Commutation relation of Hecke operators for Arakawa lifting

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Abstract

T. Arakawa, in his unpublished note, constructed and studied a theta lifting from elliptic cusp forms to automorphic forms on the real symplectic group $Sp(1,q)_{\mathbb{R}}$ of signature $(1+, q-)$. The second named author proved that such a lifting provides bounded (or cuspidal) automorphic forms on $Sp(1,q)_{\mathbb{R}}$ generating quaternionic discrete series. In this paper, restricting ourselves to the case of $q = 1$, we reformulate Arakawa's theta lifting as a theta correspondence for similitude groups $(GL_2 \times B^\times) \times GSp(1,1)$ in the adelic setting and give a commutation relation of Hecke operators satisfied by the lifting. Here $B^\times$ denotes the multiplicative group of a definite quaternion algebra B over $\mathbb{Q}$. As an application we show that the theta lift $\mathcal{L}(f, f')$ of a Hecke eigenform (f, f') on $GL_2 \times B^\times$ is also a Hecke eigenform on $GSp(1,1)$. We furthermore provide all finite local factors of the spinor L-function attached to $\mathcal{L}(f, f')$ in terms of Hecke eigenvalues of (f, f').

1 Introduction.

The prototype of our study in this paper is the classical work [E] by M. Eichler on the commutation relation of Hecke operators for theta series associated with spherical polynomials on a definite quaternion algebra over $\mathbb{Q}$. After this, several generalizations of it were given. For example, H. Yoshida constructed a theta lifting from a pair of automorphic forms on a multiplicative group of a definite quaternion algebra to holomorphic Siegel modular forms of degree two, and gave a commutation relation for his lift. S. Kudla [Ku] considered such a relation for a theta lifting from elliptic cusp forms to holomorphic automorphic forms on $SU(2,1)$. Moreover we note that S. Rallis [Ra] investigated in great generality a commutation relation via the Weil representation for symplectic-orthogonal dual pairs. Our concern here is a theta lifting from elliptic cusp forms to automorphic forms on the real symplectic group $Sp(1,q)_{\mathbb{R}}$ of signature $(1+, q-)$ originally formulated by T. Arakawa. We study this lifting for the case of $q = 1$ along the same line as [Y] and [Ku].

Let us recall that Arakawa formulated the theta lifting mentioned above by considering the restriction of a theta correspondence of $SL_2(\mathbb{R}) \times SO(4, 4q)$ to $SL_2(\mathbb{R}) \times Sp(1,q)_{\mathbb{R}}$ (cf. [Ar-1], [N-1] and [N-3]). We henceforth confine ourselves to the case of $q = 1$. It turned out that

Arakawa's formulation is not appropriate for proving a commutation relation, since we do not have sufficient Hecke operators. Following T. Ikeda's suggestion, we formulate our lift as a theta correspondence between $(GL_2 \times B^\times)$ and $GSp(1,1)$, where B is a definite quaternion algebra over $\mathbb{Q}$. This amounts to the same as taking a certain *average* of original Arakawa's lifts over the ideal classes of B. In this setting, we have Hecke operators enough to show a good commutation relation. On the other hand, we note that our lift can be viewed as a theta correspondence of similitude groups. For references in this direction see [G], [H-K], [Ro], [Sz] and [W] etc.

We now explain more precisely our reformulation of the lifting, which is given in the adelic setting. Let $\kappa > 6$ be an even integer and let D be a divisor of the discriminant d_B of B. We denote by $S_\kappa(D)$ the space of elliptic cusp forms on $GL_2(\mathbb{A})$ of weight κ and level D, and let $\mathcal{A}_\kappa$ be the space of automorphic forms on $B_{\mathbb{A}}^\times$ (for definitions of $S_\kappa(D)$ and $\mathcal{A}_\kappa$ see Definition 2.2). Furthermore, using a metaplectic representation of $GSp(1,1)_{\mathbb{A}} \times GL_2(\mathbb{A}) \times B_{\mathbb{A}}^\times$ (cf. §3, §4.1), we construct a theta kernel θ^κ on $GSp(1,1)_{\mathbb{A}} \times GL_2(\mathbb{A}) \times B_{\mathbb{A}}^\times$ under a special choice of a test function (cf. (4.1)). For $(f, f') \in S_\kappa(D) \times \mathcal{A}_\kappa$ we then construct an automorphic form $\mathcal{L}(f, f')$ on $GSp(1,1)_{\mathbb{A}}$ by integrating (f, f') against θ^κ (cf. (4.2)). This $\mathcal{L}(f, f')$ belongs to the space $\mathcal{S}_\kappa$ of bounded (or cuspidal) automorphic forms on $GSp(1,1)_{\mathbb{A}}$ given in Definition 2.1 (cf. Theorem 4.1), which turn out to generate at the infinite place a quaternionic discrete series in the sense of Gross and Wallach [G-W] (cf. [N-2, Theorem 8.7]).

