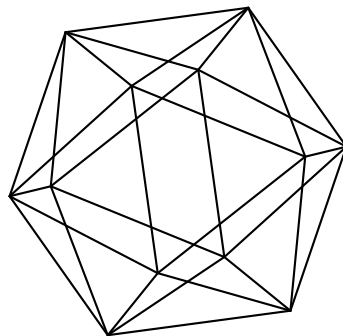# Max-Planck-Institut für Mathematik Bonn

Coverings preserving the bottom of the spectrum

by

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Abstract

We prove that if a Riemannian covering preserves the bottom of the spectrum of a Schrödinger operator, which belongs to the discrete spectrum of the operator on the base manifold, then the covering is amenable.

1 Introduction

The spectrum of the Laplacian on a Riemannian manifold is a natural isometric invariant. However, its behavior under maps between Riemannian manifolds, which respect the geometry of the manifolds to some extent, remains largely unclear. In this paper, we study the behavior of the bottom (that is, the minimum) of the spectrum under Riemannian coverings.

Let $p\colon M_2 \to M_1$ be a Riemannian covering, $S_1 = \Delta + V$ a Schrödinger operator on M_1, with V smooth and bounded from below, and $S_2 = \Delta + V \circ p$ its lift on M_2. Then the bottoms of their spectra always satisfy the inequality $\lambda_0(S_1) \leq \lambda_0(S_2)$. It is natural to examine when the equality holds. Brooks [7] proved that if the underlying manifold is closed (that is, compact without boundary), then a normal covering p preserves the bottom of the spectrum of the Laplacian if and only if p is amenable.

This theorem motivated the study of the behavior of the bottom of the spectrum under amenable coverings. Extending some former results [3, 6], it was proved in [2] that amenable Riemannian coverings preserve the bottom of the spectrum of Schrödinger operators, without any topological or geometric assumptions on the manifolds. In [14], it was proved that if, in addition, M_1 is complete, then the spectra of the operators satisfy $\sigma(S_1) \subset \sigma(S_2)$. If, in addition, the covering is infinite sheeted, then $\sigma(S_1) \subset \sigma_{\mathrm{ess}}(S_2)$, where σ_{ess} stands for the essential spectrum of the operator.

Although amenability of the covering is a natural assumption for the preservation of the bottom of the spectrum, it is not clear to what extent it is optimal. In this direction, Brooks [6], and Roblin and Tapie [15], proved that under some quite restrictive

assumptions, if the bottom of the spectrum of the Laplacian is preserved, then the covering is amenable. These assumptions involve the spectrum of fundamental domains of the covering and in particular, imply that the bottom of the spectrum of the Laplacian on M_1 belongs to its discrete spectrum (that is, the bottom is an isolated point of the spectrum and in particular, an eigenvalue). Moreover, in both results, the covering is assumed to be normal, with finitely generated deck transformations group. Recently, in [1], these conditions were replaced with some more natural geometric assumptions. More precisely, it was proved that if the manifolds are complete, without boundary, with Ricci curvature bounded from below, V and grad V are bounded, the bottom of the spectrum is preserved, and belongs to the discrete spectrum of S_1, then the covering is amenable. A question raised in [1] is whether the assumption on the Ricci curvature is necessary. In this paper, we deal with this question and establish a generalization of all the above results.

Initially, using the result of [1] we prove the following analogue of Brooks' theorem [7], involving the bottom of the Neumann spectrum of the Laplacian on manifolds with (smooth) boundary.

Theorem 1.1. *Let $p \colon M_2 \to M_1$ be a Riemannian covering, with M_1 compact with boundary. Then $\lambda_0^N(M_2) = 0$ if and only if p is amenable.*

The fact that amenable coverings preserve the bottom of the Neumann spectrum was essentially established in [14]. The main point in the above theorem is the converse implication, which is the first result providing amenability of a covering of manifolds with boundary. This turns out to be quite useful in the study of arbitrary Riemannian coverings. More precisely, as an application of this theorem, we prove the following result.

Theorem 1.2. *Let $p \colon M_2 \to M_1$ be a Riemannian covering. Let S_1 be a Schrödinger operator on M_1, with $\lambda_0(S_1) \notin \sigma_{\mathrm{ess}}(S_1)$, and S_2 its lift on M_2. Then $\lambda_0(S_2) = \lambda_0(S_1)$ if and only if p is amenable.*

It is worth to point out that we do not impose any topological or geometric assumptions on the manifolds. Hence, Theorem 1.2 is more general than the results of [1, 6, 15], since their assumptions imply that $\lambda_0(S_1) \notin \sigma_{\mathrm{ess}}(S_1)$. Examining the optimality of the assumption $\lambda_0(S_1) \notin \sigma_{\mathrm{ess}}(S_1)$ in this theorem, we show that it cannot be replaced with $\lambda_0(S_1)$ being an eigenvalue.

Since the manifolds in the above theorem may be non-complete, we obtain immediately the corresponding result for Dirichlet spectra of Schrödinger operators on manifolds with boundary. For sake of completeness, we also establish the corresponding result for the Neumann spectra, obtaining a generalization of Theorem 1.1. In particular, we obtain analogues of Brooks' result [7] for Schrödinger operators on manifolds with boundary. Namely, it follows that a Riemannian covering of a compact manifold is amenable if and only if it preserves the bottom of the Dirichlet/Neumann spectrum of some/any Schrödinger operator. In virtue of [14, Theorem 1], this is actually equivalent to the inclusion of the Dirichlet (Neumann) spectrum of a Schrödinger operator on M_1, in the

Dirichlet (Neumann, respectively) spectrum of its lift on M_2. The corresponding statement for manifolds without boundary has been established in [14, Theorem 1.5].

Finally, as another application of Theorem 1.1, we prove that if an infinite sheeted Riemannian covering preserves the bottom of the spectrum of a Schrödinger operator, then the bottom of the spectrum belongs to the essential spectrum of the operator on the covering space. This was observed for the Laplacian in [1]. For sake of completeness, we establish the analogous result for the Dirichlet and Neumann spectra of Schrödinger operators on manifolds with boundary.

The paper is organized as follows: In Section 2, we give some preliminaries. In Section 3, we present some properties of the spectrum of Schrödinger operators. In Section 4, we study Riemannian coverings of compact manifolds and establish Theorem 1.1. In Section 5, we study arbitrary Riemannian coverings and prove Theorem 1.2 and the corresponding results for manifolds with boundary. In Section 6, we show the aforementioned application of Theorem 1.1 for infinite sheeted coverings.

2   Preliminaries

We begin by recalling some definitions and standard facts from functional analysis, which may be found for instance, in [13] and [17, Appendix A].

Let $L\colon \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$, over $\mathbb{R}$ or $\mathbb{C}$. The *spectrum* of L is given by

$$\sigma(L) := \{\lambda \in \mathbb{R} : (L - \lambda)\colon \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H} \text{ not invertible}\}.$$

The *essential spectrum* of L is defined as

$$\sigma_{\mathrm{ess}}(L) := \{\lambda \in \mathbb{R} : (L - \lambda)\colon \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H} \text{ not Fredholm}\}.$$

Recall that an operator is called *Fredholm* if its kernel is finite dimensional and its range is closed and of finite codimension. The *discrete spectrum* of L is given by $\sigma_d(L) := \sigma(L) \smallsetminus \sigma_{\mathrm{ess}}(L)$, and consists of isolated eigenvalues of L of finite multiplicity.

The spectrum of a self-adjoint operator is a closed subset of $\mathbb{R}$. If $\sigma(L)$ is bounded from below, then its minimum is called the *bottom* of the spectrum of L and is denoted by $\lambda_0(L)$. The following characterization is due to Rayleigh.

Proposition 2.1. *If $\sigma(L)$ is bounded from below, then the bottom of the spectrum of L is given by*

$$\lambda_0(L) = \inf_{v \in \mathcal{D}(L) \smallsetminus \{0\}} \frac{\langle Lv, v \rangle_{\mathcal{H}}}{\|v\|_{\mathcal{H}}^2}.$$

Let $T\colon \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined, symmetric linear operator. Assume that T is *bounded from below*, that is, there exists $c \in \mathbb{R}$, such that

$$\langle Tv, v \rangle_{\mathcal{H}} \geq c\|v\|_{\mathcal{H}}^2, \tag{1}$$

for all $v \in \mathcal{D}(T)$. Fix such a $c \in \mathbb{R}$ (not necessarily the supremum of all c for which (1) holds) and consider the inner product

$$\langle v_1, v_2 \rangle_{\mathcal{H}_1} := \langle Tv_1, v_2 \rangle_{\mathcal{H}} + (1-c)\langle v_1, v_2 \rangle_{\mathcal{H}}$$

on $\mathcal{D}(T)$. Let $\mathcal{H}_1$ be the completion of $\mathcal{D}(T)$ with respect this inner product. Then $\mathcal{H}_1$ can be identified with a dense subspace of $\mathcal{H}$, via a continuous injection. The domain of the Friedrichs extension $T^{(F)}$ of T is given by

$$\mathcal{D}(T^{(F)}) := \{v \in \mathcal{H}_1 : \text{there exists } v' \in \mathcal{H}, \text{ such that } \langle v', w \rangle_{\mathcal{H}} = \langle v, w \rangle_{\mathcal{H}_1}, \text{ for all } w \in \mathcal{H}_1\}.$$

For $v \in \mathcal{D}(T^{(F)})$, we define $T^{(F)}v := v' + (c-1)v$. Then $T^{(F)}$ is a self-adjoint extension of T and is called the *Friedrichs extension* of T.

Proposition 2.2. *The bottom of the spectrum of the Friedrichs extension of T is given by*

$$\lambda_0(T^{(F)}) = c - 1 + \inf_{v \in \mathcal{H}' \smallsetminus \{0\}} \frac{\|v\|_{\mathcal{H}_1}^2}{\|v\|_{\mathcal{H}}^2},$$

where the infimum may be taken over any subspace $\mathcal{H}'$, with $\mathcal{D}(T) \subset \mathcal{H}' \subset \mathcal{H}_1$.

Proof: Evidently, for a non-zero $v \in \mathcal{D}(T^{(F)})$, we have

$$c - 1 + \frac{\|v\|_{\mathcal{H}_1}^2}{\|v\|_{\mathcal{H}}^2} = \frac{\langle T^{(F)}v, v \rangle_{\mathcal{H}}}{\|v\|_{\mathcal{H}}^2}.$$

From Proposition 2.1, we obtain the asserted equality, where the infimum is taken over all $v \in \mathcal{D}(T^{(F)}) \smallsetminus \{0\}$. From the definition of $\mathcal{H}_1$, it is easy to see that we obtain the same infimum for $v \in \mathcal{D}(T) \smallsetminus \{0\}$ and for $v \in \mathcal{H}_1 \smallsetminus \{0\}$. ∎

2.1 Schrödinger operators

Throughout this paper, manifolds are assumed to be connected with not necessarily connected, possibly empty, smooth boundary, unless otherwise stated. Let M be a possibly non-connected Riemannian manifold. A *Schrödinger operator* on M is an operator of the form $S = \Delta + V$, where Δ is the (non-negative definite) Laplacian and $V\colon M \to \mathbb{R}$ is smooth and bounded from below. On the space $C_c^\infty(M)$ consider the inner product

$$\langle f, g \rangle_{H_V(M)} := \int_M (\langle \operatorname{grad} f, \operatorname{grad} g \rangle + (V - \inf_M V + 1)fg).$$

If M has empty boundary, let $H_V(M)$ be the completion of $C_c^\infty(M)$ with respect to this inner product. If M has non-empty boundary, let $H_V(M)$ be the completion of $\{f \in C_c^\infty(M) : \nu(f) = 0 \text{ on } \partial M\}$ with respect to this inner product, where ν is the inward pointing normal to ∂M. It is clear that $H_V(M)$ can be identified with a dense subspace of $L^2(M)$, via a continuous injection.

