EXISTENCE AND UNIQUENESS OF REGULAR SOLUTION FOR EQUATION OF TURBULENT FILTRATION

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1. INTRODUCTION

In the study of water infiltration through three dimensional porous media in turbulent regimes (i.e., in the case when in view of large velocities the classical Darcy law fails to be true) several authors (see [1]) consider equation

$$\mathcal{L}[\theta] \doteqdot \partial\theta/\partial t - \operatorname{div}[| \nabla \varphi(\theta) - K(\theta)Z|^{m-2}(\nabla\varphi(\theta) - K(\theta)Z)] = 0 \tag{1.1}$$

where θ is a volumetric moisture content, $K(\theta)$ is the hydraulic conductivity, $\varphi(\theta) = \int_0^\theta D(\xi)d\xi$, $D(\theta) = K(\theta)\Phi'(\theta)$, $\Phi(\theta)$ is a hydrostatical potential, $Z = (0,0,1)$ $(n = 3)$, $m \in (1,2)$. For $m = 2$ equation (1.1) coincides with the equation of filtration in the case of laminar regimes. In the latter case Gilding and Peletier ([2]) motivated that for many porous media a reasonable choice for D and K would be

$$D = D_0\theta^{s-1},\ K = K_0\theta^r,\ 1 < s \leq r,\ D_0, K_0 > 0. \tag{1.2}$$

Extending this choice to the case $m \in (1,2)$ we can derive from (1.1), (1.2) equation

$$F_0[u] \doteqdot \partial u/\partial t - \operatorname{div}[|u|^\ell| \nabla u - c_0|u|^\kappa Z|^{m-2}(\nabla u - c_0|u|^\kappa Z)] = 0 \tag{1.3}$$

where $Z = (0,0,1)$, $c_0 = \text{const} > 0$ and

$$m \in (1,2),\ \ell = (s-1)(m-1) > 0,\ \kappa = r - s + 1 \geq 1. \tag{1.4}$$

Equation (1.3), (1.4) is a particular case of equation

$$F[u] \doteqdot \partial u/\partial t - \operatorname{div} a(u, \nabla u) = f \tag{1.5}$$

where $f = f(x,t)$ is a given function, $\nabla u = (\partial u/\partial x_1, \dots, \partial u/\partial x_n)$, $a = (a^1, \dots, a^n)$ and functions $a^i(u,p)$, $i = 1, \dots, n$, are continuous and satisfy for all $u \in \mathbb{R}$, $p \in \mathbb{R}^n$ inequalities

$$\begin{aligned} &a(u,p)\cdot p \geq \nu_0|u|^\ell|p|^m - \phi_0(u),\ \nu_0 > 0,\\ &|a(u,p)| \leq \mu_1|u|^\ell|p|^{m-1} + \phi_1(u),\ m \in (1,2),\ \ell \geq 0,\ i = 0,1. \end{aligned} \tag{1.6}$$

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Really we have in the case of equation (1.3)

$$a(u,p) = |u|^\ell |p - c_0|u|^\kappa Z|^{m-2}(p - c_0|u|^\kappa Z). \tag{1.7}$$

Using inequality $|b-c|^m \geq 2\nu_0|b|^m - |c|^m$ for $b, c \in \mathbb{R}^n$ with $2\nu_0 = 2^{1-m}$ we can estimate in the case (1.7)

$$a(u,p)\cdot p \geq \nu_0 |u|^\ell |p|^m - c|u|^{\ell+m\kappa} \tag{1.8}$$

and

$$|a(u,p)| \leq |u|^\ell |p|^m + c|u|^{\ell+(m-1)\kappa} \tag{1.9}$$

with some $c = c(c_0)$. So conditions (1.6) for equation (1.3) are fulfilled with

$$\phi_0(u) = c|u|^{m\kappa+\ell},\ \phi_1(u) = c|u|^{(m-1)\kappa+\ell},\ c = c(c_0). \tag{1.10}$$

We shall call equation (1.3), (1.4) as equation of turbulent filtration. In accordance with classification given in [3] equation (1.5), (1.6) is of the type of

slow diffusion	if	$m+\ell > 2$,
normal diffusion	if	$m+\ell = 2$,
fast diffusion	if	$m+\ell < 2$.