Our main result is a formula for Hecke eigenvalues of $\mathcal{L}(f, f')$ stated as Theorem 5.1. For all non-Archimedean primes we provide such formula in terms of Hecke eigenvalues of f and f'. This follows from our formula for the commutation relation of Hecke operators in Proposition 6.1 and Proposition 6.2. Then we discuss an application of this formula to the spinor L-functions of $\mathcal{L}(f, f')$'s. We define an Euler factor of the spinor L-function at a prime $p \nmid d_B$ (resp. $p | d_B$) using the formula for the denominator of the Hecke series by G. Shimura [Shim-1, Theorem 2] (resp. T. Hina and T. Sugano [H-S, §4], [Su, (1-34)]). To be precise we give our formulas of the spinor L-functions under some normalization of the Hecke eigenvalues. Among such formulas, the case $D = 1$ is the most interesting. In fact, if we assume that f and f' are Hecke eigenforms, the spinor L-function $L(\mathcal{L}(f, f'), spin, s)$ of $\mathcal{L}(f, f')$ for that case admits the following simple decomposition (cf. Corollary 5.2)

$$L(\mathcal{L}(f, f'), spin, s) = L(\bar{f}, s) L^{d_B}(f', s),$$

where $L(\bar{f}, s)$ (resp. $L^{d_B}(f', s)$) denotes Hecke's classical L-function for $\bar{f}$ (some partial L-function for f' whose Euler factors range only over $p \nmid d_B$).

This paper is organized as follows. In §2 we define the automorphic forms we need after giving basic notations. In §3 we introduce a metaplectic representation of $GSp(1,1) \times GL_2 \times B^\times$ over local fields. Then we define a global metaplectic representation of the adele group and provide the adelic reformulation of the Arakawa lifting for the case of $q = 1$ in §4. The section 5 is devoted to the statement of our main results, i.e., Hecke eigenvalues and spinor

L-functions for the lifting. In §6 we state our result on the commutation relation of Hecke operators, from which our main results are deduced immediately. In §7 and §8 we prove the commutation relation. More precisely the case of unramified finite places (resp. ramified finite places) is considered in §7 (resp. §8).

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Notation

For an algebraic group $\mathcal{G}$ over $\mathbb{Q}$, $\mathcal{G}_v$ stands for the group of $\mathbb{Q}_v$-points of $\mathcal{G}$, where $\mathbb{Q}_v$ denotes the p-adic field (resp. the field of real numbers) when $v = p$ is a finite prime (resp. $v = \infty$). By $\mathcal{G}_{\mathbb{A}}$ (resp. $\mathcal{G}_{\mathbb{A},f}$), we denote the adelization of $\mathcal{G}$ (resp. the group of finite adeles in $\mathcal{G}_{\mathbb{A}}$). Let ψ be the additive character of $\mathbb{Q}_{\mathbb{A}}/\mathbb{Q}$ such that $\psi(x_\infty) = \mathbf{e}(x_\infty)$ for $x_\infty \in \mathbb{R}$, where we put $\mathbf{e}(z) = \exp(2\pi i z)$ for $z \in \mathbb{C}$. We denote by ψ_v the restriction of ψ to $\mathbb{Q}_v$ for a prime v of $\mathbb{Q}$.

