On the fundamental groups of complements to the dual hypersurfaces of projective curves

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1 Introduction

Let C be a compact Riemann surface of genus $g \geq 1$. We embed C into a projective space $\mathbb{P}^{n-g}$ by a very ample line bundle L of degree $n \geq 2g+1$:

$$\Phi_{|L|} : C \hookrightarrow \mathbb{P}^{n-g}.$$

We denote by C_L the image of $\Phi_{|L|}$. Let $(\mathbb{P}^{n-g})\check{}$ be the dual projective space of $\mathbb{P}^{n-g}$, and let $\check{C}_L \subset (\mathbb{P}^{n-g})\check{}$ be the dual hypersurface of C_L; that is,

$$\check{C}_L := \{H \in (\mathbb{P}^{n-g})\check{}\,;\ H \text{ does not intersect } C_L \text{ transversely}\}.$$

The purpose of this paper is to calculate the fundamental group of the complement to this dual hypersurface. The idea of the calculation stems from [Z], where the fundamental group of such complements was calculated in the case $g = 1$.

Let $\mathrm{Pic}^n(C)$ be the Picard variety of line bundles of degree n on C, and let $S^n(C)$ be the symmetric product of n-copies of C, which parameterizes all effective divisors on C of degree n. Then there exists a natural homomorphism

$$\phi : S^n(C) \longrightarrow \mathrm{Pic}^n(C)$$

which maps a divisor D to the associated line bundle $\mathcal{O}_C(D)$. Let $V \subset S^n(C)$ be the image of the big diagonal;

$$V := \{(x_1, \ldots, x_n) \in S^n(C);\ x_i = x_j \text{ for some } i \neq j\ \}.$$

The fundamental group of the complement $S^n(C) \setminus V$ is, by definition, the braid group $B(g,n) \cong \pi_1 B_{0,n} C$ (in the notation of [B]) of C with n strings.

Theorem 1 *For a general line bundle $L \in \mathrm{Pic}^n(C)$ of degree n, the fundamental group $\pi_1((\mathbb{P}^{n-g})\check{} \setminus \check{C}_L)$ is isomorphic to the kernel of the natural homomorphism*

$$\phi'_* \quad : \quad \pi_1(S^n(C) \setminus V) \quad \longrightarrow \quad \pi_1(\mathrm{Pic}^n(C)) \cong H_1(C; \mathbf{Z})$$

induced by the restriction ϕ' of ϕ to the complement $S^n(C) \setminus V$.

We denote the kernel of ϕ'_* by $G_{g,n}$.

Theorem 2 *The group $G_{g,n}$, $n \geq 2g+1$, is generated by $n + 3g - 1$ generators. Denote these generators by*

$$\begin{array}{llll} c_2\,,\ c_4\,, & \cdots & ,\ c_{2g-4}\,,\ c_{2g-2}\,; & \\ g_{2g}\,,\ g_{2g+1}\,, & \cdots & ,\ g_{n-2}\,,\ g_{n-1}\,; & \\ g_{1,i,j}\,,\ g_{3,i,j}\,, & \cdots & ,\ g_{2g-3,i,j}\,,\ g_{2g-1,i,j}\,, & \qquad i\,,\,j = 0\,,\,1\,. \end{array}$$

The set of defining relations consists of

$$
\begin{array}{lll}
[c_{2k}, c_{2l}] = 1\,, & & |\, k-l \,| \neq 1\,; \\
[c_{2k}, g_{l,i,j}] = 1\,, & i,\, j = 0,\, 1\,, & 2k \neq l \pm 1\,; \\
[c_{2k}, g_l] = 1\,, & & (2k, l) \neq (2g-2, 2g)\,; \\
\left[(c_{2k-2}c_{2k}c_{2k-2}^{-1}), g_{2k-1,i,j}\right] = 1\,, & i,\, j = 0,\, 1\,, & 2 \leq k \leq g-1\,; \\
\left[(c_{2g-2}g_{2g}c_{2g-2}^{-1}), g_{2g-1,i,j}\right] = 1\,, & & i,\, j = 0,\, 1\,; \\
[g_{2k-1,i,j}, g_{2l-1,i,j}] = 1\,, & i,\, j = 0,\, 1\,, & k \neq l\,; \\
[g_{2k-1,i,j}, g_l] = 1\,, & i,\, j = 0,\, 1\,, & l \geq 2k+1\,; \\
[g_k, g_l] = 1\,, & & |\, k-l \,| \neq 1\,; \\
c_{2i}c_{2i+2}c_{2i} = c_{2i+2}c_{2i}c_{2i+2}\,, & & 1 \leq i \leq g-2; \\
c_{2k}g_{2k\pm1,i,j}c_{2k} = g_{2k\pm1,i,j}c_{2k}g_{2k\pm1,i,j}\,, & i,\, j = 0,\, 1\,, & 1 \leq k \leq g-1; \\
c_{2g-2}g_{2g}c_{2g-2} = g_{2g}c_{2g-2}g_{2g}\,; & & \\
g_{2g}g_{2g-1,i,j}g_{2g} = g_{2g-1,i,j}g_{2g}g_{2g-1,i,j}\,, & & i,\, j = 0,\, 1; \\
g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}\,, & & 2g \leq i \leq n-2; \\
(g_{2j\pm1,1,0}g_{2j\pm1,0,0}c_{2j})^2 = (c_{2j}g_{2j\pm1,1,0}g_{2j\pm1,0,0})^2\,, & & 1 \leq j \leq g-1; \\
(g_{2j\pm1,0,1}g_{2j\pm1,1,1}c_{2j})^2 = (c_{2j}g_{2j\pm1,0,1}g_{2j\pm1,1,1})^2\,, & & 1 \leq j \leq g-1; \\
(g_{2j\pm1,0,1}g_{2j\pm1,0,0}c_{2j})^2 = (c_{2j}g_{2j\pm1,0,1}g_{2j\pm1,0,0})^2\,, & & 1 \leq j \leq g-1; \\
(g_{2g-1,1,0}g_{2g-1,0,0}g_{2g})^2 = (g_{2g}g_{2g-1,1,0}g_{2g-1,0,0})^2\,; & & \\
(g_{2g-1,0,1}g_{2g-1,1,1}g_{2g})^2 = (g_{2g}g_{2g-1,0,1}g_{2g-1,1,1})^2\,; & & \\
(g_{2g-1,0,1}g_{2g-1,0,0}g_{2g})^2 = (g_{2g}g_{2g-1,0,1}g_{2g-1,0,0})^2\,; & & \\
\multicolumn{3}{l}{c_2 \cdots c_{2g-2}g_{2g} \cdots g_{n-1}g_{n-1} \cdots g_{2g}(g_{2g-1,0,1}g_{2g-1,1,1}g_{2g-1,1,0}g_{2g-1,0,0})c_{2g-2} \quad \cdots} \\
\multicolumn{3}{l}{\cdots (g_{3,0,1}g_{3,1,1}g_{3,1,0}g_{3,0,0})c_2(g_{1,0,1}g_{1,1,1}g_{1,1,0}g_{1,0,0}) = 1\,.}
\end{array}
$$

Proof of Theorem 2 is based essentially on the ideas contained in section 2 of [Z]. By this reason, we advise to look through section 2 in [Z] before reading the proof of this Theorem.

Let $pr : C_L \to \mathbb{P}^2$ be a general projection, and denote by C'_L its image. Then the dual curve $(C'_L)^\vee \subset (\mathbb{P}^2)^\vee$ of C'_L is nothing but a general plane section of $\check{C}_L$. Therefore, we have the following theorem as an easy consequence:

Theorem 3 *For a general line bundle $L \in \mathrm{Pic}^n(C)$ of degree n, the fundamental group $\pi_1(\mathbb{P}^2 \setminus (C'_L)^\vee)$ has the same presentation as that of $G_{g,n}$ in Theorem 2.*

The contents of this paper are as follows. In section 1, we prove Theorem 1. The main idea is to apply an analogue of [Sh, Theorem 1] to the pull-back of ϕ' by the universal covering of $\mathrm{Pic}^n(C)$. In section 2, we recall some properties of the presentations of the braid group $B(g,n)$. In section 3, we prove Theorem 2 by applying Reidemeister-Schreier method and by reducing general case to the case considered in [Z].

We would like to thank Max-Planck-Institut für Mathematik in Bonn for providing us with excellent research environment.

2 Proof of Theorem 1

Since $n \geq 2g+1$, the morphism ϕ is a fiber bundle with fibers isomorphic to $\mathbb{P}^{n-g}$. For $L \in \mathrm{Pic}^n(C)$, we denote by $\mathbb{P}(L)$ the fiber $\phi^{-1}(L)$, which is canonically isomorphic to the projective space $\mathbb{P}_*(H^0(C,L))$ of all lines in $H^0(C,L)$ passing through the origin. The embedding morphism $\Phi_{|L|}$ is, by definition, a morphism into the dual projective space $\mathbb{P}(L)^\vee = \mathbb{P}_*(H^0(C,L))^\vee$. Therefore, we can consider the dual hypersurface $\check{C}_L$ to be a hypersurface in the projective space $\mathbb{P}(L)$ in a natural way. It is obvious that

$$\check{C}_L = \mathbb{P}(L) \cap V. \tag{2.1}$$

By Nori's Lemma [N, Lemma 1.5 (C)], we have an exact sequence

$$\pi_1(\mathbb{P}(L) \setminus \check{C}_L) \longrightarrow \pi_1(S^n(C) \setminus V) \longrightarrow \pi_1(\mathrm{Pic}^n(C)) \longrightarrow \{1\}$$

for a general $L \in \mathrm{Pic}^n(C)$. Therefore, the point of the proof is to show the injectivity of the homomorphism $\pi_1(\mathbb{P}(L) \setminus \check{C}_L) \to \pi_1(S^n(C) \setminus V)$ induced by the inclusion of a general fiber of ϕ'. Let $u : \mathbb{C}^g \to \mathrm{Pic}^n(C)$ be the universal covering of $\mathrm{Pic}^n(C)$. We define $\Sigma^n(C)$ and $\mathcal{V}$ by the following fiber products:

$$\begin{array}{ccc} \Sigma^n(C) & \xrightarrow{\bar{\phi}} & \mathbb{C}^g \\ \downarrow & \square & \downarrow{\scriptstyle u} \\ S^n(C) & \xrightarrow[\phi]{} & \mathrm{Pic}^n(C) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathbb{C}^g \\ \downarrow & \square & \downarrow{\scriptstyle u} \\ V & \xrightarrow[\phi|_V]{} & \mathrm{Pic}^n(C). \end{array}$$

