λ-UNKNOTTING NUMBER ONE KNOTS NEED NOT BE PRIME

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ABSTRACT. It is an old conjecture (see for example [Ad] p. 61) that the ordinary unknotting number $u(K)$ of a knot K is additive on connected sums of knots. This is known to be true (M. Scharlemann in [Sch]: "Unknotting Number One Knots are Prime") when $u(K_1\#K_2) = 1$. The general case is an open problem. In this note we show that additivity does not hold for a host of unknotting operations λ. In fact the results in this paper (modulo the issue of u_Δ) seem to indicate that the property "Unknotting number one knots are prime" may characterize the classical unknotting operation. In general $u_\lambda(K_1\#K_2) \leq u_\lambda(K_1) + u_\lambda(K_2)$. So we provide examples where the inequality is trict.

1. Introduction

It is an old conjecture (see for example [Ad] p. 61) that the ordinary unknotting number $u(K)$ (one of the most naive of knot invariants) of a knot K is additive on connected sums of knots. This is known to be true ([Sch]) in the form of: "Unknotting Number One Knots are Prime") when $u(K_1\#K_2) = 1$. There is a variety of unknotting operations[1] developed by various authors all of which result in defining their own unknotting numbers which are knot invariants just like the ordinary one. The issue therefore arises, given an unknotting operation λ, of the bahavior of its unknotting number u_λ with respect to connected sums of knots. In general if λ stands for any unknotting operation then $u_\lambda(K_1\#K_2) \leq u_\lambda(K_1) + u_\lambda(K_2)$. We will therefore supply examples of unknotting operations for which this inequality is strict. In order to simplify exposition we will introduce some notation.

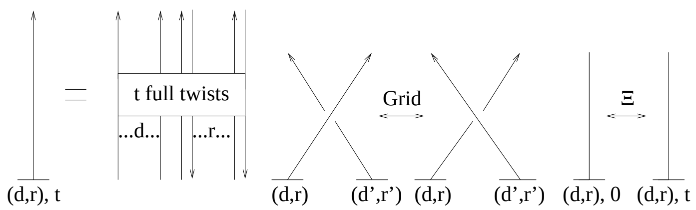

FIGURE 1. Left: Notation, Center: A grid move of type $(d, r) \times (d^{'}, r^{'})$, Right: Ξ-move of type $(d, r)\Xi(t)$

We will refer to the ordinary (single crossing change) unknotting operation as the X-move. Also we will write $\#_{(d,r)\times(d^{'},r^{'})}$ for a grid move of type $(d, r) \times (d^{'}, r^{'})$

Key words and phrases. local moves, unknotting operations, prime knots.

[1]For an account of unknotting operations the reader may want to read [Ka]. The author is working on a manuscript where the associated data of an unknotting operation will be discussed in more detail.

abbreviating it to $\#_{d\times d'}$ when $r = r' = 0$. Also a $(d,r)\Xi(t)$ will be abbreviated to $d\Xi$ when $t-1 = r = 0$. We will simply write $\#$ when $d = 2 = d', r = 0 = r'$. The unknotting number of an unknotting operation λ will be written as u_λ. Notice that an X-move is a 1×1 move as well as a 2Ξ- and a $(0,2)\Xi(1)$-move. What is known about these moves is then as follows:

Theorem 1.1. *A $\#_{(d,r)\times(d',r')}$-move is known to be an unknotting operation in the following cases:*

- *$d = 1$, $r = r' = 0$ and any d' (the X-move),*
- *$d = 1$, $r = 0$ and any d', r' (follows from* [Su]*)*
- *$d = 2 = d'$ and $r = r' = 0$ (*[M]*)*
- *$d = 2$ and $r = 0$ or $d = 3,4$ $r = r' = 0$ any d' modulo the fact that a $4\times 2x$-grid preserves the Arf invariant (*[Ask1], [Ask2]*) and in fact (*[Ask3]*):*
- *$r = r'$ and* $\gcd(d, d') = 1, 2, 3, 4$, *modulo the fact that a $4a\times 2b$-grid preserves the Arf invariant.*

A $(d,r)\Xi(t)$-move is an unknotting operation is the following case:

- *$t = 1, d = 2, r = 0$ (the X-move)*
- *$t = 1, d = 3, r = 0$ (*[Oh]*)*
- *$t = 1, d = 4, r = 0$ (*[Ask2]

