A jump theorem with uniform estimates for $\bar{\partial}_b$-closed forms on real hypersurfaces

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1 The main result and reduction of the proof to estimates for $\bar{\partial}$

Let $\Omega \subset\subset \mathbb{C}^n$ be a C^2-domain and let $\varrho : U_{\bar{\Omega}} \to \mathbb{R}$ be a C^2-function defined in a neighbourhood $U_{\bar{\Omega}}$ of $\bar{\Omega}$ such that $\Omega = \{z \in U_{\bar{\Omega}} : \varrho(z) < 0\}$ and $d\varrho(z) \neq 0$ for all $z \in b\Omega$. Suppose that for some integer $0 \leq q \leq n-1$ the following convexity condition is fulfilled: For all $z \in b\Omega$ the Levi form of ϱ at z has at least $q+1$ positive eigenvalues.

Further let $M = \{z \in U_{\bar{\Omega}} : \varrho_0 = 0\}$ be a real C^2-hypersurface in $U_{\bar{\Omega}}$ defined by a second C^2-function $\varrho_0 : U_{\bar{\Omega}} \to \mathbb{R}$ with $d\varrho_0(z) \neq 0$ for all $z \in M$ such that the intersection $M \cap b\Omega$ is transversal in the real sense ($d\varrho_0(z) \wedge d\varrho(z) \neq 0$ for all $z \in M \cap b\Omega$).

Set

$$\Omega_+ = \{z \in \Omega : \varrho_0 < 0\} \quad \text{and} \quad \Omega_- = \{z \in \Omega : \varrho_0 > 0\}.$$

In this paper we prove the following

Theorem 1.1 *For each continuous closed (n,r)-form f on $M \cap \bar{\Omega}$ with $r \geq n-q-1$, there exist continuous closed (n,r)-forms f_+ on Ω_+ and f_- on Ω_- such that, for some constant $C > 0$ which is independent of f,*

$$\|f_\pm(z)\| \leq C(1 + |\ln \operatorname{dist}(z,M)|^3) \max_{\zeta \in M \cap \bar{\Omega}} \|f(\zeta)\|, \; z \in \Omega_\pm, \tag{1}$$

and

$$(-1)^{n+r} \int_{M\cap\Omega} f \wedge \varphi = \int_{\Omega_+} f_+ \wedge d\varphi + \int_{\Omega_-} f_- \wedge f\varphi \tag{2}$$

for each $C^\infty_{0,n-r-1}$-form φ with compact support in Ω, where M carries the orientation of $b\Omega_+$. Equation (2) means that $f = f_+ - f_-$ in the sense of distributions.

For the definition of the norm of a differential form at a point which appears in (1) see for instance Section 1.6.3 in [H/Le 1].

In the paper [La/Le] it was proved that under certain additional convexity conditions on M, this theorem together with the main result of [La/Le] leads to uniform estimates for the tangential Cauchy–Riemann equation on $M \cap \bar{\Omega}$. This is the motivation for the present article.

In [La/Le] it was also observed that the essence of Theorem 1.1 is contained in a special uniform estimate for the $\bar{\partial}$-equation on Ω which will be stated in Theorem 1.2 below. Let us repeat the corresponding arguments from [La/Le].

Suppose f is as in Theorem 1.1. Further let

$$B(z,\zeta)=\frac{(n-1)!}{(2\pi i)^n}\sum_{j=1}^{n}(-1)^{j-1}(d\bar{\zeta}_1-d\bar{z}_1)\wedge ..\hat{j}..\wedge(d\bar{\zeta}_n-d\bar{z}_n)\wedge dz_1\wedge\ldots\wedge dz_n$$

be the Martinelli–Bochner–Koppelman kernel. Set

$$\tilde{f}_\pm(z)=(-1)^r\int\limits_{\zeta\in M\cap\Omega} f(\zeta)\wedge B(z,\zeta)\quad\text{for}\quad z\in\Omega_\pm$$

and

$$F(z)=(-1)^r\int\limits_{\zeta\in M\cap b\Omega} f(\zeta)\wedge B(z,\zeta)\quad\text{for}\quad z\in\Omega.$$

