble, as is usually argued !

We seem to be excluding from consideration the (conformal) spin $(1-J)-J$ version of the $b-c$ system (J is a positive integer or half-integer) which is usually considered in the literature. That is, however, not the case. For indeed let us take $deg(\alpha) = 2J(g-1)$. The Riemann-Roch theorem tells us that α has holomorphic sections ("zero modes") if $J \geq 1$. Theorem 1.2 asserts that in that case *the two point function $< bc >$ must have extra singularities not coming from the OPE (1.2) and these extra singularities must be such that we obtain a new $b-c$ system which does satisfy the conditions of the theorem.* One way is to introduce points $x_1, \ldots, x_I$, where $I = (2J-1)(g-1)$ and let D denote the divisor $x_1 + \cdots + x_I$. Define $\tilde{\alpha} \equiv \alpha \otimes \mathcal{O}(-D), \tilde{\beta} \equiv \beta \otimes \mathcal{O}(D)$. Then for $\tilde{\alpha}$ and $\tilde{\beta}$ we have the required properties $deg(\tilde{\alpha}) = g-1 = deg(\tilde{\beta})$ and $\tilde{\alpha} \otimes \tilde{\beta} = \alpha \otimes \beta = K$. Condition (iii) of Theorem 1.2 will also be satisfied for a generic choice of the points $\{x_i, i = 1, \ldots, I\}$. This is *effectively* how physicists handle the spin $(1-J)-J$ case of the $b-c$ system. For complete details we refer to [3].

Another interesting consequence of Theorem 1.2 is that it provides a proof of one of the folk theorems of the physics literature, viz. a kind of "charge conservation theorem" for "spin fields" (more generally, for "twist fields"). We are given pairs of points and rational numbers $\{x_i, \mu_i \mid 1 \leq i \leq N_+\}$, $\{y_j, -\nu_j \mid 1 \leq j \leq N_-\}$, where the x_i, y_j are points on M. The μ_i, ν_j are positive rational numbers which satisfy the constraint $\sum_{i=1}^{N_+} \mu_i - \sum_{j=1}^{N_-} \nu_j = \ell$ (ℓ is a positive or negative integer called the "total twist") and which describe the monodromy of the b and c fields near the corresponding points :

$$\begin{aligned} b(z) &\sim (z-x_i)^{-\mu_i} & c(z) &\sim (z-x_i)^{\mu_i} \\ &\sim (z-y_j)^{\nu_j} & &\sim (z-y_j)^{-\nu_j} \end{aligned} \tag{1.5}$$

Then with the help of techniques from algebraic geometry we can reduce the problem of studying the $b-c$ system in the presence of such a "twist structure" to the generalised system of Theorem 1.2 on a finite cyclic covering $\tilde{M} \to M$, defined by a positive divisor D of M and a line bundle $\mathcal{L}$ such that $\mathcal{L}^{\otimes d} = \mathcal{O}_M(D)$, where d is the degree of the cyclic covering. *Theorem 1.2 then implies that the total twist ℓ must be zero.* In the case of spin fields this immediately implies that for a nonzero two point function we must have as many spin fields with a positive square-root behaviour as with negative, a well known folk theorem [4]. For details of the construction and proofs we refer to [2].

2 Field correlation functions

In the previous section we saw that algebraic geometry helped us to achieve a rather detailed qualitative understanding of the $b-c$ system from its OPE (1.2). In fact, algebraic geometry enables us to do much more and we will need no more input from physics (apart from the question of statistics). We shall from now on assume that the line bundles α and β satisfy the conditions of Theorem 1.2, i.e. that $\alpha \in Pic^{g-1}(M)$ and has no holomorphic sections and that $\beta = K \otimes \alpha^{-1}$. We shall not give any further discussion of the spin $(1-J)-J$ system, for which we refer to [3]. We can now obtain an explicit expression for the two point function $< bc >$ with the help of the following lemma :

Lemma 2.1 *Let $\mathcal{M}_\zeta(1,1) \equiv p_1^*(K \otimes \zeta^{-1}) \otimes p_2^*(\zeta) \otimes \mathcal{O}(\Delta)$, where $deg(\zeta) = g-1$. Then:*
(i) if $g=0$, $\mathcal{M}_\zeta(1,1)$ is the trivial line bundle on $M \times M$,
(ii) if $g \geq 1$, $\mathcal{M}_\zeta(1,1) = \pi_\zeta^*(\mathcal{O}(\Theta))$ where $\pi_\zeta : M \times M \to Pic^{g-1}(M)$ is given by $(Q,P) \mapsto \mathcal{O}(Q-P) \otimes \zeta$. Here Θ denotes the canonical theta divisor (in $Pic^{g-1}(M)$). (This becomes a translate of the usual theta divisor by the Riemann constant once a marking is chosen on M, which defines a Riemann matrix in canonical form).

We also need the concept of the "prime form" $E(Q,P)$ for which we have found it convenient to introduce a new *algebro-geometric* definition :

Definition 2.2 *We define the prime form to be the image of the canonical element $1 \in \mathcal{O}_{M\times M}$ in the exact sequence :*

$$0 \longrightarrow \mathcal{O}_{M\times M} \overset{1 \mapsto E(Q,P)}{\longrightarrow} \mathcal{O}_{M\times M}(\Delta) \tag{2.1}$$

This definition of the prime form $E(Q,P)$ can be related to the usual function-theoretic definitions in various genera to be found in the books of Fay[7] and Mumford[8] with the help of Lemma 2.1. Then by once again using Lemma 2.1, we can obtain the two point function $< bc >$ explicitly and for $g \geq 1$ it coincides with the *Szegö kernel for a compact Riemann surface*, which was introduced by Hawley and Schiffer [5]. We refer to [2] and [6] for details of our approach.