This $\mathcal{V}$ is an analytic divisor of $\Sigma^n(C)$. Then we have

$$\pi_1(\Sigma^n(C) \setminus \mathcal{V}) \cong \mathrm{Ker}(\, \phi'_* : \pi_1(S^n(C) \setminus V) \to \pi_1(\mathrm{Pic}^n(C))\,). \tag{2.2}$$

Claim 1 *For all $L \in \mathrm{Pic}^n(C)$, the hypersurface $\check{C}_L$ is reduced of constant degree $2(n+g-1)$.*

To prove this claim, we choose a linear subspace $\mathbb{P}^{n-g-3}$ in $(\mathbb{P}(L))\check{}$ of codimension 3 which is in general position with respect to C_L. Consider the projection pr of C_L to $\mathbb{P}^2$ with the center being this $\mathbb{P}^{n-g-3}$. We fix a general point on $\mathbb{P}^2$ and take the pencil $\mathcal{P}$ of lines passing through this point. This pencil $\mathcal{P}$ yields a line in $\mathbb{P}(L)$ whose point corresponds to a hyperplane of $(\mathbb{P}(L))\check{}$ spanned by the $\mathbb{P}^{n-g-3}$ and a member of $\mathcal{P}$. The intersection points of this line with $\check{C}_L$ correspond to the lines in $\mathcal{P}$ which are tangent to the image $pr(C_L)$ of C_L by the projection. Therefore the degree of $\check{C}_L$ is equal with the degree of the dual curve of $pr(C_L)$. Since $n \geq 2g+1$, C_L is non-singular. Since pr is a general projection, $pr(C_L)$ is a curve of degree n with nodes as its only singularities. The number of nodes is $(n-1)(n-2)/2 - g$. Thus, by Plücker formula, its dual is of degree $2(n+g-1)$.

Now the holomorphic map $\bar{\phi} : \Sigma^n(C) \to \mathbb{C}^g$ is a fiber bundle with fibers isomorphic to $\mathbb{P}^{n-g}$. Therefore, there exists a global trivialization

$$\Sigma^n(C) \cong \mathbb{P}^{n-g} \times \mathbb{C}^g \tag{2.3}$$

over $\mathbb{C}^g$. We fix this analytic isomorphism once and for all. Let $\mathcal{W}$ be the analytic divisor of $\mathbb{P}^{n-g} \times \mathbb{C}^g$ corresponding to $\mathcal{V}$ via this isomorphism. For a point λ of $\mathbb{C}^g$, we denote by $W(\lambda)$ the intersection of $\mathcal{W}$ with $\mathbb{P}^{n-g} \times \{\lambda\}$, and consider it as a hypersurface in $\mathbb{P}^{n-g}$. It is obvious that $W(\lambda)$ is projectively isomorphic to $\check{C}_{u(\lambda)}$.

Now we shall prove that, for a general $\lambda \in \mathbb{C}^g$, the inclusion induces an isomorphism

$$\pi_1(\mathbb{P}^{n-g} \setminus W(\lambda)) \cong \pi_1((\mathbb{P}^{n-g} \times \mathbb{C}^g) \setminus \mathcal{W}). \tag{2.4}$$

This isomorphism, combined with (2.2), gives us the hoped-for isomorphism.

The proof of the isomorphism (2.4) is quite similar to the proof of [Sh, Theorem 1]. The reason why we cannot apply [Sh, Theorem 1] to our situation is that the divisor $\mathcal{W}$ on $\mathbb{P}^{n-g} \times \mathbb{C}^g$ is not algebraic but only analytic. Hence we need to modify some parts of the proof in [Sh].

To be compatible with the notation of [Sh], we denote by A the affine space $\mathbb{C}^g$, and by p the projection from $(\mathbb{P}^{n-g} \times A) \setminus \mathcal{W}$ to A. As in [Sh, p.518], we construct the following data;

- a closed real semi-*analytic* subset $\Omega \subset A$ of real codimension ≥ 3,
- a sequence of classically open subsets $U_1 \subset U_2 \subset \cdots$ such that $\cup_{i=1}^{\infty} U_i = A \setminus \Omega$, and
- sections $s_i : U_i \to p^{-1}(U_i)$ of p over U_i.

For a point $a \in A$ and a closed subset $\Gamma \subset A$, we use the symbols $R_a(\Gamma) \subset A$ and $\widetilde{R}_a(\Gamma) \subset S_a$ in the same meaning as in [Sh, p.519]. Suppose that Γ is a closed analytic subset of complex codimension $\geq c$ in A. Then $R_a(\Gamma)$ is a closed real semi-analytic subset of real codimension $\geq 2c-1$ in A, while $\widetilde{R}_a(\Gamma)$ may fail even to be closed in the $(2g-1)$-sphere S_a, and this latter is the main reason why we have to rewrite the proof in [Sh, §2].

For a positive real number r and a point $b \in A$, we denote by $\Gamma\langle b, r\rangle$ the intersection of Γ with the closed ball of radius r with the center b. Then $\widetilde{R}_a(\Gamma\langle b, r\rangle)$ is a closed real semi-analytic subset of real codimension $\geq 2c-1$ in S_a for any $r \in \mathbb{R}_{>0}$ and $b \in A$.

Since the projection $S^n(C) \setminus V \to \mathrm{Pic}^n(C)$ is algebraic, there exists a Zariski closed subset $\Delta \subset \mathrm{Pic}^n(C)$ of codimension 1 such that $S^n(C) \setminus V \to \mathrm{Pic}^n(C)$ is locally trivial (in the category of differentiable manifolds) over $\mathrm{Pic}^n(C) \setminus \Delta$. Let $D \subset A$ be the pull-back of Δ by the universal covering $u : A \to \mathrm{Pic}^n(C)$. For a line $\Lambda \subset \mathbb{P}^{n-g}$ and a point $x \in \Lambda$, we put

$$\begin{aligned} D_\Lambda &:= \{\lambda \in A\,;\ \Lambda \text{ does } \textit{not} \text{ intersect } W(\lambda) \text{ transeversely}\}, \\ D_x &:= \{\lambda \in A\,;\ x \in W(\lambda)\}. \end{aligned}$$

Then both of D_Λ and D_x are closed analytic subsets of A of codimension 1 or possibly 0. We shall prove the following:

Claim 2 *If x, Λ and a point $o \in A$ are chosen appropriately, then $R_o(D) \cap R_o(D_\Lambda) \cap R_o(D_x)$ is a closed real semi-analytic subset of real codimension ≥ 3 in A.*

After proving this claim, we can construct the hoped-for data by applying the argument in [Sh, p.521-522] verbatim.

Proof of Claim 2. It is enough to prove that, if x, Λ and o are chosen appropriately, then $\widetilde{R}_o(D\langle o, r\rangle) \cap \widetilde{R}_o(D_\Lambda\langle o, r\rangle) \cap \widetilde{R}_o(D_x\langle o, r\rangle)$ is a closed real semi-analytic subset of real codimension ≥ 3 in S_o for all $r \in \mathbb{R}_{>0}$.

The number of the irreducible components of D is at most countable. Let D_1, D_2, ... be the irreducible components of D, and let λ_i be a point on D_i. By Baire's category theorem, $\mathbb{P}^{n-g} \setminus (\cup_i W(\lambda_i))$ is non-empty. Let y be a point of $\mathbb{P}^{n-g} \setminus (\cup_i W(\lambda_i))$, and put

$$G_y := \{\Lambda \in \mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^{n-g})\,;\ y \in \Lambda\}.$$

Since $\lambda_i \notin D_y$, D_y is a closed analytic subset of codimension 1 in A.

The number of the irreducible components of D_y is at most countable. Let $D_{y,1}$, $D_{y,2}$, ... be the irreducible components of D_y, and let $\lambda_{y,j}$ be a point of $D_{y,j}$. We put

$$\Gamma_{y,j} := \{\Lambda \in G_y\,;\ \Lambda \subset W(\lambda_{y,j})\}.$$

Then $\Gamma_{y,j}$ is a Zariski closed subset of codimension ≥ 1 in G_y. We also put

$$\Gamma_i := \{\Lambda \in G_y;\ \Lambda \text{ does } \textit{not} \text{ intersect } W(\lambda_i) \text{ transeversely }\}.$$

Since $y \notin W(\lambda_i)$ and $W(\lambda_i)$ is reduced by Claim 1, Γ_i is a Zariski closed subset of codimension ≥ 1 in G_y. Hence, by Baire's theorem again, the set

$$G_y \setminus (\bigcup_i \Gamma_i \cup \bigcup_j \Gamma_{y,j})$$

is non-empty. We choose a line Λ from this set. By the definition of Γ_i, D_Λ does not contain λ_i for any i. Hence $D_\Lambda \cap D$ is of codimension ≥ 2 in A. By the definition of $\Gamma_{y,j}$, $\Lambda \cap W(\lambda_{y,j})$ consists of finite number of points for all j. Hence there exists a point z on $\Lambda \setminus (\cup_j W(\lambda_{y,j}))$. Then D_z does not contain $\lambda_{y,j}$ for any j. Hence $D_z \cap D_y$ is a closed analytic subset of codimension ≥ 2 in A. This implies that

$$\Xi_\Lambda := \{\lambda \in A\,;\ \Lambda \subset W(\lambda)\}$$

is contained in a closed analytic subset of codimension ≥ 2 in A.

Since $D_\Lambda \cap D$ is of codimension ≥ 1 in D, there exists a set $\{a_1, a_2, \ldots\}$ of countably many points on $D \setminus D_\Lambda$ which is dense in D. Let $E_\nu\langle r\rangle$ be the union of all affine lines in A passing through a_ν and intersecting $D_\Lambda\langle a_\nu, r\rangle$. Let E_ν be the union $\cup_{r\in\mathbb{R}_{>0}} E_\nu\langle r\rangle$. Each $E_\nu\langle r\rangle$ is a closed subset of A which is real semi-analytic of real codimension ≥ 1. Hence, by Baire's theorem again, we have

$$A \setminus \cup_\nu E_\nu \quad = \quad A \setminus \cup_\nu (\cup_{n=1}^{\infty} E_\nu\langle n\rangle) \quad \neq \quad \emptyset.$$

Let o be a point of $A \setminus \cup_\nu E_\nu$. Then $\widetilde{R}_o(D\langle o,r\rangle) \cap \widetilde{R}_o(D_\Lambda\langle o,r\rangle)$ is a closed real semi-analytic subset of real codimension ≥ 2 in S_o for all r, because $\widetilde{R}_o(D_\Lambda\langle o,r\rangle)$ does not contain the image of a_ν by the projection $\omega : A \setminus \{o\} \to S_o$, and the set $\{\omega(a_\nu); a_\nu \in D\langle o,r\rangle\}$ is dense in $\widetilde{R}_o(D\langle o,r\rangle)$.