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Let B be a definite quaternion algebra over $\mathbb{Q}$. In what follows, we fix an identification between $B_\infty := B \otimes_{\mathbb{Q}} \mathbb{R}$ and the Hamilton quaternion algebra $\mathbb{H}$, and an embedding $\mathbb{H} \hookrightarrow M_2(\mathbb{C})$. Let $B \ni b \mapsto \bar{b} \in B$ be the main involution of B, and put $\mathrm{tr}(b) := b + \bar{b}$ and $n(b) := b\bar{b}$ for $b \in B$. Let $B^\times := B \setminus \{0\}$ be the multiplicative group of B. The center $Z(B^\times)$ of $B^\times$ is $\mathbb{Q}^\times \cdot 1$. Let d_B be the discriminant of B. By definition, d_B is the product of finite primes p such that $B_p := B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division algebra.

We let $G = GSp(1,1)$ be an algebraic group over $\mathbb{Q}$ defined by

$$G_{\mathbb{Q}} = \{g \in M_2(B) \mid {}^t\bar{g}Qg = \nu(g)Q,\ \nu(g) \in \mathbb{Q}^\times\},$$

where $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Denote by Z_G the center of G.

The Lie group $G^1_\infty := \{g \in G_\infty \mid \nu(g) = 1\}$ acts on the hyperbolic 4-space $\mathcal{X} := \{z \in \mathbb{H} \mid \mathrm{tr}(z) > 0\}$ by linear fractional transformations

$$g \cdot z := (az+b)(cz+d)^{-1}, \quad (g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G^1_\infty,\ z \in \mathcal{X}).$$

Let $\mu : G^1_\infty \times \mathcal{X} \to \mathbb{H}^\times$ be the automorphy factor given by $\mu(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z) := cz + d$. The stabilizer subgroup K_∞ of $z_0 := 1 \in \mathcal{X}$ in G^1_∞ is a maximal compact subgroup of G^1_∞, which is isomorphic to $Sp^*(1) \times Sp^*(1)$, where $Sp^*(1) := \{z \in \mathbb{H} \mid n(z) = 1\}$.

Let κ be a positive integer. Denote by $(\sigma_\kappa, V_\kappa)$ the representation of $\mathbb{H}$ given as

$$\mathbb{H} \hookrightarrow M_2(\mathbb{C}) \to \mathrm{End}(V_\kappa),$$

where the second arrow indicates the κ-th symmetric power representation of $M_2(\mathbb{C})$. Then

$$\tau_\kappa(k_\infty) := \sigma_\kappa(\mu(k_\infty, z_0)), \quad (k_\infty \in K_\infty)$$

gives rise to an irreducible representation of K_∞ of dimension $\kappa + 1$.

Define $\omega_\kappa : G_\infty^1 \to \mathrm{End}(V_\kappa)$ by

$$\omega_\kappa(g) := \sigma_\kappa(D(g))^{-1} n(D(g))^{-1}, \quad (g \in G_\infty^1),$$

where $D(g) := \frac{1}{2}(g \cdot z_0 + 1)\mu(g, z_0)$. It is known that ω_κ is a matrix coefficient of the discrete series representation with minimal K_∞-type (τ_κ, V_κ) (cf. [Ar-2, §2.6]). That discrete series is a quaternionic discrete series in the sense of B. Gross and N. Wallach [G-W]. We note that ω_κ is integrable if $\kappa > 4$.

Throughout the paper, we fix a maximal order $\mathcal{O}$ of B. We also fix a two-sided ideal $\mathfrak{A}$ of $\mathcal{O}$ satisfying the following conditions:

(i) If $p \not| d_B$, then $\mathfrak{A}_p = \mathcal{O}_p$.

(ii) If $p | d_B$, then $\mathfrak{A}_p = \mathfrak{P}_p^{e_p}$ with $e_p \in \{0, 1\}$, where $\mathfrak{P}_p$ is the maximal ideal of $\mathcal{O}_p$.

We set

$$D = \prod_{p | d_B, e_p = 0} p.$$

Note that $D = 1$ if and only if $e_p = 1$ for any $p | d_B$. Let $L := {}^t(\mathcal{O} \oplus \mathfrak{A}^{-1})$, which is a maximal lattice of $B^{\oplus 2}$. For a finite prime p, $K_p = \{k \in G_p \mid kL_p = L_p\}$ is a maximal compact subgroup of G_p, where $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We set $K_f := \prod_{p<\infty} K_p$.