Since $d_zB(z,\zeta)=-\bar{\partial}_\zeta B(z,\zeta)$ we get $d\tilde{f}_\pm=F$ on $\Omega_\pm$ and, by the hypothesis on the Levi form of ϱ, it follows from the Andreotti–Grauert theory (see [A/G]) that $F=du$ for some continuous (n,r)-form u on Ω. Setting

$$f_\pm=\tilde{f}_\pm-u$$

we obtain closed continuous (n,r)-forms $f_\pm$ on $\Omega_\pm$. Then the relation (2) follows from the Martinelli–Bochner–Koppelman representation of the form $d\varphi$ It is clear that

$$||\tilde{f}_\pm(z)||\le C_0(1+|\ln\operatorname{dist}(z,M)|)\max_{\xi\in M\cap\Omega}||f(\xi)||\quad\text{for}\quad z\in\Omega_\pm,\tag{3}$$

where $C_0>0$ is a constant which does not depend on f. Moreover for each compact set $K\subset\subset\Omega$ there is a constant $C_K>0$ (independent of f) with

$$||u(z)||\le C_K\max_{\xi\in M\cap\Omega}||f(\xi)||\quad\text{for}\quad z\in K.\tag{4}$$

Hence, except of the validity of estimate (1) near $M\cap b\Omega$, the assertion of Theorem 1.1 may be considered as well-known. To obtain the complete Theorem 1.1 we have to estimate the solution u of $du=F$ near $M\cap b\Omega$. For that we introduce the following abbreviations: $N=M\cap b\Omega$, $\delta(z)=\operatorname{dist}(z,N)$ and $\gamma(z)=||\partial\varrho_0(z)\wedge\partial\varrho(z)||$. Since f is of maximal holomorphic degree we get

$$||f(\zeta)|_N||\le C(\gamma(z)+|\zeta-z|)\max_{\xi\in M\cap\Omega}||f(\xi)||\tag{5}$$

for all $\zeta\in N$, $z\in\Omega$, and therefore

$$||F(z)||\le C'(\frac{\gamma(z)}{\delta(z)}+1+|\ln\delta(z)|)\max_{\xi\in M\cap\Omega}||f(\xi)||,\tag{6}$$

for all $z\in\Omega$, where C and C' are positive constants which do not depend on f. Therefore Theorem 1.1 is a consequence of the following

Theorem 1.2 *Let g be a closed continuous (n,r)-form with $r \geq n-q$ on Ω such that*

$$||g(z)|| \leq \frac{\gamma(z)}{\delta(z)} + \frac{1}{\sqrt{\delta(z)}} \quad \textit{for} \quad z \in \Omega. \tag{7}$$

Then there exists an $(n, r-1)$-form u which is Hölder continuous with exponent $1/2$ on $\bar{\Omega}\backslash N$[1] such that $du = g$ on Ω and

$$||u(z)|| \leq C(1 + |\ln \delta(z)|^3) \quad \textit{for} \quad z \in \Omega. \tag{8}$$

where $C > 0$ is a constant which is independent of g.

Note that in [BF] already a similar result is proved for the more general situation where instead of N appears a submanifold of arbitrary codimension in $b\Omega$. However in the special case of codimension 1 this result is not strong enough to obtain Theorem 1.1. The main point is that in [BF] the "angle" $\gamma(z)$ between the complex tangent planes of $b\Omega$ and M is not considered. The proof of Theorem 1.2 given in the present paper is a development of the arguments from [BF] taking into account the role of $\gamma(z)$.

