THE LENGTH-2 MONAD FOR INSTANTONS OVER $\mathbb{P}^3$ WITH $c_2 = 3$.

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1. Introduction

A *mathematical instanton vector bundle* (or simply *instanton*) over $\mathbb{P}^3$ is a stable 2-dimensional vector bundle F with $c_1(F) = 0$ which is *orthogonal* to $\mathcal{O}_{\mathbb{P}^3}(-2)$ (that is $\mathbf{H}^\bullet(\mathbb{P}^3, F \otimes \mathcal{O}_{\mathbb{P}^3}(-2)) = 0$). There is a well-known description of instantons over $\mathbb{P}^3$. Namely, one can show (see e.g. [9], II.3.2. Ex. 5) that any instanton F is a cohomology of a monad

$$\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus n} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus(2n+2)} \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus n},$$

where $n = c_2(F)$. This description allows to construct a moduli space $\mathbf{MI}_n(\mathbb{P}^3)$ of instantons with $c_2 = n$ (see [9], II.4.1.6).

However, with this description the investigation of such problems as rationality and smoothness is very complicated. A very little is known about it. Namely, this problems have been solved only for the values $n = 1$ (see [1]), $n = 2$ (see [7]) and $n = 3$ (see [6]).

In this paper we show that the generic instanton vector bundle over $\mathbb{P}^3$ with $c_2 = 3$ is the cohomology bundle of the following monad of the length 2

$$\mathcal{R} \to \Omega_{\mathbb{P}^3}(3)^{\oplus 3}, \tag{1}$$

where $\mathcal{R}$ is a certain exceptional vector bundle over $\mathbb{P}^3$ of rank 11 (see 3.1). Then we get a very nice description of such instantons in terms of nets of quadrics in the dual projective space $\mathbb{P}^{3\vee}$ (see Theorem 2). This immediately implies rationality of $\mathbf{MI}_3(\mathbb{P}^3)$ since the space of nets of quadrics in $\mathbb{P}^{3\vee}$ is just the Grassmannian $\mathbf{Gr}(3, 10)$. It would be extremely good to get a similar description (i.e. as the cohomology bundle of a monad of the length 2) of generic instanton with arbitrary value of c_2.

The paper is organized as follows. In Section 2 we recollect necessary facts about nets of quadrics and *rigid plane curves* (that is a plane curve with a non-degenerate theta-characteristic). In 2.1we reproduce a theory relating nets of quadrics in $\mathbb{P}^{3\vee}$ to rigid plane quartics (an explicit presentation can be found in [10] or [11]). In 2.2–2.3 we somehow develop this theory. In particular, we define in 2.2.8 *the skew web* of the rigid quartic and in 2.3.1 we deduce the relation between the net of quadrics and the skew web of the rigid quartic.

In Section 3 we show that there is a one-one correspondence between the spaces of *non-Lüroth* (i.e. corresponding to non-Lüroth quartics) nets of quadrics in $\mathbb{P}^{3\vee}$ and the space $\mathbf{MI}_3^{\mathcal{T}}(\mathbb{P}^3)$ of instantons F with $c_2 = 3$ satisfying the additional orthogonality condition $\mathbf{Hom}^\bullet(F, \mathcal{T}) = 0$, where $\mathcal{T}$ stands for the tangent bundle for $\mathbb{P}^3$. As a byproduct we prove that any instanton $F \in \mathbf{MI}_3^{\mathcal{T}}(\mathbb{P}^3)$ is the cohomology bundle of a monad (1) and that $\mathbf{MI}_3^{\mathcal{T}}(\mathbb{P}^3)$ is an open non-empty subspace of $\mathbf{MI}_3(\mathbb{P}^3)$. In 3.1 we explain how given a net of quadrics one can construct an instanton in $\mathbf{MI}_3^{\mathcal{T}}(\mathbb{P}^3)$. In 3.2 we explain how given an instanton $F \in \mathbf{MI}_3^{\mathcal{T}}(\mathbb{P}^3)$ one can reconstruct the net of quadrics A_F. In 3.3 we show how the quartic corresponding to the net A_F arises in the geometry of non-stable hyperplanes and multiple jumping lines of F.

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2. Plane quartics and nets of quadrics.

2.1. Recall that a net of quadrics in the projective space $\mathbb{P}(V)$ is a 3-dimensional subspace in $\mathbf{S}^2V^*$ (see [10], [11]). Let W, N be fixed 4-dimensional and 3-dimensional vector spaces and let $A : N \to \mathbf{S}^2W$ be a net of quadrics in the dual projective space $\mathbb{P}(W^*)$.

Given such a net of quadrics one can consider its *Hesse curve*

$$H_A = \{n \in \mathbb{P}(N) \mid A(n) \text{ is a degenerate quadric}\} \subset \mathbb{P}(N)$$

The curve H_A is a plane quartic (because $\mathbf{dim}\, W = 4$). We denote by $\alpha : H_A \to \mathbb{P}(N)$ its embedding in the plane.

The map $\beta : H_A \to \mathbb{P}(W^*)$ defined as follows

$$\beta : n \in H_A \mapsto \mathbf{Ker}(A(n)) \in \mathbb{P}(W^*)$$

is called *the Steiner map* and its image $S_A = \beta(H_A) \subset \mathbb{P}(W^*)$ is called *the Steiner curve* of the net A.

Let θ_A be a coherent sheaf of $\mathcal{O}_{H_A}$-modules defined as follows

$$\theta_A = \beta^*\mathcal{O}_{\mathbb{P}(W^*)}(1) \otimes \alpha^*\mathcal{O}_{\mathbb{P}(N)}(-1). \tag{2}$$

The sheaf θ_A is a so-called *generalized non-degenerate theta-characteristic.* This means that we have

$$\mathcal{H}om_{\mathcal{O}_{H_A}}(\theta_A, \omega_{H_A}) \cong \theta_A \qquad \text{and} \qquad \mathbf{H}^\bullet(H_A, \theta_A) = 0. \tag{3}$$

where ω_{H_A} stands for the dualizing sheaf of H_A. Note that since H_A is a plane quartic we have

$$\omega_{H_A} = \alpha^*\mathcal{O}_{\mathbb{P}(N)}(1). \tag{4}$$

The pair (H_A, θ_A) is called *the absolute invariant* of the net A. It is well known (see e.g. [2], [8], [10], [11]) that a net A can be uniquely reconstructed from its absolute invariant. In particular, we have the canonical identifications

$$N = \mathbf{H}^0(C, \omega_{H_A})^*, \qquad W = \mathbf{H}^0(C, \theta_A \otimes \omega_{H_A}).$$

2.1.1. The net $A : N \to \mathbf{S}^2W$ is called *regular* if it contains no quadrics of corank 2 and it is called *strictly regular* if any common point P of the net is a regular point of all quadrics in the net.

Let **Nets** denote the subspace of all regular nets of quadrics in $\mathbb{P}(W^*)$ and **Qu** denote the space of pairs (C, θ) where C is a plane quartic and θ is a theta-characteristics on C which are the absolute invariants of regular nets of quadrics. An element (C, θ) of **Qu** we will call *the rigid quartic.* We denote by $\alpha : C \to \mathbb{P}(N)$ and $\beta : C \to \mathbb{P}(W^*)$ the Hesse and Steiner embeddings of C. The regular net of quadrics, corresponding to the rigid quartic (C, θ) will be denoted by A_C.

Repeating once more:

$$\textit{the maps } (C, \theta) \mapsto A_C, \quad A \mapsto (H_A, \theta_A) \textit{ are mutually inverse isomorphisms } \mathbf{Qu} \leftrightarrows \mathbf{Nets}. \tag{5}$$

2.1.2. There is a natural question, what kind of singularities can have the Hesse curve H_A of the regular net $A \in$ **Nets**. The answer is as follows:

the smooth case: H_A is smooth;
the singular case: H_A has singularities of the second order (and H_A may be reducible);
the non-reduced case: H_A is a double smooth conic.

Example. Let $S \subset \mathbb{P}(W^*)$ be a rational twisted cubic curve. Then we have $\mathbf{dim}\, \mathbf{H}^0(\mathbb{P}(W^*), J_S(2)) = 3$, where J_S stands for the sheaf of ideals of the curve S. Identifying $A : N \xrightarrow{\sim} \mathbf{H}^0(\mathbb{P}(W^*), J_S(2)) \subset \mathbf{H}^0(\mathbb{P}(W^*), \mathcal{O}_{\mathbb{P}(W^*)}(2)) = \mathbf{S}^2W$ we get a net of quadrics. One can easily show that its Hesse curve is a double conic and its Steiner curve is a curve S with additional nilpotents in all points $x \in S$ in the direction of the touching plane to S at x. One can prove that this case is the only case when the Hesse curve of the net of quadrics is not reduced in all points.