In this paper we study existence and uniqueness of Cauchy-Dirichlet problem for equation of turbulent filtration

$$F_0[u] = f \quad \text{in} \quad Q_T,\ u = \psi \quad \text{on} \quad \Gamma_T \tag{1.11}$$

where $Q_T = \Omega \times (0,T]$, Ω is a bounded open set, $\Gamma_T = S_T \cup [\bar{\Omega} \times \{t=0\}]$, $S_T = \partial\Omega \times (0,T]$, f and ψ are given functions.

We are able to establish existence and uniqueness of some Hölder continuous generalized solution of (1.11) assuming additionally to (1.4) that

$$\max\left(1, \frac{3-m}{4m-3}\right) < s \leq 1 + \frac{2-m\kappa}{m-1}. \tag{1.12}$$

Remark that in view of (1.4) the second inequality in (1.12) is equivalent to condition

$$m\kappa + \ell \leq 2 \tag{1.13}$$

and hence equation (1.3), (1.4), (1.12) is an equation of the type of fast or normal diffusion. Remark that investigations of some qualitative properties of equations of the type of slow, normal, and fast diffusion are given in [4]. In particular from results of [4] it follows that homogeneous ($f \equiv 0$, $\phi_i(u) \equiv 0$, $i = 0,1$), equations (1.5), (1.6) have an infinite speed of propagation if $m + \ell \leq 2$..

The remainder of this paper is structured as follows. In section 2 we discuss uniqueness results established in [5] for equations (1.5), (1.6) with any $m > 1$, $\ell > 1 - m$. In section 3 we state theorem on existence of a regular solution of Cauchy-Dirichlet problem for equations (1.6), (1.7) in the case $m \in (1,2)$, $\ell \geq 0$ which can be derived from results of [6]. In the case $m + \ell \geq 2$, $\max\left(1, \frac{2n}{n+2}\right) < m < 2$, $\ell \geq 0$ this theorem follows also from results of [5]. In section 4 we apply results of previous sections to the equation (1.3), (1.4), (1.12).

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2. Uniqueness of quasistrong solution

Consider equation (1.5) assuming that for any $u, v \in \mathbb{R}$ and any $p, q \in \mathbb{R}^n$

(G) $|a(u,p)| \leq \mu(|u|^\ell |p|^{m-1} + 1)$, $\mu \geq 0$;

(M) $[a(u,p) - a(u,q)] \cdot (p - q) \geq 0$;

and let for any $u, v \in [\epsilon, M]$, $\epsilon > 0$, $M > \epsilon$, for any $p \in \mathbb{R}^n$

(L_{loc}) $|a(u,p) - a(v,p)| \leq \Lambda |u - v|(|p|^{m-1} + 1)$, $\quad \Lambda = \Lambda(\epsilon, M) \geq 0$

where $m > 1$, $\ell > 1 - m$. Consider Cauchy-Dirichlet problem

$$F[u] = f \quad \text{in} \quad Q_T,\ u = \psi \quad \text{on} \quad \Gamma_T \tag{2.1}$$

where $f \in L_1(Q_T)$, $\psi \in W_1^1(Q_T), \psi \geq 0$ in Q_T.