Definition 2.1. For an even integer $\kappa > 4$, let $\mathcal{S}_\kappa$ be the space of smooth functions on $F : G_{\mathbb{A}} \to V_\kappa$ satisfying the following conditions:

1. $F(z\gamma g k_f k_\infty) = \tau_\kappa(k_\infty)^{-1} F(g) \quad \forall (z, \gamma, g, k_f, k_\infty) \in Z_{G,\mathbb{A}} \times G_{\mathbb{Q}} \times G_{\mathbb{A}} \times K_f \times K_\infty$,
2. F is bounded,
3. $c_\kappa \int_{G_\infty^1} \omega_\kappa(h_\infty^{-1} g_\infty) F(g_f h_\infty) dh_\infty = F(g_f g_\infty)$ for any fixed $(g_f, g_\infty) \in G_{\mathbb{A},f} \times G_\infty$,

where $c_\kappa := 2^{-4}\pi^{-2}\kappa(\kappa - 1)$.

Here we remark that this automorphic form has been verified to be cuspidal (cf. [Ar-2, Proposition 3.1]) and generate a quaternionic discrete series at the infinite place (cf. [N-2, Theorem 8.7]).

Next let H and H' be algebraic groups over $\mathbb{Q}$ defined by $H_{\mathbb{Q}} = GL_2(\mathbb{Q})$ and $H'_{\mathbb{Q}} = B^\times$ respectively, and denote by Z_H and $Z_{H'}$ the center of H and H' respectively. We define an action of $SL_2(\mathbb{R})$ on the complex upper half plane $\mathfrak{h} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ as usual. Let $U_\infty := \{h \in SL_2(\mathbb{R}) \mid h \cdot i = i\} = SO(2)$ and $U'_\infty := \{h' \in \mathbb{H} \mid n(h') = 1\} = Sp^*(1)$. Moreover, we put $U_f = \prod_{p<\infty} U_p$ and $U'_f = \prod_{p<\infty} U'_p$, where $U_p := \{u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z}_p) \mid c \in D\mathbb{Z}_p\}$ and $U'_p := \mathcal{O}_p^\times$.

Definition 2.2. (1) Let $S_\kappa(D)$ be the space of smooth functions f on $H_{\mathbb{A}}$ satisfying the following conditions:

1. $f(z\gamma h u_f u_\infty) = j(u_\infty, i)^{-\kappa} f(h) \quad \forall (z, \gamma, h, u_f, u_\infty) \in Z_{H,\mathbb{A}} \times H_{\mathbb{Q}} \times H_{\mathbb{A}} \times U_f \times U_\infty$,
2. For any fixed $h_f \in H_{\mathbb{A},f}$, $\mathfrak{h} \ni h_\infty \cdot i \mapsto j(h_\infty, i)^\kappa f(h_f h_\infty)$ is holomorphic for $h_\infty \in SL_2(\mathbb{R})$,
3. f is bounded,

where $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) := c\tau + d$ denotes the standard $\mathbb{C}$-valued automorphy factor of $SL_2(\mathbb{R}) \times \mathfrak{h}$.
(2) Furthermore, $\mathcal{A}_\kappa$ stands for the space of smooth V_κ-valued functions f' on $H'_{\mathbb{A}}$ such that

$$f'(z'\gamma' h' u'_f u'_\infty) = \sigma_\kappa(u'_\infty)^{-1} f(h') \quad \forall (z', \gamma', h', u'_f, u'_\infty) \in Z_{H',\mathbb{A}} \times H'_{\mathbb{Q}} \times H'_{\mathbb{A}} \times U'_f \times U'_\infty.$$

3 Metaplectic representation

In this section, we fix a prime v of $\mathbb{Q}$. When $v = p$ is a finite prime (resp. $v = \infty$), $|*|_v$ denotes the p-adic valuation (resp. the usual absolute value for $\mathbb{R}$). For $X = \begin{pmatrix} x \\ y \end{pmatrix} \in B_v^{\oplus 2}$, we put $X^* := (\bar{x}, \bar{y})$. For a finite prime p, let $\mathbb{V}_p$ be the space of functions on $B_p^{\oplus 2} \times \mathbb{Q}_p^\times$ generated by $\varphi_1(X)\varphi_2(t)$, where φ_1 (resp. φ_2) is a locally constant and compactly supported function on $B_p^{\oplus 2}$ (resp. $\mathbb{Q}_p^\times$). We also let $\mathbb{V}_\infty$ be the space of smooth functions φ on $B_\infty^{\oplus 2} \times \mathbb{Q}_\infty^\times = \mathbb{H}^{\oplus 2} \times \mathbb{R}^\times$ such that, for any fixed $t \in \mathbb{R}^\times$, $X \mapsto \varphi(X, t)$ is rapidly decreasing on $\mathbb{H}^{\oplus 2}$.