We will also pay some attention to the Δ-unknotting operations. The symbols U, D, R, L will stand respectively for Up, Down, Right, Left. Let $i, j \in \{U, D, R, L\}$. Then a Δ_{ij} move is one which exchanges the corresponding arrangements in the figure below:

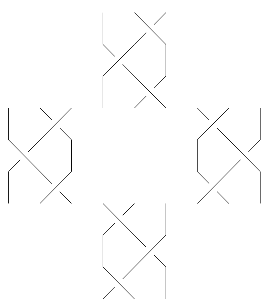

FIGURE 2. Δ-shaped arrangements.

These are all unknotting operations and if $I = \{(U,R),(L,D),(U,L),(R,D)\}$, $II = \{(U,D),(R,L)\}$ then $\Delta_{i,j}$, $(i,j)\in I$ are mutually locally equivalent. Same for $\Delta_{i,j}$, $(i,j)\in II$. These statements are due to Nakanishi ([Na2]). Also their variants with orientations are mutually locally equivalent. We will refer to the ones in family II as the Δ-move.

In [M-Y1] a certain $\#^p$ unknotting operation is defined.To facilitate the discussion in our language define the union $M = M_1 \cup M_2$ of two local moves to be that local move in which both moves M_1 and M_2 are allowed. Then we have the following formula:

$$\#^p = \bigcup_{p|d_1,d_2} \#_{(d_1,r_1)\times(d_2,r_2)}, \quad u_{\#^p} \le u_{\#_{(d_1,r_1)\times(d_2,r_2)}}, \quad p|d_1, d_2$$

2. The non-additivity

In this section we discuss the issue of the additivity of the unknotting operations mentioned in the introduction. Before offering our examples we dispose of the easy cases. It is easy to see that the unknotting number of a grid unknotting operation of type $(1,0)\times(1,r)$ is non-additive with respect to connected sums because it follows from [Su] that any knot has unknotting number one in infinitely many grid unknotting operations of type $(1,0)\times(d,r)$. It is interesting to observe that all grids of type $(1,0)\times(d,r)$ are locally equivalent to the X-move and we have the following formulas:

$$d+r\geq d^{'}+r^{'} \implies u_{(1,0)\times(d',r')}\leq u_{(1,0)\times(d,r)}$$
$$u_X(K)=u_{1\times 1}=\min\{2d+r\colon u_{(1,0)\times(d,r)}(K)=1\}$$

The non-additivity of the unknotting number of $\#^p$, $(p=2,3,4)$ will follow from the corresponding statements regarding grid unknotting operations. It is interesting to contrast this with: if K is the figure eight knot then $u_{\#^p}(\#_{i=1}^{n}K)\to\infty$ as $n\to\infty$ ([M-Y1], Prop. 1.6).

The $H(n)$-move ([H-N-T]) can easily be seen to at least contain non-additive unknotting numbers due to a certain limiting property: $u_{H(n)}(K)\to 1$ as $n\to\infty$.

It follows from [Oh] that the Δ-moves in family I are locally equivalent to $\Xi(3)$ (each Δ-move can be expressed as one $\Xi(3)$ move and vice versa). From this it follows that the unknotting numbers of the Δ-unknotting operations in I are non-additive since the one for the $\Xi(3)$-move is not as we will show below.

We should mention here that there are examples in the literature of composite knots trivialized by one twist (see for example [M-Y2] and its references [8], [13]).

Theorem 2.1. *Let $\lambda=\#_{2\times b},\#_{3\times 3b},\#_{4\times 4b},3\Xi,4\Xi$. Then u_λ is not additive with respect to connected sums: there exist non-prime knots of λ-unknotting number* 1.