Remark 1. Let $n \in H_A$ and $\varphi = \beta(n) \in S_A \subset \mathbb{P}(W^*)$. Then it is easy to see that

$$\mathcal{T}_n H_A = \{n' \in N \mid A(n', \varphi, \varphi) = 0\}/\langle n\rangle,$$

and

$$\mathcal{T}_\varphi S_A = \{\psi \in W^* \mid A(n, \psi) = A(n', \varphi) \text{ for some } n' \in N\}/\langle\varphi\rangle$$

(see *loc. cit.*). This means that the isomorphisms of (5) identify the subspace $\mathbf{Qu}^s \subset \mathbf{Qu}$ of smooth quartics with the subspace $\mathbf{Nets}^s \subset \mathbf{Nets}$ of strictly regular nets.

2.1.3. Given a net $A : N \to \mathbf{S}^2 W$ let $\tilde{A} : W^* \otimes N^* \to N \otimes W$ denote the induced map.

Lemma. *The regular nets A with degenerate $\tilde{A}$ correspond under the isomorphism of* (5) *to the subspace $\mathbf{Qu}^L \subset \mathbf{Qu}$ of Lüroth quartics.*

Proof. See [2, p. 84] (the condition of degenerating $\tilde{A}$ is equivalent to the Barth's ($\alpha 3$) condition). □

Definition. *Accordingly to the Lemma we will call the nets A with degenerate $\tilde{A}$ the Lüroth nets.*

The isomorphism (5) identify the subspace $\mathbf{Qu}^L \subset \mathbf{Qu}$ of Lüroth quartics with subspace $\mathbf{Nets}^L \subset \mathbf{Nets}$ of Lüroth nets.

2.2. The net of quadrics $A_C \in N^* \otimes \mathbf{S}^2 W$ corresponding to the rigid quartic C induces a certain section $s_C \in \mathbf{H}^0(\mathbb{P}(N) \times \mathbb{P}(W^*), W \otimes \mathcal{O}_{\mathbb{P}(N)\times\mathbb{P}(W^*)}(1,1)) = W \otimes N^* \otimes W$. Consider the zero locus Z_{s_C} of s_C.

Lemma. $Z_{s_C} = (\alpha \times \beta)(C) \subset \mathbb{P}(N) \times \mathbb{P}(W^*)$

Proof. First we will check the equality set-theoretically and then we will prove that the differential of $\alpha \times \beta$ induces an isomorphism of tangent spaces of C and Z_{s_C}.

Assume that the point $(n, \varphi) \in \mathbb{P}(N) \times \mathbb{P}(W^*)$ is in the zero locus. This means that $A_C(n, \varphi) = 0$.

But this means that the quadric $A_C(n)$ is degenerate and $\varphi = \mathbf{Ker}(A_C(n))$ is its kernel, hence $(n, \varphi) = (\alpha \times \beta)(x)$, $x \in C$. By the same reasons $(\alpha \times \beta)(C) \subset Z_{s_C}$.

Now consider a point $(n, \varphi) = (\alpha \times \beta)(x)$, $x \in C$. We have

$$\mathcal{T}_{(n,\varphi)}\Big(\mathbb{P}(N) \times \mathbb{P}(W^*)\Big) = \mathcal{T}_n\mathbb{P}(N) \oplus \mathcal{T}_\varphi\mathbb{P}(W^*) \cong N/n \oplus W^*/\varphi,$$

where $\mathcal{T}$ stands for the tangent space.

On the other hand the tangent space of the zero locus $\mathcal{T}_{(n,\varphi)}Z_{s_C}$ is nothing but the kernel of the differential $d_{s_c} : \mathcal{T}_{(n,\varphi)}\Big(\mathbb{P}(N)\times\mathbb{P}(W^*)\Big) \to W$. Its restrictions on summands N/n and W^*/φ are following:

$$d_{s_C\,|\mathcal{T}_n\mathbb{P}(N)} = f_\varphi := A_C(\cdot, \varphi) : N/n \to W, \qquad d_{s_C\,|\mathcal{T}_\varphi\mathbb{P}(W^*)} = g_n := A_C(n, \cdot) : W^*/\varphi \to W.$$

Since $A_C(n)$ is a quadric of corank 1 the second map is an isomorphism $W^*/\varphi \xrightarrow{\sim} \mathbf{Ann}\,\varphi \subset W$. This means that

$$\mathcal{T}_{(n,\varphi)}Z_{s_C} = \mathbf{Ker}\, d_{s_C} \cong f_\varphi^{-1}(\mathbf{Ann}\,\varphi) = \{n' \in N \mid A(n', \varphi, \varphi) = 0\}/\langle n\rangle$$

and Lemma follows from the Remark 1. □

2.2.1. The Lemma 2.2 implies that we have the following (Koszul) resolvent of the sheaf $(\alpha\times\beta)_*\mathcal{O}_C$ on $\mathbb{P}(N)\times\mathbb{P}(W^*)$:

$$0\to\Lambda^4W^*(-4,-4)\xrightarrow{\wedge s_C}\Lambda^3W^*(-3,-3)\xrightarrow{\wedge s_C}\Lambda^2W^*(-2,-2)\xrightarrow{\wedge s_C}$$
$$\xrightarrow{\wedge s_C}W^*(-1,-1)\xrightarrow{s_C}\mathcal{O}_{\mathbb{P}(N)\times\mathbb{P}(W^*)}\to(\alpha\times\beta)_*\mathcal{O}_C\to 0. \quad (6)$$

Taking the push-forward of this resolvent onto factors of $\mathbb{P}(N)\times\mathbb{P}(W^*)$ we get the following resolvents of $\alpha_*\mathcal{O}_C$ and $\beta_*\mathcal{O}_C$:

$$\begin{aligned} 0\to\mathcal{O}_{\mathbb{P}(N)}(-4)\to\mathcal{O}_{\mathbb{P}(N)}\to\alpha_*\mathcal{O}_C\to 0,\\ 0\to N\otimes\mathcal{O}_{\mathbb{P}(W^*)}(-4)\to W\otimes\mathcal{O}_{\mathbb{P}(W^*)}(-3)\to\mathcal{O}_{\mathbb{P}(W^*)}\to\beta_*\mathcal{O}_C\to 0.\end{aligned} \quad (7)$$

Here we have applied an isomorphisms $\Lambda^3N\cong\mathbb{C}$ and $\Lambda^3W^*\cong W$. The first of these resolvents is the usual resolvent of the structure sheaf of the plane quartic.

2.2.2. The equations (2) and (3) imply that we have $\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(2)\cong\alpha^*\mathcal{O}_{\mathbb{P}(N)}(3)$. The resolvents (7) imply that

$$\begin{aligned}\mathbf{S}^3N^* &= \mathbf{H}^0(\mathbb{P}(N),\mathcal{O}_{\mathbb{P}(N)}(3)) &&\cong \mathbf{H}^0(C,\alpha^*\mathcal{O}_{\mathbb{P}(N)}(3)),\\ \mathbf{S}^2W &= \mathbf{H}^0(\mathbb{P}(W^*),\mathcal{O}_{\mathbb{P}(W^*)}(2)) &&\cong \mathbf{H}^0(C,\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(2)).\end{aligned}$$

Finally we get the natural isomorphism $\mathbf{S}^3N^*\cong\mathbf{S}^2W$.

2.2.3. Now we will construct a certain 2-dimensional vector bundle on C.

Consider the pull-back via β of the second sequence of (7) twisted by $\mathcal{O}_{\mathbb{P}(W^*)}(3)$. It gives the following exact sequence on C:

$$0\to\mathbf{Tor}^2(\beta_*\mathcal{O}_C,\beta_*\mathcal{O}_C)\otimes\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(3)\to N\otimes\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(-1)\to$$
$$\to W\otimes\mathcal{O}_C\to\mathbf{Tor}^1(\beta_*\mathcal{O}_C,\beta_*\mathcal{O}_C))\otimes\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(3)\to 0.$$

The natural isomorphism

$$\mathbf{Tor}^2(\beta_*\mathcal{O}_C,\beta_*\mathcal{O}_C)\otimes\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(3)\cong\Lambda^2\mathcal{N}^*_{\beta(C)/\mathbb{P}(W^*)}\otimes\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(3)\cong$$
$$\cong\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(-4)\otimes\omega_C^*\otimes\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(3)\cong\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(-1)\otimes\omega_C^*$$

then implies that the quotient sheaf $\dfrac{N\otimes\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(-1)}{\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(-1)\otimes\omega_C^*}$ is naturally embedded in the trivial vector bundle $W\otimes\mathcal{O}_C$. This gives rise to the Grassmannian embedding $\gamma: C\to\mathbf{Gr}(2,W)$ such that

$$0\to\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(-1)\otimes\omega_C^*\to N\otimes\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(-1)\to\gamma^*U\to 0, \quad (8)$$

where U stands for the tautological subbundle over $\mathbf{Gr}(2,W)$.