Lemma 3.1. *There exists a smooth representation $r = r_v$ of $G_v \times H_v \times H'_v$ on $\mathbb{V}_v$ given as follows:*
For $\varphi \in \mathbb{V}_v$, $X \in B_v^{\oplus 2}$ and $t \in \mathbb{Q}_v^\times$,

$$r(g,1,1)\varphi(X,t) = |\nu(g)|_v^{-\frac{3}{2}}\varphi(g^{-1}X,\nu(g)t), \quad (g \in G_v), \tag{3.1}$$

$$r(1,\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right),1)\varphi(X,t) = \psi_v(\frac{bt}{2}\operatorname{tr}(X^*QX))\varphi(X,t), \quad (b \in \mathbb{Q}_v), \tag{3.2}$$

$$r(1,\left(\begin{smallmatrix}a & 0\\ 0 & a'\end{smallmatrix}\right),1)\varphi(X,t) = |a|_v^{\frac{7}{2}}|a'|_v^{-\frac{1}{2}}\varphi(aX,(aa')^{-1}t), \quad (a,a' \in \mathbb{Q}_v^\times), \tag{3.3}$$

$$r(1,\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right),1)\varphi(X,t) = |t|_v^4\int_{B_v^{\oplus 2}}\psi_v(t\operatorname{tr}(Y^*QX))\varphi(Y,t)d_QY, \tag{3.4}$$

$$r(1,1,z)\varphi(X,t) = |n(z)|_v^{\frac{3}{2}}\varphi(Xz,n(z)^{-1}t), \quad (z \in B_v^\times). \tag{3.5}$$

Here d_QY is the Haar measure on $B_v^{\oplus 2}$ self-dual with respect to the pairing

$$B_v^{\oplus 2} \times B_v^{\oplus 2} \ni (Y,Y') \mapsto \psi_v(\operatorname{tr}(Y^*QY')).$$

We define a partial Fourier transform $\mathcal{I}$ by

$$\mathcal{I}\varphi(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},t) = \int_{B_v}\psi_v(-t\operatorname{tr}(\bar{y}x_1))\varphi(\begin{pmatrix} y \\ x_2 \end{pmatrix},t)dy, \quad (\varphi \in \mathbb{V}_v),$$

where dy is the Haar measure on B_v self-dual with respect to the pairing $B_v \times B_v \ni (x,y) \mapsto \psi_v(\operatorname{tr}(\bar{x}y))$. The proof of the following fact is straightforward and we omit it.

Lemma 3.2. *For $h \in H_v$, we have*

$$(\mathcal{I}\cdot r(1,h,1)\varphi)(X,t) = |\det h|_v^{-\frac{1}{2}}\mathcal{I}\varphi((\det h)\cdot h^{-1}X,(\det h)^{-1}t), \quad (\varphi \in \mathbb{V}_v,\ X \in B_v^{\oplus 2},\ t \in \mathbb{Q}_v^\times).$$

When $v = p < \infty$, we put

$$\varphi_{0,p}(X,t) := \operatorname{char}_{L_p}(X)\operatorname{char}_{\mathbb{Z}_p^\times}(t),$$

where $\operatorname{char}_{L_p}$ (resp. $\operatorname{char}_{\mathbb{Z}_p^\times}$) is the characteristic function of $L_p = {}^t\left(\mathcal{O}_p \oplus \mathfrak{A}_p^{-1}\right)$ (resp. $\mathbb{Z}_p^\times$). When $v = \infty$, we put

$$\varphi_{0,\infty}^\kappa(X,t) := \begin{cases} t^{\frac{\kappa+3}{2}}\sigma_\kappa((1,1)X)\mathbf{e}(\frac{it}{2}\operatorname{tr}(X^*X)) & (t > 0) \\ 0 & (t < 0) \end{cases}.$$

Lemma 3.3. *Let $v = p < \infty$. Then we have*

$$r(k_p,u_p,u'_p)\varphi_{0,p} = \varphi_{0,p}$$

for $k_p \in K_p$, $u_p \in U_p$ and $u'_p \in U'_p$.