The question of determining the higher point field correlation functions of the system is, of course, not meaningful until we have specified the *statistics* of the system. In [2] we have analysed the possible statistics of the system from an axiomatic analysis of the OPE (1.2). We shall not go into that here but merely report the conclusion that the usual Fermi/Bose dichotomy holds (if we weaken our requirements then some more exotic possibilities do exist [2]). Of the two cases the fermionic one turns out to be more interesting and we shall confine our attention to that case, although the bosonic case requires only a simple modification.

Our claim that the fermionic case is more interesting than the bosonic one only holds if we implement the condition of fermionic statistics in a special way, viz. by adding to the OPE (1.2) the following OPE's for two b or two c fields:

$$b(z)b(w) \sim O(z-w), \qquad c(z)c(w) \sim O(z-w) \tag{2.2}$$

whose meaning is that the correlation function $C(m,n)$ should vanish when the arguments of two b or two c fields coincide. We can now write down simple axioms for all the field correlation functions $C(m,n)$:

Axioms 2.3 *Each field correlation function* $C(m,n) =< b(Q_1)\ldots b(Q_m)$ $c(P_1)\ldots c(P_n) >$ *is a meromorphic section of the holomorphic line bundle*

$$\mathcal{F}_\alpha(m,n) \equiv p_1^*(K\otimes\alpha^{-1})\otimes\cdots\otimes p_m^*(K\otimes\alpha^{-1})\otimes p_{m+1}^*(\alpha)\otimes\cdots\otimes p_{m+n}^*(\alpha) \tag{2.3}$$

on M^{m+n} *having :*

($\mathcal{A}1$) *a simple zero for* $Q_i = Q_j$ *or* $P_i = P_j$,

($\mathcal{A}2$) *a simple pole for* $Q_i = P_j$,

($\mathcal{A}3$) *no singularities other than those required by the second axiom.*

Axioms ($\mathcal{A}1$) and ($\mathcal{A}2$) define *divisors* (formal sums with integral coefficients of codimension 1 subvarieties) of M^{m+n} which we respectively denote by $D_z(m,n)$ and $D_p(m,n)$ and the total divisor is $D(m,n) = D_z(m,n) - D_p(m,n)$, where we follow the usual convention of putting a plus sign for zeros and a minus sign for poles (see [9] for an introduction to this concept for physicists). Then by ($\mathcal{A}3$) we conclude that $C(m,n)$ defines a *holomorphic* section of the line bundle

$$\mathcal{M}_\alpha(m,n) = \mathcal{F}_\alpha(m,n)\otimes\mathcal{O}(-D(m,n)) \tag{2.4}$$

Thus, $C(m,n)$ defines an element of $H^0(M^{m+n},\mathcal{M}_\alpha(m,n))$ and we can pose a precise mathematical question, viz. what is the dimension of the space of holomorphic sections of $\mathcal{M}_\alpha(m,n)$? The answer is given by the following theorem (see [6] for the proof) :

Theorem 2.4 *(i) If $m \neq n$,* $\quad dim H^0(M^{m+n}, \mathcal{M}_\alpha(m,n)) = 0$
(ii) If $m = n$, $\quad dim H^0(M^{2n}, \mathcal{M}_\alpha(n,n)) = 1$

This theorem has significant implications for the physics of the $b-c$ system, for part (i) implies that $C(m,n) = 0$ for $m \neq n$, which is in perfect agreement with the physicist's argument that this must happen due to the requirement of *charge conservation*. This latter argument would normally be based on the lagrangian of the system, but here we see that the OPE's serve equally well. Part (ii) of the theorem is even more remarkable, as it implies the validity of the *Wick representation*

$$C(n,n) \quad = \quad \det(< b(Q_i)c(P_j) >)\Big|_{i,j=1}^{n} \tag{2.5}$$

since it is clear that the determinant of two point functions on the right-hand side of (2.5) satisfies all our axioms. This result shows that indeed our method of introducing the condition of fermionic statistics through the OPE's (2.2) was correct. Moreover, it also leads to - rigorous proofs of- interesting identities once we can write down the correlation functions explicitly. For this all we need is the following lemma and our previous observations on the prime form :

Lemma 2.5 *(i) If $g = 0$, then $\mathcal{M}_\alpha(n,n))$ is the trivial line bundle;*
(ii) for $g \geq 1$, $\mathcal{M}_\alpha(n,n)) = \pi_\alpha^{n*}(\mathcal{O}(\Theta))$ where $\pi_\alpha^n : M^{2n} \to Pic^{g-1}(M)$ is given by $(Q_1,\ldots,Q_n,P_1,\ldots,P_n) \mapsto \mathcal{O}(\sum_1^n(Q_i - P_i)) \otimes \alpha$.