If M has empty boundary, we are interested in the Friedrichs extension of the operator

$$S \colon C_c^\infty(M) \subset L^2(M) \to L^2(M).$$

If M has non-empty boundary, we are interested in the Neumann extension of S, that is, the Friedrichs extension of

$$S \colon \{f \in C_c^\infty(M) : \nu(f) = 0 \text{ on } \partial M\} \subset L^2(M) \to L^2(M).$$

In any of these cases, we denote this Friedrichs extension by S^N and its domain by $\mathcal{D}(S^N)$. It is worth to point out that the space $H_V(M)$ plays the role of $\mathcal{H}_1$ in the discussion of the Friedrichs extension in the beginning of this section (where we consider the lower bound $c := \inf_M V$ for the operator).

The spectrum and the essential spectrum of S^N are denoted by $\sigma^N(S)$ and $\sigma_{\text{ess}}^N(S)$, respectively, and their bottoms (that is, their minimums) by $\lambda_0^N(S)$ and $\lambda_0^{N,\text{ess}}(S)$, respectively. These sets and quantities for the Laplacian are denoted by $\sigma^N(M)$, $\sigma_{\text{ess}}^N(M)$ and $\lambda_0^N(M)$, $\lambda_0^{N,\text{ess}}(M)$, respectively. If M has empty boundary, we sometimes drop the superscript "N" in the notation of the spectrum, the essential spectrum and their bottoms.

If M has non-empty boundary, the Dirichlet extension S^D of S is the Friedrichs extension of the operator

$$S \colon \{f \in C_c^\infty(M) : f = 0 \text{ on } \partial M\} \subset L^2(M) \to L^2(M).$$

We denote by $\lambda_0^D(S)$ the bottom of the spectrum of this operator. The bottom of the spectrum of the Dirichlet extension of the Laplacian is denoted by $\lambda_0^D(M)$. In virtue of the next remark, Dirichlet extensions of Schrödinger operators are closely related to Schrödinger operators on non-complete manifolds without boundary.

Remark 2.3. If M is a Riemannian manifold with non-empty boundary, then any $f \in C_c^\infty(M)$ vanishing on ∂M, can be approximated in $H^1(M)$ with smooth functions, compactly supported in the interior of M. Therefore, if S is a Schrödinger operator on M, then the Dirichlet extension of S coincides with the Friedrichs extension of S viewed as an operator in the interior of M.

We end this subsection with some already known properties of the spectrum, that will be used in the sequel. Since they will be used only for complete manifolds without boundary, we do not state them in their most general forms.

The next proposition characterizes the bottom of the spectrum of a Schrödinger operator as the maximum of its positive spectrum, and may be found in [8, Theorem 7], [12, Theorem 1] and [16, Theorem 2.1].

Proposition 2.4. *Let S be a Schrödinger operator on a complete Riemannian manifold M without boundary. Then $\lambda_0(S)$ is the maximum of all $\lambda \in \mathbb{R}$, such that there exists a positive $\varphi \in C^\infty(M)$, with $S\varphi = \lambda\varphi$.*

It is worth to point out that the positive functions involved in this proposition are not required to be square-integrable. The next expression of the bottom of the essential spectrum follows from the Decomposition Principle [9, Proposition 2.1].

Proposition 2.5 ([4, Proposition 3.2]). *Let S be a Schrödinger operator on a complete Riemannian manifold M without boundary. Let $(K_n)_{n\in\mathbb{N}}$ be an exhausting sequence of M consisting of compact sets. Then the bottom of the essential spectrum of S is given by*

$$\lambda_0^{\mathrm{ess}}(S) = \lim_n \lambda_0(S, M \smallsetminus K_n),$$

where $\lambda_0(S, M \smallsetminus K_n)$ stands for the bottom of the spectrum of S on $M \smallsetminus K_n$.

2.2 Amenable coverings

In this subsection, we present the definition and some basic properties of amenable coverings. A right action of a countable group Γ on a countable set X is called *amenable* if there exists a Γ-invariant mean on $L^\infty(X)$.

Throughout the paper manifolds are assumed to be connected, unless otherwise stated. In particular, Riemannian coverings are assumed to be between connected manifolds, unless otherwise stated. For reasons that will become clear in the sequel, we must consider possibly non-connected covering spaces at some points.

Let $p\colon M_2 \to M_1$ be a Riemannian covering, with M_2 possibly non-connected, that is, M_2 has countably many connected components, the restriction of p on any component is a Riemannian covering over M_1, and any point of M_1 has a neighborhood that is evenly covered with respect to the restriction of p on any connected component of M_2. Fix $x \in M_1^\circ$ (that is, the interior of M_1) and consider the fundamental group $\pi_1(M_1)$ of M_1 with base point x. For $g \in \pi_1(M_1)$, let $\gamma_g\colon [0,1] \to M_1$ be a representative loop based at x. For $y \in p^{-1}(x)$, lift γ_g to a path $\tilde{\gamma}_g$, with $\tilde{\gamma}_g(0) = y$. We define $y \cdot g := \tilde{\gamma}_g(1)$. In this way, we obtain a right action of $\pi_1(M_1)$ on $p^{-1}(x)$. The covering p is called *amenable* if this right action is amenable.

This definition coincides with the definition presented in [1,2,14] in terms of the right cosets of $\pi_1(M_2)$ in $\pi_1(M_1)$, when M_2 is connected. However, this definition allows us to extend the notion of amenable coverings in case M_2 is non-connected.

For instance, consider a Riemannian covering $p\colon M_2 \to M_1$, where M_2 has countably many connected components $M_2^{(n)}$, $n \in \mathbb{N}$. If the restriction $p\colon M_2^{(n)} \to M_1$ is amenable, for some $n \in \mathbb{N}$, then the covering $p\colon M_2 \to M_1$ is amenable. Indeed, if there exists a $\pi_1(M_1)$-invariant mean μ_n on $L^\infty(p^{-1}(x) \cap M_2^{(n)})$, for some $n \in \mathbb{N}$, then the linear functional $\mu\colon L^\infty(p^{-1}(x)) \to \mathbb{R}$, defined by

$$\mu(f) := \mu_n(f|_{p^{-1}(x)\cap M_2^{(n)}}),$$

for any $f \in L^\infty(p^{-1}(x))$, is a $\pi_1(M_1)$-invariant mean on $L^\infty(p^{-1}(x))$. However, the covering $p\colon M_2 \to M_1$ may be amenable, even when the restriction $p\colon M_2^{(n)} \to M_1$ is non-amenable, for any $n \in \mathbb{N}$.

The following characterization of amenable coverings follows from Følner's criterion (cf. [3, Section 2]).

Proposition 2.6. *The covering p is amenable if and only if for any finite $G \subset \pi_1(M_1)$ and $\varepsilon > 0$, there exists a finite $F \subset p^{-1}(x)$, such that*

$$\#(F \smallsetminus Fg) < \varepsilon \#(F),$$

for all $g \in G$.

In particular, the covering is amenable if and only if the right action of any finitely generated subgroup of $\pi_1(M_1)$ on $p^{-1}(x)$ is amenable. For a smoothly bounded, compact and connected neighborhood K of x, we denote by $i_*\pi_1(K)$ the image of the fundamental group of K in $\pi_1(M_1)$.

It is clear that $p\colon p^{-1}(K) \to K$ is a Riemannian covering of manifolds with boundary, where $p^{-1}(K)$ is possibly non-connected. Evidently, the covering $p\colon p^{-1}(K) \to K$ is amenable if and only if the right action of $i_*\pi_1(K)$ on $p^{-1}(x)$ is amenable.

Proposition 2.7. *The covering $p\colon M_2 \to M_1$ is amenable if and only if the covering $p\colon p^{-1}(K) \to K$ is amenable, for any smoothly bounded, compact and connected neighborhood K of x.*

Proof: From Proposition 2.6, it suffices to prove that for any finite subset G of $\pi_1(M_1)$, there exists a smoothly bounded, compact and connected neighborhood K of x, such that $G \subset i_*\pi_1(K)$. Let G be a finite subset of $\pi_1(M_1)$ and consider a representative loop $\gamma_g\colon [0,1] \to M_1^\circ$, for each $g \in G$. Let C be the union of the images of these loops and let U be a relatively compact, open neighborhood of C that does not intersect the boundary of M (if non-empty). Consider $\chi \in C^\infty(M_1)$, with $0 \le \chi \le 1$, $\chi = 1$ in C and supp $\chi \subset U$. From Sard's Theorem, it follows that for almost any $t \in (0,1)$, the level set $\{\chi = t\}$ is a smooth hypersurface of M_1. Consider such a t, and the smoothly bounded, compact set $K' := \{\chi \ge t\}$. Then for the connected component K of K' containing x, we have $G \subset i_*\pi_1(K)$. ∎

2.3 Manifolds with Ricci curvature bounded from below

In this subsection we recall the main result of [1] and point out that its proof, with some slight modifications, establishes this result for possibly non-connected covering spaces.

A non-connected Riemannian manifold M is complete if all of its connected components are complete. The distance of points of different connected components of M is considered to be infinite. In particular, any bounded subset of M is contained in a connected component of M.

Theorem 2.8 ([1, Theorem 4.1]). *Let* $p\colon M_2 \to M_1$ *be a Riemannian covering, with* M_2 *possibly non-connected. Assume that* M_1 *is complete, without boundary, and with Ricci curvature bounded from below. Let* $S_1 := \Delta + V$ *be a Schrödinger operator on* M_1, *with* V *and* $\mathrm{grad}\, V$ *bounded, and let* S_2 *be its lift on* M_2. *If* $\lambda_0(S_2) = \lambda_0(S_1) \neq \lambda_0^{\mathrm{ess}}(S_1)$, *then the covering is amenable.*

We begin with some definitions and remarks from [1]. Let M be a possibly non-connected Riemannian manifold without boundary. A positive $\varphi \in C^\infty(M)$ *satisfies a Harnack estimate* if there exists a constant $c_\varphi \geq 1$, such that

$$\sup_{B(x,r)} \varphi^2 \leq c_\varphi^{r+1} \inf_{B(x,r)} \varphi^2,$$

for all $x \in M$ and $r > 0$. Assume that M is complete, with Ricci curvature bounded from below, and let $S = \Delta + V$ be a Schrödinger operator on M, with V and $\mathrm{grad}\, V$ bounded. From [8, Theorem 6], if a positive function $\varphi \in C^\infty(M)$ satisfies $S\varphi = \lambda\varphi$, for some $\lambda \in \mathbb{R}$, then φ satisfies a Harnack estimate.

The *modified Cheeger's constant* of M is defined as

$$h_\varphi(M) := \inf_A \frac{\int_{\partial A} \varphi^2}{\int_A \varphi^2},$$

where the infimum is taken over all bounded domains A of M with smooth boundary.

Lemma 2.9. *Let* M *be a possibly non-connected, complete Riemannian manifold, without boundary and with Ricci curvature bounded from below. Let* $\varphi \in C^\infty(M)$ *be a positive function, which satisfies a Harnack estimate. If* $h_\varphi(M) = 0$, *then for any* $\varepsilon, r > 0$, *there exists a bounded open subset* A *of* M, *such that*

$$\int_{A^r \smallsetminus A} \varphi^2 < \varepsilon \int_A \varphi^2,$$

where $A^r := \{y \in M : d(y, A) < r\}$.