For the proof of Theorm 1.2 we use as in [BF] a version of the classical integral operator constructed by GRAUERT and LIEB [G/L], HENKIN [H] and W. FISCHER and LIEB [WF/L]. A certain technical difference to [BF] consists in the following: In [BF] only the strictly pseudoconvex case ($q = n-1$) is considered and threrfore global integral formulas can be used. In the case of a general q we have only local operators which immediately give only the following local version of Theorem 1.2:

Theorem 1.3 *For each $\xi \in b\Omega$ there exists a neighbourhood U of ξ such that Theorem 1.2 becomes true after replacing Ω by $U \cap \Omega$.*

By the well-known arguments which are known as GRAUERT's "Beulenmethode" (see, e.g., the proof of Theorem 2.3.5 in [H/Le 1]), Theorem 1.3 and the global results without estimates from the Abdreotti–Grauert theory [A/G] lead to Theorem 1.2. We omit these arguments.

Remark 1.4 Let $\tilde{\varrho}$, $\tilde{\varrho}_0 : U_{\bar{\Omega}} \to \mathbb{R}$ two other C^2-functions with

$$N = \{z \in U_{\bar{\Omega}} : \tilde{\varrho}(z) = \tilde{\varrho}_0(z) = 0\}$$

and

$$d\varrho_0(z) \wedge d\varrho(z) \neq 0 \quad \text{for} \quad z \in N.$$

Then in Theorems 1.2 and 1.3 the function $\gamma(z)$ can be replaced by

$$\tilde{\gamma}(z) := ||\partial\tilde{\varrho}_0(z) \wedge \partial\tilde{\varrho}(z)||.$$

[1]This means u is Hölder continuous with exponent 1/2 on each compact set $K \subset\subset \bar{\Omega}\backslash N$ with a Hölder constant depending on K.

In fact then there is a non-vanishing function φ on N such that $\partial\varrho_0(z) \wedge \partial\varrho(z) = \varphi(z)\partial\tilde{\varrho}_0(z) \wedge \partial\tilde{\varrho}(z)$ for $z \in N$. Hence we get

$$\tilde{\gamma}(z) \leq K(\gamma(z) + \delta(z)) \quad \text{and} \quad \gamma(z) \leq K(\tilde{\gamma}(z) + \delta(z))$$

with some constant $K > 0$ for all $z \in \Omega$, and therefore

$$\frac{1}{2K}\left(\frac{\tilde{\gamma}(z)}{\delta(z)} + \frac{1}{\sqrt{\delta(z)}}\right) \leq \frac{\gamma(z)}{\delta(z)} + \frac{1}{\sqrt{\delta(z)}} \leq 2K\left(\frac{\tilde{\gamma}(z)}{\delta(z)} + \frac{1}{\sqrt{\delta(z)}}\right)$$

for all $z \in \Omega$.

2 Proof of Theorem 1.3

For the proof of Theorem 1.3 we shall use the same integral operator as in [BF/Le]. Recall the following definition from Section 3 in [BF/Le]: $D \subset\subset \mathbb{C}^n$ will be called a local q-convex C^2-domain, $0 \leq q \leq n-1$, if there exist a biholomorphic map h from some neighbourhood of $\bar{D}$ onto an open set $W \subseteq \mathbb{C}^n$ as well as a C^2-function $\varphi : W \to \mathbb{R}$ such that

(i) $h(D) = \{z \in W : \varphi(z) < 0\}$;

(ii) $d\varphi(z) \neq 0 \quad \text{for} \quad z \in h(bD)$;

(iii) φ is strictly convex with respect to $z_1, ..., z_{q+1}$.

Repeating the proof of Lemmas 3.1 in [BF/Le] one obtains: *If Ω, M and q are as in Theorem 1.3 then for each $\xi \in b\Omega$ one can find a neighbourhood U of ξ such that $U \cap \Omega$ is a local q-convex C^2-domain and moreover the intersection $M \cap b(U \cap \Omega)$ is transversal.* Therefore and by Theorem 4.1 in [BF/Le], Theorem 1.3 is a consequence of the following

Theorem 2.1 *Suppose that*

(i) $D \subset\subset \mathbb{C}^n$ is a local q-convex C^2-domain, $0 \leq q \leq n-1$;