Let $\widetilde{R}_o(D, D_\Lambda, r)$ be the union of the irreducible components of $\widetilde{R}_o(D, D_\Lambda, r)$ which are of real codimension 2 in S_o. Recall that Ξ_Λ is contained in a closed analytic subset of codimension ≥ 2 in A. Hence $\widetilde{R}_o(\Xi_\Lambda\langle o,r\rangle)$ is contained in a closed real semi-analytic subset of real codimension ≥ 3 in S_o. Thus there exists a set $\{b_1, b_2, \ldots\}$ of countably many points of $\widetilde{R}_o(D, D_\Lambda, r) \setminus \widetilde{R}_o(\Xi\langle o,r\rangle)$ which is dense in $\widetilde{R}_o(D, D_\Lambda, r)$. Let σ_μ be the real semi-line in A passing through o and b_μ with the end-point o. Then the intersection

$$\Lambda \cap (\cup_{\lambda\in\sigma_\mu\langle o,t\rangle} W(\lambda))$$

is a closed real semi-analytic subset of Λ of real codimension ≥ 1 for all $t \in \mathbb{R}_{>0}$. Hence $\Lambda \setminus \cup_\mu(\cup_{\lambda\in\sigma_\mu} W(\lambda))$ is a non-empty set, from which we choose a point x. Then $\widetilde{R}_o(D_x)$ contains none of b_μ. This implies that $\widetilde{R}_o(D\langle o,r\rangle) \cap \widetilde{R}_o(D_\Lambda\langle o,r\rangle) \cap \widetilde{R}_o(D_x\langle o,r\rangle)$ is a closed real semi-analytic subset of real codimension ≥ 3. □

Thus the construction of the hoped-for data is completed.

The projection $(\mathbb{P}^{n-g} \times A) \setminus \mathcal{W} \to A$ is locally trivial (in the category of differentiable manifolds) over $A \setminus D$. Moreover, when we are given a continuous map $f_0 : I^2 \to A$ such that $f_0(\partial I^2) \cap D = \emptyset$, then we can perturb f_0 to $f_\epsilon : I^2 \to A$ homotopically relative to ∂I^2 so that $f_\epsilon^{-1}(f_\epsilon(I^2) \cap D)$ consists of finitely many points in I^2.

Now we can apply the argument in the first paragraph of [Sh, p.519], and follow the proof of [Sh, Corollary] to obtain the isomorphism (2.4). The assumption (C.1) in [Sh, Corollary] follows from Claim 1. The assumption (S) in [Sh, p.511] follows from the above construction. The assumptions (2.1), (2.2) and (3.1) in [Sh, Theorems 2 and 3] hold obviously. The assumption (3.2) in [Sh, Theorem 3] does not hold in our case, at least literally, because we have left the category of algebraic varieties when we take the universal covering of $\mathrm{Pic}^n(C)$. This assumption, however, is used only in [Sh, §1.3]. All we have to do is to replace $\mathbb{P}^M$ in [Sh, p.517] by the first factor $\mathbb{P}^{n-g}$ of the product $\mathbb{P}^{n-g} \times \mathbb{C}^g$, and to replace Zariski open subsets of B by classically open subsets of B. □

3 The braid groups $B(g,n)$

Consider the braid group $B(g,n)$ of n strings on a surface S_g of genus g. We shall assume that $n \geq 2g+1$. The presentation of $B(g,n)$ was obtained in [Sc] . The sets of generators and defining relations of the presentation in [Sc] (after correction misprints) can be reduced to the following presentation of $B(g,n)$. The generators of $B(g,n)$ are

$$\rho_{i,j}\,, \qquad 1 \leq i \leq n\,, 1 \leq j \leq 2g\,;$$
$$\sigma_1\,, \sigma_2\,, \quad \cdots \quad , \sigma_{n-1}.$$

The set of defining relations consists of

$$[\rho_{i,j}, \rho_{k,l}] = 1\,, \qquad i < k\,,\, j < l,\ (j,l) \neq (2t-1, 2t)\,; \tag{1}$$
$$[\rho_{i,j}, \sigma_k] = 1\,, \qquad i \neq k \ \text{ nor } \ k-1\,; \tag{2}$$
$$\rho_{k,j} = \sigma_k \rho_{k+1,j} \sigma_k^{-1}\,, \qquad 1 \leq k \leq n-1\,; \tag{3}$$
$$(\rho_{i,j}\sigma_i^{-1})^2 = (\sigma_i^{-1}\rho_{i,j})^2\,, \qquad 1 \leq i \leq n-1, 1 \leq j \leq 2g\,; \tag{4}$$
$$[\sigma_i, \sigma_j] = 1\,, \qquad |\, i-j \,| \neq 1\,; \tag{5}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\,, \qquad 1 \le i \le n-2; \tag{6}$$

$$\left[(\sigma_i\rho_{j,2t}\sigma_i^{-1}), \rho_{j,2t-1}^{-1}\right] = \sigma_i^2\,, \qquad j = i\,, \text{ or } i+1\,; \tag{7}$$

$$\sigma_1\sigma_2\cdot\ldots\cdot\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdot\ldots\cdot\sigma_1 = \left[\rho_{1,1}, \rho_{1,2}^{-1}\right]\left[\rho_{1,3}, \rho_{1,4}^{-1}\right]\cdots\left[\rho_{1,2g-1}, \rho_{1,2g}^{-1}\right]\,. \tag{8}$$

Note that we read all words contained in the presentation given in [Sc] from right to left and write down them from left to right.

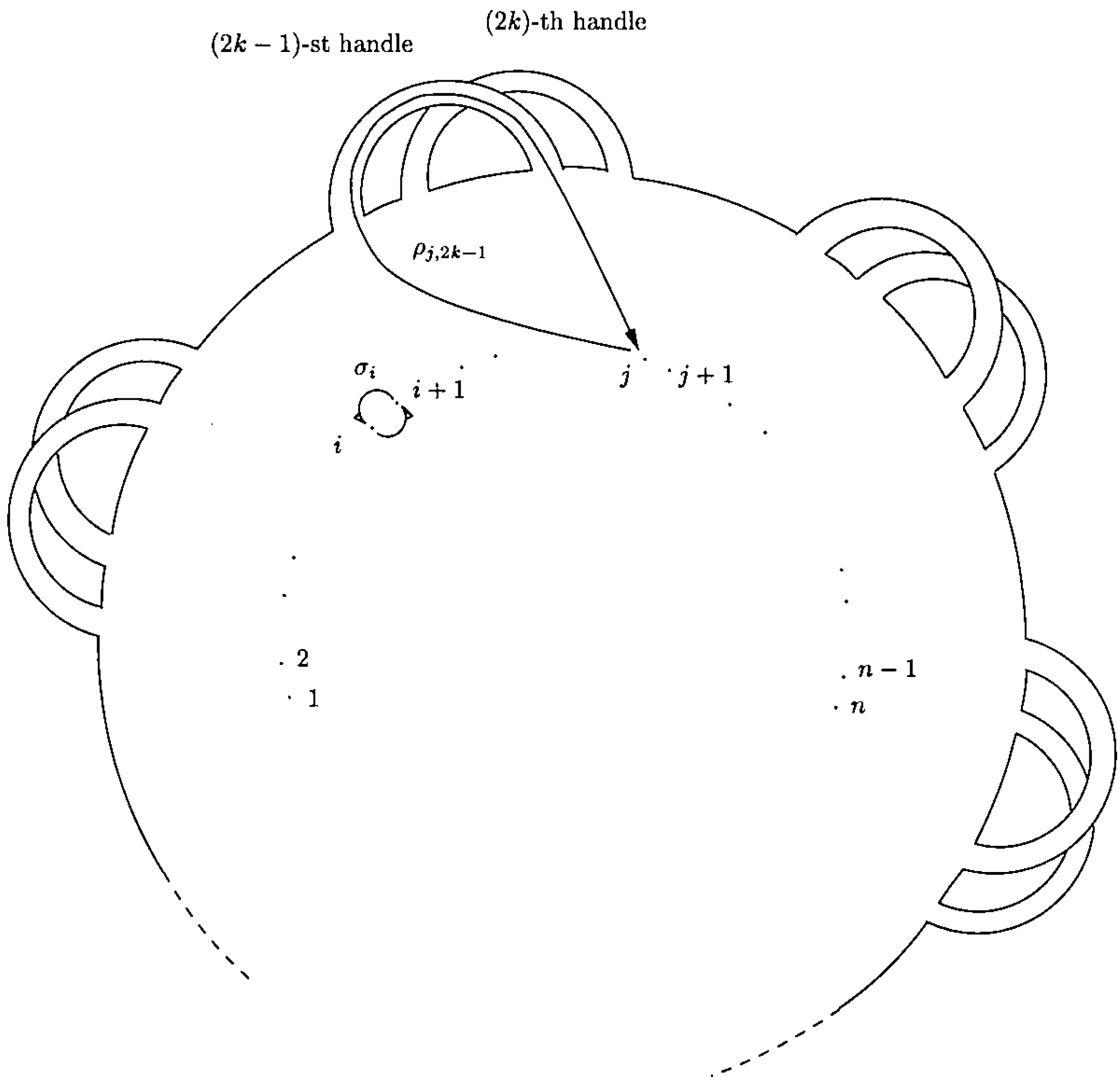

Figure 1

The generators $\rho_{i,j}$ and σ_k have the following geometrical meaning: S_g minus a 2-disc can be thought as a 2-disc Δ union $2g$ untwisted 1-handles. For each r the $(2r-1)$st and $(2r)$th handles are linked and no other pair of handles is linked. We number the handles reading from left to right. We shall assume that n fixed points lie on a circle which is the boundary of a smaller disc in Δ. We choose one of these points, say x, and number them (starting from x) consecutively moving along the circle in clockwise direction. The elements $\rho_{j,k}$ and σ_i are drawn in Figure 1.