Proof: We may renormalize the Riemannian metric of M, so that $\mathrm{Ric}_M \geq 1 - m$, where m is the dimension of M. Since $h_\varphi(M) = 0$, for any $\varepsilon, r > 0$, there exists a non-empty, bounded domain A of M satisfying the estimate (3.2) of [1]. Evidently, A is contained in a connected component of M and the arguments of the proof of [1, Lemma 3.1] can be carried out in this connected component of M, establishing the asserted claim. ∎

Lemma 2.10. *In the setting of Theorem 2.8, there exists a compact set* $K \subset M_1$, *such that for any* $\varepsilon, r > 0$, *there exists* $z \in K$ *and a bounded open subset* A *of* M_2, *such that*

$$\#(p^{-1}(z) \cap (A^r \smallsetminus A)) < \varepsilon \#(p^{-1}(z) \cap A).$$

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Proof: Since $\lambda_0(S_1) \notin \sigma_{\text{ess}}(S_1)$, from Proposition 2.5, there exists a compact $K \subset M_1$, such that $\lambda_0(S_1, M_1 \smallsetminus K) > \lambda_0(S_1)$. The proof is identical to the one of [1, Lemma 4.5], taking into account that [1, Lemma 3.1] has been extended to possibly non-connected manifolds in Lemma 2.9. ∎

Proof of Theorem 2.8: Fix $x \in M_1$ and consider the fundamental group $\pi_1(M_1)$ with base point x. Consider a compact set $K \subset M_1$ as in Lemma 2.10, and let $R > 0$, such that $K \subset B(x, R)$. Let $\varepsilon > 0$ and G be a finite subset of $\pi_1(M_1)$. For each $g \in G$, consider a smooth representative loop γ_g based at x, and let

$$r > \max_{g \in G} \ell(\gamma_g) + 2R,$$

where $\ell(\cdot)$ stands for the length of a curve. From Lemma 2.10, there exists $z \in K$ and a bounded open subset A of M_2, such that

$$\#(p^{-1}(z) \cap (A^r \smallsetminus A)) < \varepsilon \#(p^{-1}(z) \cap A).$$

Consider a smooth path $\gamma \colon [0, 1] \to M_1$ from x to z, of length less than R. For $y \in p^{-1}(x)$, lift γ to a path $\tilde{\gamma} \colon [0, 1] \to M_2$, with $\tilde{\gamma}(0) = y$, and define $\Phi(y) := \tilde{\gamma}(1)$. Then the map $\Phi \colon p^{-1}(x) \to p^{-1}(z)$ is bijective. Let $F := \Phi^{-1}(p^{-1}(z) \cap A)$ and consider $y \in F \smallsetminus Fg$, for some $g \in G$. Then $\Phi(y) \in A$ and $\Phi(y \cdot g^{-1}) \notin A$. Evidently, we have

$$d(\Phi(y), \Phi(y \cdot g^{-1})) \leq d(y, y \cdot g^{-1}) + 2\ell(\gamma) \leq \ell(\gamma_g) + 2R < r.$$

Therefore, $\Phi(y \cdot g^{-1}) \in A^r \smallsetminus A$. Since Φ is bijective, it is clear that

$$\begin{aligned}
\#(F \smallsetminus Fg) &= \#\{y \cdot g^{-1} : y \in F \smallsetminus Fg\} = \#\{\Phi(y \cdot g^{-1}) : y \in F \smallsetminus Fg\} \\
&\leq \#(p^{-1}(z) \cap (A^r \smallsetminus A)) < \varepsilon \#(p^{-1}(z) \cap A) = \varepsilon \#(F).
\end{aligned}$$

From Proposition 2.6, it follows that the covering is amenable. ∎

3 Properties of the Neumann spectrum

In this section, we establish some properties of the Neumann spectrum that will be used in the sequel. Let M be a possibly non-connected Riemannian manifold and $S = \Delta + V$ a Schrödinger operator on M. It is worth to point out that we do not require M to have non-empty boundary, which yields that the following results also hold for manifolds without boundary (and most of them are already known in this case). If M has non-empty boundary, we denote by ν the inward pointing normal to ∂M. Throughout this section, we denote by $H_V(M)$ the space defined in Subsection 2.1

First, we establish some convenient expressions for the bottom of the Neumann spectrum, and derive some straightforward applications to Riemannian coverings.

Proposition 3.1. *Any compactly supported smooth function belongs to $H_V(M)$. Moreover, any compactly supported Lipschitz function is in $H_V(M)$.*

Proof: If M has empty boundary, then any compactly supported Lipschitz function f belongs to $H_0^1(M)$. Since V is smooth, it is easy to see that any such f also belongs to $H_V(M)$. Therefore, it remains to prove the proposition for manifolds with non-empty boundary.

Let $f \in C_c^\infty(M)$. Then there exists a compact $K \subset \partial M$ and $\delta > 0$, such that the map $\Phi \colon K \times [0, \delta) \to M$, defined by $\Phi(x, t) := \exp_x(t\nu)$, is a diffeomorphism onto its image W_δ, and $\operatorname{supp} f \cap W_\delta \subset W_\delta^\circ$. For $0 < \delta_0 < \delta$, consider the Lipschitz function f_{δ_0}, which is equal to f outside W_{δ_0}, and $f_{\delta_0}(\Phi(x, t)) = f(\Phi(x, \delta_0))$ in W_{δ_0}. Let K_1 be a compact neighborhood of $\Phi(K \times \{\delta_0\})$ and K_2 a compact neighborhood of K_1, that does not intersect ∂M. Consider $\chi \in C_c^\infty(M)$, with $\chi = 1$ in K_1 and $\operatorname{supp} \chi \subset K_2$. Since χf_{δ_0} is Lipschitz and compactly supported in the interior of M, it follows that $\chi f_{\delta_0} \in H_V(M)$. Moreover, $(1 - \chi)f_{\delta_0} \in C_c^\infty(M)$ and $\nu(f) = 0$ on ∂M. Therefore, $(1 - \chi)f_{\delta_0} \in H_V(M)$, which yields that $f_{\delta_0} \in H_V(M)$. It is clear that $f_{\delta_0} \to f$ in $H_V(M)$, as $\delta_0 \to 0$, and in particular, $f \in H_V(M)$.

Let f be a compactly supported Lipschitz function on M. Consider a Riemannian manifold N of the same dimension, without boundary, containing M (for instance, glue cylinders along ∂M). Extend f to a compactly supported Lipschitz function f' in N and let K be a smoothly bounded, compact neighborhood of $\operatorname{supp} f'$. Then there exists $(g_n)_{n \in \mathbb{N}} \subset C_c^\infty(N)$, with $\operatorname{supp} g_n \subset K$ and $g_n \to f'$ in $H_0^1(K)$. Then $h_n := g_n|_M \in C_c^\infty(M)$ and from the first statement, it follows that $h_n \in H_V(M)$. Evidently, we have that $h_n \to f$ in $H_V(M)$, and in particular, $f \in H_V(M)$. ∎

For $f \in \operatorname{Lip}_c(M) \smallsetminus \{0\}$, the *Rayleigh quotient* of f with respect to S, is defined as

$$\mathcal{R}_S(f) := \frac{\int_M (\|\operatorname{grad} f\|^2 + V f^2)}{\int_M f^2}.$$

Proposition 3.2. *The bottom of the spectrum of S^N is given by*

$$\lambda_0^N(S) = \inf_{f \in C_c^\infty(M) \smallsetminus \{0\}} \mathcal{R}_S(f) = \inf_{f \in \operatorname{Lip}_c(M) \smallsetminus \{0\}} \mathcal{R}_S(f).$$

Proof: It is clear that for any non-zero $f \in \operatorname{Lip}_c(M)$, we have

$$\mathcal{R}_S(f) = \inf_M V - 1 + \frac{\|f\|_{H_V(M)}^2}{\|f\|_{L^2(M)}^2},$$

and the asserted equalities follow from Proposition 2.2. ∎

Proposition 3.3. *Let $p \colon M_2 \to M_1$ be a Riemannian covering, with M_2 possibly non-connected. Let S_1 be a Schrödinger operator on M_1 and consider its lift S_2 on M_2. Then $\lambda_0^N(S_1) \leq \lambda_0^N(S_2)$.*

Proof: Let $f \in C_c^\infty(M_2) \smallsetminus \{0\}$ and consider its pushdown

$$g(z) := \Big(\sum_{y \in p^{-1}(z)} f(y)^2 \Big)^{1/2}$$

on M_1. Then $g \in \mathrm{Lip}_c(M_1)$, $\|g\|_{L^2(M_1)} = \|f\|_{L^2(M_2)}$ and $\mathcal{R}_{S_1}(g) \leq \mathcal{R}_{S_2}(f)$ (cf. [2, Section 4]). The statement follows from Proposition 3.2. ∎

Theorem 3.4. *Let* $p \colon M_2 \to M_1$ *be a Riemannian covering. Let* S_1 *be a Schrödinger operator on* M_1 *and consider its lift* S_2 *on* M_2. *If* p *is infinite sheeted and amenable, then* $\lambda_0^N(S_1) = \lambda_0^{N,\mathrm{ess}}(S_2)$.

Proof: Follows from [14, Theorem 1.2] and Corollary 3.3. ∎

Next, we study properties of eigenfunctions corresponding to the bottom of the spectrum and minimizing sequences for the Rayleigh quotient of Schrödinger operators on connected Riemannian manifolds.

Proposition 3.5. *Let* $S = \Delta + V$ *be a Schrödinger operator on a Riemannian manifold* M, *and consider* $(f_n)_{n \in \mathbb{N}} \subset \mathrm{Lip}_c(M)$, *with* $\|f_n\|_{L^2(M)} = 1$ *and* $\mathcal{R}_S(f_n) \to \lambda_0^N(S)$. *If* $\lambda_0^N(S) \notin \sigma_{\mathrm{ess}}^N(S)$, *then there exists a subsequence* $(f_{n_k})_{k \in \mathbb{N}}$, *such that* $f_{n_k} \to \varphi$ *in* $L^2(M)$, *for some* $\lambda_0^N(S)$-*eigenfunction* φ *of* S^N.

Proof: From Proposition 3.1, there exists $(f_n')_{n \in \mathbb{N}} \subset C_c^\infty(M) \cap \mathcal{D}(S^N)$, with $\|f_n'\|_{L^2(M)} = 1$ and $\|f_n - f_n'\|_{H_V(M)} \leq 1/n$, for any $n \in \mathbb{N}$. Evidently, $\mathcal{R}_S(f_n') \to \lambda_0^N(S)$ and it suffices to prove the asserted statement for $(f_n')_{n \in \mathbb{N}}$.