(ii) H is the integral operator constructed for D in Section 4 of [BF/Le];

(iii) θ_D is a neighbourhood of $\bar{D}$;

(iv) $\varrho : \theta_D \to \mathbb{R}$ is a C^2-function with $D = \{z \in \theta_D : \varrho(z) < 0\}$ and $d\varrho(z) \neq 0$ for $z \in bD$;

(v) $\varrho_0 : \theta_D \to \mathbb{R}$ is a second C^2-function with $d\varrho_0(z) \wedge d\varrho(z) \neq 0$ for all $z \in bD$ with $\varrho_0(z) = 0$;

(vi) $N := \{z \in \bar{D} : \varrho_0(z) = \varrho(z) = 0\}$;

(vii) $\delta(z) := \inf_{\zeta \in N} |\zeta - z| \quad$ for $\quad z \in \theta_D$;

(viii) $\gamma(z) := ||\partial\varrho_0(z) \wedge \partial\varrho(z)||$ *for* $z \in \theta_D$.

Then for each continuous differential form f *on* D *with*

$$||f(z)|| \leq \frac{\gamma(z)}{\delta(z)} + \frac{1}{\sqrt{\delta(z)}} \quad \textit{for} \quad z \in D \tag{9}$$

the form Hf *is Hölder continuous with exponent* $1/2$ *on* $\bar{D}\backslash N$ *and moreover*

$$||Hf(z)|| \leq C(1 + |\ln\delta(z)|^3) \quad \textit{for} \quad z \in D, \tag{10}$$

where C *is a positive constant which is independent of* f.

Proof. Let f be a continuous differential form on D satisfying estimate (9). In the following we denote by C and C' positive constants which are independent of f, where the same letter C or C' in different places may denote different constants.

In view of Remark 1.4 we may assume that ϱ is the same function as in Section 4 of [BF/Le]. Let also G, V_D, h, $\Phi(z,\zeta)$ and $t(z,\zeta) = \operatorname{Im}\Phi(z,\zeta)$ be as in Section 4 of [BF/Le]. Then we may moreover asssume that $D = G$, $\theta_D = V_D$ and h is the identical map.

Since $d\varrho(z) \neq 0$ for $z \in bD$ there is a neighbourhood θ_{bD} of bD with

$$d_\zeta t(z,\zeta)\Big|_{\zeta = z} \wedge d\varrho(z) \neq 0 \quad \text{for} \quad z \in \theta_{bD}. \tag{11}$$

Since $d\varrho_0(z) \wedge d\varrho(z) \neq 0$ for $z \in N$ we can find a neighbourhood $\theta_N \subset\subset \theta_{bD}$ with

$$d\varrho_0(z) \wedge d\varrho(z) \neq 0 \quad \text{for} \quad z \in \theta_N, \tag{12}$$

and

$$C'\delta(z) \leq |\varrho_0(z)| + |\varrho(z)| \leq C\delta(z) \quad \text{for} \quad z \in \theta_N. \tag{13}$$

Since

$$d_\zeta t(z,\zeta)\Big|_{\zeta = z} = i\bar{\partial}\varrho(z) - i\partial\varrho(z),$$

we have the estimate

$$C'\gamma(z) \leq ||d\varrho_0(z) \wedge d\varrho(z) \wedge d_\zeta t(z,\zeta)\Big|_{\zeta = z}|| \leq C\gamma(z) \quad \text{for} \quad z \in \theta_N. \tag{14}$$

From (13) in [BF/Le] one obtains that

$$|\Phi(z,\zeta)| \geq C(|t(z,\zeta)| + |\varrho(\zeta)| + |\zeta - z|^2) \quad \text{for} \quad z,\zeta \in D. \tag{15}$$

Further it is clear that

$$|\gamma(\zeta) - \gamma(z)| \leq C|\zeta - z| \quad \text{for} \quad z,\zeta \in D, \tag{16}$$

and

$$||d_\zeta t(z,\zeta) - d_\zeta t(z',\zeta)|| \leq C|z - z'| \quad \text{for} \quad z,z' \in D. \tag{17}$$