Definition 2.1. Any nonnegative bounded in Q_T function u is a weak solution of equation (1.5) if

a) $u \in C([0,T]; L_2(\Omega))$, $\nabla u^{\sigma+1} \in L_m(Q_T; \mathbb{R}^n)$, $\sigma = \frac{\ell}{m-1}$;

b) for any $\phi \in C^1(\bar{Q}_T)$, $\phi = 0$ on S_T, and any $t_1, t_2 \in [0,T]$

$$\int_\Omega u\phi dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega [-u\phi_t + a(u, u_x) \cdot \nabla\phi - f\phi] dx dt = 0. \tag{2.2}$$

Definition 2.2. Function u is a weak solution of Cauchy-Dirichlet problem (2.1) if u is a weak solution of equation (1.5) and $u = \psi$ onΓ_T.

Remark that any weak solution of (1.5) and any $\psi \in W_1^1(Q_T)$ have traces on Γ_T; so Definition 2.2 has sense.

Definition 2.3. Let $\inf(\psi, \Gamma_T) > 0$. We say that u is a strong solution of Cauchy-Dirichlet problem (2.1) if u is a weak solution of (2.1) and moreover

$$\inf(u, Q_T) > 0 \qquad \text{(and hence} \qquad u \in W_m^{1,0}(Q_T)).$$

Definition 2.4. Let $\psi \in \overset{\circ}{W}{}_1^1(Q_T)$. We say that u is a quasistrong solution of Cauchy-Dirichlet problem (2.1) if u is a weak solution of (2.1) and moreover there exists a sequence of strong solutions of problems

$$F[u_n] = f_n \quad \text{in} \quad Q_T,\, u = \psi_n \quad \text{on} \quad \Gamma_T$$

such that

$$u_n \to u \quad \text{in} \quad C([0,T]; L_1(\Omega));\; f_n \in L_1(Q_T),\; f_n \to f \quad \text{in} \quad L_1(Q_T);$$
$$\psi_n = \psi + \epsilon_n(x,t),\; \epsilon_n \in W_1^1(Q_T) \cap C(\bar{Q}_T),\; \inf(\epsilon_n, \Gamma_T) > 0,\; \sup(\epsilon_n, \Gamma_T) \to 0.$$

Theorem 2.1 (uniqueness of quasistrong solution ([5])). *Let assumptions* (G), (M) *and* (L_{loc}) *hold. Then there is at most one quasistrong solution of Cauchy-Dirichlet problem (2.1).*

Remark 2.1. Theorem 2.1 is proved in [5]. In some sense definition of quasistrong solution and Theorem 2.1 are similar to definition of "the limit of strong solutions" and the correspondent uniqueness theorem of paper [7] by Bamberger for equation $\partial u/\partial t - \operatorname{div}[|u|^\ell |\nabla u|^{m-2} \nabla u] = f$, $m > 1$, $\ell > 0$. However instead of condition $\inf(u, Q_T) > 0$ as in Definition 2.1 Bamberger used condition "u has $\frac{\partial u}{\partial t} \in L_1(Q_T)$".

Remark 2.2. The meaning of Theorem 2.1 is the fact that it is possible (see sect. 3) to prove existence of Cauchy-Dirichlet problem (2.1) in some subclass of quasistrong solutions.

3. Existence of regular solution

In this section we consider Cauchy-Dirichlet problem (2.1) assuming that

(Ω) $\exists \rho_0 > 0\, \exists \alpha_0 \in (0,1)\, \forall x_0 \in \partial\Omega\, \forall \rho \in (0, \rho_0) : |B_\rho(x_0) \cap \Omega| \le (1 - \alpha_0)|B_\rho(x_0)|$;

(BI) $\psi \ge 0$, $\psi \in \overset{0}{W}{}_1^1(Q_T) \cap C_{\beta, \beta/m}(\Gamma_T)$;

(RHS) $f \ge 0$, $f \in L_\infty(Q_T)$.

Definition 3.1. We say that function u is a regular solution of Cauchy-Dirichlet problem (2.1) if u is Hölder continuous in $\bar{Q}_T$ and u is a quasistrong solution of this problem.