Proof. This is verified by a direct calculation. When $p|D$ we need Lemma 3.2 to show the U_p-invariance. $\square$

Lemma 3.4. *Let $v = \infty$. Then we have*

$$r(k_\infty, u_\infty, u'_\infty)\varphi^\kappa_{0,\infty} = j(u_\infty, i)^{-\kappa}\tau_\kappa(k_\infty)^{-1} \cdot \varphi^\kappa_{0,\infty} \cdot \sigma_\kappa(u'_\infty)$$

for $k \in K_\infty$, $u \in U_\infty$ and $u' \in U'_\infty$.

Proof. The transformation law with respect to the U_∞-action follows immediately from [N-3, Lemma 3.8]. The other transformation laws are checked in a straightforward way. $\square$

4 Arakawa lift

4.1

Let $\mathbb{V}_\mathbb{A}$ be the restricted tensor product of $\mathbb{V}_v$ with respect to $\{\varphi_{0,p}\}_{p<\infty}$. By $r_\mathbb{A}$ we denote a smooth representation of $G_\mathbb{A} \times H_\mathbb{A} \times H'_\mathbb{A}$ on $\mathbb{V}_\mathbb{A}$ given as

$$r_\mathbb{A}(g, h, h')\varphi := \underset{v}{\otimes} r_v(g_v, h_v, h'_v)\varphi_v$$

for $\varphi = \underset{v}{\otimes}\varphi_v \in V_\mathbb{A}$ and $(g = (g_v), h = (h_v), h' = (h'_v)) \in G_\mathbb{A} \times H_\mathbb{A} \times H'_\mathbb{A}$. Define a function $\varphi^\kappa_0 \in \mathbb{V}_\mathbb{A}$ by

$$\varphi^\kappa_0(X, t) := \varphi^\kappa_{0,\infty}(X_\infty, t_\infty) \prod_{p<\infty} \varphi_{0,p}(X_p, t_p)$$

for $X = (X_v) \in B^{\oplus 2}_\mathbb{A}$ and $t = (t_v) \in \mathbb{Q}^\times_\mathbb{A}$.

Set

$$\theta^\kappa(g, h, h') := \sum_{(X,t)\in B^{\oplus 2}\times\mathbb{Q}^\times} r_\mathbb{A}(g, h, h')\varphi^\kappa_0(X, t), \quad ((g, h, h') \in G_\mathbb{A} \times H_\mathbb{A} \times H'_\mathbb{A}). \tag{4.1}$$

It is easily verified that the series (4.1) is uniformly convergent on any compact subset of $G_\mathbb{A} \times H_\mathbb{A} \times H'_\mathbb{A}$, and that

$$\theta^\kappa(\gamma g k_f k_\infty, \gamma_1 h u_f u_\infty, \gamma_2 h' u'_f u'_\infty) = \tau_\kappa(k_\infty)^{-1} j(u_\infty, i)^{-\kappa}\theta(g, h, h')\sigma_\kappa(u'_\infty)$$

for $(\gamma, g, k_f, k_\infty) \in G_\mathbb{Q} \times G_\mathbb{A} \times K_f \times K_\infty$, $(\gamma_1, h, u_f, u_\infty) \in H_\mathbb{Q} \times H_\mathbb{A} \times U_f \times U_\infty$ and $(\gamma_2, h', u'_f, u'_\infty) \in H'_\mathbb{Q} \times H'_\mathbb{A} \times U'_f \times U'_\infty$. Now note that $r_\infty(g_\infty, h_\infty, h'_\infty)\varphi_{0,\infty}(X_\infty, t_\infty)$ is $Z_{G,\infty} \times Z_{H,\infty} \times Z_{H',\infty}$-invariant. We also then see that θ^κ is $Z_{G,\mathbb{A}} \times Z_{H,\mathbb{A}} \times Z_{H',\mathbb{A}}$-invariant, since $\mathbb{Q}^\times_\mathbb{A} = \mathbb{Q}^\times \cdot \mathbb{R}_{>0} \cdot \mathbb{Z}^\times_f$.