Taking the determinant we get

$$\gamma^*\Lambda^2U\cong\Lambda^3N\otimes\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(-3)\otimes\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(1)\otimes\omega_C\cong$$
$$\cong\beta^*\mathcal{O}_{\mathbb{P}(W^*)}(-2)\otimes\omega_C\cong\alpha^*\mathcal{O}_{\mathbb{P}(N)}(-3)\otimes\omega_C\cong{\omega_C^*}^2. \quad (9)$$

2.2.4. Now we will introduce a geometrical description of the embedding γ. To this end recall, that the map $N\otimes\mathcal{O}_{\mathbb{P}(W^*)}(-4)\to W\otimes\mathcal{O}_{\mathbb{P}(W^*)}(-3)$ in (7) over the point $\varphi\in\mathbb{P}(W^*)$ is nothing but the map $f_\varphi = A_C(\cdot,\varphi): N\to W$. Hence

$$\gamma(x)=\mathbf{Im}\, f_{\beta(x)}\subset W.$$

Example. In the non-reduced case (see 2.1.2) one easily sees that the curve $\gamma(C)_{\mathbf{red}}$ is but the curve of the tangent lines to S.

2.2.5. Below we will need the following technical Lemma.

Lemma. *Let $L = \gamma(x) \subset \mathbb{P}(W)$ be a line. If $A_C(N, \varphi) \subset L$, then $\varphi = \beta(x)$.*

Proof. Since $\mathbf{dim}\, L = 2 < 3 = \mathbf{dim}\, N$ we have $\varphi = \beta(y)$ for some $y \in C$. Assume that $y \neq x$. Then $\beta(y) \neq \beta(x)$ and for any $\psi \in \langle \beta(x), \beta(y) \rangle \subset W^*$ we evidently have $A_C(N, \psi) \subset \gamma(x)$. Hence the line $M := \langle \beta(x), \beta(y) \rangle$ is contained in the Steiner curve $\beta(C)$. We have three possibilities, either $M \cap L^\perp = 0$, or $M \cap L^\perp = \langle \psi \rangle$, or $M = L^\perp$.

If $M \cap L^\perp = 0$ then we have $W^* = M \oplus L^\perp$ and the net $A_C : N \to \mathbf{S}^2 W$ is the direct sum $A_C = A'_C \oplus A''_C$, where $A' : N \to \mathbf{S}^2 M^*$ and $A'' : N \to \mathbf{S}^2 (L^\perp)^*$. If $n \in \mathbb{P}(N)$ is a common point of the Hesse curve of A' and of the Hesse curve of A'' then $\mathbf{rank}\, A_C(n) \leq 2$ which contradicts the regularity of the net A_C.

If $M \cap L^\perp = \langle \psi \rangle$ then we have $A_C(N, \psi, L^\perp) = 0$ (because $\psi \in M$), $A_C(N, \psi, M) = A_C(N, M, \psi) = 0$ (because $\psi \in L^\perp$) hence $A_C(N, \psi, M + L^\perp) = 0$. But $\mathbf{dim}(M + L^\perp) = 3$, hence the net A_C contains a pencil of quadrics with common kernel ψ. This means that there are three quadrics of corank 2 in this pencil, which again contradicts the regularity of the net A_C.

Finally, let $M = L^\perp$. Then we have $W^* = M \oplus L^*$, $W = M^* \oplus L$ and the net A_C induces maps $A' : N \to \mathbf{Hom}(M, L)$ and $A'' : N \to \mathbf{S}^2 L$ such that $A_C(n, \psi) = A'(n)(m) + (A'(n))^*(\lambda) + A''(n)(\lambda)$, where $\psi = m + \lambda$, $m \in M$, $\lambda \in L^*$ is the decomposition of ψ, and $A'(n)^* : L^* \to M^*$ is the dual operator of $A'(n) : M \to L$.

Let $H = \{n \in \mathbb{P}(N) \mid A'(n) \text{ is degenerate}\} \subset \mathbb{P}(N)$, $\sigma : n \in H \mapsto \mathbf{Ker}\, A'(n) \in \mathbb{P}(M)$, $\tau : n \in H \mapsto \mathbf{Im}\, A'(n) \in \mathbb{P}(L)$, $\sigma' : n \in H \mapsto \mathbf{Ker}(A'(n)^*) \in \mathbb{P}(L^*)$ and $\tau' : n \in H \mapsto A''(n)(\sigma'(n)) \in \mathbb{P}(L)$. Let $n \in H$ be a point such that $\tau(n) = \tau'(n)$ (that is a point of intersection of the curve $(\tau \times \tau')(H) \subset \mathbb{P}(L) \times \mathbb{P}(L)$ with diagonal). Then we have $A_C(n, \langle \sigma(n), \sigma'(n) \rangle) = 0$ that is $\mathbf{rank}\, A_C(n) \leq 2$, which again contradicts the regularity of the net A_C. The Lemma is proved. $\square$

2.2.6. The net of quadrics A_C corresponding to the rigid quartic C induces a certain section $t_C \in \mathbf{H}^0(\mathbb{P}(W^*) \times \mathbf{Gr}(2, W)), N^* \otimes \mathcal{O}_{\mathbb{P}(W^*)}(1) \boxtimes W/U) = N^* \otimes W \otimes W$. Consider the zero locus Z_{t_C} of t_C.

Lemma. $Z_{t_C} = (\beta \times \gamma)(C) \subset \mathbb{P}(W^*) \times \mathbf{Gr}(2, W)$.

Proof. Assume that the point $(\varphi, L) \in \mathbb{P}(W^*) \times \mathbf{Gr}(2, W)$ is in the zero locus. This means that $\mathbf{Im}\, f_\varphi \subset L$, hence $\varphi = \beta(x)$ and $L = \gamma(x)$ (see Lemma 2.2.5). The set-theoretical equality follows, hence it is remained to show that the induced map of tangent spaces is an isomorphism.

To this end consider a point $(\varphi, L) = (\beta \times \gamma)(x)$, $x \in C$. Then we have

$$\mathcal{T}_{(\varphi, L)} \Big(\mathbb{P}(W^*) \times \mathbf{Gr}(2, W) \Big) = \mathcal{T}_\varphi \mathbb{P}(W^*) \oplus \mathcal{T}_L \mathbf{Gr}(2, W) \cong W^*/\varphi \oplus \mathbf{Hom}(L, W/L).$$

On the other hand, the tangent space of the zero locus $\mathcal{T}_{(\varphi, L)} Z_{t_C}$ is nothing but the kernel of the differential $d_{t_C} : \mathcal{T}_{(\varphi, L)} \Big(\mathbb{P}(W^*) \times \mathbf{Gr}(2, W) \Big) \to N^* \otimes W/L = \mathbf{Hom}(N, W/L)$. Its restrictions on the first summand W^*/φ is following:

$$d_{t_C \,|\, \mathcal{T}_\varphi \mathbb{P}(W^*)} = h_L := \pi_L \cdot A_C : W^*/\varphi \to \mathbf{Hom}(N, W/L),$$

where $\pi_L : W \to W/L$ is the natural projection, $A_C : W^* \to \mathbf{Hom}(N, W)$ is the polarization of A_C. Evidently that A_C maps $\varphi \in W^*$ to $\mathbf{Hom}(N, L) \subset \mathbf{Hom}(N, W)$ hence $\pi_L \cdot A_C$ maps φ to zero and the above composition is well-defined.

The restriction of d_{t_C} on the second summand $\mathbf{Hom}(L, W/L)$ is following:

$$d_{t_C \,|\, \mathcal{T}_L \mathbf{Gr}(2, W)} = k_\varphi := (\cdot) \cdot f_\varphi : \mathbf{Hom}(L, W/L) \to \mathbf{Hom}(N, W/L).$$

Here we have used the fact that the image of $f_\varphi : N \to W$ is L (the map f_φ is defined in the proof of Lemma 2.2) hence we can take its composition with any map $L \to W/L$.