That Hf is Hölder continuous with exponent $1/2$ on $\bar{D}\backslash N$ follows by the same arguments as in the beginning of the proof of Theorem 4.3 in [BF/Le]. It remains to prove estimate (10). As usual (cf., e.g., Section 3.2.7 in [H/Le 1]) one obtains that

$$||Hf(z)|| \leq C \sum_{k=0}^{n} \int_{\zeta \in D} \frac{||f(\zeta)|| d\lambda(\zeta)}{|\Phi(z,\zeta)|^k |\zeta - z|^{2n-1-k}} \quad \text{for} \quad z \in D, \tag{18}$$

where $d\lambda$ is the Lebesgue measure on D. For each open set $W \subseteq D$, $z \in D$ and $k = 0, 1, 2$ we set

$$I_k^1(W,z) := \int_{\zeta \in W} \frac{\gamma(\zeta) d\lambda(\zeta)}{(|t(z,\zeta)| + |\varrho(\zeta)| + |\zeta - z|^2)^k |\zeta - z|^{2n-k-1} \delta(\zeta)}$$

and

$$I_k^2(W,z) := \int_{\zeta \in W} \frac{d\lambda(\zeta)}{(|t(z,\zeta)| + |\varrho(\zeta)| + |\zeta - z|^2)^k |\zeta - z|^{2n-k-1} \sqrt{\delta(\zeta)}}.$$

Then it follows from (9), (15) and (18) that

$$||Hf(z)|| \leq C \sum_{\nu=1}^{2} \sum_{k=0}^{2} I_k^\nu(D,z) \quad \text{for} \quad z \in D. \tag{19}$$

Now we choose some neighbourhood θ_N^0 of N with $\theta_N^0 \subset\subset \theta_N$. Then it is clear that $I_k^\nu(D\backslash\theta_N^0, z)$ is bounded by

$$C \int_{\zeta \in D} \frac{d\lambda(\zeta)}{(|t(z,\zeta)| + |\varrho(\zeta)| + |\zeta - z|^2)^k |\zeta - z|^{2n-k-1}}.$$

Therefore, integrating with respect to $t(z,\cdot)$ and ϱ which is possible by (11), one obtains that

$$I_k^\nu(D\backslash\theta_N^0, z) \leq C \quad \text{for} \quad z \in D, \tag{20}$$

for $k = 0, 1, 2$ and $\nu = 1, 2$. Also it is clear that

$$I_k^\nu(D \cap \theta_N^0, z) \leq C \quad \text{for} \quad z \in D\backslash\theta_N, \tag{21}$$

for $k = 0, 1, 2$ and $\nu = 1, 2$. In view of (19)-(21) it remains to prove that

$$I_k^\nu(D \cap \theta_N^0, z) \leq C(1 + |\ln \delta(z)|^3) \quad \text{for} \quad z \in D \cap \theta_N, \tag{22}$$

for $k = 0, 1, 2$ and $\nu = 1, 2$. For all $z \in D \cap \theta_N$ we set

$$W^1(z) = \{\zeta \in D \cap \theta_N^0 : |\zeta - z| < \delta(z)/2\}$$

and

$$W^2(z) = \{\zeta \in D \cap \theta_N^0 : |\zeta - z| > \delta(z)/2\}.$$

To prove (22) now it is sufficient to show that

$$I_k^\nu(W^m(z),z) \le C(1+|\ln\delta(z)|^3) \tag{23}$$

for all $z \in D\cap\theta_N$, $k=0,1,2$, $\nu=1,2$ and $m=1,2$.