Since $\lambda_0^N(S)$ is not in the essential spectrum, it is an isolated eigenvalue of finite multiplicity. Let E be the eigenspace corresponding to $\lambda_0^N(S)$, and g_n be the projection of f_n' on E with respect to the $L^2(M)$-inner product, $n \in \mathbb{N}$. Since E is finite dimensional, after passing to a subsequence, we may assume that $g_n \to \varphi$ in $L^2(M)$, for some $\varphi \in E$. Consider $h_n := f_n' - g_n \in \mathcal{D}(S^N)$. Since h_n is perpendicular to E, from the Spectral Theorem (cf. for instance [18, Chapter 8]), it follows that there exists $c_0 > 0$, such that

$$\|h_n\|_{H_V(M)}^2 - (1 - \inf_M V)\|h_n\|_{L^2(M)}^2 = \langle S^N h_n, h_n \rangle_{L^2(M)} \geq (\lambda_0^N(S) + c_0)\|h_n\|_{L^2(M)}^2, \quad (2)$$

for any $n \in \mathbb{N}$. It is clear that

$$\begin{aligned}
\langle h_n, g_n \rangle_{H_V(M)} &= \langle h_n, S^N g_n \rangle_{L^2(M)} + (1 - \inf_M V)\langle h_n, g_n \rangle_{L^2(M)} \\
&= (\lambda_0^N(S) + 1 - \inf_M V)\langle h_n, g_n \rangle_{L^2(M)} = 0.
\end{aligned}$$

Let $\varepsilon > 0$. Then, for n sufficiently large, we have $\mathcal{R}_S(f_n) \leq \lambda_0^N(S) + \varepsilon$, and thus

$$\begin{aligned}
\|h_n\|_{H_V(M)}^2 - (1 - \inf_M V)\|h_n\|_{L^2(M)}^2 &= (\|f_n'\|_{H_V(M)}^2 - (1 - \inf_M V)\|f_n'\|_{L^2(M)}^2) \\
&\quad - (\|g_n\|_{H_V(M)}^2 - (1 - \inf_M V)\|g_n\|_{L^2(M)}^2) \\
&\leq (\lambda_0^N(S) + \varepsilon)\|f_n'\|_{L^2(M)}^2 - \lambda_0^N(S)\|g_n\|_{L^2(M)}^2 \\
&= \varepsilon + \lambda_0^N(S)\|h_n\|_{L^2(M)}^2.
\end{aligned}$$

From (2), this yields that $h_n \to 0$ in $L^2(M)$. Therefore, $f_n' \to \varphi$ in $L^2(M)$. ∎

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Lemma 3.6. *Let S be a Schrödinger operator on a (connected) Riemannian manifold M and let $\varphi \in C^\infty(M) \setminus \{0\}$ be a non-negative function satisfying $S\varphi = \lambda\varphi$, for some $\lambda \in \mathbb{R}$. Then φ is positive in the interior of M. If, in addition, M has non-empty boundary, and $\nu(\varphi) = 0$ on ∂M, then φ is positive on ∂M.*

Proof: Assume that there exists a point x in the interior of M, such that $\varphi(x) = 0$. Let $\delta > 0$, such that $\exp_x \colon B(0, 2\delta) \subset T_x M \to M$ is a diffeomorphism onto its image. Then $B(x, \delta)$ may be considered as a geodesic ball of radius δ in a complete Riemannian manifold without boundary. In $B(x, \delta)$, for any $\varepsilon > 0$, we have

$$|\Delta(\varphi + \varepsilon)| \leq (\varphi + \varepsilon) \sup_{B(x,\delta)} |\lambda - V|$$

and

$$\| \operatorname{grad} \Delta(\varphi + \varepsilon)\| \leq \| \operatorname{grad}(\varphi + \varepsilon)\| \sup_{B(x,\delta)} |\lambda - V| + (\varphi + \varepsilon) \sup_{B(x,\delta)} \| \operatorname{grad} V\|.$$

From [8, Theorem 6], it follows that there exists $c > 0$, independent from ε, such that

$$\sup_{B(x,\delta/2)} (\varphi + \varepsilon) \leq c \inf_{B(x,\delta/2)} (\varphi + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, it follows that if $\varphi(x) = 0$, then $\varphi = 0$ in $B(x, \delta/2)$. In particular, the set $\{x \in M^\circ : \varphi(x) = 0\}$ is open and closed. Since M is connected and φ is not identically zero, it follows that φ is positive in M°.

Assume that M has non-empty boundary and $\nu(\varphi) = 0$ on ∂M. Assume that there exists $x \in \partial M$, such that $\varphi(x) = 0$. Since $\nu(\varphi) = 0$ on ∂M and $\varphi|_{\partial M}$ attains a minimum at x, it follows that $\operatorname{grad} \varphi(x) = 0$. Consider a coordinate system $\Phi \colon U := B(0, r) \cap \mathbb{H}^m \to M$, with $\Phi(0) = x$, where m is the dimension of M and $\mathbb{H}^m$ is the upper half-space of dimension m. Consider $c_0 \in \mathbb{R}$, with $c_0 \geq -\inf_M V$ and $c_0 \geq -\lambda$. Then $\phi := \varphi \circ \Phi$ is non-negative, smooth and satisfies

$$L\phi := -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{\det g} \frac{\partial \phi}{\partial x_j} \right) + (V + c_0)\phi = (\lambda + c_0)\phi \geq 0.$$

Since $V + c_0 \geq 0$, $\phi(0) = 0 < \phi(y)$ for all $y \in U^\circ$, and U satisfies the interior ball condition at the origin, from Hopf's Lemma (cf. for instance [11, p. 330]), it follows that

$$\frac{\partial \phi}{\partial x_m}(0) \neq 0,$$

which is a contradiction, since $\operatorname{grad} \phi(0) = 0$. Therefore, φ is positive on ∂M. ∎

Proposition 3.7. *Let M be a (connected) Riemannian manifold and $S = \Delta + V$ a Schrödinger operator on M. If $\varphi \in \mathcal{D}(S^N) \setminus \{0\}$ is a $\lambda_0^N(S)$-eigenfunction of S^N, then φ is smooth and nowhere vanishing. Moreover, if M has non-empty boundary, then $\nu(\varphi) = 0$ on ∂M.*

Proof: Since $\varphi \in \mathcal{D}(S^N)$, there exists $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M)$, such that $f_n \to \varphi$ in $H_V(M)$. Clearly, $|f_n|$ is Lipschitz and compactly supported. From Proposition 3.1, it follows that $|f_n| \in H_V(M)$. From Rademacher's Theorem, $|f_n|$ is almost everywhere differentiable. Therefore, we have $\|\operatorname{grad}|f_n|\| = \|\operatorname{grad} f_n\|$ almost everywhere, and in particular, $\mathcal{R}_S(|f_n|) = \mathcal{R}_S(f_n)$. Since $(|f_n|)_{n\in\mathbb{N}}$ is bounded in $H_V(M)$, it has a weakly convergent subsequence in $H_V(M)$. Since $|f_n| \to |\varphi|$ in $L^2(M)$, it follows that $|\varphi| \in H_V(M)$, and after passing to a subsequence, we have that $|f_n| \rightharpoonup |\varphi|$ in $H_V(M)$. Hence, $\mathcal{R}_S(|\varphi|) = \lambda_0^N(S)$.

In particular, for any $f \in C_c^\infty(M)$, the function $t \mapsto \mathcal{R}_S(|\varphi| + tf)$, with $|t| < \varepsilon$, is differentiable and attains minimum for $t = 0$. This yields that

$$\int_M (\langle \operatorname{grad}|\varphi|, \operatorname{grad} f\rangle + V|\varphi|f) = \lambda_0^N(S) \int_M |\varphi|f, \tag{3}$$

for any $f \in C_c^\infty(M)$. From Elliptic Regularity Theory, it follows that $|\varphi| \in C^\infty(M^\circ)$ and $S|\varphi| = \lambda_0^N(S)|\varphi|$ in M°. From Lemma 3.6, $|\varphi|$ is nowhere vanishing in the interior of M, and so is φ. If M has empty boundary, this completes the proof.

If M has non-empty boundary, then without loss of generality, we may assume that φ is positive in the interior of M. Since $\varphi \in \mathcal{D}(S^N)$ and $S^N\varphi = \lambda_0^N(S)\varphi$, from Elliptic Regularity Theory, it follows that $\varphi \in C^\infty(M)$. Moreover, from (3) we have that

$$\int_{\partial M} \nu(\varphi)f = \int_M fS\varphi - \int_M (\langle \operatorname{grad}\varphi, \operatorname{grad} f\rangle + V\varphi f) = 0,$$

for any $f \in C_c^\infty(M)$. Therefore, $\nu(\varphi) = 0$ on ∂M, and from Lemma 3.6, it follows that φ is positive on ∂M. ∎

Proposition 3.8. *Let S be a Schrödinger operator on a (connected) Riemannian manifold M, with $\lambda_0^N(S) \notin \sigma_{\mathrm{ess}}^N(S)$. Then for any compact $K \subset M$ of positive measure, we have*

$$\inf_f \mathcal{R}_S(f) > \lambda_0^N(S),$$

where the infimum is taken over all non-zero $f \in \operatorname{Lip}_c(M)$, with $\operatorname{supp} f \cap K = \emptyset$.

Proof: Assume to the contrary that there exists a compact subset K of M of positive measure, such that for any $\varepsilon > 0$, there exists a non-zero $f \in \operatorname{Lip}_c(M)$, with $\mathcal{R}_S(f) < \lambda_0^N(S) + \varepsilon$ and $\operatorname{supp} f \cap K = \emptyset$. Evidently, there exists $(f_n)_{n\in\mathbb{N}} \subset \operatorname{Lip}_c(M)$, with $\|f_n\|_{L^2(M)} = 1$, $\operatorname{supp} f_n \cap K = \emptyset$ and $\mathcal{R}_S(f_n) \to \lambda_0^N(S)$. From Proposition 3.5, after passing to a subsequence, we have that $f_n \to \varphi$ in $L^2(M)$, for some $\lambda_0^N(S)$-eigenfunction φ of S^N. Since $\|\varphi\|_{L^2(M)} = 1$, from Proposition 3.7, it follows that φ is nowhere vanishing in M. This is a contradiction, since

$$\|\varphi - f_n\|_{L^2(M)}^2 \geq \int_K \varphi^2 > 0,$$

while $f_n \to \varphi$ in $L^2(M)$. This proves the asserted claim. ∎

We end this section with the notion of renormalized Schrödinger operators. This notion was introduced for the Laplacian in [6] and for Schrödinger operators on manifolds without boundary in [14].

Let M be a possibly non-connected Riemannian manifold and $S = \Delta + V$ a Schrödinger operator on M. Let $\varphi \in C^\infty(M)$ be a positive function, satisfying $S\varphi = \lambda\varphi$, for some $\lambda \in \mathbb{R}$. If M has non-empty boundary, assume that $\nu(\varphi) = 0$ on ∂M. Consider the space $L^2_\varphi(M) := \{[v] : \varphi v \in L^2(M)\}$, where two measurable functions are equivalent if they are almost everywhere equal, endowed with the inner product $\langle v_1, v_2 \rangle_{L^2_\varphi(M)} := \int_M v_1 v_2 \varphi^2$. Then the map $\mu_\varphi \colon L^2_\varphi(M) \to L^2(M)$, defined by $\mu_\varphi v := \varphi v$, is an isometric isomorphism. In particular, $L^2_\varphi(M)$ is a separable Hilbert space. The *renormalization* S_φ of S with respect to φ is defined by

$$S_\varphi v := \mu_\varphi^{-1}(S^N - \lambda)(\mu_\varphi v), \text{ for all } v \in \mathcal{D}(S_\varphi) := \mu_\varphi^{-1}(\mathcal{D}(S^N)).$$

It is clear that $S_\varphi \colon \mathcal{D}(S_\varphi) \colon L^2_\varphi(M) \to L^2_\varphi(M)$ is self-adjoint and $\sigma(S_\varphi) = \sigma^N(S) - \lambda$. For a non-zero $f \in \mathrm{Lip}_c(M)$, the *Rayleigh quotient* of f with respect to S_φ is defined as

$$\mathcal{R}_{S_\varphi}(f) := \frac{\int_M \|\operatorname{grad} f\|^2 \varphi^2}{\int_M f^2 \varphi^2}.$$

Proposition 3.9 ([14, Proposition 7.1],[1, Subsection 2.1])**.** *In the above situation, if M has empty boundary, then the bottom of the spectrum of S_φ is given by*

$$\lambda_0(S) - \lambda = \lambda_0(S_\varphi) = \inf_{f \in C^\infty_c(M) \smallsetminus \{0\}} \mathcal{R}_{S_\varphi}(f) = \inf_{f \in \mathrm{Lip}_c(M) \smallsetminus \{0\}} \mathcal{R}_{S_\varphi}(f).$$

Proposition 3.10. *In the above situation, if M has non-empty boundary, then the bottom of the spectrum of S_φ is given by*

$$\lambda_0^N(S) - \lambda = \lambda_0(S_\varphi) = \inf_f \mathcal{R}_{S_\varphi}(f),$$

where the infimum is taken over all non-zero $f \in C^\infty_c(M)$, with $\nu(f) = 0$ on ∂M.