Let $n = \alpha(x) \in N$. We have the following exact sequence:

$$0 \to \mathbf{Hom}(N/n, W/L) \to \mathbf{Hom}(N, W/L) \xrightarrow{\mathbf{ev}_n} W/L \to 0.$$

The third map here is evaluation at the point $n \in N$.

Since f_φ induces an isomorphism $N/n \to L$ the map k_φ is also an isomorphism $\mathbf{Hom}(L, W/L) \xrightarrow{\sim} \mathbf{Hom}(N/n, W/L)$, therefore

$$\mathcal{T}_{(\varphi,L)} Z_{t_C} = \mathbf{Ker}\, d_{t_C} \cong {h_L}^{-1}(\mathbf{Hom}(N/n, W/L)) = \{\psi \in W^* \mid A_C(n, \psi) \in L\}/\langle\varphi\rangle$$

and the Lemma follows from the Remark 1 □

2.2.7. The Lemma 2.2.6 implies that we have the following (Koszul) resolvent of the sheaf $(\beta \times \gamma)_* \mathcal{O}_C$:

$$\ldots \to \Lambda^2(N \otimes \mathcal{O}_{\mathbb{P}(W^*)}(-1) \boxtimes U^\perp) \to N \otimes \mathcal{O}_{\mathbb{P}(W^*)}(-1) \boxtimes U^\perp \xrightarrow{t_C} \mathcal{O}_{\mathbb{P}(W^*) \times \mathbf{Gr}(2,W)} \to (\beta \times \gamma)_* \mathcal{O}_C \to 0, \tag{10}$$

where $U^\perp := (W/U)^*$.

2.2.8. The cohomology spectral sequence of (10) multiplied by $\Lambda^2 U^*$ shows that the restriction map $\mathbf{H}^0(\mathbf{Gr}(2,W), \Lambda^2 U^*) \to \mathbf{H}^0(C, \gamma^* \Lambda^2 U^*)$ is an isomorphism. Therefore we have

$$\begin{aligned}\Lambda^2 W^* = \mathbf{H}^0(\mathbf{Gr}(2,W), \Lambda^2 U^*) \xrightarrow{\sim} \mathbf{H}^0(C, \gamma^* \Lambda^2 U^*) = \\ = \mathbf{H}^0(C, \omega_C^2) = \mathbf{H}^0(C, \alpha^* \mathcal{O}_{\mathbb{P}(N)}(2)) = \mathbf{H}^0(\mathbb{P}(N), \mathcal{O}_{\mathbb{P}(N)}(2)) = \mathbf{S}^2 N^*.\end{aligned}$$

Here we used the equality $\gamma^* \Lambda^2 U^* = \omega_C^2$ which follows from (9).

The dual map of the above composition of isomorphisms

$$B_C : \mathbf{S}^2 N \xrightarrow{\sim} \Lambda^2 W$$

we will call *the skew web* of the rigid plane quartic C.

Remark 2. The skew web $B_C : \mathbf{S}^2 N \to \Lambda^2 W$ can be described on the abstract tensor language as the following composition:

$$\mathbf{S}^2 N \to N^* \otimes \mathbf{S}^3 N \xrightarrow{\sim} \Lambda^2 N \otimes \mathbf{S}^2 W^* \xrightarrow{\Lambda^2 A_C} \Lambda^2 W.$$

Here the first map is dual to the natural pairing $N \otimes \mathbf{S}^3 N^* \to \mathbf{S}^2 N^*$, the second map is the tensor product of isomorphisms $N^* \cong \Lambda^2 N$, $\mathbf{S}^3 N \cong \mathbf{S}^2 W^*$ (see 2.2.2) and the third map is the exterior square of the map $A_C : N \otimes W^* \to W$. However, in this definition it is absolutely not clear that it is an isomorphism.

2.2.9. **Lemma.** The quartic C is non-Lüroth iff $\mathbf{H}^1(C, \gamma^* W/U) = 0$.

Proof. The cohomology spectral sequence of (10) multiplied by W/U shows that the space $\mathbf{H}^1(C, \gamma^* W/U)$ is isomorphic to the cokernel of a map $\tilde{A}_C : N^* \otimes W^* \to N \otimes W$. □

2.2.10. **Corollary.** The quartic C is non-Lüroth iff $\mathbf{Hom}(\gamma^* W/U, \omega_C) = 0$.

Proof. The Serre duality implies that

$$\mathbf{Hom}(\gamma^* W/U, \omega_C) \cong \mathbf{H}^0(C, (\gamma^* W/U)^* \otimes \omega_C)) \cong \mathbf{H}^1(C, \gamma^* W/U))^*.$$

and it is remained only to apply the above Lemma. □

2.3. **The relation between tensors.**

Since the skew web $B_C : \mathbf{S}^2 N \to \Lambda^2 W$ is an isomorphism we can consider its inverse map

$$B_C^{-1} : \Lambda^2 W \to \mathbf{S}^2 N.$$

Let $\hat{B}_C : N^* \otimes \Lambda^2 W \to N$ denote the map induced by B_C^{-1}.

2.3.1. Consider also the map

$$\hat{A}_C : W^* \otimes W \xrightarrow{A_C \otimes \mathbf{id}} N^* \otimes W \otimes W \xrightarrow{\mathbf{id} \otimes \mathbf{Alt}} N^* \otimes \Lambda^2 W,$$

where **Alt** stands for the alternation.

Proposition. *The composition $\hat{B}_C \cdot \hat{A}_C : W^* \otimes W \to N$ vanishes.*

Proof. Let $\varphi = \beta(x)$ for some $x \in C$ and $\gamma(x) = \langle w, w' \rangle$. Chose a basis (n_1, n_2, n_3) in N such that $n_1 = \alpha(x)$ (i.e. $A(n_1, \varphi) = 0$) and $A(n_2, \varphi) = w$, $A(n_3, \varphi) = w'$. Let (n^1, n^2, n^3) be a dual basis in N^*. By the definition of $\hat{A}_C$ we have

$$\hat{A}_C(\varphi \otimes w) = n^1 \otimes A(n_1, \varphi) \wedge w + n^2 \otimes A(n_2, \varphi) \wedge w + n^3 \otimes A(n_3, \varphi) \wedge w = n^3 \otimes w' \wedge w.$$

Since image of $\gamma(x)$ in $\mathbb{P}(\Lambda^2 W)$ is but $w' \wedge w$ (see 2.2.4) we have

$$B_C^{-1}(w' \wedge w) = n_1 \cdot n_1 \in \mathbf{S}^2 N$$

(up to multiplcative constant). But this means that

$$\hat{B}_C(\hat{A}_C(\varphi \otimes w)) = \hat{B}_C(n^3 \otimes w' \wedge w) = \langle n^3, n_1 \cdot n_1 \rangle = 2\langle n^3, n_1 \rangle n_1 = 0,$$

where $\langle \cdot , \cdot \rangle$ stands for the natural pairing. Now the Proposition follows from the following Lemma. $\square$

2.3.2. **Lemma.** The tensors of the type $\varphi \otimes w$, where $\varphi = \beta(x)$, $w \in \gamma(x)$ and x runs over C generate all space $W^* \otimes W$.

Proof. Assume that they do not. Then there exists a linear map $M : W^* \to W^*$ such that for all $x \in C$, $\varphi = \beta(x)$, $w \in \gamma(x)$ we have $M(\varphi)(x) = 0$. But this means that the composition

$$\mathcal{O}_{\mathbb{P}(W^*)}(-1) \boxtimes \mathcal{O}_{\mathbf{Gr}(2,W)} \to W^* \otimes \mathcal{O}_{\mathbb{P}(W^*) \times \mathbf{Gr}(2,W)} \xrightarrow{M} W^* \otimes \mathcal{O}_{\mathbb{P}(W^*) \times \mathbf{Gr}(2,W)} \to \mathcal{O}_{\mathbb{P}(W^*)} \boxtimes U^*$$

vanishes over $(\beta \times \gamma)(C)$, that is the restriction map

$$W \otimes W^* = \mathbf{H}^0(\mathbb{P}(W^*) \times \mathbf{Gr}(2, W), \mathcal{O}_{\mathbb{P}(W^*)}(1) \boxtimes U^*) \to \mathbf{H}^0(C, (\beta \times \gamma)^*(\mathcal{O}_{\mathbb{P}(W^*)}(1) \boxtimes U^*))$$

sends M to zero. But the cohomology spectral sequence of (10) multiplied by $\mathcal{O}_{\mathbb{P}(W^*)}(1) \boxtimes U^*$ shows that this restriction map is an isomorphism and we arrived at the contradiction. $\square$

2.3.3. **Lemma.** We have $\mathbf{Ker}\, \hat{A}_C = \mathbb{C} = \langle \mathbf{id} \rangle \subset \mathbf{Hom}(W^* \otimes W^*) = W^* \otimes W$ is the subspace of all homotheties.