The case $m=1$*:* Since $|\zeta-z|<\delta(z)/2$ and therefore $\delta(\zeta)>\delta(z)/2$ for $\zeta\in W^1(z)$ it follows from (11) that

$$\begin{aligned}
I_k^\nu(W^1(z),z) &\le \frac{C}{\delta(z)}\int\limits_{\substack{x\in\mathbb{R}^{2n}\\ |x|<\delta(z)/2}} \frac{dx_1\wedge\ldots\wedge dx_{2n}}{(|x_1|+|x_2|+|x|^2)^k|x|^{2n-k-1}}\\
&\le \frac{C}{\delta(z)}\int\limits_{\substack{x\in\mathbb{R}^{2n-k}\\ |x|<\delta(z)/2}} \frac{1+|\ln|x||}{|x|^{2n-k-1}}dx_1\wedge\ldots\wedge dx_{2n-k}\\
&\le C(1+|\ln\delta(z)|),
\end{aligned}$$

for all $z\in D\cap\theta_N$, $k=0,1,2$ and $\nu=1,2$. Hence (23) holds for $m=1$.

The case $m=2$ *and* $\nu=1$*:* In view of (13),(14) and (17) the integrals $I_k^1(W^2(z),z)$ ($z\in D\cap\theta_N$, $k=0,1,2$) are bounded by

$$\begin{aligned}
&C\int\limits_{\zeta\in W^2(z)}\frac{||d\varrho_0(\zeta)\wedge d\varrho(\zeta)\wedge d_\zeta t(z,\zeta)||d\lambda(\zeta)}{(|t(z,\zeta)|+|\varrho(\zeta)|+|\zeta-z|^2)^k|\zeta-z|^{2n-k-1}(|\varrho_0(\zeta)|+|\varrho(\zeta)|)}\\
&+\ C\int\limits_{\zeta\in W^2(z)}\frac{d\lambda(\zeta)}{(|t(z,\zeta)|+|\varrho(\zeta)|+|\zeta-z|^2)^k|\zeta-z|^{2n-k-2}(|\varrho_0(\zeta)|+|\varrho(\zeta)|)}.
\end{aligned}$$

By (12), ϱ_0 and ϱ are local coordinates on $\bar\theta_N$. Therefore we can use the Range–Siu trick (see the proof of Proposition (3.7) in [R/S]) which allows us to consider ϱ_0, ϱ and $t(z,\cdot)$ as coordinates. In this way it follows that the integrals $I_k^1(W^2(z),z)$ ($z\in D\cap\theta_N$, $k=0,1,2$) are bounded by

$$\begin{aligned}
&C\int\limits_{\substack{x\in\mathbb{R}^{2n}\\ \delta(z)/2<|x|<C'}}\frac{dx_1\wedge\ldots\wedge dx_{2n}}{(|x_3|+|x_2|+|x|^2)^k|x|^{2n-k-1}(|x_1|+|x_2|)}\\
&+\ C\int\limits_{\substack{x\in\mathbb{R}^{2n}\\ \delta(z)/2<|x|<C'}}\frac{dx_1\wedge\ldots\wedge dx_{2n}}{(|x_2|+|x|^2)^k|x|^{2n-k-2}(|x_1|+|x_2|)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
I_2^1(W^2(z),z) &\le C\int\limits_{\substack{x\in\mathbb{R}^{2n-3}\\ \delta(z)/2<|x|<C'}}\frac{1+|\ln|x||^2}{|x|^{2n-3}}dx_1\wedge\ldots\wedge dx_{2n-3}\\
&\quad +C\int\limits_{\substack{x\in\mathbb{R}^{2n-2}\\ \delta(z)/2<|x|<C'}}\frac{1+|\ln|x||}{|x|^{2n-2}}dx_1\wedge\ldots\wedge dx_{2n-2}\\
&\le C(1+|\ln\delta(z)|^3)
\end{aligned}$$