Proof: Let $f \in C^\infty_c(M) \smallsetminus \{0\}$, with $\nu(f) = 0$ on ∂M. Since φ is smooth and $\nu(\varphi) = 0$ on ∂M, it follows that $f \in \mathcal{D}(S_\varphi)$. It is easy to see that

$$S_\varphi f = \Delta f - \frac{2}{\varphi}\langle \operatorname{grad}\varphi, \operatorname{grad} f\rangle.$$

Hence, we have

$$
\begin{aligned}
\langle S_\varphi f, f \rangle_{L^2_\varphi(M)} &= \int_M (\varphi^2 f \Delta f - 2f\varphi\langle \operatorname{grad} f, \operatorname{grad}\varphi\rangle) \\
&= \int_M (\langle \operatorname{grad}(\varphi^2 f), \operatorname{grad} f\rangle - 2f\varphi\langle \operatorname{grad} f, \operatorname{grad}\varphi\rangle) + \int_{\partial M} \varphi^2 f \nu(f) \\
&= \int_M \|\operatorname{grad} f\|^2 \varphi^2,
\end{aligned}
$$

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where we used that $\nu(f) = 0$ on ∂M. In particular, we have that

$$\mathcal{R}_{S_\varphi}(f) = \frac{\langle S_\varphi f, f \rangle_{L^2_\varphi(M)}}{\|f\|^2_{L^2_\varphi(M)}}.$$

From Proposition 2.1, it follows that $\mathcal{R}_{S_\varphi}(f) \geq \lambda_0(S_\varphi)$. From Proposition 2.2, there exists $(g_n)_{n \in \mathbb{N}} \subset C^\infty_c(M) \smallsetminus \{0\}$, with $\nu(g_n) = 0$ on ∂M and $\mathcal{R}_S(g_n) \to \lambda_0^N(S)$. Consider $f_n := \mu_\varphi^{-1} g_n$. Then $f_n \in C^\infty_c(M)$, $\nu(f_n) = 0$ on ∂M, and $\mathcal{R}_{S_\varphi}(f_n) \to \lambda_0(S_\varphi)$. This proves the asserted equality. ∎

4   Coverings of compact manifolds

Throughout this section, for simplicity of notation, we denote by $\mathcal{R}(f)$ the Rayleigh quotient of a Lipschitz function f with respect to the Laplacian. The aim of this section is to prove Theorem 1.1. Since a part of it follows from Theorem 3.4, it remains to prove the converse implication. For reasons that will become clear in the sequel, we need to establish it also for non-connected covering spaces.

Theorem 4.1. *Let $p\colon M_2 \to M_1$ be a Riemannian covering, with M_1 compact with non-empty boundary, and M_2 possibly non-connected. If $\lambda_0^N(M_2) = 0$, then p is amenable.*

Let ν_i be the inward pointing normal to ∂M_i, $i = 1, 2$. Then there exists $\delta > 0$, such that the map $\Phi\colon \partial M_1 \times [0, 2\delta) \to M_1$, defined by $\Phi(x, t) := \exp_x(t\nu_1)$, is a diffeomorphism onto its image. By definition, any point of M_1 has an evenly covered neighborhood with respect to the restriction of p on any connected component of M_2. Therefore, we may assume that δ is sufficiently small, so that for any $x \in \partial M_1$ and $y_1, y_2 \in p^{-1}(x)$, with $y_1 \neq y_2$, we have $d(y_1, y_2) \geq 2\delta$. It is worth to point out that we consider the distance of points of different connected components of M_2 to be infinite.

Lemma 4.2. *The map $\Psi\colon \partial M_2 \times [0, \delta) \to M_2$, defined by $\Psi(y, t) := \exp_y(t\nu_2)$, is a diffeomorphism onto its image.*

Proof: Since $(p \circ \Psi)(y, t) = \Phi(p(y), t)$, for any $y \in \partial M_2$ and $t \in [0, \delta)$, it is clear that Ψ is a local diffeomorphism. So, it suffices to prove that it is injective. Consider $y_1, y_2 \in \partial M_2$ and $t_1, t_2 \in [0, \delta)$, such that $\Psi(y_1, t_1) = \Psi(y_2, t_2) =: z$. Then $d(y_i, z) < \delta$, $i = 1, 2$, which yields that $d(y_1, y_2) < 2\delta$. Moreover, it follows that $\Phi(p(y_1), t_1) = \Phi(p(y_2), t_2)$. Since Φ is a diffeomorphism onto its image, this yields that $t_1 = t_2$, $p(y_1) = p(y_2)$, and in particular, $y_1 = y_2$. ∎

Lemma 4.3. *There exists a Riemannian metric g' on M_1, such that Φ restricted to $\partial M_1 \times [0, \delta)$ is an isometry onto its image.*

Proof: Let g_c be the push-forward of the product metric of $\partial M_1 \times [0, 2\delta)$ via Φ. Denote by g the original Riemannian metric of M_1. Consider a smooth $\tau\colon [0, 2\delta) \to [0, 1]$, with $\tau(t) = 1$ for $t \leq \delta$, and $\tau(t) = 0$ for $t \geq 3\delta/2$. Consider the function $\tau' \in C^\infty(M_1)$, defined by $\tau'(\Phi(x, t)) = \tau(t)$ in $\Phi(\partial M_1 \times [0, 2\delta))$, and $\tau' = 0$ otherwise. Evidently, the Riemannian metric

$$g' := \tau' g_c + (1 - \tau') g.$$

on M_1 satisfies the desired property. ∎

Consider M_1 and M_2 endowed with g' and its lift, respectively. Since $(p \circ \Psi)(y, t) = \Phi(p(y), t)$, for any $y \in \partial M_2$ and $t \in [0, \delta)$, it follows that Ψ restricted on $\partial M_2 \times [0, \delta)$ is a local isometry, with respect to the lift of g'. From Lemma 4.2, this map is also injective, which yields that it is an isometry onto its image. Denote by U_t the open set $\Psi(\partial M_2 \times [0, t))$, and by C_t the closed set $\Psi(\partial M_2 \times \{t\})$.

Evidently, there exist $c_1, c_2 > 0$, such that for any $f \in C^\infty(M_2)$, the norms of the gradients of f with respect to the lifts of g and g', are related by

$$c_1 \|\mathrm{grad}_g f\|_g \leq \|\mathrm{grad}_{g'} f\|_{g'} \leq c_2 \|\mathrm{grad}_g f\|_g.$$

Moreover, there exists a positive, smooth $\mathcal{V}\colon M_1 \to \mathbb{R}$, such that the volume elements induced by the lifts of g and g', satisfy

$$\frac{d\mathrm{Vol}_{g'}}{d\mathrm{Vol}_g} = \mathcal{V} \circ p.$$

Therefore, for any non-zero $f \in C_c^\infty(M_2)$, the Rayleigh quotients of f with respect to the Laplacians induced by the lifts of g and g', satisfy

$$\mathcal{R}_{g'}(f) = \frac{\int_{M_2} \|\mathrm{grad}_{g'} f\|_{g'}^2 d\mathrm{Vol}_{g'}}{\int_{M_2} f^2 d\mathrm{Vol}_{g'}} \leq c_2^2 \frac{\max \mathcal{V}}{\min \mathcal{V}} \mathcal{R}_g(f).$$

From Proposition 3.2, since $\lambda_0^N(M_2) = 0$ with respect to the lift of g, it follows that $\lambda_0^N(M_2) = 0$ with respect to the lift of g'. From now on, we will be working with g' and its lift. It is worth to point out that the maps Φ and Ψ are defined in terms of the exponentials with respect to the original Riemannian metrics.

Lemma 4.4. *For any $\varepsilon > 0$, there exists $f \in \mathrm{Lip}_c(M_2)$, smooth on $M_2 \smallsetminus C_{t_0}$, for one $t_0 \in (0, \delta)$, with $f|_{\partial M_2}$ non-zero, such that $\mathcal{R}(f) \leq \varepsilon$ and*

$$\frac{\int_{\partial M_2} \|\mathrm{grad} f\|^2}{\int_{\partial M_2} f^2} \leq \varepsilon.$$

Proof: Since $\lambda_0^N(M_2) = 0$, from Proposition 3.2, there exists $(f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M_2)$, with $\|f_n\|_{L^2(M_2)} = 1$, such that $\mathcal{R}(f_n) \to 0$. Assume that there exists $\varepsilon > 0$, such that for any $n \in \mathbb{N}$ and $t \in [0, \delta)$, we have

$$\int_{C_t} \|\mathrm{grad} f_n\|^2 > \varepsilon \int_{C_t} f_n^2. \tag{4}$$

Then
$$\int_{U_\delta} \|\operatorname{grad} f_n\|^2 > \varepsilon \int_{U_\delta} f_n^2,$$
which yields that $\int_{U_\delta} f_n^2 \to 0$ and $\int_{M_2 \smallsetminus U_\delta} f_n^2 \to 1$. Let $\chi \in C^\infty(M_1)$, with $\chi(x) = 1$ for $d(x, \partial M_1) \geq \delta$, and $\chi(x) = 0$ for $d(x, \partial M_1) < \delta/2$. Let $\tilde{\chi} \in C^\infty(M_2)$ be the lift of χ. Then $\tilde{\chi} = 0$ in $U_{\delta/2}$ and $\tilde{\chi} = 1$ outside U_δ. For $g_n := \tilde{\chi} f_n \in C_c^\infty(M_2)$, we have

$$\|g_n\|_{L^2(M_2)}^2 = \int_{U_\delta} \tilde{\chi}^2 f_n^2 + \int_{M_2 \smallsetminus U_\delta} f_n^2 \to 1,$$

and

$$\int_{M_2} \|\operatorname{grad} g_n\|^2 \leq 2 \int_{U_\delta} (\tilde{\chi}^2 \|\operatorname{grad} f_n\|^2 + f_n^2 \|\operatorname{grad} \tilde{\chi}\|^2) + \int_{M_2 \smallsetminus U_\delta} \|\operatorname{grad} f_n\|^2 \to 0.$$

In particular, we have that $\mathcal{R}(g_n) \to 0$. Since g_n is supported in the interior of M_2, for any $n \in \mathbb{N}$, from Proposition 3.2 and Remark 2.3, it follows that $\lambda_0^D(M_2) = 0$. This is a contradiction, since from Proposition 3.3 and Remark 2.3, we have $\lambda_0^D(M_2) \geq \lambda_0^D(M_1) > 0$.