Proof. In the figure below we supply examples of composite knots with λ-unknotting number equal to one for $\lambda=\#,\#_{3\times 3},\#_{4\times 4},3\Xi,4\Xi$. The examples for $\lambda=\#_{2\times b},\#_{3\times 3b},\#_{4\times 4b}$ are then easily generated from these. $\square$

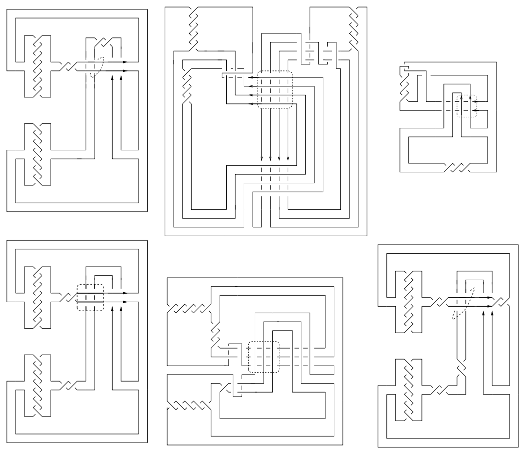

FIGURE 3. Non-trivial connected sums with $u_\lambda=1$.

It seems at this point reasonable to make the following definition.

Definition 2.1. *If $u_\lambda(K_1\#K_2) \leq u_\lambda(K_1)$ for some unknotting operation λ and non trivial knots K_1, K_2 then we call K_2 a λ-antiadditive companion of K_1.*

Questions-Remarks 2.1. *The following questions arise naturally from the discussion above:*

- *The issues of which knots have λ-antiadditive companions, the difference $u_\lambda(K_1) - u_\lambda(K_1\#K_2) > 0$, the number of nontrivial summands of λ-unknotting number one knots deserve further investigations. It should be observed that for unknotting operations λ for which a q_λ exists such that $q_\lambda u_\lambda \geq u_X$ these issues are related to the conjectured additivity of u_X.*
- *Is u_Δ additive? The author believes the answer to be affirmative but has been so far unable to prove it. The Δ-move differs from the other moves in that a $u_\Delta = 1$ knot has to have $a_2(K) = 1$ (*[Ok]*). Early attempts of the author to look for a composite knot with $u_\Delta = 1$ were failing in interesting enough ways to lead him to believe in the additivy of u_Δ.*

The following theorem seems in its simplicity to be new:

Theorem 2.2. *Let $g^\star(K)$ denote the 4-ball genus of the knot K. Then $g^\star(K) \leq u_\Delta$*

Proof. The Borromean rings bound a properly embedded torus in B^4. □

Remark 2.1. *Theorem 2.2 is a consequence of a more general fact. Suppose that X_K is the set of all pair of non negative integers (p, n) such that K can be unknotted by changing p (resp. n) positive (resp. negative) crossings in some projection of K (then of course $u_X(K)$ is the minimum $p + n$ over all such pairs). Then $g^\star(K) \leq max\{p, n \colon p + n = u_X(K)\}$ by a standard 4-dimensional trick. Theorem 2.2 implies that in Table F.3 of Knot Invariants of* [Ka] *the A in the column of $g^\star$ can be substituted by 1 whenever the u_Δ column contains a 1 i.e. $g^\star = 1$ for $8_{16}, 8_{18}, 9_{32}, 9_{40}, 9_{47}$. Similarly $g^\star = 1$ for $9_{17}, 9_{31}, 9_{32}, 9_{40}, 9_{47}$ because they can be unknotted by changing a pair of crossings of opposite sign.*

It follows from work of Nakanishi in [Na1] that a Δ-move is locally expressible in terms of certain (in our language) Ξ-moves as in the figure below.

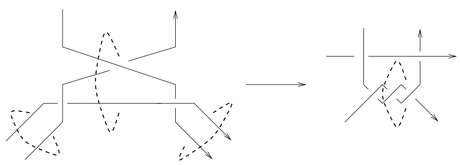

FIGURE 4. Apply from left to right: $(2,0)\Xi(2), -(3,0)\Xi(2), (2,0)\Xi(2), (2,0)\Xi(2)$

Since a $(2,0)\Xi(2)$ move is expressible as a 3Ξ move we have that one Δ move can be locally expressed as three 3Ξ-moves and one -3Ξ. Also since a 3Ξ move is locally expressible as one #-move we have that a Δ-move can be locally expressed as 5 #-moves. Here are then some properties of $u_\Delta =$ knots:

- $g^\star = 1$
- $|a_2| = 1$
- $u \leq 2$

- $\sigma \leq 2$
- As a consequence of theorem 2.2 we get: $u_{\Delta}(K) = 1 \implies K \notin \mathcal{K}_{(M^4,\alpha)}$ if $g(\alpha) \geq 2$
- Arf=1
- $u_{\Xi(3)} \leq 5$
- $u_{2\times 2} \leq 5$.

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