for $z \in D \cap \theta_N$,

$$
\begin{aligned}
I_1^1(W^2(z),z) &\leq C \int\limits_{\substack{x\in\mathbb{R}^{2n-2}\\ \delta(z)/2<|x|<C'}} \frac{1+|\ln|x||^2}{|x|^{2n-2}} dx_1 \wedge \dots \wedge dx_{2n-2} \\
&\quad + C \int\limits_{\substack{x\in\mathbb{R}^{2n-2}\\ \delta(z)/2<|x|<C'}} \frac{1+|\ln|x||^2}{|x|^{2n-3}} dx_1 \wedge \dots \wedge dx_{2n-2} \\
&\leq C(1+|\ln\delta(z)|^3)
\end{aligned}
$$

for $z \in D \cap \theta_N$, and

$$
\begin{aligned}
I_0^1(W^2(z),z) &\leq C \int\limits_{\substack{x\in\mathbb{R}^{2n}\\ \delta(z)/2<|x|<C'}} \frac{dx_1 \wedge \dots \wedge dx_{2n}}{|x|^{2n-1}(|x_1|+|x_2|)} \\
&\leq C \int\limits_{\substack{x\in\mathbb{R}^{2n}\\ \delta(z)/2<|x|<C'\\ |x_1|+|x_2|>|x|/2}} \frac{dx_1 \wedge \dots \wedge dx_{2n}}{|x|^{2n}} + C \int\limits_{\substack{x\in\mathbb{R}^{2n-2}\\ \delta(z)/2<|x|<C'}} \frac{dx_1 \wedge \dots \wedge dx_{2n-2}}{|x|^{2n-2}} \\
&\leq C(1+|\ln\delta(z)|)
\end{aligned}
$$

for $z \in D \cap \theta_N$. Hence (23) is proved for $m = 2$ and $\nu = 1$.

The case $m = 2$ and $\nu = 2$: From (11) it follows that

$$
I_k^2(W^2(z),z) \leq C \int\limits_{\substack{x\in\mathbb{R}^{2n}\\ \delta(z)/2<|x|<C'}} \frac{dx_1 \wedge \dots \wedge dx_{2n}}{(|x_1|+|x_2|+|x|^2)^k |x|^{2n-k-1}\sqrt{|x_1|}}
$$

for all $z \in D \cap \theta_N$ and $k = 0, 1, 2$. This implies that

$$
\begin{aligned}
I_2^2(W^2(z),z) &\leq C \int\limits_{\substack{x\in\mathbb{R}^{2n-1}\\ \delta(z)/2<|x|<C'}} \frac{dx_1 \wedge \dots \wedge dx_{2n-1}}{(|x_1|+|x|^2)^{3/2}|x|^{2n-3}} \\
&\leq C \int\limits_{\substack{x\in\mathbb{R}^{2n-2}\\ \delta(z)/2<|x|<C'}} \frac{dx_1 \wedge \dots \wedge dx_{2n-2}}{|x|^{2n-2}} \\
&\leq C(1+|\ln\delta(z)|)
\end{aligned}
$$

for $z \in D \cap \theta_N$,

$$
\begin{aligned}
I_1^2(W^2(z),z) &\leq C \int\limits_{\substack{x\in\mathbb{R}^{2n-1}\\ \delta(z)/2<|x|<C'}} \frac{dx_1 \wedge \dots \wedge dx_{2n-1}}{|x|^{2n-1}} \\
&\leq C(1+|\ln\delta(z)|)
\end{aligned}
$$

for $z \in D \cap \theta_N$, and

$$I_0^2(W^2(z), z) \leq C \int\limits_{\substack{x \in \mathbb{R}^{2n} \\ \delta(z)/2 < |x| < C' \\ |x_1| > |x|/2}} \frac{dx_1 \wedge \ldots \wedge dx_{2n}}{|x|^{2n-1/2}} + C \int\limits_{\substack{x \in \mathbb{R}^{2n-1} \\ \delta(z)/2 < |x| < C'}} \frac{dx_1 \wedge \ldots \wedge dx_{2n-1}}{|x|^{2n-3/2}}$$

$$\leq C$$

for $z \in D \cap \theta_N$. Hence (23) is proved also for $m = 2$ and $\nu = 2$. ■

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