Lemma 1 *Put*

$$\sigma_n = \sigma_0 = \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1} \cdots \sigma_1^{-1}. \tag{9}$$

and define σ_k for all $k \in \mathbf{Z}$ assuming $\sigma_k = \sigma_{n+k}$. Let

$$A_k = \sigma_k\sigma_{k+1} \cdot \ldots \cdot \sigma_{k+n-3}\sigma_{k+n-2}^2\sigma_{k+n-3} \cdot \ldots \cdot \sigma_k. \tag{10}$$

Then the following relations

$$\begin{aligned}
\sigma_n\sigma_1\sigma_n &= \sigma_1\sigma_n\sigma_1\,; && && (11)\\
\sigma_{n-1}\sigma_n\sigma_{n-1} &= \sigma_n\sigma_{n-1}\sigma_n\,; && && (12)\\
\sigma_n\sigma_k &= \sigma_k\sigma_n\,, && 2 \le k \le n-2\,; && (13)\\
A_{k+1} &= \sigma_k A_k \sigma_k^{-1}\,, && k \in \mathbf{Z}\,; && (14)\\
A_k\sigma_l &= \sigma_l A_k\,, && l \not\equiv k \text{ nor } k-1 (mod\, n) && (15)
\end{aligned}$$

are consequences of (5) and (6).

Proof follows from the same assertion for the braid group of n strings on a disc. □

Lemma 2 *The presentation (1) - (8) of $B(g,n)$ is equivalent to the following presentation. The generators of $B(g,n)$ are*

$$\begin{matrix} \rho_{1,1},\, \rho_{1,2},\, \rho_{3,3},\, \rho_{3,4}, & \cdots, & \rho_{2g-1,2g-1},\, \rho_{2g-1,2g}\,, \\ \sigma_1\,, \sigma_2\,, & \cdots, & \sigma_{n-1}. \end{matrix}$$

The set of defining relations consists of

$$\begin{aligned}
[\rho_{i,*},\rho_{j,*}] &= 1\,,\ i \neq j\,; && (16)\\
[\rho_{i,*},\sigma_j] &= 1\,,\ \ j \neq i \quad nor \quad i-1\,; && (17)\\
\left[(\sigma_{i-1}\rho_{i,*}\sigma_{i-1}^{-1}),\sigma_i\right] &= 1\,,\ \ i = 1, 3, \ldots, 2g-1\,; && (18)\\
(\rho_{2i-1,j}\sigma_{2i-1}^{-1})^2(\sigma_{2i-1}^{-1}\rho_{2i-1,j})^{-2} &= 1\,,\ \ j = 2i-1 \quad or \quad 2i\,; && (19)\\
[\sigma_i,\sigma_j] &= 1\,,\ \ |\, i-j \,| \neq 1\,; && (20)\\
\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} &= 1\,,\ \ 1 \le i \le n-2; && (21)\\
\left[(\sigma_j\rho_{2i+1,2i}\sigma_j^{-1}),\rho_{2i+1,2i-1}^{-1}\right]\sigma_j^{-2} &= 1\,,\ \ j = 2i \quad or \quad 2i+1\,; && (22)
\end{aligned}$$

$$\begin{aligned}
&\sigma_1\sigma_2 \cdot \ldots \cdot \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdot \ldots \cdot \sigma_{2g-1}\left[\rho_{2g-1,2g-1},\rho_{2g-1,2g}^{-1}\right]^{-1}\sigma_{2g-2}\sigma_{2g-3} \quad \cdot \\
&\cdot \left[\rho_{2g-3,2g-3},\rho_{2g-3,2g-2}^{-1}\right]^{-1} \cdot \ldots \cdot \sigma_4\sigma_3\left[\rho_{3,3},\rho_{3,4}^{-1}\right]^{-1}\sigma_2\sigma_1\left[\rho_{1,1},\rho_{1,2}^{-1}\right]^{-1} \ = \ 1.
\end{aligned} \tag{23}$$

Proof. To obtain relations (1) - (8), we define $\rho_{i,l}$ by induction using (3). After that, to verify relations (2), we need to show by induction that if relations (17), (18) hold for $\rho_{i,l}$, then the similar relations also hold for $\rho_{i\pm1,l}$. The checking is the following.

$$[\rho_{i-1,l},\sigma_j] = [\sigma_{i-1}\rho_{i,l}\sigma_{i-1}^{-1},\sigma_j] = 1$$

for $j \neq i-2$ nor $i-1$ by assumption of induction and by (20).

$$\begin{aligned}
&\left[(\sigma_{i-2}\rho_{i-1,l}\sigma_{i-2}^{-1}),\sigma_{i-1}\right] = \left[(\sigma_{i-2}\sigma_{i-1}\rho_{i,l}\sigma_{i-1}^{-1}\sigma_{i-2}^{-1}),\sigma_{i-1}\right] = \\
&\sigma_{i-2}\sigma_{i-1}\rho_{i,l}\sigma_{i-1}^{-1}\sigma_{i-2}^{-1}\underline{\sigma_{i-1}\sigma_{i-2}\sigma_{i-1}}\rho_{i,l}^{-1}\sigma_{i-1}^{-1}\sigma_{i-2}^{-1}\sigma_{i-1}^{-1} = \\
&\sigma_{i-2}\sigma_{i-1}\rho_{i,l}\sigma_{i-1}^{-1}\sigma_{i-2}^{-1}\underline{\sigma_{i-2}\sigma_{i-1}\sigma_{i-2}}\rho_{i,l}^{-1}\sigma_{i-1}^{-1}\sigma_{i-2}^{-1}\sigma_{i-1}^{-1} = \\
&\sigma_{i-2}\sigma_{i-1}\rho_{i,l}\sigma_{i-2}\rho_{i,l}^{-1}\sigma_{i-1}^{-1}\sigma_{i-2}^{-1}\sigma_{i-1}^{-1} = \\
&\sigma_{i-2}\sigma_{i-1}\sigma_{i-2}\rho_{i,l}\rho_{i,l}^{-1}\sigma_{i-1}^{-1}\sigma_{i-2}^{-1}\sigma_{i-1}^{-1} = \sigma_{i-2}\sigma_{i-1}\sigma_{i-2}\sigma_{i-1}^{-1}\sigma_{i-2}^{-1}\sigma_{i-1}^{-1} = 1.
\end{aligned}$$

The detailed check of the remaining relations is left to the reader. □

Denote $c_{2i} = \sigma_{2i+1}^{-1}\sigma_{2i}\sigma_{2i+1}$ for $1 \le i \le g-1$.

Lemma 3 *The group $B(g,n)$ is generated by*

$$\rho_{1,1},\ \rho_{1,2},\ \rho_{3,3},\ \rho_{3,4},\ \cdots,\ \rho_{2g-1,2g-1},\ \rho_{2g-1,2g}\ , \tag{24}$$

$$\sigma_1\ , c_2\ , \sigma_3\ , c_4\ \cdots, \sigma_{2g-3}\ , c_{2g-2}\ , \sigma_{2g-1}\ , \sigma_{2g}\ , \cdots, \sigma_{n-1}\ . \tag{25}$$

The set of defining relations consists of

$$\begin{aligned}
R_{1,i,j} &:= & [\rho_{i,*}, \rho_{j,*}] = 1, \qquad i \neq j\,; && (26)\\
R_{2,i,j} &:= & [\rho_{i,*}, \sigma_j] = 1, \qquad i \neq j\,; && (27)\\
R_{3,i,j} &:= & [\rho_{2i+1,*}, c_{2j}] = 1, \qquad 0 \leq i \leq g-1\,,\ 1 \leq j \leq g-1\,; && (28)\\
R_{4,i,i} &:= & (\rho_{2i-1,*}\sigma_{2i-1}^{-1})^2(\sigma_{2i-1}^{-1}\rho_{2i-1,*})^{-2} = 1, \qquad 1 \leq i \leq g\,; && (29)\\
R_{5,i,j} &:= & [\sigma_i, \sigma_j] = 1, \qquad \mid i-j \mid \neq 1\,; && (30)\\
R_{5,2j,2j-2} &:= & [c_{2j}, (c_{2j-2}^{-1}\sigma_{2j-1}c_{2j-2})] = 1\,; && (31)\\
R_{6,i,j} &:= & [\sigma_i, c_j] = 1, \qquad i \neq j \pm 1\,; && (32)\\
R_{7,2g,2g-2} &:= & [\sigma_{2g}, (c_{2g-2}^{-1}\sigma_{2g-1}c_{2g-2})] = 1\,; && (33)\\
R_{8,i,j} &:= & [c_{2i}, c_{2j}] = 1, \qquad \mid i-j \mid \neq 1\,; && (34)\\
R_{9,i,i} &:= & \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} = 1, \qquad 1 \leq i \leq n-2; && (35)\\
R_{10,i,i} &:= & c_{2i}c_{2i+2}c_{2i}c_{2i+2}^{-1}c_{2i}^{-1}c_{2i+2}^{-1} = 1, \qquad 1 \leq i \leq g-2; && (36)\\
R_{11,i,i} &:= & c_{2i}\sigma_{2i\pm1}c_{2i}\sigma_{2i\pm1}^{-1}c_{2i}^{-1}\sigma_{2i\pm1}^{-1} = 1, \qquad 1 \leq i \leq g-1; && (37)\\
R_{12,i,j} &:= & [(\sigma_{2i-1}\rho_{2i-1,2i}\sigma_{2i-1}^{-1}), \rho_{2i-1,2i-1}^{-1}]\,\sigma_{2i-1}^{-2} = 1, \qquad 1 \leq i \leq g\,; && (38)\\
R_{13} &:= & c_2c_4\cdots c_{2g-2}\sigma_{2g}\sigma_{2g+1}\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_{2g}\,\cdot &&\\
&& \cdot\,(\sigma_{2g-1}\,[\cdot]_{2g-1}^{-1}\,\sigma_{2g-1})c_{2g-2}\cdots(\sigma_3\,[\cdot]_3^{-1}\,\sigma_3)c_2(\sigma_1\,[\cdot]_1^{-1}\,\sigma_1) = 1\,, && (39)
\end{aligned}$$

where $[\cdot]_{2i-1} = [\rho_{2i-1,2i-1}, \rho_{2i-1,2i}^{-1}]$.