With the help of this lemma we can write down the $2n$-point function $C(n,n)$ directly as a product of the unique section of $\mathcal{M}_\alpha(n,n)$ (1 for $g = 0$ and a theta function for $g \geq 1$) and the canonical meromorphic section of $\mathcal{O}(D(n,n))$ (a ratio of products of prime forms). On the other hand, by the Wick representation (2.5), *which we have proved*, it can also be expressed as a determinant of two point functions. In this way we obtain, with complete mathematical rigour, interesting identities. These identities are usually referred to as the *bosonization identities* in the physics literature because of the way they first appeared in physics, viz. they provided a proof that two methods of obtaining the correlation functions of the $b-c$ system were consistent [10]. Our analysis, on the other hand, traces their physics origin to the OPE's of the system and is more powerful in that it provides a *proof* of the identities instead of using them to show consistency. Of course, the bosonization viewpoint can also be made rigorous in the Grassmannian formulation[11]. The identities take different forms in different genera and were first obtained by the mathematicians whose names are attached to them :

(i) **Cauchy's bialternant identity** ($g = 0$) :

$$\frac{\Pi_{1\le i<j\le n}(Q_i - Q_j)(P_j - P_i)}{\Pi_{1\le i,j\le n}(Q_i - P_j)} = \det\left(\frac{1}{(Q_i - P_j)}\right)\Big|_{i,j=1}^{n} \qquad (2.6)$$

(ii) **Frobenius' identity** ($g = 1$) :

$$\frac{\sigma(\alpha + \sum_1^n(Q_i - P_i))}{\sigma(\alpha)} \quad \frac{\Pi_{1\le i<j\le n}\sigma(Q_i - Q_j)\,\sigma(P_j - P_i)}{\Pi_{1\le i,j\le n}\sigma(Q_i - P_j)} = \det\left(\frac{\sigma(\alpha + Q_i - P_j)}{\sigma(\alpha)\,\sigma(Q_i - P_j)}\right)\Big|_{i,j=1}^{n} \qquad (2.7)$$

(iii) **Fay's identity** ($g \ge 2$) :

$$\frac{\theta[\alpha](\sum_1^n(Q_i - P_j))}{\theta[\alpha](0)} \quad \frac{\Pi_{1\le i<j\le n}E(Q_i\,,\,Q_j)\,E(P_j\,,\,P_i)}{\Pi_{1\le i,j\le n}E(Q_i\,,\,P_j)} = \det\left(\frac{\theta[\alpha](Q_i - P_j)}{\theta[\alpha](0)\,E(Q_i\,,\,P_j)}\right)\Big|_{i,j=1}^{n} \qquad (2.8)$$

Note that in (2.7) and (2.8) the conditions that $\sigma(\alpha) \ne 0$ and $\theta[\alpha](0) \ne 0$ hold if and only if α has no holomorphic sections, which is ($\mathcal{A}3$) of our axioms. The case $n = 2$ of (2.8) is usually known in the mathematics literature as the *trisecant identity* for geometrical reasons into which we shall not go here [8]. Detailed proofs of the results of this section can be found in [6] and [2]. An exposition of our proof for mathematicians, in which the connections with physics have been eliminated, has also appeared in the treatise [12].

Before we conclude our discussion of the field correlation functions of the $b-c$ system, let us expand on a remark in [2] concerning a variant of the system dicussed above, which sometimes appears in the literature (see the paper of the Verlindes in [10]). The only difference is that the defining line bundle α is now taken to be an *odd theta characteristic*. More generally, we take $deg(\alpha) = g - 1$, $h^0(M,\alpha) = 1$, i.e. α is a smooth point of the canonical theta divisor Θ in $Pic^{g-1}(M)$. Of course, Theorem 1.2 says that this system has no two point function in the sense of Definition 1.1, but in the physics context what is of interest is the question of the existence of higher point functions satisfying Axioms 2.3. By a careful analysis of the proof [6] of part (ii) of Theorem 2.4 we obtain :

Theorem 2.6 *$dim H^0(M^{2n}, \mathcal{M}_\alpha(n,n)) = 1$ $(n = 1,2,\ldots) \iff$ either $\alpha \in Pic^{g-1}(M) - \Theta$ or α is a smooth point of Θ.*

With this theorem we can not only show that the system has $2n-$point functions (for $n \ge 2$) uniquely determined by Axioms 2.3, but with this we can also give a *direct proof* of a corollary to Fay's identity (2.8), which Fay [7] obtains by a limiting argument from (2.8) (see equation following eqn.43 on p.33 of [7]). Thus we see that this case of the $b - c$ system is also covered by our approach.

3 Currents

We shall now describe some recent work [13](details will appear in [14]) on an algebraic geometry approach to the current correlation functions. The heuristic definition of the current $j(z)$ is through point-splitting and subtracting the leading singularity :

$$j(z) \quad = \quad \lim_{Q \to P = z} b(Q)c(P) \quad - \quad \frac{1}{Q \ - \ P} \tag{3.1}$$

This definition does not lend itself in any obvious way to a geometric formulation, but we shall show (see [13]) that in fact the modern Grothendieck formulation of algebraic geometry provides us with the necessary concepts to achieve this. To understand the problem, let us first consider the one point function $< j(z) >$. According to the heuristic definition (3.1), $< j(z) >$ should be identified with the coefficient of $(Q - P)$ in an expansion of $< b(Q)c(P) > \ E(Q,P) - 1$ about the diagonal Δ of $M \times M$. This definition suggests that $< j(z) >$ is a holomorphic one form on M, but this procedure does not offer a global geometric definition. It involves subtracting sections of (at least for $g \geq 1$) two different line bundles, viz. $\mathcal{M}_\alpha(1,1)$ and the trivial line bundle on $M \times M$, and performing a Taylor series expansion.

Our solution to this problem is based on the observation that we do not require $\mathcal{M}_\alpha(1,1)$ to be trivialisable on the whole of $M \times M$, which in any case is not true for $g \geq 1$, but only on the *first infinitesimal neighbourhood of* Δ *in* $M \times M$, which is defined through the concept of a *ringed space.* Since ringed spaces are not very familiar to physicists, let us first consider the variety Δ itself as a ringed space. It is defined as a pair $(\Delta,\ \mathcal{O}_\Delta)$ consisting of the topological space Δ and its structure sheaf of holomorphic functions $\mathcal{O}_\Delta$, which is the quotient of the sheaf of holomorphic functions on $M \times M$ modulo those vanishing on Δ. The restriction of $\mathcal{O}_\Delta$ to an affine open set $\mathcal{U}$ is of the form $k[x,y]/\Im(x-y)$, where $k[x,y]$ is the polynomial ring in the variables x and y and $\Im(x-y)$ is the ideal in $k[x,y]$ generated by $(x-y)$. This quotient is of the form of a polynomial ring in *one* variable $k[t]$, where $t = x + y$, which is as it should be for it to be the structure sheaf of the one dimensional variety Δ, which is simply a copy of M.