Proof. Consider action of the group $\mathbf{GL}(W^*)$ on the space $W \otimes W$ defined by the formula

$$g\Phi(\phi, \psi) := \Phi(g\phi, \psi), \qquad \phi, \psi \in W^*$$

where $\Phi \in W \otimes W$ is considered as a bilinear form on W^*.

Remark 3. If both Φ and $g\Phi$ are symmetic forms then it is evident that the kernel of Φ is g-invariant.

Given a subspace $N \subset \mathbf{S}^2 W$ (i.e. a net of quadrics in $\mathbb{P}(W^*)$) one can consider the subgroup $G \subset \mathbf{GL}(W^*)$ of all g such that $gN \subset \mathbf{S}^2 W$ (in other words $\mathbf{Alt}(gN) = 0$). Considering the net A_C we can interpret the kernel of the map $\hat{A}_C$ as the tangent Lie algebra of G. But the above Remark implies that the curve $\beta(C)$ is the G fixed-point set, hence G can contain only dilations and the Lemma follows. $\square$

2.3.4. The Lemma 2.3.3 and the Proposition 2.3.1 imply that we have the following left-exact complex

$$0 \to \mathfrak{sl}(W) \xrightarrow{\hat{A}_C} N^* \otimes \Lambda^2 W \xrightarrow{\hat{B}_C} N \to 0, \tag{11}$$

where $\mathfrak{sl}(W)$ stands for the $(W^* \otimes W)/\mathbb{C}$.

Lemma. *The complex* (11) *is exact.*

Proof. Since $\mathbf{dim}\, N^* \otimes \Lambda^2 W = 18 = 15 + 3 = \mathbf{dim}\, \mathfrak{sl}(W) + \mathbf{dim}\, N$ it suffices to prove the surjectivity of the map $\hat{B}_C$. But this easy follows since B_C^{-1} is isomorphism. □

3. Instanton bundles

3.1. Let $\mathcal{R}$ be an exceptional vector bundle on $\mathbb{P}(W)$ obtained by the right mutation of $\mathcal{O}_{\mathbb{P}(W)}(1)$ through $\mathcal{T}_{\mathbb{P}(W)}$ (for the definition of exceptional bundles and their mutations consult [4]), that is a cokernel of the natural map

$$0 \to \mathcal{O}_{\mathbb{P}(W)}(1) \to W^* \otimes \mathcal{T}_{\mathbb{P}(W)} \to \mathcal{R} \to 0. \tag{12}$$

This sequence implies that $\mathbf{rank}(\mathcal{R}) = 11$, $c_1(\mathcal{R}) = 15$, $c_2(\mathcal{R}) = 105$ and $c_3(\mathcal{R}) = 455$.

3.1.1. One can easily prove that $\mathbf{Hom}(\mathcal{R}, \Omega_{\mathbb{P}(W)}(3))$ is naturally isomorphic to $\mathbf{S}^2 W$. Actually it evidently follows from the fact that both $\mathcal{R}$ and $\Omega_{\mathbb{P}(W)}(3)$ are the members of the certain exceptional collection over $\mathbb{P}(W)$ (see 3.2.4).

Given a rigid quartic (C, θ) its net of quadrics $A_C : N \to \mathbf{S}^2 W$ then gives rise to the natural homomorphism $\mu_C : \mathcal{R} \to N^* \otimes \Omega_{\mathbb{P}(W)}(3)$.

Proposition. *The homomorphism μ_C is surjective iff C is non-Lüroth.*

Proof. The definition of $\mathcal{R}$ implies that it has the following resolvent

$$0 \to W^* \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathfrak{sl}(W) \otimes \mathcal{O}_{\mathbb{P}(W)}(1) \to \mathcal{R} \to 0.$$

On the other hand, $N^* \otimes \Omega_{\mathbb{P}(W)}(3)$ has the following resolvent

$$0 \to N^* \otimes \mathcal{O}_{\mathbb{P}(W)}(-1) \to N^* \otimes W \otimes \mathcal{O}_{\mathbb{P}(W)} \to N^* \otimes \Lambda^2 W \otimes \mathcal{O}_{\mathbb{P}(W)}(1) \to N^* \otimes \Omega_{\mathbb{P}(W)}(3) \to 0.$$

The map μ_C can be extended to the map of resolvents and we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & W^* \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[r] \arrow[d] & \mathfrak{sl}(W) \otimes \mathcal{O}_{\mathbb{P}(W)}(1) \arrow[r] \arrow[d] & \mathcal{R} \arrow[r] \arrow[d, "\mu_C"] & 0 \\
0 \arrow[r] & N^* \otimes \mathcal{O}_{\mathbb{P}(W)}(-1) \arrow[r] & N^* \otimes W \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[r] & N^* \otimes \Lambda^2 W \otimes \mathcal{O}_{\mathbb{P}(W)}(1) \arrow[r] & N^* \otimes \Omega_{\mathbb{P}(W)}(3) \arrow[r] & 0
\end{tikzcd}$$

The map $\mathfrak{sl}(W) \otimes \mathcal{O}_{\mathbb{P}(W)}(1) \to N^* \otimes \Lambda^2 W^* \otimes \mathcal{O}_{\mathbb{P}(W)}(1)$ here is induced from the map $\hat{A}_C$, hence its cokernel is but $N \otimes \mathcal{O}_{\mathbb{P}(W)}(1)$. Therefore, it suffices to prove that the arising map

$$N^* \otimes W \otimes \mathcal{O}_{\mathbb{P}(W)} \to N \otimes \mathcal{O}_{\mathbb{P}(W)}(1)$$

is surjective. Assume that it is not surjective. Then there exists a point $w \in W$ and a point $\xi \in N^*$ such that the map

$$N^* \otimes W \xrightarrow{\mathbf{id} \otimes (\wedge w)} N^* \otimes \Lambda^2 W \xrightarrow{\mathbf{id} \otimes B_C^{-1}} N^* \otimes \mathbf{S}^2 N \to N \xrightarrow{\xi} \mathbb{C}$$

vanishes. This means that B_C^{-1} maps the subspace $(W/w) \wedge w \subset \Lambda^2 W$ on the subspace $\mathbf{S}^2\, \mathbf{Ann}\, \xi \subset \mathbf{S}^2 N$. But since B_C^{-1} is an isomorphism (because B_C is) and the dimension of both subspaces is 3, it follows that the map induces an isomorphism of these subspaces, i.e. we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (W/w)\wedge w & \longrightarrow & \Lambda^2 W & \longrightarrow & \Lambda^2(W/w) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle B_C^{-1}} & & \downarrow{\scriptstyle B_C^{-1}} & & \downarrow{\scriptstyle B_C^{-1}} & & \\ 0 & \longrightarrow & \mathbf{S}^2\,\mathbf{Ann}\,\xi & \longrightarrow & \mathbf{S}^2 N & \longrightarrow & N\otimes N/\mathbf{Ann}\,\xi & \longrightarrow & 0 \end{array}$$

and all vertical arrows here are isomorphisms. Consider the inverse dual diagram, that is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda^2\,\mathbf{Ann}\,w & \longrightarrow & \Lambda^2 W^* & \xrightarrow{w} & \mathbf{Ann}\,w & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle B_C^*} & & \downarrow{\scriptstyle B_C^*} & & \downarrow{\scriptstyle B_C^*} & & \\ 0 & \longrightarrow & N^*\otimes\xi & \longrightarrow & \mathbf{S}^2 N^* & \longrightarrow & \mathbf{S}^2(N^*/\xi) & \longrightarrow & 0 \end{array}$$

Here all vertical arrows are also isomorphisms. It has the following geometrical meaning. Consider all hyperplanes in $\mathbb{P}(\Lambda^2 W)$ which correspond to the subspace $\Lambda^2\,\mathbf{Ann}\,w \subset \Lambda^2 W^* = \mathbf{H}^0(\mathcal{O}_{\mathbb{P}(\Lambda^2 W)}(1))$. Then all divisors in the linear system $|2K_C|$ intersected on the curve $\gamma(C) \subset \mathbf{Gr}(2,W) \subset \mathbb{P}(\Lambda^2 W)$ by these hyperplanes form the subsystem $\xi_{|C} + |K_C|$ corresponding to the subspace $N^*\otimes\xi \subset \mathbf{S}^2 N^* = \mathbf{H}^0(C,\omega_C^2)$. In other words, the map

$$\gamma^* W/U \xrightarrow{\wedge w} \gamma^*\Lambda^2(W/U) = \gamma^*\mathcal{O}_{\mathbf{Gr}(2,W)}(1) \cong \omega_C^2$$

factors as

$$\gamma^* W/U \to \omega_C \xrightarrow{\xi} \omega_C^2$$

hence $\mathbf{Hom}(\gamma^* W/U, \omega_C) \neq 0$ which by the Corollary 2.2.10 means that C is Lüroth.