Hence, (4) cannot hold, that is, for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ and $t \in [0, \delta)$, such that

$$\int_{C_t} \|\operatorname{grad} f_n\|^2 \leq \varepsilon \int_{C_t} f_n^2. \tag{5}$$

Let $0 < \varepsilon < \lambda_0^D(M_2)$ and consider $f_n \in C_c^\infty(M_2)$, with $\|f_n\|_{L^2(M_2)} = 1$, $\mathcal{R}(f_n) < \varepsilon$, satisfying (5) for some $t \in [0, \delta)$. Let t_0 be the minimum of all $t \in [0, \delta)$, for which (5) holds. If $t_0 = 0$, then f_n is the desired function. Otherwise, define $f \in C_c(M_2)$ by $f = f_n$ outside U_{t_0}, and $f(\Psi(x, t)) = f_n(\Psi(x, t_0))$ for $t \leq t_0$. It is clear that $f \in \operatorname{Lip}_c(M)$ and is smooth on $M_2 \smallsetminus C_{t_0}$.

Since $\mathcal{R}(f_n) < \lambda_0^D(M_2)$, from Proposition 3.2 and Remark 2.3, it follows that f_n is not identically zero on U_{t_0}. Since $\mathcal{R}(f_n) < \varepsilon$, from the definition of t_0, it follows that f_n is not identically zero on $M_2 \smallsetminus U_{t_0}$. In particular, this yields that f is non-zero. Since (5) holds for $t = t_0$, we have

$$\int_{\partial M_2} \|\operatorname{grad} f\|^2 = \int_{C_{t_0}} \|\operatorname{grad}(f_n|_{C_{t_0}})\|^2 \leq \varepsilon \int_{C_{t_0}} f_n^2 = \varepsilon \int_{\partial M_2} f^2.$$

Furthermore, we have

$$
\begin{aligned}
\mathcal{R}(f) &= \frac{\int_0^{t_0} \int_{C_t} \|\operatorname{grad} f\|^2 + \int_{M_2 \smallsetminus U_{t_0}} \|\operatorname{grad} f_n\|^2}{\int_0^{t_0} \int_{C_t} f^2 + \int_{M_2 \smallsetminus U_{t_0}} f_n^2} \\
&\leq \frac{\varepsilon \int_0^{t_0} \int_{C_t} f^2 + \int_{M_2 \smallsetminus U_{t_0}} \|\operatorname{grad} f_n\|^2}{\int_0^{t_0} \int_{C_t} f^2 + \int_{M_2 \smallsetminus U_{t_0}} f_n^2} \\
&\leq \max \left\{ \varepsilon, \frac{\int_{M_2 \smallsetminus U_{t_0}} \|\operatorname{grad} f_n\|^2}{\int_{M_2 \smallsetminus U_{t_0}} f_n^2} \right\}.
\end{aligned}
\tag{6}
$$

It is clear that

$$
\begin{aligned}
\varepsilon \;>\; \mathcal{R}(f_n) &= \frac{\int_0^{t_0}\int_{C_t}\|\operatorname{grad} f_n\|^2 + \int_{M_2\setminus U_{t_0}}\|\operatorname{grad} f_n\|^2}{\int_0^{t_0}\int_{C_t} f_n^2 + \int_{M_2\setminus U_{t_0}} f_n^2}\\[2mm]
&\geq\; \min\left\{\frac{\int_0^{t_0}\int_{C_t}\|\operatorname{grad} f_n\|^2}{\int_0^{t_0}\int_{C_t} f_n^2},\; \frac{\int_{M_2\setminus U_{t_0}}\|\operatorname{grad} f_n\|^2}{\int_{M_2\setminus U_{t_0}} f_n^2}\right\}.
\end{aligned}
$$

From the definition of t_0, the first term is greater than ε, which yields that the second term is smaller than ε. From (6), it follows that $\mathcal{R}(f) \leq \varepsilon$. Since $\varepsilon < \lambda_0^D(M_2)$, from Remark 2.3 and Proposition 3.2, it is clear that f cannot vanish identically on ∂M_2. ∎

Glue the cylinder $\partial M_1 \times [0, +\infty)$, with the product metric, along ∂M_1, so that $\partial/\partial t$ is the outward pointing normal to ∂M_1. Denote by N_1 the obtained Riemannian manifold. The covering $p\colon M_2 \to M_1$ can be extended to a Riemannian covering $p\colon N_2 \to N_1$, where N_2 is the Riemannian manifold obtained by gluing $\partial M_2 \times [0, +\infty)$ along ∂M_2 in the analogous way. Evidently, $p\colon M_2 \to M_1$ is amenable if and only if $p\colon N_2 \to N_1$ is amenable. Points in $N_i \setminus M_i^{\circ}$ will be written in the form (x, t), with $x \in \partial M_i$ and $t \geq 0$, $i = 1, 2$.

Consider a positive smooth $\phi\colon [0, +\infty) \to \mathbb{R}$, with $\phi(t) = 1$ for $t \leq 1/2$, and $\phi(t) = e^{-t}$ for $t \geq 1$. Let $\varphi \in C^{\infty}(N_1)$ be the square-integrable function defined by $\varphi = 1$ in M_1, and $\varphi(x, t) = \phi(t)$ in $N_1 \setminus M_1$. Consider the function $V \in C^{\infty}(N_1)$, defined by $V = 0$ in M_1, and $V(x, t) = \phi''(t)/\phi(t)$ in $N_1 \setminus M_1$. It is worth to point out that outside the compact set $M_1 \cup (\partial M_1 \times [0, 1])$, we have that $V = 1$ and in particular, V is bounded from below. Consider the Schrödinger operator $S_1 = \Delta + V$ on N_1 and its lift S_2 on N_2. It is clear that $S_1\varphi = 0$.

Remark 4.5. Evidently, N_1 is complete, without boundary and with Ricci curvature bounded from below. Since $V = 1$ outside the compact set $M_1 \cup (\partial M_1 \times [0, 1])$, from Propositions 2.5 and 3.2, it follows that $\lambda_0^{\mathrm{ess}}(S_1) \geq 1$. Moreover, it is clear that V and $\operatorname{grad} V$ are bounded.

Lemma 4.6. *The function φ belongs to the domain of the Friedrichs extension of S_1 and in particular, $\lambda_0(S_1) = 0$.*

Proof: For $T > 0$, consider the compactly supported Lipschitz function χ_T defined by $\chi_T = 1$ in M_1, $\chi_T(x, t) = 1$ for $t \leq T$, $\chi_T(x, t) = T + 1 - t$ for $T \leq t \leq T + 1$, and $\chi_T(x, t) = 0$ for $t \geq T$. Then $\chi_T\varphi \in H_0^1(N_1)$, for any $T > 0$, and

$$
\|\varphi - \chi_T\varphi\|_{L^2(N_1)}^2 \leq \int_{\partial M_1 \times [T, +\infty)} \varphi^2.
$$

Moreover, we have

$$\int_{N_1} \|\operatorname{grad}(\varphi - \chi_T\varphi)\|^2 \;\leq\; 2\int_{N_1} \left((1-\chi_T)^2\|\operatorname{grad}\varphi\|^2 + \varphi^2\|\operatorname{grad}(1-\chi_T)\|^2\right)$$

$$\leq\; 2\int_{\partial M_1 \times [T,+\infty)} \|\operatorname{grad}\varphi\|^2 + 2\int_{\partial M_1 \times [T,T+1]} \varphi^2.$$

Evidently, $\varphi(x,t) = \|\operatorname{grad}\varphi(x,t)\| = e^{-t}$ for $t \geq 1$, which yields that $\chi_T\varphi \to \varphi$ in $H_0^1(N_1)$, as $T \to +\infty$. Since V is bounded, it follows that $\varphi \in H_V(N_1)$. Since $S_1\varphi = 0$, it is clear that φ is an eigenfunction of the Friedrichs extension of S_1, which yields that $\lambda_0(S_1) \leq 0$. From Proposition 2.4, since φ is positive, it follows that $\lambda_0(S_1) = 0$. ∎

Denote by $\tilde{\varphi}$ the lift of φ on N_2 and consider the renormalization $S_{\tilde{\varphi}}$ of S_2 with respect to $\tilde{\varphi}$. Let $g \in \operatorname{Lip}_c(M_2)$, such that g restricted on ∂M_2 is non-zero and smooth, and $h\colon [0,+\infty) \to \mathbb{R}$ be a compactly supported, smooth function, with $h(t) = 1$ for $t \leq 1/2$. Extend g in the glued ends $\partial M_2 \times [0,+\infty)$ by

$$g(x,t) := g(x)h(t). \tag{7}$$

It is clear that $g \in \operatorname{Lip}_c(N_2)$, and in the glued ends, we have

$$\operatorname{grad} g(x,t) = g(x)h'(t)\frac{\partial}{\partial t} + h(t)\operatorname{grad} g(x).$$

In particular, it follows that

$$\|\operatorname{grad} g(x,t)\|^2 = g^2(x)h'(t)^2 + h^2(t)\|\operatorname{grad} g(x)\|^2,$$

which yields that

$$\frac{\int_{N_2\smallsetminus M_2} \|\operatorname{grad} g\|^2 \tilde{\varphi}^2}{\int_{N_2\smallsetminus M_2} g^2\tilde{\varphi}^2} \;=\; \frac{\int_{\partial M_2}\int_0^{+\infty} \|\operatorname{grad} g\|^2\tilde{\varphi}^2}{\int_{\partial M_2}\int_0^{+\infty} g^2\tilde{\varphi}^2}$$

$$=\; \frac{\int_{\partial M_2} \|\operatorname{grad} g\|^2}{\int_{\partial M_2} g^2} + \frac{\int_0^{+\infty}(h')^2\phi^2}{\int_0^{\infty} h^2\phi^2}, \tag{8}$$

where we used that in the glued ends $\partial M_2 \times [0,+\infty)$, we have $\tilde{\varphi}(x,t) = \phi(t)$.