Proof. The elements σ_{2i} can be expressed through σ_{2i+1} and c_{2i} :

$$\sigma_{2i} = \sigma_{2i+1}c_{2i}\sigma_{2i+1}^{-1}.$$

Since $\sigma_{2i}\sigma_{2i+1}\sigma_{2i} = \sigma_{2i+1}\sigma_{2i}\sigma_{2i+1}$, it is easy to check that

$$c_{2i}\sigma_{2i+1}c_{2i} = \sigma_{2i+1}c_{2i}\sigma_{2i+1} \tag{40}$$

and

$$\sigma_{2i} = c_{2j}\sigma_{2i+1}c_{2i}^{-1}. \tag{41}$$

If we substitute these expressions into (16) - (23), then we obtain relations (26) - (39). For example, relations (18) (applying (21) and (17)) gives rise to (28). In fact, for $j = i$

$$\begin{aligned}
\sigma_{2j+1}\sigma_{2j}\rho_{2j+1,*}\sigma_{2j}^{-1} &= \sigma_{2j}\rho_{2j+1,*}\sigma_{2j}^{-1}\sigma_{2j+1} \Rightarrow\\
\sigma_{2j}^{-1}\sigma_{2j+1}\sigma_{2j}\rho_{2j+1,*} &= \rho_{2j+1,*}\sigma_{2j}^{-1}\sigma_{2j+1}\sigma_{2j} \Rightarrow\\
\sigma_{2j}^{-1}(\sigma_{2j+1}\sigma_{2j}\sigma_{2j+1})\sigma_{2j+1}^{-1}\rho_{2j+1,*} &= \rho_{2j+1,*}\sigma_{2j}^{-1}(\sigma_{2j+1}\sigma_{2j}\sigma_{2j+1})\sigma_{2j+1}^{-1} \Rightarrow \text{ (by (21))}\\
\sigma_{2j}^{-1}(\sigma_{2j}\sigma_{2j+1}\sigma_{2j})\sigma_{2j+1}^{-1}\rho_{2j+1,*} &= \rho_{2j+1,*}\sigma_{2j}^{-1}(\sigma_{2j}\sigma_{2j+1}\sigma_{2j})\sigma_{2j+1}^{-1} \Rightarrow\\
c_{2j}\rho_{2j+1,*} &= \rho_{2j+1,*}c_{2j}.
\end{aligned}$$

If $j \neq i$, then (28) is a consequence of (18) , since σ_{2j} and σ_{2j+1} are commutative with $\rho_{2i+1,*}$.

Conversely, if we substitute $\sigma_{2i-1}^{-1}\sigma_{2i}\sigma_{2i-1}$ in (26) - (39) instead of c_{2i} we obtain relations (16) - (23). The detailes are left to the reader. □

For the presentation of $B(g,n)$ given in Lemma 3, the following elements will be called the additional generators: σ_{2i} defined by (41), $1 \leq i \leq g-1$; σ_n defined by (9); A_k defined by (10); $\rho_{i,j}$ recurrently defined by (3), $(i,j) \neq (2t-1, 2t-1)$ nor $(2t-1, 2t)$; $B_{i,j} = [\rho_{i,2j-1}, \rho_{i,2j}]$; and $c_0 = \sigma_1^{-1}\sigma_n\sigma_1$. It is easy to check that the following relations hold.

$$\sigma_n = c_0\sigma_1 c_0^{-1}\,; \tag{42}$$
$$c_0\sigma_1 c_0 = \sigma_1 c_0 \sigma_1\,; \tag{43}$$
$$c_0\sigma_n c_0 = \sigma_n c_0 \sigma_n\,; \tag{44}$$
$$[\sigma_j, c_0] = 1, \qquad 2 \le j \le n-1\,; \tag{45}$$
$$[\rho_{2j-1,*}, c_0] = 1, \qquad 1 \le j \le g\,; \tag{46}$$
$$[B_{2i-1,j}, c_{2k}] = 1, \qquad \text{for all } i,\ j,\ k; \tag{47}$$
$$[B_{i,j}, \sigma_k] = 1\,, \qquad i \neq k \ \text{ nor } \ k-1\,; \tag{48}$$
$$B_{k,j} = \sigma_k B_{k+1,j}\sigma_k^{-1}\,, \qquad 1 \le k \le n-1\,; \tag{49}$$

The following lemma is a corollary from Lemmas 1 - 3.

Lemma 4 *Relations (2), (11) - (15), (42) - (49) are consequences of (27), (28), (30) - (37), (39).*

Denote relations (2), (11) - (15), (42) - (49), respectively, by $\widetilde{R}_1, \ldots, \widetilde{R}_{15}$.

4 Proof of Theorem 2

In the sequel we use presentation (24) - (39) of $B(g,n)$. Consider the homomorphism

$$\alpha : B(g,n) \to \mathbf{Z}^{2g}$$

sending $\rho_{2i-1,j}$ to $\bar{1}_j = (0,\ldots,0,1,0\ldots,0)$, where 1 is in jth place, and sending all σ_i and c_{2j} to zero. Obviously, $\alpha \simeq \phi'_*$. Denote by $G = G_{g,n}$ the kernel of α. Put $\rho_j = \rho_{2i-1,j}$, where $\rho_{2i-1,j}$ are the generators of $B(g,n)$ from presentation (24) - (39).

By Reidemeister - Schreier Theorem [R], [Sch], the following elements are generators of G :

$$a_{k,\bar{I}} \;=\; (\rho^{\bar{I}})\rho_k(\rho^{\bar{I}+\bar{1}_k})^{-1}, \qquad 1 \le k \le 2g\,; \tag{50}$$
$$c_{2j,\bar{I}} \;=\; (\rho^{\bar{I}})c_{2j}(\rho^{\bar{I}})^{-1}, \qquad 1 \le j \le g-1\,; \tag{51}$$
$$g_{l,\bar{I}} \;=\; (\rho^{\bar{I}})\sigma_l(\rho^{\bar{I}})^{-1}, \qquad l = 1, 3, \ldots, 2g-1, 2g, \ldots, n-1\,, \tag{52}$$

where $\bar{I} = (i_1, \ldots, i_{2g})$ and

$$\rho^{\bar{I}} = \rho_1^{i_1} \cdot \ldots \cdot \rho_{2g}^{i_{2g}}.$$

The defining relations of G are

$$R^{\bar{I}}_{k,i,j} = (\rho^{\bar{I}})R_{k,i,j}(\rho^{\bar{I}})^{-1}, \qquad k = 1, \ldots, 13, \tag{53}$$

where each $R^{\bar{I}}_{k,i,j}$ is written as a word in the generators a_*, c_* and g_*.

Remark 1 *If a relation R is a consequence of relations $R_1 \ldots, R_k$, then for fixed $\bar{I}$ the relation $R^{\bar{I}}$ is a consequence of the relations $R_1^{\bar{I}} \ldots, R_k^{\bar{I}}$.*

Decrease the number of generators of G. It follows from (26) that

$$a_{2j,\bar{I}} = 1 \tag{54}$$

for $1 \le i \le g$ and all $\bar{I}$. Similarly,

$$a_{2j-1,i_1,\ldots i_{2j-1},0,i_{2j+1},\ldots,i_{2g}} = 1 \tag{55}$$

for all sets of integers $(i_1, \ldots i_{2j-1}, i_{2j+1}, \ldots, i_{2g})$.

Relations (26) give rise to

$$a_{2j-1,I}a_{2l,I+\bar{1}_{2j-1}}a^{-1}_{2j-1,I+\bar{1}_{2l}}a^{-1}_{2l,I}=1\,,\quad j\neq l\,, \tag{56}$$

and

$$a_{2l-1,I}a_{2j-1,I+\bar{1}_{2l-1}}a^{-1}_{2l-1,I+\bar{1}_{2j-1}}a^{-1}_{2j-1,I}=1\,,\quad j\neq l\,. \tag{57}$$

It follows from (54) - (57) that

$$a_{2j-1,i_1,\dots,i_{2j-2},i_{2j-1},i_{2j},i_{2j+1},\dots,i_{2g}}=a_{2j-1,0,\dots,0,i_{2j-1},i_{2j},0,\dots,0}=a_{2j-1,i_{2j-1},i_{2j}}\,, \tag{58}$$

that is, $a_{2j-1,i_1,\dots,i_{2j-2},i_{2j-1},i_{2j},i_{2j+1},\dots,i_{2g}}$ does not depend on $i_1,\dots,i_{2j-2},i_{2j+1},\dots,i_{2g}$. In particular, by (55),

$$a_{2j-1,i_{2j-1},0}=1\,. \tag{59}$$

Similarly, it follows from (27) and (28) that

$$g_{2j-1,i_1,\dots,i_{2j-2},i_{2j-1},i_{2j},i_{2j+1},\dots,i_{2g}}=g_{2j-1,0,\dots,0,i_{2j-1},i_{2j},0,\dots,0}=g_{2j-1,i_{2j-1},i_{2j}}\,,\quad j\leq g; \tag{60}$$

$$g_{j,i_1,\dots,i_{2g}}=g_{j,0,\dots,0}=g_j\,,\quad j\geq 2g; \tag{61}$$

$$c_{2j,i_1,\dots,i_{2g}}=c_{2j,0,\dots,0}=c_{2j}\,, \tag{62}$$

that is, $g_{j,i_1,\dots,i_{2g}}$, $j\geq 2g$, and $c_{2j,i_1,\dots,i_{2g}}$ do not depend on $i_1,\dots,i_{2g}$.

Similarly, it follows from (46) that the generators $c_{0,I}$ corresponding to the additional generator c_0 do not depend on I, that is, $c_{0,I}=c_0$. By (41), the generators $g_{2j,I}$ corresponding to the additional generator σ_{2j} :

$$g_{2j,I}=c_{2j}g_{2j-1,i_{2j-1},i_{2j}}c^{-1}_{2j} \tag{63}$$

do not depend on $i_1,\dots,i_{2j-2}$, $i_{2j+1},\dots,i_{2g}$, and it follows from (42) that the generators $g_{n,I}$ corresponding to the additional generator σ_n :

$$g_{n,I}=c_0g_{i,i_1,i_2}c_0^{-1} \tag{64}$$

do not depend on $i_3,\dots,i_{2g}$.