The same arguments show that if C is Lüroth then μ_C is not surjective. □

Remark 4. The fact that for generic $N \subset \mathbf{S}^2 W$ the corresponding map $\mathcal{R} \to N^*\otimes\Omega_{\mathbb{P}(W)}(3)$ is surjective is rather surprising, because counting a parameters it seems that this map should degenerate in codimension 3.

3.1.2. Recall that an *instanton vector bundle* over $\mathbb{P}(W)$ is by definition a 2-dimensional stable vector bundle F with $c_1(F) = 0$ and $\mathbf{H}^\bullet(\mathbb{P}(W), F\otimes\mathcal{O}_{\mathbb{P}(W)}(-2)) = 0$. Let $\mathbf{MI}_3(\mathbb{P}(W))$ denote the space of instanton vector bundles over $\mathbb{P}(W)$ with $c_2 = 3$.

Definition. *Let C be a non-Lüroth rigid quartic. Denote by $\mathcal{R}_C$ the kernel of the surjective map μ_C : $\mathcal{R} \to N^*\otimes\Omega_{\mathbb{P}(W)}(3)$.*

Theorem 1. *$\mathcal{R}_C$ is an instanton bundle over $\mathbb{P}(W)$ with $c_2 = 3$.*

The proof of the Theorem consists of the following three Lemmas.

3.1.3. **Lemma.** $\mathcal{R}_C$ is a 2-dimensional vector bundle on $\mathbb{P}(W)$ with $c_1(\mathcal{R}_C) = 0$, $c_3(\mathcal{R}_C) = 3$.

Proof. By Definition 3.1.2 and Proposition 3.1.1 we have the following exact sequence

$$0 \to \mathcal{R}_C \to \mathcal{R} \xrightarrow{\mu_C} N^*\otimes\Omega_{\mathbb{P}(W)}(3) \to 0 \tag{13}$$

which implies the Lemma. □

3.1.4. **Lemma.** The bundle $\mathcal{R}_C$ is stable.

Proof. We need to show that $\mathcal{R}_C$ does not contain an invertible subsheaves of zero degree. This is equivalent to $\mathbf{H}^0(\mathbb{P}(W),\mathcal{R}_C)=0$.

By the Serre duality it suffices to check that $\mathbf{H}^3(\mathbb{P}(W),\mathcal{R}_C^*\otimes\mathcal{O}_{\mathbb{P}(W)}(-4))=0$. Since $c_1(\mathcal{R}_C)=0$ the bundle $\mathcal{R}_C$ is self-dual, hence we need to compute $\mathbf{H}^3(\mathbb{P}(W),\mathcal{R}_C\otimes\mathcal{O}_{\mathbb{P}(W)}(-4))$. This can be done applying (13). Namely, it suffices to note that

$$\mathbf{H}^3(\mathbb{P}(W),\mathcal{R}(-4))=\mathbf{H}^2(\mathbb{P}(W),\Omega_{\mathbb{P}(W)}(-1))=\mathbf{H}^3(\mathbb{P}(W),\Omega_{\mathbb{P}(W)}(-1))=0.$$

□

3.1.5. **Lemma.** We have $\mathbf{H}^\bullet(\mathbb{P}(W),\mathcal{R}_C(-2))=0$.

Proof. Follows from $\mathbf{H}^\bullet(\mathbb{P}(W),\mathcal{R}(-2))=0$ and $\mathbf{H}^\bullet(\mathbb{P}(W),\Omega_{\mathbb{P}(W)}(1))=0$. □

3.1.6. Here we will state an additional property of $\mathcal{R}_C$ which as we will see below describes the subset in $\mathbf{MI}_3(\mathbb{P}(W))$ consisting of instantons $F=\mathcal{R}_C$, $C\in\mathbf{Qu}\setminus\mathbf{Qu}^L$.

Lemma. *We have* $\mathbf{Hom}^\bullet(\mathcal{R}_C,\mathcal{T}_{\mathbb{P}(W)})=0$.

Proof. Follows from $\mathbf{Hom}^\bullet(\mathcal{R},\mathcal{T}_{\mathbb{P}(W)})=0$ and $\mathbf{Hom}(\Omega_{\mathbb{P}(W)}(3),\mathcal{T}_{\mathbb{P}(W)})=0$. □

3.2. Now we will study all instantons which satisfy the orthogonality property of Lemma 3.1.6.

Let $\mathbf{MI}_3^{\mathcal{T}}(\mathbb{P}(W))\subset\mathbf{MI}_3(\mathbb{P}(W))$ be subspace parameterizing all instantons F with

$$\mathbf{Hom}^\bullet(F,\mathcal{T}_{\mathbb{P}(W)})=0. \tag{14}$$

Remark 5. Evidently (14) is an open condition, hence $\mathbf{MI}_3^{\mathcal{T}}(\mathbb{P}(W))\subset\mathbf{MI}_3(\mathbb{P}(W))$ is an open subspace.

Remark 6. Note that $c_1(F)=0$ implies the self-duality of F

$$F^*\cong F$$

Here isomorphism is unique up to dilations, because F is stable.

3.2.1. Let $F\in\mathbf{MI}_3^{\mathcal{T}}(\mathbb{P}(W))$.

Lemma. *The following table contains the dimensions* $h^i(F(q)):=\mathbf{dim}\,\mathbf{H}^i(\mathbb{P}(W),F\otimes\mathcal{O}_{\mathbb{P}(W)}(q))$ *of the cohomology spaces of twisted bundles* $F(q):=F\otimes\mathcal{O}_{\mathbb{P}(W)}(q)$.

q	-5	-4	-3	-2	-1	0	1
$h^3(F(q))$	0	0	0	0	0	0	0
$h^2(F(q))$	1	4	3	0	0	0	0
$h^1(F(q))$	0	0	0	0	3	4	1
$h^0(F(q))$	0	0	0	0	0	0	0

Proof. Actually, we only need to show that $\mathbf{H}^0(\mathbb{P}(W),F(1))=0$. Then the proposition follows by the usual arguments (see [3]) and Riemann-Roch Theorem.

To this end consider the Euler exact sequence on $\mathbb{P}(W)$ multiplied by F

$$0\to F\to W\otimes F(1)\to F\otimes\mathcal{T}_{\mathbb{P}(W)}\to 0.$$

Since $\mathbf{H}^\bullet(\mathbb{P}(W),F\otimes\mathcal{T}_{\mathbb{P}(W)})\cong\mathbf{H}^\bullet(\mathbb{P}(W),F^*\otimes\mathcal{T}_{\mathbb{P}(W)})\cong\mathbf{Hom}^\bullet(F,\mathcal{T}_{\mathbb{P}(W)})=0$ we have canonical isomorphisms

$$\mathbf{H}^\bullet(\mathbb{P}(W),F)\cong W\otimes\mathbf{H}^\bullet(\mathbb{P}(W),F(1)).$$

In particular, $\mathbf{H}^0(\mathbb{P}(W),F(1))=0$ since $\mathbf{H}^0(\mathbb{P}(W),F)=0$ (because F is stable). □

3.2.2. The proof of the above Lemma gives us natural identification

$$\mathbf{H}^1(\mathbb{P}(W), F) \cong W \otimes \mathbf{H}^1(\mathbb{P}(W), F(1)) = W \otimes \mathbb{C} = W.$$

Let $N := \mathbf{H}^1(F(-1))$. Then the Serre duality gives us the following identification of the cohomology spaces of $F(q)$.

q	-5	-4	-3	-2	-1	0	1
$\mathbf{H}^3(\mathbb{P}(W), F(q))$	0	0	0	0	0	0	0
$\mathbf{H}^2(\mathbb{P}(W), F(q))$	$\mathbb{C}$	W^*	N^*	0	0	0	0
$\mathbf{H}^1(\mathbb{P}(W), F(q))$	0	0	0	0	N	W	$\mathbb{C}$
$\mathbf{H}^0(\mathbb{P}(W), F(q))$	0	0	0	0	0	0	0

3.2.3. Consider the following collection of vector bundles on $\mathbb{P}(W)$

$$\mathfrak{E} = (\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3, \mathfrak{E}_4) := (\Omega_{\mathbb{P}(W)}(-4), \mathcal{O}_{\mathbb{P}(W)}(-5), \mathcal{O}_{\mathbb{P}(W)}(-3), \mathcal{O}_{\mathbb{P}(W)}(-2)).$$