Proposition 4.7. *The renormalized operator $S_{\tilde{\varphi}}$ satisfies $\lambda_0(S_{\tilde{\varphi}}) = 0$, which yields that $\lambda_0(S_2) = \lambda_0(S_1)$.*

Proof: Let $\varepsilon > 0$. From Lemma 4.4, there exists $g \in \operatorname{Lip}_c(M_2)$, smooth on $M_2 \smallsetminus C_{t_0}$, for one $t_0 \in (0,\delta)$, not vanishing identically on the boundary, such that

$$\frac{\int_{M_2} \|\operatorname{grad} g\|^2}{\int_{M_2} g^2} < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{\int_{\partial M_2} \|\operatorname{grad} g\|^2}{\int_{\partial M_2} g^2} < \frac{\varepsilon}{2}.$$

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Let $T > 1$ and consider a compactly supported, smooth $h\colon [0, +\infty) \to \mathbb{R}$, with $h(t) = 1$ for $t \leq T$, $h(t) = 0$ for $t \geq T + 1$, and $|h'| \leq 2$. Extend $g \in \mathrm{Lip}_c(M_2)$ to the compactly supported $g \in \mathrm{Lip}_c(N_2)$ as in (7). Evidently, we have

$$\frac{\int_0^{+\infty} (h')^2 \phi^2}{\int_0^{+\infty} h^2 \phi^2} \leq 4\frac{\int_T^{T+1} e^{-2t} dt}{\int_1^T e^{-2t} dt} = 4\frac{1 - e^2}{e^2 - e^{2T}} < \frac{\varepsilon}{2},$$

for some sufficiently large T. From (8), it follows that

$$\frac{\int_{N_2 \smallsetminus M_2} \|\operatorname{grad} g\|^2 \tilde{\varphi}^2}{\int_{N_2 \smallsetminus M_2} g^2 \tilde{\varphi}^2} < \varepsilon.$$

Hence, we have

$$
\begin{aligned}
\mathcal{R}_{S_{\tilde{\varphi}}}(g) &= \frac{\int_{M_2} \|\operatorname{grad} g\|^2 + \int_{N_2 \smallsetminus M_2} \|\operatorname{grad} g\|^2 \tilde{\varphi}^2}{\int_{M_2} g^2 + \int_{N_2 \smallsetminus M_2} g^2 \tilde{\varphi}^2} \\
&\leq \max \left\{ \frac{\int_{M_2} \|\operatorname{grad} g\|^2}{\int_{M_2} g^2}, \frac{\int_{N_2 \smallsetminus M_2} \|\operatorname{grad} g\|^2 \tilde{\varphi}^2}{\int_{N_2 \smallsetminus M_2} g^2 \tilde{\varphi}^2} \right\} < \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, from Proposition 3.9, it follows that $\lambda_0(S_{\tilde{\varphi}}) = 0$ and in particular, $\lambda_0(S_2) = \lambda_0(S_1)$. ∎

Proof of Theorem 4.1: Consider a Riemannian metric on M_1 as in Lemma 4.3 and its lift on M_2. Glue cylinders along the boundaries and extend the covering $p\colon M_2 \to M_1$ to a Riemannian covering $p\colon N_2 \to N_1$ as above. From Remark 4.5, N_1 is complete, without boundary, and with Ricci curvature bounded from below. Consider the Schrödinger operator $S_1 = \Delta + V$ on N_1, as above, and its lift S_2 on N_2. From Remark 4.5, we have that V and $\operatorname{grad} V$ are bounded. From Lemma 4.6 and Proposition 4.7, we obtain that $\lambda_0(S_2) = \lambda_0(S_1) = 0$, and Remark 4.5 yields that $\lambda_0^{\mathrm{ess}}(S_1) \geq 1$. From Theorem 2.8, it follows that the covering $p\colon N_2 \to N_1$ is amenable, and so is the covering $p\colon M_2 \to M_1$. ∎

Proof of Theorem 1.1: Follows from Theorems 4.1 and 3.4. ∎

5 Arbitrary Riemannian coverings

In this section, we prove Theorem 1.2 and present some immediate consequences of it. As stated in the Introduction, we establish the following more general version of this theorem, involving manifolds with possibly non-empty boundary.

Theorem 5.1. *Let $p\colon M_2 \to M_1$ be a Riemannian covering. Let S_1 be a Schrödinger operator on M_1, with $\lambda_0^N(S_1) \notin \sigma_{\mathrm{ess}}^N(S_1)$, and S_2 its lift on M_2. Then $\lambda_0^N(S_2) = \lambda_0^N(S_1)$ if and only if the covering is amenable.*

The following lemma, which is a consequence of Theorem 4.1, is essential for the proof of this theorem.

Lemma 5.2. *Let $p\colon M_2 \to M_1$ be a non-amenable Riemannian covering. Let S_1 be a Schrödinger operator on M_1, with $\lambda_0^N(S_1)$ being an eigenvalue of S_1^N, and S_2 its lift on M_2. If $\lambda_0^N(S_2) = \lambda_0^N(S_1)$, then there exists a compact $K \subset M_1$ with non-empty interior, and $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M_2)$, with $\|f_n\|_{L^2(M_2)} = 1$, $\operatorname{supp} f_n \cap p^{-1}(K) = \emptyset$, for any $n \in \mathbb{N}$, and $\mathcal{R}_{S_2}(f_n) \to \lambda_0^N(S_2)$.*

Proof: If M_1 has non-empty boundary, then we denote by ν_i the inward pointing normal to ∂M_i, $i = 1, 2$. From Proposition 2.7, since $p\colon M_2 \to M_1$ is non-amenable, there exists a smoothly bounded, compact domain K', with non-empty interior, such that the covering $p\colon p^{-1}(K') \to K'$ is non-amenable, where $p^{-1}(K')$ may be non-connected. From Theorem 4.1, it follows that $\lambda_0^N(p^{-1}(K')) > 0$.

Since $\lambda_0^N(S_1)$ is an eigenvalue of S_1^N, from Proposition 3.7, there exists a positive function $\varphi \in C^\infty(M_1)$, with $S_1\varphi = \lambda_0^N(S_1)\varphi$ and $\nu_1(\varphi) = 0$ on ∂M_1 (if non-empty). Consider the lift $\tilde\varphi$ of φ on M_2 and the renormalization $S_{\tilde\varphi}$ of S_2 with respect to $\tilde\varphi$. Since $\lambda_0^N(S_2) = \lambda_0^N(S_1)$, from Propositions 3.9 and 3.10, it follows that

$$0 = \lambda_0(S_{\tilde\varphi}) = \inf_f \frac{\int_{M_2} \|\operatorname{grad} f\|^2 \tilde\varphi^2}{\int_{M_2} f^2 \tilde\varphi^2},$$

where the infimum is taken over all non-zero $f \in C_c^\infty(M_2)$, with $\nu_2(f) = 0$ on ∂M_2 (if non-empty). In particular, there exists $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M_2)$, with $\|f_n\|_{L^2_{\tilde\varphi}(M_2)} = 1$, $\mathcal{R}_{S_{\tilde\varphi}}(f_n) \to 0$ and $\nu_2(f_n) = 0$ on ∂M_2 (if non-empty).

Since φ is smooth and positive and K' is compact, there exist $c_1, c_2 > 0$, such that $c_1 \le \varphi \le c_2$ in K'. From Proposition 3.2, it follows that

$$\frac{\int_{p^{-1}(K')} \|\operatorname{grad} f\|^2 \tilde\varphi^2}{\int_{p^{-1}(K')} f^2 \tilde\varphi^2} \ge \frac{c_1^2}{c_2^2} \lambda_0^N(p^{-1}(K')) > 0,$$

for any $f \in C_c^\infty(p^{-1}(K')) \smallsetminus \{0\}$. Since $\|f_n\|_{L^2_{\tilde\varphi}(M_2)} = 1$ and $\mathcal{R}_{S_{\tilde\varphi}}(f_n) \to 0$, it follows that

$$\int_{p^{-1}(K')} f_n^2 \tilde\varphi^2 \to 0 \text{ and } \int_{M_2 \smallsetminus p^{-1}(K')} f_n^2 \tilde\varphi^2 \to 1.$$

Let $K \subset M_2^\circ$ be a compact set, with non-empty interior, contained in the interior of K'. Let $\chi \in C_c^\infty(M_1)$, with $\chi = 1$ in a neighborhood of K, and $\operatorname{supp} \chi \subset K' \cap M_2^\circ$. Consider the lift $\tilde\chi$ of χ on M_2, and let $g_n := (1 - \tilde\chi)f_n \in C_c^\infty(M_2)$. It is clear that if M_1 has non-empty boundary, then $\nu_2(g_n) = 0$ on ∂M_2. Moreover, we have

$$\|g_n\|_{L^2_{\tilde\varphi}(M_2)}^2 = \int_{p^{-1}(K')} (1 - \tilde\chi)^2 f_n^2 \tilde\varphi^2 + \int_{M_2 \smallsetminus p^{-1}(K')} f_n^2 \tilde\varphi^2 \to 1$$

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and

$$\int_{M_2} \|\operatorname{grad} g_n\|^2 \tilde{\varphi}^2 \;\leq\; 2 \int_{p^{-1}(K')} (f_n^2 \|\operatorname{grad} \tilde{\chi}\|^2 + (1 - \tilde{\chi})^2 \|\operatorname{grad} f_n\|^2) \tilde{\varphi}^2$$
$$+ \int_{M_2 \smallsetminus p^{-1}(K')} \|\operatorname{grad} f_n\|^2 \tilde{\varphi}^2 \to 0.$$

Therefore, $\mathcal{R}_{S_{\tilde{\varphi}}}(g_n) \to 0$ and $\operatorname{supp} g_n \cap p^{-1}(K) = \emptyset$. We may normalize g_n in $L^2_{\tilde{\varphi}}(M)$, so that $\|g_n\|_{L^2_{\tilde{\varphi}}(M_2)} = 1$, for any $n \in \mathbb{N}$.

Consider $h_n := \tilde{\varphi} g_n \in C^\infty_c(M_2)$. If M_2 has non-empty boundary, since $\nu_2(\tilde{\varphi}) = 0$ and $\nu_2(g_n) = 0$, it follows that $\nu_2(h_n) = 0$ on ∂M_2. Evidently, $\|h_n\|_{L^2(M_2)} = \|g_n\|_{L^2_{\tilde{\varphi}}(M_2)} = 1$. Moreover, from the definition of the renormalized Schrödinger operator, it is clear that

$$\begin{aligned}
\mathcal{R}_{S_2}(h_n) &= \langle S_2 h_n, h_n \rangle_{L^2(M_2)} = \langle S_{\tilde{\varphi}} g_n, g_n \rangle_{L^2_{\tilde{\varphi}}(M_2)} + \lambda_0^N(S_2) \\
&= \mathcal{R}_{S_{\tilde{\varphi}}}(g_n) + \lambda_0^N(S_2) \to \lambda_0^N(S_2),
\end{aligned}$$

which completes the proof. ∎

Proof of Theorem 5.1: From Theorem 3.4, if the covering is infinite sheeted and amenable, then $\lambda_0^N(S_1) = \lambda_0^N(S_2)$. If the covering is finite sheeted, then for $f \in C^\infty_c(M_1)$, we have that $f \circ p \in C^\infty_c(M_2)$, and the equality of the bottoms follows from Proposition 3.2 and Corollary 3.3. Hence, it remains to prove the converse implication.

Assume to the contrary that the covering is non-amenable. Since $\lambda_0^N(S_2) = \lambda_0^N(S_1) \notin \sigma_{\mathrm{ess}}^N(S_1)$, from Lemma 5.2 there exists a compact $K \subset M_1$ with non-empty interior, and $(f_n)_{n \in \mathbb{N}} \subset C^\infty_c(M_2)$, with $\|f_n\|_{L^2(M_2)} = 1$, $\operatorname{supp} f_n \cap p^{-1}(K) = \emptyset$, for any $n \in \mathbb{N}$, and $\mathcal{R}_{S_2}(f_n) \to \lambda_0^N(S_2)$. For $n \in \mathbb{N}$, consider the pushdown g_n of f_n, defined by

$$g_n(z) := \Big(\sum_{y \in p^{-1}(z)} f_n(y)^2 \Big)^{1/2},$$

for any $z \in M_1$. Then $g_n \in \operatorname{Lip}_c(M_1)$, $\|g_n\|_{L^2(M_1)} = 1$ and $\mathcal{R}_{S_1}(g_n) \leq \mathcal{R}_{S_2}(f_n)$, for any $n \in \mathbb{N}$ (cf. [2, Section 4]). From Proposition 3.2, since $\lambda_0^N(S_2) = \lambda_0^N(S_1)$, it follows that $\mathcal{R}_{S_1}(g_n) \to \lambda_0^N(S_1)$. From Proposition 3.8, since $\lambda_0^N(S_1) \notin \sigma_{\mathrm{ess}}^N(S_1)$ and $\operatorname{supp} g_n \cap K = \emptyset$, this is a contradiction. Hence, the covering is amenable. ∎

Proof of Theorem 1.2: Follows from Theorem 5.1, since the manifolds involved may have empty boundary. ∎

Remark 5.3. In Theorem 1.2, the manifolds do not have to be complete. Therefore, from Remark 2.3, we obtain the corresponding result for the Dirichlet spectrum of Schrödinger operators on manifolds with boundary.