Denote by $A_{k,I}$, $\rho_{j,k,I}$, and $B_{j,k,I}$ the generators corresponding respectively to the additional generators A_k, $\rho_{j,k}$, and $B_{j,k}$. The relations defining $B_{j,k}$ give rise to the relations

$$B_{j,k,I}=\rho_{j,2k-1,I}\rho_{j,2k,I+\bar{1}_{2k-1}}\rho^{-1}_{j,2k-1,I+\bar{1}_{2k}}\rho^{-1}_{j,2k,I},$$

in particular,

$$B_{2k-1,k,I}=a_{2k-1,i_{2k-1},i_{2k}}a^{-1}_{2k-1,i_{2k-1},i_{2k}-1}, \tag{65}$$

and relations (47) and (48) yield the following relations

$$[B_{l,j,I},c_{k,I}]=1,\qquad \text{for all } l,\ j,\ k; \tag{66}$$

$$[B_{l,j,I},\sigma_{k,I}]=1\,,\qquad l\neq k \text{ nor } k-1. \tag{67}$$

Let us write down relations (53).

$$R^I_{1,j,l}:=\quad [a_{2j-1,*},a_{2l-1,*}]=1,\qquad j\neq l\,; \tag{68}$$

$$R^I_{2,j,l}:=\quad [a_{2j-1,*},g_{2l-1,*}]=1,\qquad j\neq l\,; \tag{69}$$

$$R^I_{2,j,l}:=\quad [a_{2j-1,*},g_l]=1,\qquad l\geq 2g\,; \tag{70}$$

$$R^I_{3,j,l}:=\quad [a_{2j-1,*},c_{2l}]=1,\qquad 1\leq j\leq g,\ 1\leq l\leq g-1\,; \tag{71}$$

$$R^I_{4,2j-1,2j-1}:=\quad a_{2j-1,i_{2j-1},i_{2j}}g^{-1}_{2j-1,i_{2j-1}+1,i_{2j}}a_{2j-1,i_{2j-1}+1,i_{2j}}g^{-1}_{2j-1,i_{2j-1}+2,i_{2j}}$$
$$a^{-1}_{2j-1,i_{2j-1}+1,i_{2j}}g_{2j-1,i_{2j-1}+1,i_{2j}}a^{-1}_{2j-1,i_{2j-1},i_{2j}}g_{2j-1,i_{2j-1},i_{2j}}=1,\quad 1\leq j\leq g\,; \tag{72}$$

$$R^I_{4,2j-1,2j} := \quad g^{-1}_{2j-1,i_{2j-1},i_{2j}+1} g^{-1}_{2j-1,i_{2j-1},i_{2j}+2} g_{2j-1,i_{2j-1},i_{2j}+1} g_{2j-1,i_{2j-1},i_{2j}} = 1, \quad 1 \le j \le g\,; \tag{73}$$

$$R^I_{5,i,j} := \quad [g_i, g_j] = 1, \qquad | i-j | \neq 1\,,\ i,j \ge 2g\,; \tag{74}$$

$$R^I_{5,j,l} := \quad \left[g_{2j-1,i_{2j-1}i_{2j}}, g_{2l-1,i_{2l-1},i_{2l}}\right] = 1, \quad l \neq j\,; \tag{75}$$

$$R^I_{5,2g-1,j} := \quad \left[g_{2g-1,i_{2g-1},i_{2g}}, g_j\right] = 1, \quad j > 2g\,; \tag{76}$$

$$R^I_{5,2j,2j-2} = \quad \left[c_{2j}, (c^{-1}_{2j-2} g_{2j-1,i_{2j-1},i_{2j}} c_{2j-2})\right] = 1\,, \quad 1 \le j \le g-1\,; \tag{77}$$

$$R^I_{6,j,l} := \quad \left[g_{2j-1,i_{2j-1},i_{2j}}, c_{2l}\right] = 1, \quad j \neq l \ \text{ and } \ j \neq l+1\,; \tag{78}$$

$$R^I_{7,2g,2g-2} = \quad \left[g_{2g}, (c^{-1}_{2g-2} g_{2g-1,i_{2g-1},i_{2g}} c_{2g-2})\right] = 1\,; \tag{79}$$

$$R^I_{8,i,j} := \quad [c_{2i}, c_{2j}] = 1, \qquad | i-j | \neq 1\,; \tag{80}$$

$$R^I_{9,j,j} := \quad g_j g_{j+1} g_j = g_{j+1} g_j g_{j+1}\,, \qquad 2g \le j \le n-2; \tag{81}$$

$$R^I_{9,2g-1,2g} := \quad g_{2g-1,i_{2g-1},i_{2g}} g_{2g} g_{2g-1,i_{2g-1},i_{2g}} = g_{2g} g_{2g-1,i_{2g-1},i_{2g}} g_{2g}\,; \tag{82}$$

$$R^I_{10,i,i} := \quad c_{2i} c_{2i+2} c_{2i} = c_{2i+2} c_{2i} c_{2i+2}\,, \quad 1 \le i \le g-2; \tag{83}$$

$$R^I_{11,j,j} := \quad c_{2j} g_{2j\pm1,i_{2j\pm1},i_{2j\pm1}+1} c_{2j} = g_{2j\pm1,i_{2j\pm1},i_{2j\pm1}+1} c_{2j} g_{2j\pm1,i_{2j\pm1},i_{2j\pm1}+1}\,, \quad j \le g-1; \tag{84}$$

$$R^I_{11,g,g} := \quad c_{2g-2} g_{2g} c_{2g-2} = g_{2g} c_{2g-2} g_{2g}\,; \tag{85}$$

$$R^I_{12,j,1} := \quad a_{2j-1,i_{2j-1},i_{2j}+1} = g_{2j-1,i_{2j-1},i_{2j}+1} g^{-1}_{2j-1,i_{2j-1},i_{2j}} a_{2j-1,i_{2j-1},i_{2j}} g^{-1}_{2j-1,i_{2j-1}+1,i_{2j}} g^{-1}_{2j-1,i_{2j-1}+1,i_{2j}+1}\,; \tag{86}$$

$$R^I_{13} := \quad c_2 c_4 \cdots c_{2g-2} g_{2g} g_{2g+1} \cdots g_{n-2} g^2_{n-1} g_{n-2} \cdots g_{2g} \cdot \cdot (g_{2g-1,i_{2g-1},i_{2g}} (a_{2g-1,i_{2g-1},i_{2g}} a^{-1}_{2g-1,i_{2g-1},i_{2g}-1})^{-1} g_{2g-1,i_{2g-1},i_{2g}}) c_{2g-2} \cdots \cdots (g_{3,i_3,i_4} (a_{3,i_3,i_4} a^{-1}_{3,i_3,i_4-1})^{-1} g_{3,i_3,i_4}) c_2 (g_{1,i_1,i_2} (a_{1,i_1,i_2} a^{-1}_{1,i_1,i_2-1})^{-1} g_{1,i_1,i_2}) = 1\,. \tag{87}$$

Each relation depends on at most two parameters and the set of relations is similar to the relations in [Z] in the case $g = 1$. Now we shall show how to obtain a finite presentation of G using the arguments of [Z].

Relations (73) imply that $g_{2j-1,i_{2j-1},i_{2j}+1} g_{2j-1,i_{2j-1},i_{2j}}$ is independent of i_{2j}. Let for brevity,

$$g_{2j-1,i_{2j-1},i_{2j}+1} g_{2j-1,i_{2j-1},i_{2j}} = s_{2j-1,i_{2j-1}}\,. \tag{88}$$

The recurrence relations (86) allow us to express all $a_{2j-1,i_{2j-1},i_{2j}}$'s in terms of the $g_{2j-1,i_{2j-1},i_{2j}}$'s, since $a_{2j-1,i_{2j-1},0} = 0$ by (55). We obtain

$$a_{2j-1,i_{2j-1},i_{2j}} = g_{2j-1,i_{2j-1},i_{2j}} g_{2j-1,i_{2j-1},o} s^{-i_{2j}}_{2j-1,i_{2j-1}+1}\,. \tag{89}$$

Substituting these expressions of $a_{2j-1,i_{2j-1},i_{2j}}$'s into relation (87) and taking into account (88) we find in a straightforward manner that relations (87) can be replaced by the following relations:

$$c_2 \cdots c_{2g-2} g_{2g} \cdots g_{n-1} g_{n-1} \cdots g_{2g} (g_{2g-1,i_{2g-1},1} g_{2g-1,i_{2g-1}+1,1} g_{2g-1,i_{2g-1}+1,0} g_{2g-1,i_{2g-1},0}) c_{2g-2} \cdots \cdots (g_{3,i_3,1} g_{3,i_3+1,1} g_{3,i_3+1,0} g_{3,i_3,0}) c_2 (g_{1,i_1,1} g_{1,i_1+1,1} g_{1,i_1+1,0} g_{1,i_1,0}) = 1\,. \tag{90}$$

By (55), relation (72) for $i_{2j} = 0$ yields the following relation

$$g_{2j-1,i_{2j-1}+2,0} g_{2j-1,i_{2j-1}+1,0} = g_{2j-1,i_{2j-1}+1,0} g_{2j-1,i_{2j-1},0}\,. \tag{91}$$

Since, by (90), the product

$$g_{2j-1,i_{2j-1},1} g_{2j-1,i_{2j-1}+1,1} g_{2j-1,i_{2j-1}+1,0} g_{2j-1,i_{2j-1},0}$$

is independent of $i_1, \ldots, i_{2g-1}$, we deduce, as a cosequence of (91), the following relation

$$g_{2j-1,i_{2j-1},1} g_{2j-1,i_{2j-1}+1,1} = g_{2j-1,i_{2j-1}-1,1} g_{2j-1,i_{2j-1},1}\,. \tag{92}$$

Lemma 5 *The defining relations (72) are consequences of the set of relations (68) - (71), (73) - (87), (91), (92), where the elements $a_{2j-1,i_{2j-1},i_{2j}}$ are defined by (88) and (89).*

Proof. Denote by

$$\tau_{2j-1} = (g_{2j-1,i_{2j-1},1}g_{2j-1,i_{2j-1}+1,1}g_{2j-1,i_{2j-1}+1,0}g_{2j-1,i_{2j-1},0})^{-1}. \tag{93}$$

or, equivalently,

$$\tau_{2j-1} = g^{-1}_{2j-1,i_{2j-1},i_{2j}}(a_{2j-1,i_{2j-1},i_{2j}}a^{-1}_{2j-1,i_{2j-1},i_{2j}-1})g^{-1}_{2j-1,i_{2j-1},i_{2j}} \tag{94}$$

By (90), we have

$$\tau_{2j-1} = c_{2j-2}\tau^{-1}_{2j-3}\cdots c_2\tau_1^{-1}c_2\cdots c_{2g-2}g_{2g}\cdots g_{n-2}g^2_{n-1}g_{n-2}\cdots g_{2g}\tau^{-1}_{2g-1}c_{2g-2}\cdots\tau^{-1}_{2j+1}c_{2j} \tag{95}$$

By (69) - (71) and due to (93) and the previous equation, each element $a_{2l-1,i_{2l-1},i_{2l}}$, $1 \le l \le g$, and τ_{2j-1} commute, hence τ_{2j-1} and $a_{2j-1,i2j-1,i_{2j}}a^{-1}_{2j-1,i2j-1,i_{2j}-1}$ commute, i.e. in view of (94), τ_{2j-1} and $g_{2j-1,i_{2j-1},i_{2j}}\tau_{2j-1}g_{2j-1,i_{2j-1},i_{2j}}$ commute:

$$(\tau_{2j-1}g_{2j-1,i_{2j-1},i_{2j}})^2 = (g_{2j-1,i_{2j-1},i_{2j}}\tau_{2j-1})^2.$$

The rest of the proof of Lemma coincides with the proof of the same assertion in the case $g = 1$ and is contained in [Z] pp. 347 - 349 (starting from equation (14) in [Z]). □

Lemma 6 *The set of relations (68) - (71), (73) - (87), (91), (92) is equivalent to the set (73) - (85), (90), (91), (92), where the elements $a_{2j-1,i_{2j-1},i_{2j}}$ are defined by (88) and (89).*

Proof. Relations (68) and (69) are cosequences of (75) due to (89).