This is full exceptional collection (see [4]). The dual exceptional collection $\hat{\mathfrak{E}}$ of $\mathfrak{E}$ is following

$$\hat{\mathfrak{E}} = (\hat{\mathfrak{E}}_1, \hat{\mathfrak{E}}_2, \hat{\mathfrak{E}}_3, \hat{\mathfrak{E}}_4) := (\mathcal{T}_{\mathbb{P}(W)}, \mathcal{R}, \Omega_{\mathbb{P}(W)}(3), \mathcal{O}_{\mathbb{P}(W)}(2)).$$

Proposition. *There is an exact sequence of vector bundles on* $\mathbb{P}(W)$

$$0 \to F \to \mathcal{R} \xrightarrow{\mu_F} N^* \otimes \Omega_{\mathbb{P}(W)}(3) \to 0. \tag{15}$$

Proof. Consider the Beilinson spectral sequence of F with respect to the pair $(\mathfrak{E}, \hat{\mathfrak{E}})$ of dual exceptional collections

$$E_1^{pq} = \mathbf{H}^q(F \otimes \mathfrak{E}_p) \otimes \hat{\mathfrak{E}}_p \Rightarrow F.$$

Since

$$\mathbf{H}^q(\mathbb{P}(W), F \otimes \mathfrak{E}_1) = \mathbf{H}^q(\mathbb{P}(W), F \otimes \Omega_{\mathbb{P}(W)}(-4)) \cong \mathbf{H}^{3-q}(F^* \otimes \mathcal{T}_{\mathbb{P}(W)})^* \cong \mathbf{Hom}^{3-q}(F, \mathcal{T}_{\mathbb{P}(W)})^* = 0;$$
$$\mathbf{H}^q(\mathbb{P}(W), F \otimes \mathfrak{E}_2) = \mathbf{H}^q(\mathbb{P}(W), F \otimes \mathcal{O}_{\mathbb{P}(W)}(-5)) = \begin{cases} \mathbb{C}, & \text{if } q = 2 \\ 0, & \text{in other cases;} \end{cases}$$
$$\mathbf{H}^q(\mathbb{P}(W), F \otimes \mathfrak{E}_3) = \mathbf{H}^q(\mathbb{P}(W), F \otimes \mathcal{O}_{\mathbb{P}(W)}(-3)) = \begin{cases} N^*, & \text{if } q = 2 \\ 0, & \text{in other cases;} \end{cases}$$
$$\mathbf{H}^q(\mathbb{P}(W), F \otimes \mathfrak{E}_4) = \mathbf{H}^q(\mathbb{P}(W), F \otimes \mathcal{O}_{\mathbb{P}(W)}(-2)) = 0$$

the spectral sequence degenerates into (15) and the Proposition follows. □

3.2.4. Since $\hat{\mathfrak{E}}$ is dual to $\mathfrak{E}$ we have

$$\mathbf{Hom}(\mathcal{R}, \Omega_{\mathbb{P}(W)}(3)) = \mathbf{Hom}(\hat{\mathfrak{E}}_2, \hat{\mathfrak{E}}_3) \cong \mathbf{Hom}(\mathfrak{E}_2, \mathfrak{E}_3)^* =$$
$$= \mathbf{Hom}(\mathcal{O}_{\mathbb{P}(W)}(-5), \mathcal{O}_{\mathbb{P}(W)}(-3))^* \cong \mathbf{H}^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(2))^* \cong \mathbf{S}^2 W$$

Therefore the surjective map $\mu_F : \mathcal{R} \to N^* \otimes \Omega_{\mathbb{P}(W)}(3)$ gives rise to the embedding $A_F : N \to S^2 W$ which we can interpret as a net of quadrics in $\mathbb{P}(W^*)$.

Lemma. *The net A_F is a regular non-Lüroth net.*

Proof. Assume that for some $0 \neq n \in N$ the quadric $Q := A_F(n)$ is the quadric of corank 2. Let $L \subset W$ be the image of the map $W^* \to W$ induced by Q. Consider the composition $\mu_Q : \mathcal{R} \xrightarrow{\mu_F} N^* \otimes \Omega_{\mathbb{P}(W)}(3) \xrightarrow{\mathbf{ev}_n \otimes \mathbf{id}} \Omega_{\mathbb{P}(W)}(3)$. It should be surjective, because μ_F and $\mathbf{ev}_n$ are.

On the other hand this composition is nothing but the map, induced by the quadric $Q \in \mathbf{S}^2 W = \mathbf{Hom}(\mathcal{R}, \Omega_{\mathbb{P}(W)}(3))$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & W^* \otimes \mathcal{O}_{\mathbb{P}(W)} & \longrightarrow & \mathfrak{sl}(W) \otimes \mathcal{O}_{\mathbb{P}(W)}(1) & \longrightarrow & \mathcal{R} & \longrightarrow & 0 \\
 & & & & \big\downarrow{\scriptstyle Q} & & \big\downarrow & & \big\downarrow{\scriptstyle \mu_Q} & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}(W)}(-1) & \longrightarrow & W \otimes \mathcal{O}_{\mathbb{P}(W)} & \longrightarrow & \Lambda^2 W \otimes \mathcal{O}_{\mathbb{P}(W)}(1) & \longrightarrow & \Omega_{\mathbb{P}(W)}(3) & \longrightarrow & 0
\end{array}$$

Here the map $\mathfrak{sl}(W) \otimes \mathcal{O}_{\mathbb{P}(W)}(1) \to \Lambda^2 W \otimes \mathcal{O}_{\mathbb{P}(W)}(1)$ is induced from the composition

$$\mathfrak{sl}(W) \xrightarrow{\mathbf{id} \otimes Q} \mathfrak{sl}(W) \otimes \mathbf{S}^2 W \to W \otimes W^* \otimes W \otimes W \xrightarrow{\mathbf{id} \otimes \langle\cdot,\cdot\rangle \otimes \mathbf{id}} W \otimes W \xrightarrow{\mathbf{Alt}} \Lambda^2 W.$$

One easily sees that the cokernel of this map is $\Lambda^2(W/L)$. The above commutative diagram implies that the cokernel of the map μ_Q is equal to the cokernel of the arising map

$$W/L \otimes \mathcal{O}_{\mathbb{P}(W)} \to \Lambda^2(W/L) \otimes \mathcal{O}_{\mathbb{P}(W)}(1)$$

This map evidently vanishes over $\mathbb{P}(L) \subset \mathbb{P}(W)$. But if the corank of Q is greater then 1 then $\mathbf{dim}\, W/L \geq 2$, hence the second term above is not zero, hence the map is not surjective and the regularity follows.

Now since the map μ_F is surjective the Proposition 3.1.1 implies that A_F is non-Lüroth. $\square$

3.2.5. The above results can be summarized as follows.

Theorem 2. *The maps*

$$(C,\theta) \mapsto \mathcal{R}_C, \qquad F \mapsto A_F, \qquad A \mapsto (H_A, \theta_A)$$

define an isomorphisms

$$\mathbf{Qu} \setminus \mathbf{Qu}^L \cong \mathbf{MI}_3^{\mathcal{T}}(\mathbb{P}(W)) \cong \mathbf{Nets} \setminus \mathbf{Nets}^L .$$

3.2.6. **Corollary.** The space $\mathbf{MI}_3(\mathbb{P}(W))$ is rational.

Proof. Because $\mathbf{Nets} \setminus \mathbf{Nets}^L$ is an open subset of the Grassmann variety $\mathbf{Gr}(3, \mathbf{S}^2 W)$ and $\mathbf{MI}_3^{\mathcal{T}}(\mathbb{P}(W))$ is an open subset of $\mathbf{MI}_3(\mathbb{P}(W))$. $\square$

Remark 7. The rationality of $\mathbf{MI}_3(\mathbb{P}(W))$ originally was proved by G. Ellingsrud and S. A. Stromme in [6].

3.3. In this Subsection we will demonstrate how the curve C naturally arises while studying the stability of restriction of $F = \mathcal{R}_C$ on hyperplanes and lines in $\mathbb{P}(W)$.

3.3.1. The hyperplane $H \subset \mathbb{P}(W)$ is called *non-stable* for F if $F_{|H}$ is not stable (i.e. if $\mathbf{H}^0(H, F_{|H}) \neq 0$).