The first infinitesimal neighbourhood 2Δ of Δ consists of the pair $(\Delta,\ \mathcal{O}_{2\Delta})$, where Δ is, as before, the topological space, but with a new structure sheaf $\mathcal{O}_{2\Delta}$. This latter is the quotient of the sheaf of holomorphic functions on $M \times M$ modulo those with a *double zero* on Δ. The restriction of this to an affine open set $\mathcal{U}$ is of the form $k[x,y]/\Im(x-y)^2$, where $\Im(x-y)^2$ is the square of the ideal generated by $\Im(x-y)$. This quotient is of the form $k[t] \oplus k[t]dt$, where $t = x + y$ and $dt = x - y$ so that $(dt)^2 = 0$, i.e. *the ringed space* $(\Delta,\ \mathcal{O}_{2\Delta})$ *contains nilpotents.* The global geometric way of describing this is that

$$p_{1*}(\mathcal{O}_{2\Delta}) \quad = \quad \mathcal{O}_M \ \oplus \ K_M, \tag{3.2}$$

which says that the direct image of the structure sheaf of $\mathcal{O}_{2\Delta}$ to the first factor of $M \times M$ is the direct sum of the trivial line bundle and the cotangent bundle of M (or rather their associated sheaves). The important point is that the decomposition (3.2) is *canonical*. This means that we can in a natural way find the component of an element of the l.h.s. of (3.2) in each factor on the r.h.s.

4 Field-current correlation functions

We shall now see how the concepts introduced in the previous section enable us to compute not merely $< j(z) >$, but also the general field-current correlation function $< b(Q_1)\ldots b(Q_{n-1})c(P_1)\ldots c(P_{n-1})j(z) >$. By (3.1), this field-current correlation function is the coefficient of $(Q_n - P_n)$ in an expansion of $C(n,n)E(Q_n,P_n)$ about the diagonal $Q_n = P_n = z$ of $M_n \times M_{2n}$. This suggests that the field-current correlation function is a meromorphic one form for fixed $\{Q_i,\ P_i,\ 1 \le i \le n-1\}$. As explained in the last section, our proposal is to study $C(n,n)E(Q_n,P_n)$ on *the first infinitesimal neighbourhood* 2Δ of Δ in $M_n \times M_{2n}$ and to use the *canonical global splitting* of (3.2) above to determine this meromorphic one form.

Now $C(n,n)E(Q_n,P_n)$ is a meromorphic section of $\mathcal{F}_\alpha(n,n)\otimes\mathcal{O}(D_{n,2n})$, where D_{ij} denotes the divisor of M^{2n} defined by the diagonal of $M_i \times M_j$. Since the only relevant variables of $C(n,n)E(Q_n,P_n)$ are Q_n and P_n, we can restrict $\mathcal{F}_\alpha(n,n) \otimes \mathcal{O}(D_{n,2n})$ to general position on each M_i for $i \neq n, 2n$ to get the line bundle $\mathcal{L} = \mathcal{M}_\gamma(1,1)\otimes\mathcal{G}$ on $M_n \times M_{2n}$, where $\mathcal{G} = p_n^*(\mathcal{O}(D)) \otimes p_{2n}^*(\mathcal{O}(-D))$, $\gamma = \alpha \otimes \mathcal{O}(D)$ and $D = \sum_{i=1}^{n-1}(Q_i - P_i)$. As explained in the last section, our approach will only make sense if $\mathcal{L}$ is trivialisable on the ringed space 2Δ. While it is simple to see that $\mathcal{L}$ is trivialisable on Δ, this is a very subtle question on 2Δ, but nevertheless true :

Proposition 4.1 *The line bundles $\mathcal{M}_\gamma(1,1)$ and $\mathcal{G}$ are both trivialisable on 2Δ and thus $\mathcal{L}$ is so as well.*

We can now use the canonical splitting (3.2) to compute the image of $C(n,n)E(Q_n,P_n)$ in (3.2), where for $g \ge 1$ we take the classical trivialisation of $\mathcal{M}_\gamma(1,1)$ by Riemann's theta function $\theta[\alpha](\sum_1^{n-1}(Q_i - P_i) + \vec{u})/\theta[\alpha](\sum_1^{n-1}(Q_i - P_i))$. Here α in square brackets denotes theta characteristics determined by α (after having chosen a marking on M, a symplectic homology basis and the dual basis of holomorphic one forms $w_i, 1 \le i \le g$), $\vec{u}$ is in $\mathbb{C}^g$ and the sums are Abel sums. We also need to determine what happens to the natural meromorphic section of $\mathcal{G}$, which is a product of prime forms.