Relations (71), $l \ne j$ and $l \ne j+1$, are cosequences of (78) in view of (89). Deduce (71) from (73) - (84), (90), (91), (92) in the case $l = j$. By (87) (which is cosequence of (90), (88) and (89)),

$$a_{2j-1,i_{2j-1},i_{2j}}a^{-1}_{2j-1,i_{2j-1},i_{2j}-1} =$$
$$g_{2j-1,i_{2j-1},i_{2j}}c_{2j-2}\tau^{-1}_{2j-3}\cdots c_2\tau_1^{-1}c_2\cdots c_{2g-2}g_{2g}\cdots g^2_{n-1}\cdots g_{2g}\tau^{-1}_{2g-1}c_{2g-2}\cdots\tau^{-1}_{2j+1}c_{2j}g_{2j-1,i_{2j-1},i_{2j}}.$$

Since $a_{2j-1,i_{2j-1},0} = 1$, it is sufficient to deduce that c_{2j} and $a_{2j-1,i_{2j-1},i_{2j}}a^{-1}_{2j-1,i_{2j-1},i_{2j}-1}$ commute. Note that relations (77) and (79), in view of (82) and (84), are equivalent respectively to

$$\left[c_{2j},(g_{2j-1,i_{2j-1},i_{2j}}c_{2j-2}g^{-1}_{2j-1,i_{2j-1},i_{2j}})\right] = 1 \tag{96}$$

and

$$\left[g_{2g},(g_{2g-1,i_{2g-1},i_{2g}}c_{2g-2}g^{-1}_{2g-1,i_{2g-1},i_{2g}})\right] = 1. \tag{97}$$

We have

$$c_{2j}(a_{2j-1,i_{2j-1},i_{2j}}a^{-1}_{2j-1,i_{2j-1},i_{2j}-1}) =$$
$$\underline{c_{2j}g_{2j-1,i_{2j-1},i_{2j}}c_{2j-2}}\tau^{-1}_{2j-3}\cdots c_2\tau_1^{-1}c_2\cdots c_{2g-2}g_{2g}\cdots$$
$$\cdots g^2_{n-1}\cdots g_{2g}\tau^{-1}_{2g-1}c_{2g-2}\cdots\tau^{-1}_{2j+1}c_{2j}g_{2j-1,i_{2j-1},i_{2j}} = \text{(by (96))}$$
$$\underline{g_{2j-1,i_{2j-1},i_{2j}}c_{2j-2}(g^{-1}_{2j-1,i_{2j-1},i_{2j}}c_{2j}g_{2j-1,i_{2j-1},i_{2j}})\tau^{-1}_{2j-3}\cdots c_2\tau_1^{-1}c_2\cdots c_{2g-2}g_{2g}\cdots}$$
$$\cdots g^2_{n-1}\cdots g_{2g}\tau^{-1}_{2g-1}c_{2g-2}\cdots\tau^{-1}_{2j+1}c_{2j}g_{2j-1,i_{2j-1},i_{2j}} = \text{(by (74) - (80))}$$
$$g_{2j-1,i_{2j-1},i_{2j}}c_{2j-2}\tau^{-1}_{2j-3}\cdots c_2\tau_1^{-1}c_2\cdots\underline{(g^{-1}_{2j-1,i_{2j-1},i_{2j}}c_{2j}g_{2j-1,i_{2j-1},i_{2j}})c_{2j-2}c_{2j}}\cdots c_{2g-2}g_{2g}\cdots$$
$$\cdots g^2_{n-1}\cdots g_{2g}\tau^{-1}_{2g-1}c_{2g-2}\cdots\tau^{-1}_{2j+1}c_{2j}g_{2j-1,i_{2j-1},i_{2j}} = \text{(by (96))}$$
$$g_{2j-1,i_{2j-1},i_{2j}}c_{2j-2}\tau^{-1}_{2j-3}\cdots c_2\tau_1^{-1}c_2\cdots\underline{c_{2j-2}c_{2j}(g_{2j-1,i_{2j-1},i_{2j}})}\cdots c_{2g-2}g_{2g}\cdots$$
$$\cdots g^2_{n-1}\cdots g_{2g}\tau^{-1}_{2g-1}c_{2g-2}\cdots\tau^{-1}_{2j+1}c_{2j}g_{2j-1,i_{2j-1},i_{2j}} = \text{(by (74) - (80))}$$
$$g_{2j-1,i_{2j-1},i_{2j}}c_{2j-2}\tau^{-1}_{2j-3}\cdots c_2\tau_1^{-1}c_2\cdots c_{2j-2}c_{2j}\cdots c_{2g-2}g_{2g}\cdots g^2_{n-1}\cdots$$
$$\cdots g_{2g}\tau^{-1}_{2g-1}c_{2g-2}\cdots\tau^{-1}_{2j+1}\underline{(g_{2j-1,i_{2j-1},i_{2j}})c_{2j}g_{2j-1,i_{2j-1},i_{2j}}} = \text{(by (84))}$$
$$g_{2j-1,i_{2j-1},i_{2j}}c_{2j-2}\tau^{-1}_{2j-3}\cdots c_2\tau_1^{-1}c_2\cdots c_{2j-2}c_{2j}\cdots c_{2g-2}g_{2g}\cdots g^2_{n-1}\cdots$$
$$\cdots g_{2g}\tau^{-1}_{2g-1}c_{2g-2}\cdots\tau^{-1}_{2j+1}\underline{c_{2j}g_{2j-1,i_{2j-1},i_{2j}}c_{2j}} =$$
$$(a_{2j-1,i_{2j-1},i_{2j}}a^{-1}_{2j-1,i_{2j-1},i_{2j}-1})c_{2j}.$$

The deducting of (70) and (71) (in the case $l = j+1$) from (73) - (84), (90), (91), (92) is the same as the previous one and will be omitted. □

The relations (73):

$$g_{2j-1,i_{2j-1},i_{2j}+2} = g_{2j-1,i_{2j-1},i_{2j}+1} g_{2j-1,i_{2j-1},i_{2j}} g^{-1}_{2j-1,i_{2j-1},i_{2j}+1}, \tag{98}$$

for a fixed value of i_{2j-1}, can be considered as recurrence relations defining the elements $g_{2j-1,i_{2j-1},i_{2j}}$ in terms of the two free elements $g_{2j-1,i_{2j-1},0}$ and $g_{2j-1,i_{2j-1},1}$. Then the relations (91) and (92) can be used in order to express all the elements $g_{2j-1,i_{2j-1},0}$ and $g_{2j-1,i_{2j-1},1}$ in terms of $g_{2j-1,0,0}$, $g_{2j-1,1,0}$ and $g_{2j-1,0,1}$, $g_{2j-1,1,1}$ respectively. Consequently, our group $G_{g,n}$ is generated by $3g+n-1$ elements:

$$g_{2j-1,0,0},\ g_{2j-1,1,0},\ g_{2j-1,0,1},\ g_{2j-1,1,1}, \qquad 1 \le j \le g; \tag{99}$$

$$c_2,\ c_4,\ \ldots,\ c_{2g-2}; \tag{100}$$

$$g_{2g},\ g_{2g+1},\ \ldots,\ g_{n-1}. \tag{101}$$

Relations (75) - (79) follow from the same relations for $g_{2j-1,0,0}$, $g_{2j-1,1,0}$, $g_{2j-1,0,1}$, $g_{2j-1,1,1}$ (respectively, $g_{2g-1,0,0}$, $g_{2g-1,1,0}$, $g_{2g-1,0,1}$, $g_{2g-1,1,1}$), since all $g_{2j-1,i_{2j-1},i_{2j}}$ (respectively, $g_{2g-1,i_{2g-1},i_{2g}}$) belong to a subgroup generated by these elements, and since relations (77) (respectively, (79)) can be written as

$$\left[(c_{2j-2}c_{2j}c^{-1}_{2j-2}),\ g_{2j-1,i_{2j-1},i_{2j}}\right] = 1; \tag{102}$$

$$\left[(c_{2g-2}g_{2g}c^{-1}_{2g-2}),\ g_{2g-1,i_{2g-1},i_{2g}}\right] = 1. \tag{103}$$

Applying Zariski's Lemma ([Z], p.350), we obtain that relations (84) (for (82) the arguments are the same) are consequences of any three of them relative to three consecutive indices i_{2j}, say $i_{2j} = 0, 1, 2$. By (91) and (92), we conclude, on the basis of Zariski's Lemma, that for $i_{2j} = 0, 1$ relations (84) are consequences of three of these relations relative to three consecutive values of i_{2j-1}, say $i_{2j-1} = 0, 1, 2$. To decrease the number of relations (84) for $i_{2j} = 2$, we change, as in [Z], these relations to equivalent relations

$$(g_{2j\pm1,i_{2j\pm1},1}\, g_{2j\pm1,i_{2j\pm1},0}\, c_{2j})^2 = (c_{2j}\, g_{2j\pm1,i_{2j\pm1},1}\, g_{2j\pm1,i_{2j\pm1},0})^2. \tag{104}$$