Theorem 3.1 (existence of regular solution ([6])). *Let conditions* (Ω), (BI) *and* (RHS) *hold. Assume that the following conditions are fulfilled:*

1) $(m, \ell) \in \mathfrak{D} \backslash \omega$, $\mathfrak{E} \doteqdot \{m \in (1,2),\, \ell > 0\}$, $\omega \doteqdot \left\{(m, \ell) \in \mathfrak{E} : \frac{\sigma+1}{\sigma+2} \le \frac{1}{m} - \frac{1}{n},\, \sigma = \frac{\ell}{m-1}\right\}$;
2) *functions* $|u|^{-\alpha} a^i(u, |u|^{-\alpha} p)$, $\alpha = \frac{\ell}{m}$, *are continuous on* $\mathbb{R} \times \mathbb{R}^n$;
3) *for any* $u \in \mathbb{R}$, $p \in \mathbb{R}^n$
   $a(u,p) \cdot p \ge \nu_0 |u|^\ell |p|^m - \mu_0(|u|^\delta + 1)$, $\nu_0 > 0$, $\delta \in (0, m + \ell)$ *if* $m + \ell > 2$ *and* $\delta = 2$ *if* $m + \ell \le 2$, $|a(u,p)| \le \mu_1 |u|^\ell |p|^{m-1} + \mu(|u|)|u|^\alpha$, $\alpha = \frac{\ell}{m}$, $\mu(s) \ge 1$ *is nondecreasing on* $\mathbb{R}_+$;
4) *there exist* $\nu_1 > 0$ *and continuous vector-function* $b(u) \in \mathbb{R}^n$ *such that for any* $u \in \mathbb{R}$ *and any* $p, q \in \mathbb{R}^n$

$$[a(u,p) - a(u,q)] \cdot (p - q) \ge \nu_1 |u|^\ell |p - q|^2 [|p - b|^m + |q - b|^m]^{1 - \frac{2}{m}};$$

5) *for any* $u, v \in [\epsilon, M]$, $0 < \epsilon < M$, *and any* $p \in \mathbb{R}^n$

$$|a(u,p) - a(v,p)| \le \Lambda |u - v|(|p|^{m-1} + 1),\ \Lambda = \Lambda(\epsilon, M) \ge 0.$$

Then Cauchy-Dirichlet problem (2.1) has at least one regular solution.

Remark 3.1. Theorem 3.1 is a particular case of Theorem 2.1 of paper [6] which is an imporvement of existence result established earlier in [5] where instead of condition 1) we assumed that $\max(1, \frac{2n}{n+2}) < m < 1$ and instead of the first inequality in condition 3) we supposed that $a(u,p) \cdot p \ge \nu_0 |u|^\ell |p|^m - \mu_0(|u|^\delta + 1)$, $\delta \le m + \ell$ if $m + \ell < 2$.

Remark 3.2. Condition 1) means that point (m, ℓ) does not belong to "the bad set ω". It is easy to see that $\omega \subset \{m > 1,\ \ell \ge 0,\ m + \ell < 2\}$. The counterexample of paper [8] shows that the local boundedness (and hence the local hölderness) of generalized solutions of equations (1.5), (1.6) with $f(x,t) \equiv 0$, $\phi_i(u) = 0$, $i = 0, 1$, can not be proved if $(m, \ell) \in \omega$. In this sense condition 1) is sharp.

4. The case of equation of turbulent filtration

The main result of this paper is the following

Theorem 4.1. *Let conditions* (Ω), (BI), *and* (RHS) *hold. Then Cauchy-Dirichlet problem (1.11) for equation (1.3), (1.4), (1.12) has exactly one regular solution.*