Lemma 4.2 *Let 1_D denote the the natural section of $\mathcal{O}(D)$ and now consider $p_n^*(1_D) \otimes p_{2n}^*(1_{-D})$, which can be written in the notation of prime forms as $(\Pi_1^{n-1}E(Q_n,Q_i)E(P_n,P_i))/(\Pi_1^{n-1}E(Q_n,P_i)E(P_n,Q_i))$. Then the* *restriction of this meromorphic section to 2Δ takes, in the canonical decomposition (3.2) above, the form $1 + w_D$, where w_D is a meromorphic one form on Δ (identified with M_n) having simple poles at $Q_1, \ldots, Q_{n-1}$ with residue $+1$ and at $P_1, \ldots, P_{n-1}$ with residue -1. The signs of these residues are fixed by choosing an isomorphism between K_Δ and $\mathcal{O}(-\Delta)|\Delta$ which, when $g \geq 1$, is compatible with the Abel map.*

Lemma 4.2 is an algebro-geometric form of a well known formula for the prime form: $w_{a-b}(z) = d_z \ln(E(z,a)/E(z,b))$. Note that there is no canonical trivialisation of $\mathcal{G}$ on 2Δ. The effect of choosing a different trivialisation is to simply change w_{a-b} by the addition of a *holomorphic* one form, which does not matter since w_{a-b} was only defined up to the addition of such a one form. When we want to write the formulas for the correlation function in function- theoretic form, the correct choice will be made by the chosen marking of M and the representation of the prime form as a function on $\mathbf{U}\times\mathbf{U}$, where $\mathbf{U}$ is the universal covering space of M.

With these two results it is easy to obtain the field-current correlation function :

Theorem 4.3 *The field-current correlation function is given by :*

$$< b(Q_1)\ldots b(Q_{n-1})c(P_1)\ldots c(P_{n-1})j(z) >= C(n-1,n-1) \times$$
$$\left(w_D(z) + \sum_{j=1}^{g} w_j(z)\, \partial \ln \theta[\alpha](\sum_{i=1}^{n-1}(Q_i - P_i) + \vec{u})/\partial u^j \Big|_{\vec{u}=0} \right) \qquad (4.1)$$

where the second term is absent for $g = 0$.

Alternatively, we can compute the field-current correlation function starting from the determinantal form of the field correlation function. For this we need an algebro-geometric analogue of a simple formula given by Mumford (see part (a) of the lemma on page 3.225 of [8]) for the prime form, viz. $d_z(E(z,a)/E(z,b))|_{z=a} = 1/E(a,b)$. From the proof it is clear that Mumford actually obtains $d_z E(z,a)|_{z=a} = 1$ from which this formula follows. Our algebro-geometric analogue of the latter is that we can find the image of the canonical section "1" of the trivial line bundle $\mathcal{O}_{M\times M}$ on $M \times M$ in $\mathcal{O}_{2\Delta}(\Delta)$ by traversing the following commutative diagram in two different ways :

$$\begin{array}{ccc} \mathcal{O}_{M\times M}|\Delta & \hookrightarrow & \mathcal{O}_{2\Delta}(\Delta) \\ \uparrow & & \uparrow \\ \mathcal{O}_{M\times M} & \hookrightarrow & \mathcal{O}_{M\times M}(\Delta) \end{array} \qquad (4.2)$$

As a result of this calculation we obtain the following alternative form for the field-current correlation function :

Theorem 4.4 *From the Wick representation we obtain,*

$$< b(Q_1)\dots b(Q_{n-1})c(P_1)\dots c(P_{n-1})j(z) >= C(n-1,n-1) < j(z) >$$

$$-\sum_{k=1}^{n-1} < b(Q_1)\dots b(Q_{n-1})c(P_1)\dots c(P_k = z)\dots c(P_{n-1}) >< b(z)c(P_k) > \tag{4.3}$$

The two expressions (4.1) and (4.3) for the field-current correlation functions give us an identity, which is a simple generalisation of the "first corollary to Fay's identity" given by Mumford [8], valid in all genera. We refer to [14] for details.

5 The two point function of currents

The two point function of currents $< j(z_1)j(z_2) >$ is the most important current correlation function from the point of view of physics and so it is very important to see whether our techniques generalise to this case.

From the OPE (1.2) we see that $< j(z_1)j(z_2) >$ is given in terms of the 4-point function $C(2,2)$ by the following heuristic double limit :

$$\begin{aligned} < j(z_1)j(z_2) > = & \{C(2,2)E(Q_1,P_1)E(Q_2,P_2) \\ & -1- < j(z_1) > - < j(z_2) >\}\Big|_{Q_i \to P_i = z_i (i=1,2)} \end{aligned} \tag{5.1}$$

It may look very unlikely that we can make mathematical sense out of (5.1), since we have to make sense first of the sum of 1, $< j(z_1) >$ and $< j(z_2) >$ and then of subtracting it from the first term in (5.1) !

We expect $< j(z_1)j(z_2) >$ to be a (meromorphic) one form in each variable, i.e. that it is a meromorphic section of the *canonical bundle* $\omega_{M\times M} \equiv p_1^*(K)\otimes p_2^*(K)$ of $M \times M$. Now $C(2,2)E(Q_1,P_1)E(Q_2,P_2)$, which appears in (5.1), is a meromorphic section of the line bundle

$$\mathcal{R} \equiv \mathcal{F}_\alpha(2,2)\otimes \mathcal{O}(D_{13}+D_{24}) = \mathcal{M}_\alpha(2,2)\otimes \mathcal{A}, \tag{5.2}$$

where

$$\mathcal{A} \equiv \mathcal{O}(D_{12} + D_{34} - D_{14} - D_{23}) \tag{5.3}$$

The natural generalisation of the procedure of the previous section is to consider the restriction of the line bundle $\mathcal{R}$ to the sub-*scheme* $Z \equiv 2\Delta_{13}\times 2\Delta_{24}$ of M^4. Our basic tool for studying this restriction is the following beautiful exact sequence

$$O \longrightarrow \omega_Y \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{2Y} \longrightarrow O \tag{5.4}$$

where $Y \equiv \Delta_{13}\times\Delta_{24}$.