Proposition. *The set of all non-stable hyperplanes for $F = \mathcal{R}_C$ is $\beta(C) \subset \mathbb{P}(W^*)$.*

Proof. By definition $F_{|H}$ is not stable iff $\mathbf{H}^0(H, F_{|H}) \neq 0$. The exact sequence

$$0 \to F(-1) \xrightarrow{H} F \to F_{|H} \to 0$$

and 3.2.2 show that this is equivalent to the non-injectivity of the arising map $N \xrightarrow{H} W$. But this map is nothing but $A_C(\cdot, H)$ and the Proposition follows. $\square$

3.3.2. The line $L \subset \mathbb{P}(W)$ is called *d-jumping line* for F if $F_{|L} = \mathcal{O}_L(-d) \oplus \mathcal{O}_L(d)$.

Proposition. *If $F = \mathcal{R}_C$ then F has no 3-jumping lines. The set of all 2-jumping lines for $F = \mathcal{R}_C$ coincides with $\gamma(C) \subset \mathbf{Gr}(2, W)$.*

Proof. The line L is d-jumping line for F with $d \geq 2$ iff $\mathbf{H}^1(L, F_{|L}) \neq 0$. The exact sequence

$$0 \to F(-2) \to \mathbf{Ann}\, L \otimes F(-1) \to F \to F_{|L} \to 0$$

and 3.2.2 give the following exact sequence

$$N \otimes \mathbf{Ann}\, L \to W \to \mathbf{H}^1(L, F_{|L}) \to 0.$$

The first map here is just $A_C(\cdot, \cdot)$. Assume that this is not surjective. Then there exists $\varphi \in W^*$ such that $A_C(N, \varphi) \subset L$, hence $\varphi = \beta(x)$ for some $x \in C$ and $L = \gamma(x)$ (see Lemma 2.2.5).

To conclude L can not be a 3-jumping line for $F = \mathcal{R}_C$ because for any L we have $\mathcal{R}_{|L} = \mathcal{O}_L(1)^{\oplus 7} \oplus \mathcal{O}_L(2)^{\oplus 4}$ and $F_{|L}$ is a subbundle in $\mathcal{R}_{|L}$. □

3.3.3. Assume that C is smooth. Then θ is a line bundle.

We denote the divisor class on C corresponding to the line bundle θ by the same letter. We denote by K_C the canonical class of C. We have $\omega_C = \mathcal{O}_C(K_C)$.

Given a point $x \in C$ the Riemann-Roch implies that $|\theta + x| \neq \emptyset$. The non-degeneracy of θ means that there is a unique effective divisor $x_1 + x_2 + x_3 \in |\theta + x|$ which we will denote by $\mathbf{T}_\theta(x)$.

Let $\mathbf{T}_\theta \subset C \times C$ be the following subspace

$$\mathbf{T}_\theta = \{(x, y) \mid \mathbf{T}_\theta(x) - y \geq 0\} \subset C \times C.$$

It is known (see [5]) that $\mathbf{T}_\theta$ is a symmetric (with respect to the permutation of factors) divisor in $C \times C$ nonintersecting diagonal ($\mathbf{T}_\theta \cap \Delta = \emptyset$) and that $\mathbf{T}_\theta \sim \Delta + p_1^*\theta + p_2^*\theta$.

Proposition. *A 2-jumping line $L = \gamma(x)$ is contained in the non-stable hyperplane $H = \beta(y)$ iff $(x, y) \in \mathbf{T}_\theta \subset C \times C$.*

Proof. Let $\mathbf{T}_\theta(x) = y_1 + y_2 + y_3$ and let $H_i = \beta(y_i)$. The Riemann-Roch says that

$$\mathbf{H}^0(C, \beta^*\mathcal{O}_{\mathbb{P}(W^*)}(1) \otimes \mathcal{O}_C(-y_1 - y_2 - y_3)) = \mathbf{H}^0(C, \mathcal{O}_C(K_C + \theta - y_1 - y_2 - y_3)) = \mathbf{H}^0(C, \mathcal{O}_C(K_C - x)) = \mathbb{C}^2,$$

hence the triple of points $\beta(y_1), \beta(y_2), \beta(y_3) \in \mathbb{P}(W^*)$ lies on a line, that is $H_1 \cap H_2 \cap H_3 = L' \subset \mathbb{P}(W)$ and L' is a line. We claim that actually $L' = L$.

Indeed, chose a point $P \in L'$ and let $D_P = z_1 + z_2 + z_3 \in |K_C + \theta - y_1 - y_2 - y_3| \in |K_C - x|$ be the corresponding divisor and $l_P \subset \mathbb{P}(N)$ be the line in $\mathbb{P}(N)$ such that $D_P = \alpha^{-1}(l_P \cap \alpha(C))$. Then we have

$$A_C(\alpha(x), \beta(x), \beta(z_j)) = 0, \quad (j = 1, 2, 3).$$

On the other hand since $x, z_i, z_j \in l_P$ we can choose $a_{ij}, a'_{ij} \in \mathbb{C}$ such that $\alpha(z_i) = a_{ij}\alpha(x) + a'_{ij}\alpha(z_j)$, hence

$$A_C(\alpha(z_i), \beta(x), \beta(z_j)) = A_C(a_{ij}\alpha(x) + a'_{ij}\alpha(z_j), \beta(x), \beta(z_j)) = 0, \quad (i, j = 1, 2, 3).$$

This means that $A_C(l_P, \beta(x)) \in \mathbb{P}(W)$ is a common point of hyperplanes $\beta(z_1), \beta(z_2), \beta(z_3)$, that is $A_C(l_P, \beta(x)) = P$. Therefore $A_C(N, \beta(x)) = L'$, that is $L' = L$ and the Claim is proven.

Now we see that $\{y \mid \beta(y) \supset \alpha(x)\} = \mathbf{T}_\theta(x)$. Note that

$$\beta(y) \supset \alpha(x) \Leftrightarrow A_C(N, \beta(x), \beta(y)) = 0 \Leftrightarrow A_C(N, \beta(y), \beta(x)) = 0 \Leftrightarrow \alpha(y) \subset \beta(x),$$

and the Proposition follows. □

Remark 8. In the non-reduced case (see 2.1.2, 2.2.4) it is evident that a 2-jumping line $L = \gamma(x)$ is contained in a non-stable hyperplane $H = \beta(y)$ $(x, y \in C_{\mathbf{red}})$ iff $(x, y) \in \Delta \subset C_{\mathbf{red}} \times C_{\mathbf{red}}$.

References

[1] W. Barth. *Some properties of stable rank-2 vector bundles on* $\mathbb{P}_n$, Math. Ann. 226 (1977), pp. 125–150.

[2] W. Barth. *Moduli of vector bundles on the projective plane*, Invent. math., 42 (1977), pp. 63–91.

[3] W. Barth, G. Elencwajg. *Concernant la cohomologie des fibrés algébriques stables sur* $\mathbb{P}_n(\mathbb{C})$, In Variétés analytiques compactes, Nice 1977, Lecture Notes in Math. 683, pp. 1–24. Springer 1978.

[4] A. Bondal. *Representations of associative algebras and coherent sheaves*, Izv. Akad. Nauk SSSR, Ser. Mat., 53 (1989), pp. 25–44; English transl. in Math. USSR Izvestiya, 34 (1990).

[5] I. Dolgachev, V. Kanev. *Polar covariants of plane cubics and quartics*, Advances in Math., **98** (1993), pp. 216–301.

[6] G. Ellingsrud, S. A. Strømme. *Stable rank-2 vector bundles on* $\mathbb{P}^3$ *with* $c_1 = 0$ *and* $c_2 = 3$, Math. Ann. 255 (1981), pp. 123–135.

[7] R. Hartshorne. *Stable vector bundles of rank* 2 *on* $\mathbb{P}^3$, Math. Ann. 238 (1978), pp. 229-280.

[8] M. Maruyama. *Vector bundles on* $\mathbb{P}^2$ *and torsion sheaves on the dual plane*, In Vector bundles on algebraic varieties, Bombay 1984, Tata Inst. Fund. Res. Stud. Math., 11, Tata Inst. Fund. Res., pp. 275–340. Bombay, 1987

[9] C. Okonek, M. Schneider, H. Spindler. *Vector bundles on complex projective spaces*, Progress in Mathematics, 3. Birkhäuser, 1980.

[10] A. Tyurin. *On intersection of quadrics*, Uspekhi Mat. Nauk, **30** (1975), 6, pp. 51–99; English transl. in Russian Math. Surveys, **30**:6 (1975), pp. 51–105.

[11] Wall C.T.C. *Nets of quadrics and theta-characteristics of singular curves*, Philos. Trans. R. Soc. London, Ser. A, 289 (1978), pp. 229–269.

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