To show that these relations are equivalent to one of them, say

$$(g_{2j\pm1,0,1}\, g_{2j\pm1,0,0}\, c_{2j})^2 = (c_{2j}\, g_{2j\pm1,0,1}\, g_{2j\pm1,0,0})^2, \tag{105}$$

it is sufficient to show that the expressions

$$s_{2j\pm1,i_{2j\pm1}} = (g_{2j\pm1,i_{2j\pm1},1}\, g_{2j\pm1,i_{2j\pm1},0}\, c_{2j})^2 (c_{2j}\, g_{2j\pm1,i_{2j\pm1},1}\, g_{2j\pm1,i_{2j\pm1},0})^{-2}$$

are all transforms of each other, for $i_{2j\pm1} = 0, \pm1, \pm2, \ldots$, as a consequence of relations (74) - (84) ($i_{2j} = 0$ or 1, $1 \le j \le g-1$), (73), (87), (91), (92) (where the elements $a_{2j-1,i_{2j-1},i_{2j}}$ are defined by (88) and (89)), and additional relations defining additional generators. Hence, we shall be able to take the relations corresponding to $i_{2j-1} = 0$. For this we need, in order to apply Zariski's arguments (see the computation on p. 351 in [Z]), to show that

$$\delta_{j,\pm} = \delta_{2j\pm1,i_{2j\pm1}} = c_{2j}\, g_{2j\pm1,i_{2j\pm1},1}\, g_{2j\pm1,i_{2j\pm1}+1,1}\, g_{2j\pm1,i_{2j\pm1}+1,0}\, g_{2j\pm1,i_{2j\pm1},0}\, c_{2j}$$

are commutative respectively with $g_{2j+1,i_{2j+1},i_{2j+2}}$ and $g_{2j-1,i_{2j-1},i_{2j}}$ in the case i_{2j+2} and $i_{2j} = 0$ or 1. Let us check that $\delta_{j-1,+}$ and $g_{2j-1,i_{2j-1},i_{2j}}$ commute. For this, denote by

$A = (g_{2j-3,i_{2j-3},1}\, g_{2j-3,i_{2j-3}+1,1}\, g_{2j-3,i_{2j-3}+1,0}\, g_{2j-3,i_{2j-3},0}) c_{2j-4} \cdots c_2 (g_{1,i_1,1}\, g_{1,i_1+1,1}\, g_{1,i_1+1,0} \cdot$
$\cdot g_{1,i_1+1,0}\, g_{1,i_1,0}) c_2 \cdots c_{2j-4}$;
$B = c_{2j+2} \cdots c_{2g-2}\, g_{2g} \cdots g_{n-1}\, g_{n-1} \cdots g_{2g} (g_{2g-1,i_{2g-1},1}\, g_{2g-1,i_{2g-1}+1,1}\, g_{2g-1,i_{2g-1}+1,0}\, g_{2g-1,i_{2g-1},0}) \cdot$
$\cdot c_{2g-2} \cdots (g_{2j+1,i_{2j+1},1}\, g_{2j+1,i_{2j+1}+1,1}\, g_{2j+1,i_{2j+1}+1,0}\, g_{2j+1,i_{2j+1},0})$.

We have

$$\begin{aligned}
&g_{2j-1,i_{2j-1},i_{2j}}\delta_{j-1,+}^{-1} = && \text{(by (90))}\\
&(g_{2j-1,i_{2j-1},i_{2j}})Ac_{2j-2}c_{2j}Bc_{2j}c_{2j-2}^{-1} = && \text{(by (74) – (76))}\\
&A\underline{(g_{2j-1,i_{2j-1},i_{2j}})c_{2j-2}c_{2j}}Bc_{2j}c_{2j-2}^{-1} = && \text{(by (77), (83), (84))}\\
&A\underline{c_{2j-2}c_{2j}(c_{2j-2}^{-1}g_{2j-1,i_{2j-1},i_{2j}}c_{2j-2})}Bc_{2j}c_{2j-2}^{-1} = && \text{(by (74) – (76), (78), (80))}\\
&Ac_{2j-2}c_{2j}B(c_{2j-2}^{-1}g_{2j-1,i_{2j-1},i_{2j}}c_{2j-2})c_{2j}c_{2j-2}^{-1} = && \text{(by (77), (83), (84))}\\
&Ac_{2j-2}c_{2j}Bc_{2j}c_{2j-2}^{-1}g_{2j-1,i_{2j-1},i_{2j}} = &&\\
&\delta_{j-1,+}^{-1}g_{2j-1,i_{2j-1},i_{2j}}\,. &&
\end{aligned}$$

To prove that $g_{2j-1,i_{2j-1},i_{2j}}$ and $\delta_{j,-}^{-1}$ commute in the case $i_{2j}=0$ or 1, we need the following lemma.

Lemma 7 *For fixed $\bar{I}=(i_1,\ldots,i_{2g})$, where $i_{2j}=0$ or 1, the following relation*

$$\begin{aligned}
A_{2j-1,I} = \; & B_{2j-1,j,I}g_{2j-1,I}^{-1}g_{2j,I}^{-1}B_{2j+1,j+1,I}g_{2j+1,I}^{-1}g_{2j+2,I}^{-1}\cdot\ldots\cdot g_{2g-3,I}^{-1}g_{2g-2,I}^{-1}B_{2g-1,g,I}\cdot\\
&\cdot g_{2g-1,I}^{-1}g_{2g,I}^{-1}A_{2g+1,I}^{-1}B_{2g+1,1,I}\cdot\ldots\cdot B_{2g+1,j-1,I}A_{2g+1,\bar{I}}g_{2g,\bar{I}}\cdot\ldots\cdot g_{2j-1,\bar{I}}
\end{aligned}\tag{106}$$

is a consequence of relations (68) - (71), (73) - (85), (87) with the same set $\bar{I}$.

Proof. By (8),

$$A_1 = B_{1,1}\cdot\ldots\cdot B_{1,g}.$$

Hence,

$$A_1 = B_{1,j}\cdot\ldots\cdot B_{1,g}A_1^{-1}B_{1,1}\cdot\ldots\cdot B_{1,j-1}A_1.$$

By (14) and (49), this relation can be written in the form

$$A_{2j-1} = B_{2j-1,j}\cdot\ldots\cdot B_{2j-1,g}A_{2j-1}^{-1}B_{2j-1,1}\cdot\ldots\cdot B_{2j-1,j-1}A_{2j-1}.$$

If we substitute in the last relation $\sigma_{2j-1}^{-1}\cdots\sigma_{2k-2}^{-1}B_{2k-1,k}\sigma_{2k-2}\cdots\sigma_{2j-1}$ instead of $B_{2j-1,k}$ for $k>j$; $\sigma_{2j-1}^{-1}\cdots\sigma_{2g}^{-1}A_{2g+1}\sigma_{2g}\cdots\sigma_{2j-1}$ instead of A_{2j-1}; and for $k<j$, substitute $\sigma_{2j-1}^{-1}\cdots\sigma_{2g}^{-1}B_{2g+1,k}\sigma_{2g}\cdots\sigma_{2j-1}$ instead of $B_{2j-1,k}$, we obtain the following relation

$$\begin{aligned}
A_{2j-1} = \; & B_{2j-1,j}\sigma_{2j-1}^{-1}\sigma_{2j}^{-1}B_{2j+1,j+1}\sigma_{2j+1}^{-1}\sigma_{2j+2}^{-1}\cdot\ldots\cdot\sigma_{2g-3}^{-1}\sigma_{2g-2}^{-1}B_{2g-1,g}\cdot\\
&\cdot\sigma_{2g-1}^{-1}\sigma_{2g}^{-1}A_{2g+1}^{-1}B_{2g+1,1}\cdot\ldots\cdot B_{2g+1,j-1}A_{2g+1}\sigma_{2g}\cdot\ldots\cdot\sigma_{2j-1}
\end{aligned}\tag{107}$$

Now Lemma follows from Lemma 4 and Remark 1. □

Since, by (65), $B_{2k-1,k,I}=a_{2k-1,i_{2k-1},i_{2k}}a_{2k-1,i_{2k-1},i_{2k}-1}^{-1}$, therefore, by (89), (91), (92), and (63), relation (107) can be witten in the form

$$\begin{aligned}
\delta_{j,-}^{-1} = \; & c_{2j}^{-1}(g_{2j-1,i_{2j-1},1}g_{2j-1,i_{2j-1}+1,1}g_{2j-1,i_{2j-1}+1,0}g_{2j-1,i_{2j-1},0})^{-1}c_{2j}^{-1} =\\
& g_{2j+2,I}\cdots g_{2g-1,I}g_{2g}\cdots g_{n+2j-3,I}g_{n+2j-3,\bar{I}}\cdots g_{2g+1}A_{2g+1,I}^{-1}B_{2g+1,j-1,I}^{-1}\cdots B_{2g+1,1,I}^{-1}A_{2g+1,I}\cdot\\
&\cdot g_{2g}g_{2g-1,i_{2g-1},i_{2g}}B_{2g-1,g,I}^{-1}g_{2g-2,i_{2g-3},i_{2g-2}}\cdots g_{2j+2,i_{2j+1},i_{2j+2}}B_{2j+1,j+1,I}^{-1}g_{2j+1,i_{2j+1},i_{2j+2}}.
\end{aligned}\tag{108}$$

Now, by Lemma 4, Remark 1, and by (13) - (15), (30), (45) - (48), it is obvious that $g_{2j-1,i_{2j-1},i_{2j}}$ and $\delta_{j,-}^{-1}$ commute.

Finally, by (91) and (92), we observe that the infinite set of relations (87) reduces to one relation, say $i_{2j-1}=0$ for all j. This completes the proof of Theorem 2.

References

[B] J.S. Birman, *Braids, Links, and Mapping Class Groups*, Annals of Math. studies, vol. 82 Princeton Univ. Press, Princeton, 1975.

[N] M. Nori, *Zariski's conjecture and related problems*, Ann. Sci. École Norm. Sup., vol. 16 (1983), pp. 308-344.

[R] K. Reidemeister , *Knoten und Gruppen*, Abhandl. Math. Sem. Univ. Hamburg vol. 5, (1927), pp. 8-23.

[Sch] O. Schreier, *Die Untergruppen der freien Gruppen*, Abhandl. Math. Sem. Univ. Hamburg vol. 5, (1927), pp. 161-183.

[Sc] G.P. Scott, *Braid groups and the group of homeomorphisms of a surface*, Proc. Camb. Phil. Soc. vol. 68, (1970) pp. 605-617.

[Sh] I. Shimada, *Fundamental groups of open algebraic varieties*, Topology, vol. 34, (1995), pp. 509-531.

[Z] O. Zariski, *The topological discriminant group of a Riemann surface of genus p*, Amer. J. Math., vol. 59 (1937) pp. 335-358.

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