The exact sequence (5.4) does not seem to exist in the mathematics literature and its only *raison d'être* appears to be to enable us to make sense of the heuristic formula (5.1), for which it has just the right properties. For if pr_{12} denotes the projection $M^4 \to M_1 \times M_2$ and we take the direct image of (5.4) by it, we get a new exact sequence

$$O \longrightarrow \omega_{M_1 \times M_2} \longrightarrow pr_{12*}(\mathcal{O}_Z) \longrightarrow pr_{12*}(\mathcal{O}_{2Y}) \longrightarrow O \qquad (5.5)$$

The remarkable feature of (5.5) is that $pr_{12*}(\mathcal{O}_{2Y})$ is a rank 3 vector bundle on $M_1 \times M_2$ which is *canonically* the direct sum of three line bundles :

$$pr_{12*}(\mathcal{O}_{2Y}) \;=\; \mathcal{O}_{M_1 \times M_2} \;\oplus\; K_{M_1} \;\oplus\; K_{M_2} \qquad (5.6)$$

Note that this splitting enables us to make sense out of the formal sum $1 + < j(z_1) > \; + \; < j(z_2) >$ in (5.1). In addition, $pr_{12*}(\mathcal{O}_Z)$ is a rank 4 vector bundle on $M_1 \times M_2$ which is *canonically* the direct sum of four line bundles :

$$pr_{12*}(\mathcal{O}_Z) \;=\; \mathcal{O}_{M_1 \times M_2} \;\oplus\; K_{M_1} \;\oplus\; K_{M_2} \;\oplus\; \omega_{M_1 \times M_2} \qquad (5.7)$$

Now all we need is to know whether the line bundle $\mathcal{R}$ is trivialisable on Z.

Proposition 5.1 *The line bundles $\mathcal{M}_\alpha(2,2)$ and $\mathcal{A}$ are trivialisable on Z and hence so is $\mathcal{R}$.*

In order to be able to actually compute $< j(z_1) j(z_2) >$ we need to know what happens to $C(2,2)E(Q_1,P_1)E(Q_2,P_2)$ on restriction to Z. It is the product of the unique section of $\mathcal{M}_\alpha(2,2)$, which is easy to handle by a simple generalisation of the procedure for the case of the field-current correlation function, and the canonical meromorphic section of the line bundle $\mathcal{A}$, defined in (5.3), which we shall denote as $1_{\mathcal{A}}$. The answer in this case can be expressed in terms of what is sometimes called the *Bergmann kernel* ω_B [5] or, more appropriately, the *generalised Weierstrass* $\wp$ *function*. This concept was introduced for a compact Riemann surface in function-theoretic form by Hawley and Schiffer [5]. This way is not suitable for us and so we introduce it through an algebro-geometric definition: it is a symmetric meromorphic section of $\omega_{M \times M}$, defined by a holomorphic section of $\omega_{M \times M}(2\Delta)$ which is 1 on restriction to the diagonal Δ of $M \times M$.

Lemma 5.2 *Let $1_{\mathcal{A}}$ denote the canonical meromorphic section of the line bundle $\mathcal{A}$ defined in (5.3), which can be written as $\{E(Q_1,Q_2)E(P_2,P_1)\}$ $/\{E(Q_1,P_2)E(Q_2,P_1)\}$ in the notation of prime forms. Then its restriction to Z has the following decomposition in the canonical decomposition of eqn.(5.7) :*

$$1_{\mathcal{A}}|Z \;=\; 1 \;+\; \omega_B(z_1,z_2) \qquad (5.8)$$

Since there is no canonical trivialisation of $\mathcal{A}$ on Z, $\omega_B(z_1, z_2)$ is defined only up to the addition of a holomorphic bidifferential. However, a definite one is automatically fixed when the appropriate choices have been made, just as for $w_D(z)$ in Lemma 4.2.

If we think about the meaning of restricting $1_{\mathcal{A}}$ to Z we easily realise that Lemma 5.2 is simply an algebro-geometric proof of the following well known formula $\omega_B(x, y) = \partial^2 \ln E(x, y)/\partial x \partial y$ (see [7] and [8]). When $g = 1$ this gives the following well known formula linking two functions of Weierstrass, viz. $\wp(z) = -d^2 \ln \sigma(z)/dz^2$. The link between Lemma 5.2 and this equation is the elementary formula

$$\lim_{\substack{\epsilon\to 0\\ \delta\to 0}} \left(\frac{f(x+\epsilon, y+\delta)f(x,y)}{f(x+\epsilon, y)f(x, y+\delta)} - 1\right) = \frac{\partial^2}{\partial x \partial y} \ln f(x,y) dx\, dy. \qquad (5.9)$$

Lemma 5.2 gives a rather remarkable geometric interpretation of this formula for ω_B, for if we write down $1_{\mathcal{A}}$ on an affine subset of the Riemann sphere (case when $g = 0$), i.e. on the complex plane, we see that it is simply the *anharmonic ratio* of four points on the complex plane. *Thus $1_{\mathcal{A}}$ is a natural generalisation of the notion of the anharmonic ratio (or cross ratio) to a compact Riemann surface of arbitrary genus.* Thus the formula for the Weierstrass function $\wp(z)$ comes by coalescing two pairs of arguments in the generalised anharmonic ratio in the case of genus $g = 1$. This viewpoint accords an unexpected fundamental role to this well known formula as well as a new insight. In Lemma 5.2 we wrote down this generalised anharmonic ratio in terms of prime forms. By the dictionary we have earlier established for the prime form in [2] and [6], we find that this combination of prime forms coincides with a function-theoretic definition of a generalised anharmonic ratio proposed recently by Gunning [15].