For $f \in S_\kappa(D)$ and $f' \in \mathcal{A}_\kappa$, we set

$$\mathcal{L}(f, f')(g) := \int_{Z_{H,\mathbb{A}}H_\mathbb{Q}\backslash H_\mathbb{A}} dh \int_{Z_{H',\mathbb{A}}H'_\mathbb{Q}\backslash H'_\mathbb{A}} dh'\, \theta^\kappa(g, h, h')\overline{f(h)}f'(h') \quad (g \in G_\mathbb{A}). \tag{4.2}$$

Theorem 4.1 (Arakawa, Narita). *Suppose* $\kappa > 6$.
(*i*) *The integral* (4.2) *is absolutely convergent.*
(*ii*) $\mathcal{L}(f, f')(g) \in \mathcal{S}_\kappa$.

Proof. Since $G_{\mathbb{A}} = Z_{G,\mathbb{A}} G_{\mathbb{Q}} G^1_\infty K_f$ (cf. [Shim-2, Theorem 6.14]), it is sufficient to consider the restriction of $\mathcal{L}(f, f')$ to G^1_∞. By a standard argument, we see that $\mathcal{L}(f, f')|_{G^1_\infty}$ is a finite linear combination of original Arakawa lift (cf. [Ar-1], [N-1, §4] and [N-3, Theorem 4.1]), from which the theorem follows. □

5 Main result

5.1

To state the main result of the paper, we need to review several facts on Hecke operators.

5.2

First we consider the case where $p \nmid d_B$. We fix an isomorphism of B_p onto $M_2(\mathbb{Q}_p)$ such that $\mathcal{O}_p$ maps onto $M_2(\mathbb{Z}_p)$ and that the main involution of B_p corresponds to an involution of $M_2(\mathbb{Q}_p)$ given by

$$M_2(\mathbb{Q}_p) \ni X \mapsto w^{-1}\,{}^tXw, \quad (w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)).$$

The reduced trace tr corresponds to the trace Tr of $M_2(\mathbb{Q}_p)$. We henceforth identify B_p with $M_2(\mathbb{Q}_p)$ using the above isomorphism. Then G_p, K_p, H'_p and U'_p are identified with $GSp(J, \mathbb{Q}_p)$, $GSp(J, \mathbb{Z}_p)$, $GL_2(\mathbb{Q}_p)$ and $GL_2(\mathbb{Z}_p)$ respectively, where $GSp(J)$ is the group of similitudes of $J = \left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right)$. Note that we can identify U_p with U'_p by the isomorphism $B_p \simeq M_2(\mathbb{Q}_p)$ fixed above.

Define Hecke operators $\mathcal{T}^i_p$ $(i = 0, 1, 2)$ on $\mathcal{S}_\kappa$ by

$$\mathcal{T}^i_p F(g) = \int_{G_p} F(gx)\Phi^i_p(x)dx,$$

where Φ^0_p, Φ^1_p and Φ^2_p are the characteristic function of $K_p \operatorname{diag}(p, p, p, p)K_p$, $K_p \operatorname{diag}(p, p, 1, 1)K_p$ and $K_p \operatorname{diag}(p^2, p, p, 1)K_p$ respectively. Note that $\mathcal{T}^0_p F = F$ for any $F \in \mathcal{S}_\kappa$.

We also define Hecke operators T_p and T'_p on $S_\kappa(D)$ and $\mathcal{A}_\kappa$ by

$$T_p f(h) = \int_{H_p} f(hx)\phi_p(x)dx,$$

$$T'_p f'(h') = \int_{H'_p} f'(h'x')\phi'_p(x')dx',$$

where $\phi_p = \phi'_p$ is the characteristic function of $GL_2(\mathbb{Z}_p) \operatorname{diag}(p, 1) GL_2(\mathbb{Z}_p)$.