Proof. Inequality (1.8) shows that the first inequality in condition 3) of Theorem 3.1 is fulfilled with $\delta = m\kappa + \ell$ where $\kappa \ge 1$. So $\delta \ge m + \ell$ and we can not apply Theorem 3.1 in the case $m + \ell > 2$. Namely therefore we assume that $m\kappa + \ell \le 2$ (see (1.12), (1.13)). In this case the first inequality in condition 3) is fulfilled (with $\delta = 2$). Obviously that the second inequality in condition 3) is also fulfilled with increasing function $\mu(s) = s^\beta$, $\beta = \frac{\ell}{m'} + (m-1)\kappa$, $1/m + 1/m' = 1$. To verify condition 4) we use the following well-known inequality (see, for example, [1])

$$[|p|^{m-2}p - |q|^{m-2}q] \cdot (p - q) \ge \nu_1 |p - q|^2 \cdot \{|p|^m + |q|^m\}^{1-\frac{2}{m}},\ 1 < m < 2, \quad (4.1)$$

where $\nu_1 = \nu_1(m) > 0$. Denote

$$\tilde{p} = p - c_0|u|^k Z,\ \tilde{q} = q - c_0|u|^\kappa Z.$$

Then in view of (4.1) we have in the case (1.7)

$$[a(u,p) - a(u,q)] \cdot (p - q) \ge \nu_1 |p - q|^2 \{|p - b|^m + |q - b|^m\}^{1-\frac{2}{m}},\ b \doteqdot c_0|u|^\kappa Z.$$

Obviously that conditions 1) and 5) are trivially valid for $a(u,p)$ given by (1.7). For example in the case (1.7) we can conclude using identity $\ell = m\alpha$ that function

$$|u|^{-\alpha} a(u, |u|^{-\alpha} p) = |p - c_0|u|^{\kappa+\alpha} Z|^{m-2}(p - c_0|u|^{\kappa+\alpha} Z)$$

is continuous on $\mathbb{R} \times \mathbb{R}^n$.

Finally using that in the case $n = 3$ condition 1) is equivalent to condition

$$\frac{\sigma+1}{\sigma+2} > \frac{1}{m} - \frac{1}{3}, \ \sigma = \frac{\ell}{m-1}, \ m > 1, \ \ell \geq 0 \tag{4.2}$$

and taking into account that $\ell = (m-1)(s-1) > 0$ (see (1.4)), it is easy the see that (4.2) follows from inequality

$$s > \max\left(1, \frac{3-m}{4m-3}\right). \tag{4.3}$$

So in view of the first inequality in condition (1.12) we can derive that condition 1) of Theorem 3.1 is fulfilled. Therefore all conditions of Theorem 3.1 are fulfilled for equation (1.3), (1.4), (1.5). Because conditions (G), (M) and (L_{loc}) of Theorem 2.1 follow from conditions 3), 4) and 5) of Theorem 3.1 we can conclude that Theorem 4.1 is proved.

Rewrite result of Theorem 4.1 in terms of equation (1.1), (1.2). Consider Cauchy-Dirichlet problem

$$\mathcal{L}[\theta] = 0 \quad \text{in} \quad Q_T, \ \theta = 0 \quad \text{on} \quad S_T, \ \theta(x,0) = \theta_0(x) \tag{4.4}$$

where $\mathcal{L}[\theta]$ is defined by the left-hand side of (1.1) with functions $D(\theta)$ and $K(\theta)$ like in (1.2). Assume that initial function $\theta_0(x)$ satisfies conditions

$$\theta_0 \geq 0, \ \theta_0 \in C_\beta(\bar{\Omega}) \quad \text{for some} \quad \beta \in (0,1). \tag{4.5}$$

From Theorem 4.1 we can derive the following

Theorem 4.2. *Let conditions (Ω) and (4.5) hold. Assume that parameters m, s and r (see (1.2)) satisfy conditions*

$$\max(1, \frac{3-m}{4m-3}) < s \leq 1 + \frac{2-m\kappa}{m-1}, \ r = s + \kappa - 1,$$
$$1 \leq \kappa < \max\left(\frac{2}{m}, \frac{5m-3}{4m-3}\right), \ 1 < m < 2.$$

Then Cauchy-Dirichlet problem (4.4) has exactly one regular solution.

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