We can now compute the image of $C(2,2)E(Q_1, P_1)E(Q_2, P_2)|Z$ in (5.7) to obtain :

Theorem 5.3 *The two point function of currents is given by*

$$< j(z_1)j(z_2) > \; = \; \omega_B(z_1, z_2) + \sum_{i,j=1}^{g} w_i(z_1)w_j(z_2)\frac{\partial^2\theta[\alpha](\vec{u}+\vec{v})}{\partial u_i \partial v_j \theta[\alpha](0)}\Big|_{\vec{u}=\vec{v}=0} \qquad (5.10)$$

where the second term on the right-hand side is absent if $g = 0$.

Theorem 5.4 *From the Wick representation we obtain*

$$< j(z_1)j(z_2) > \; = \; < j(z_1) >< j(z_2) > - < b(z_1)c(z_2) >< b(z_2)c(z_1) > \qquad (5.11)$$

Equating these two expressions for $< j(z_1)j(z_2) >$ gives us only a tautology for $g = 0$, but for $g \geq 1$ it gives the "second corollary to the trisecant identity" [8]:

Theorem 5.5 *The following identity holds for all genera $g \geq 1$,*

$$\omega_B(z_1, z_2) + \sum_{i,j=1}^{g} w_i(z_1) w_j(z_2) \frac{\partial^2 \ln \theta[\alpha](0)}{\partial u_i \partial v_j} = \frac{\theta[\alpha](z_1 - z_2)\theta[\alpha](z_2 - z_1)}{(\theta[\alpha](0))^2 (E(z_1, z_2))^2} \tag{5.12}$$

It is important to note that our results for the field-current correlation functions (equs.(4.1) and (4.3)) as well as for the two point function of the currents (equs.(5.10) and (5.11)) are perfectly consistent with the standard field-current and current-current operator product expansions as found in [1] (with an adjustment for a difference of notation):

$$b(w)j(z) \sim \frac{b(w)}{z - w}, \qquad c(w)j(z) \sim -\frac{c(w)}{z - w} \tag{5.13}$$

$$j(z)j(w) \sim \frac{1}{(z - w)^2}. \tag{5.14}$$

Details of the proofs of the results of this section will appear in [14].

6 n-Point current correlation functions

We shall briefly summarise the procedure by which we can extend the methods of the previous section so as to calculate the n-point function of the currents $< j(z_1) \ldots j(z_n) >$. Details will be given in [14]. The heuristic double limit (5.1) is now replaced by the multiple limit

$$< j(z_1) \cdots j(z_n) > = \lim_{Q_i \to P_i = z_i (1 \leq i \leq n)} C(n, n) \prod_{i=1}^{n} E(Q_i, P_i)$$
$$- 1 - \sum_{i=1}^{n} < j(z_i) > - \ldots - \sum_{1 \leq i_1 < \ldots\ldots < i_{n-1} \leq n} < j(z_{i_1}) \cdots j(z_{i_{n-1}}) > \tag{6.1}$$

Our starting point is now the line bundle $\mathcal{F}_\alpha(n, n) \otimes \mathcal{O}(\Sigma_{i=1}^n D_{i,n+i})$ of which the first term on the right- hand side of (6.1) is a meromorphic section. This line bundle can be shown to be trivialisable on restriction to the product Z_n of the first order neighbourhoods $2\Delta_{i,n+i}$ of the diagonals Δ_{ij} of $M_i \times M_j$, over $1 \leq i \leq n$. Instead of the exact sequence (5.4) it turns out that we now have an *array* of $n - 1$ short exact sequences. We define the canonical projection $pr_n : M^{2n} \to M^n$ and, as before, taking the direct image of $\mathcal{O}_{Z_n}$ by this map enables us to make sense of the sum of lower order current correlation functions appearing in (6.1). We have then to compute the image of the first term on the r.h.s. of (6.1) in $pr_{n*}\mathcal{O}_{Z_n}$ and to pick out, in the canonical decomposition of the latter, the component giving a (meromorphic) section of the canonical bundle of M^n. This is a rather complicated expression (except when $g = 0$), which we do not write down, but it can be regarded as coming from the non- determinantal side of the expressions we had obtained for $C(n,n)$ in Section 2. The computation is much simpler when one starts with the determinantal expression for $C(n,n)$. In this case the n-point function of currents is given by the determinant of the $n \times n$ matrix $\mathbf{A} \equiv (a_{ij})$, where $a_{ij} = < b(z_i)c(z_j) >$ if $i \neq j$ and $a_{ii} = < j(z_i) >$. Equating these two expressions for the n-point function of currents leads to interesting identities in all genera:

Theorem 6.1 *(a) In the case of genus $g = 0$, where $< j(z_1)\cdots j(z_n) >= 0$ if n is odd, let $z_1, \ldots, z_{2m} (m \geq 2)$ be arbitrary complex numbers and define the antisymmetric matrix $\mathbf{A} \equiv \{a_{ij}\}$ by a_{ij} by $a_{ij} = 1/(z_i - z_j)$ for $i \neq j$, 0 otherwise. Then*

$$det(\mathbf{A}) = \sum \prod_{\gamma} 1/(z_{i_\gamma} - z_{j_\gamma})^2, \tag{6.2}$$

where (i_γ, j_γ) is an element of a partition of $1, \ldots, 2n$ into non-intersecting pairs with $i_\gamma < j_\gamma$, the product being over distinct pairs in a given partition and the sum over different partitions.