Corollary 5.4. *Let $p \colon M_2 \to M_1$ be a Riemannian covering, with M_1 compact. Then the covering is amenable if and only if it preserves the bottom of the Dirichlet/Neumann spectrum of some/any Schrödinger operator.*

Proof: Follows from Theorem 5.1 and Remark 5.3, since the Dirichlet and the Neumann spectrum of a Schrödinger operator on a compact manifold is discrete. ∎

The next example shows that the assumption $\lambda_0(S_1) \notin \sigma_{\text{ess}}(S_1)$ in Theorem 1.2 cannot be replaced with $\lambda_0(S_1)$ being an eigenvalue of the Friedrichs extension of S_1.

Example 5.5. Let M_1 be a two dimensional torus with a cusp attached, endowed with a Riemannian metric, such that M_1 is complete and outside a compact set, the cusp is the surface of revolution generated by $1/t^2$, with $t \geq 1$. Since M_1 has finite volume, it follows that $\lambda_0(M_1) = 0$ and constant functions are $\lambda_0(M_1)$-eigenfunctions of the Friedrichs extension of the Laplacian on M_1. Let x be a point of the torus and consider the non-negative quantity

$$\mu := -\lim_{r \to +\infty} \frac{1}{r} \ln(\text{Vol}(M_1) - \text{Vol}(B(x, r))) \leq -\lim_{r \to +\infty} \frac{1}{r} \ln(2\pi \int_{r+1}^{+\infty} \frac{1}{t^2} dt) = 0.$$

From [5, Theorem 1], it follows that $\lambda_0^{\text{ess}}(M_1) = 0$. Consider the universal covering $p \colon M_2 \to M_1$. Since $\pi_1(M_1)$ is the free group with two generators, it follows that p is non-amenable. Since the fundamental group of the cusp is amenable, from [14, Corollary 1.6], it follows that $\lambda_0(M_2) = 0$.

It is clear that Theorem 1.2 is more general than the results of $[1, 6, 15]$. For sake of completeness, we present an example demonstrating this fact. Let $p \colon M_2 \to M_1$ be a Riemannian covering, with M_1 non-compact, complete, without boundary, and with $\sigma_{\text{ess}}(M_1) = \emptyset$. Then, from Theorem 1.2, it follows that $\lambda_0(M_2) = \lambda_0(M_1)$ if and only if the covering is amenable. It is worth to point out that since we do not require the covering to be normal, the results of $[6, 15]$ cannot be applied in this case. Moreover, from [10, Theorem 3.1], it follows that the Ricci curvature of M_1 is not bounded from below. Hence, also the result of [1] cannot be applied in this case.

6 An application

The aim of this section is to prove the following proposition, which was established for the Laplacian on manifolds without boundary in [1].

Proposition 6.1. *Let $p \colon M_2 \to M_1$ be an infinite sheeted Riemannian covering. If $\lambda_0^N(S_1) = \lambda_0^N(S_2)$, then $\lambda_0^N(S_2) \in \sigma_{\text{ess}}^N(S_2)$.*

The main point of this proposition is that the covering is not required to be normal (or to have infinite deck transformations group), since in this case, according to [14, Corollary 1.4], the spectrum of S_2^N coincides with its essential spectrum. It is worth to point out that the manifolds in this proposition may have empty boundary. Moreover, since they may be non-complete, from Remark 2.3, the analogous statement holds for the Dirichlet spectrum of Schrödinger operators on manifolds with boundary.

Proposition 6.2. *Let $S = \Delta + V$ be a Schrödinger operator on a Riemannian manifold M and $(f_n)_{n \in \mathbb{N}} \subset \mathrm{Lip}_c(M)$, with $\|f_n\|_{L^2(M)} = 1$ and $\mathcal{R}_S(f_n) \to \lambda_0^N(S)$. If $\lambda_0^N(S)$ is not an eigenvalue of S^N, then there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$, such that $f_{n_k} \rightharpoonup 0$ in $L^2(M)$.*

Proof: From Proposition 3.1, there exists $(f'_n) \in C_c^\infty(M) \cap \mathcal{D}(S^N)$, with $\|f'_n\|_{L^2(M)} = 1$ and $\|f_n - f'_n\|_{H_V(M)} \le 1/n$, for any $n \in \mathbb{N}$, where $H_V(M)$ is the space defined in Subsection 2.1. It is clear that $\mathcal{R}_S(f'_n) \to \lambda_0^N(S)$ and it suffices to prove the statement for $(f'_n)_{n \in \mathbb{N}}$. From the Spectral Theorem (cf. [18, Chapter 8]), there exists a measure space X, such that $L^2(M)$ is isometrically isomorphic to $L^2(X)$, and under this identification, S^N corresponds to a multiplication operator with a measurable function $f \colon X \to \mathbb{R}$; that is, an operator of the form $\mu_f \colon \mathcal{D}(\mu_f) \subset L^2(X) \to L^2(X)$, with $\mathcal{D}(\mu_f) := \{g \in L^2(X) : fg \in L^2(X)\}$ and $\mu_f(g) = fg$, for any $g \in \mathcal{D}(\mu_f)$. The spectrum of S^N coincides with the essential range of f and in particular, $f \ge \lambda_0^N(S)$ almost everywhere.

Let $(g_n)_{n \in \mathbb{N}} \subset \mathcal{D}(\mu_f)$ be the sequence corresponding to $(f'_n)_{n \in \mathbb{N}}$ under this identification. Since $\|g_n\|_{L^2(X)} = 1$, after passing to a subsequence, we have that $g_n \rightharpoonup g$ in $L^2(X)$, for some $g \in L^2(X)$. It is clear that

$$\int_X (f - \lambda_0^N(S))g_n^2 = \langle \mu_f g_n, g_n \rangle_{L^2(X)} - \lambda_0^N(S) = \mathcal{R}_S(f'_n) - \lambda_0^N(S) \to 0.$$

For $\varepsilon > 0$, consider the measurable set $A_\varepsilon := \{f \ge \lambda_0^N(S) + \varepsilon\}$. Evidently, we have

$$\int_{A_\varepsilon} g_n^2 \le \frac{1}{\varepsilon} \int_{A_\varepsilon} (f - \lambda_0(S^N))g_n^2 \to 0.$$

Since $g_n \rightharpoonup g$ in $L^2(X)$, this yields that $g = 0$ almost everywhere in A_ε. In particular, $g = 0$ almost everywhere in $X \smallsetminus f^{-1}(\{\lambda_0^N(S)\})$, which yields that $\mu_f g = \lambda_0^N(S)g$. Since $\lambda_0^N(S)$ is not an eigenvalue of S^N, it follows that $g = 0$. Therefore, $g_n \rightharpoonup 0$ in $L^2(X)$, which yields that $f'_n \rightharpoonup 0$ in $L^2(M)$. ∎

Lemma 6.3. *Let $p \colon M_2 \to M_1$ be a Riemannian covering. If $\lambda_0^N(S_2) = \lambda_0^N(S_1) \notin \sigma_{\mathrm{ess}}^N(S_2)$, then $\lambda_0^N(S_1)$ is an eigenvalue of S_1^N.*

Proof: Assume to the contrary that $\lambda_0^N(S_1)$ is not an eigenvalue of S_1^N. From Proposition 3.7, there exists a square-integrable, $\lambda_0^N(S_2)$-eigenfunction φ of S_2^N, which is smooth and positive in M_2. Without loss of generality, we may assume that $\|\varphi\|_{L^2(M_2)} = 1$. Since $\varphi \in H_{V \circ p}(M_2)$, there exists $(f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M_2)$, with $\|f_n\|_{L^2(M_2)} = 1$ and $f_n \to \varphi$ in $H_{V \circ p}(M_2)$, where $H_{V \circ p}(M_2)$ is the space defined in Subsection 2.1. Evidently, we have that $\mathcal{R}_{S_2}(f_n) \to \lambda_0^N(S_2)$.

Consider the pushdowns

$$g_n(z) := \Big(\sum_{y \in p^{-1}(z)} f_n(y)^2 \Big)^{1/2}.$$

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on M_1, with $n \in \mathbb{N}$. Then $g_n \in \mathrm{Lip}_c(M_1)$, $\|g_n\|_{L^2(M_1)} = 1$ and $\mathcal{R}_{S_1}(g_n) \leq \mathcal{R}_{S_2}(f_n)$, for any $n \in \mathbb{N}$ (cf. [2, Section 4]). From Proposition 3.2, since $\lambda_0^N(S_1) = \lambda_0^N(S_2)$, it follows that $\mathcal{R}_{S_1}(g_n) \to \lambda_0^N(S_1)$. Since $\lambda_0^N(S_1)$ is not an eigenvalue of S_1^N, from Proposition 6.2, after passing to a subsequence, we have that $g_n \rightharpoonup 0$ in $L^2(M_1)$.

Consider a non-negative $\chi_2 \in C_c^\infty(M_2) \smallsetminus \{0\}$, and its pushdown $\chi_1 \in \mathrm{Lip}_c(M_1)$ on M_1. Then

$$
\begin{aligned}
\langle \chi_2, f_n \rangle_{L^2(M_2)} &= \int_{M_1} \sum_{y \in p^{-1}(z)} \chi_2(y) f_n(y) dz \\
&\leq \int_{M_1} \big(\sum_{y \in p^{-1}(z)} \chi_2(y)^2 \big)^{1/2} \big(\sum_{y \in p^{-1}(z)} f_n(y)^2 \big)^{1/2} dz \\
&= \langle \chi_1, g_n \rangle_{L^2(M_1)}.
\end{aligned}
$$

This is a contradiction, since $\langle \chi_1, g_n \rangle_{L^2(M_1)} \to 0$ and $\langle \chi_2, f_n \rangle_{L^2(M_2)} \to \int_{M_2} \chi_2 \varphi > 0$. Therefore, $\lambda_0^N(S_1)$ is an eigenvalue of S_1^N. ∎

Proof of Proposition 6.1: If the covering is amenable, then the claim follows from Theorem 3.4. Hence, it remains to prove the statement for p non-amenable. Assume to the contrary that $\lambda_0^N(S_2) \notin \sigma_{\mathrm{ess}}^N(S_2)$. From Lemma 6.3, it follows that $\lambda_0^N(S_1)$ is an eigenvalue of S_1^N. Since $\lambda_0^N(S_2) = \lambda_0^N(S_1)$, from Lemma 5.2, there exists a compact set $K \subset M_1$ with non-empty interior, and $(f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M_2) \smallsetminus \{0\}$, such that $\mathcal{R}_{S_2}(f_n) \to \lambda_0^N(S_2)$ and $\mathrm{supp}\, f_n \cap p^{-1}(K) = \emptyset$, for any $n \in \mathbb{N}$. From Proposition 3.8, since $\lambda_0^N(S_2) \notin \sigma_{\mathrm{ess}}^N(S_2)$ and $p^{-1}(K)$ contains compact sets of positive measure, this is a contradiction. ∎

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