(b) Let $g \geq 1$ $(n \geq 3)$ and let D_z denote the vector field $\Sigma_{i=1}^{g} w_i(z)\partial/\partial u_i$, where, as before, the w_i are a basis of holomorphic 1-forms. Then

$$D_{z_1}\cdots D_{z_n} \ln\theta[\alpha](0) = (-1)^{(n-1)}\{\sum \prod S_\alpha(z_{i_\gamma}, z_{j_\gamma})\} \tag{6.3}$$

where each term of the sum is labelled by an irreducible n-cycle (e.g. $(12)(23)\cdots(n-1\ n)(n\ 1)$) and the sum is over all such cycles. $S_\alpha(z,w)$ is the Szegö kernel, as usual.

We were unable to find the identity (6.2) in the literature, though, of course, it would be very surprising if it were new. A direct elementary proof of it was obtained by Prof. Don Zagier. The second identity (6.1) can be found in the book of Fay [7] when α is an even theta characteristic (see eqn. (40) on p.27) and $n = 4$.

There is a well known procedure by which one sees that the OPE (5.14)

$$j(z)j(w) \sim \frac{1}{(z - w)^2}$$

is equivalent to the infinite oscillator algebra

$$\{c, j_n | [j_m, j_n] = mc\delta_{m+n,0}, \ [j_n, c] = 0\}$$

It is then natural to regard the set $\{< j(z_1)\ldots j(z_n) > | n = 1, 2\ldots\}$ as a *realisation* of the oscillator algebra with $c = 1$ on an arbitrary compact Riemann surface.

7 Energy-momentum tensor

In this section we shall summarise results of [14] on the only aspect of the system that we have not discussed as yet, viz. the *energy-momentum tensor*. We shall find that once again our ideas of trivialisation play a role, but that this time we have to consider *second order neighbourhoods*. Formally, the energy momentum tensor is the normal ordered product of the currents: $T(z) = (1/2) : jj :$. It is to be calculated heuristically by the following "point splitting formula":

$$T(z) \quad = \quad \lim_{z_1 \to z_2 = z} \frac{1}{2}(< j(z_1)j(z_2) > \quad - \quad \frac{1}{(z_1 \quad - \quad z_2)^2}) \qquad (7.1)$$

This means that the one point function $< T(z) >$ should be obtained by expanding $< j(z_1)j(z_2) > (E(z_1, z_2))^2 \; - \; 1$ about the diagonal and looking at the coefficient of $(z_1 - z_2)^2$, which is of a higher order than in the corresponding situation for currents. We are thus led to consider the holomorphic line bundle $\omega_{M\times M}(2\Delta)$ restricted to the *second order neighbourhood* of the diagonal, which we shall denote as 3Δ; it represents the ringed space $(\Delta, \mathcal{O}_{3\Delta})$ where $\mathcal{O}_{3\Delta} = \mathcal{O}_{M\times M}/\mathcal{O}(-3\Delta)$. The exact sequence that we have to consider is the following canonical exact sequence:

$$O \longrightarrow K_{\Delta}^2 \longrightarrow \mathcal{O}_{3\Delta} \longrightarrow \mathcal{O}_{2\Delta} \longrightarrow O \qquad (7.2)$$

We then have the following proposition:

Proposition 7.1 *The line bundle $\omega_{M\times M}(2\Delta)$ has a canonical trivialisation on 2Δ and is trivialisable on 3Δ (in fact on $n\Delta$).*

We shall normalise the trivialisation by requiring that the section which defines it should restrict to $1/2$ on Δ.

Definition 7.2 *We define the one-point function $< T(z) >$ of the energy-momentum tensor to be a trivialisation of $\omega(2\Delta)$ on 3Δ such that its restriction to 2Δ coincides with the canonical trivialisation, taking the value $1/2$ on Δ.*

It should be noted that the exact sequence (7.2) implies that two such $< T(z) >$ differ by a holomorphic quadratic differential.

Theorem 7.3 *The above definition of $< T(z) >$ is equivalent to an assignment of quadratic differentials to each open set in a comlex analytic coordinate covering $\mathcal{U}$ of Δ, which is simply a copy of M, such that on overlaps they transform as quadratic differentials plus an inhomogeneous term, viz. $(1/12)\times$ the schwarzian of the coordinate transformation.*

According to a standard mathematical terminology [16], this implies that $12 < T(z) >$ is a *projective connection*. On the other hand, according to the definition of Belavin, Polyakov and Zamolodchikov [17], this implies that *the system has central charge* $c = 1$. *This is an algebro-geometric counterpart on a compact Riemann surface of any genus of a well known result in the theory of the Virasoro algebra* (see case (i) of Remark 4.2 on page 46 of [18]. Since we have shown that the spin $(1 - J) - J$ version of this system is a disguised version of the (twisted) conformal spin 1/2 case, it is not surprising that our methods can be extended to the other cases of the anomaly formula.

It should also be noted that we have obtained a rather elegant geometric interpretation of both $< T(z) >$ as well as of the concept of a projective connection, viz. that they are just an infinitesimal version of our generalised anharmonic ratio ! This may be regarded, perhaps, as a concrete form of the characterisation of Deligne that "Intuitivement, se donner une connection projective ... permet de définir le birapport (=rapport anharmonique) de 4 points infiniment voisins ..." [19].

8 Concluding Remarks

We have seen how every detail of the structure of the $b - c$ system is a consequence of its OPE (1.2) and statistics. Algebraic geometry provides a rigorous mathematical language for describing the system and one which is as physically natural as the language of Hilbert space is for Quantum Mechanics. Further developments are under investigation.

It is a pleasure to thank the Max Planck Institut für Mathematik, Bonn, and its director Prof. F. Hirzebruch for hospitality during the preparation of this paper. Discussions with Prof. Werner Nahm and Prof. Don Zagier were most useful and are gratefully acknowledged.

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