5.3

We next consider the case where $p|d_B$, i.e., B_p is a division algebra. In this case, we fix a prime element Π of B_p and put $\pi := n(\Pi)$. Then π is a prime element of $\mathbb{Q}_p$.

Define Hecke operators $\mathcal{T}_p^i$ $(i = 0, 1)$ on $\mathcal{S}_\kappa$ by

$$\mathcal{T}_p^i\, F(g) = \int_{G_p} F(gx)\Phi_p^i(x)dx,$$

where Φ_p^0 and Φ_p^1 are the characteristic functions of $K_p \cdot \mathrm{diag}(\Pi, \Pi) \cdot K_p$ and $K_p\,\mathrm{diag}(1, \pi)K_p$ respectively. Note that $(\mathcal{T}_p^0)^2 F = F$ for any $F \in \mathcal{S}_\kappa$. We also define Hecke operators T_p and T_p' on $S_\kappa(D)$ and $\mathcal{A}_\kappa$ by

$$T_p\, f(h) = \int_{H_p} f(hx)\phi_p(x)dx,$$
$$T_p'\, f'(h') = \int_{H_p'} f'(h'x')\phi_p'(x')dx'.$$

Here ϕ_p' is the characteristic function of $U_p'\Pi U_p' = \Pi U_p'$ and ϕ_p is defined as follows: If $p|D$, ϕ_p is the sum of the characteristic functions of $U_p(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix})U_p$ and $U_p(\begin{smallmatrix} 1 & 0 \\ 0 & \pi \end{smallmatrix})U_p$. If $p \nmid D$, ϕ_p is the characteristic function of $U_p(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix})U_p$.

5.4

We say that $F \in \mathcal{S}_\kappa$ is a *Hecke eigenform* if F is a common eigenfunction of the Hecke operators $\mathcal{T}_p^i$ for any $p < \infty$. Let $F \in \mathcal{S}_\kappa$ be a Hecke eigenform with $\mathcal{T}_p^i F = \Lambda_p^i F$ $(\Lambda_p^i \in \mathbb{C})$. We define the spinor L-function of F by

$$L(F, spin, s) = \prod_{p<\infty} L_p(F, spin, s),$$

where $L_p(F, spin, s) = Q_p(F, p^{-s})^{-1}$,

$$Q_p(F,t) = \begin{cases} 1 - p^{\kappa-3}\Lambda_p^1 t + p^{2\kappa-5}(\Lambda_p^2 + p^2 + 1)t^2 - p^{3\kappa-6}\Lambda_p^1 t^3 + p^{4\kappa-6}t^4 & \text{if } p \nmid d_B, \\ 1 - \{p^{\kappa-3}\Lambda_p^1 - p^{\kappa-3}(p^{A_p} - 1)\Lambda_p^0\}t + p^{2\kappa-3}(\Lambda_p^0)^2 t^2 & \text{if } p|d_B, \end{cases}$$

and

$$A_p = \begin{cases} 1 & \text{if } p \nmid D, \\ 2 & \text{if } p|D. \end{cases}$$

The Euler factor for $p \nmid d_B$ (resp. $p|d_B$) is given by the formula for the denominator of the Hecke series in [Shim-1, Theorem 2] (resp. [H-S, §4] and [Su, (1-34)]), under the normalization of the Hecke eigenvalues

$$\begin{cases} (\Lambda_p^0, \Lambda_p^1, \Lambda_p^2) \to (p^{2(\kappa-3)}\Lambda_p^0, p^{\kappa-3}\Lambda_p^1, p^{2(\kappa-3)}\Lambda_p^2) & (p \nmid d_B) \\ (\Lambda_p^0, \Lambda_p^1) \to (p^{\kappa-3}\Lambda_p^0, p^{\kappa-3}\Lambda_p^1) & (p | d_B) \end{cases}.$$

We say that $f \in S_\kappa$ (resp. $f' \in \mathcal{A}_\kappa$) is a *Hecke eigenform* if f (resp. f') is a common eigenfunction of T_p (resp. T'_p) for any $p < \infty$. For Hecke eigenforms $f \in S_\kappa$ and $f' \in \mathcal{A}_\kappa$ with $T_p f = \lambda_p f$ and $T'_p f' = \lambda'_p f'$ ($\lambda_p,\ \lambda'_p \in \mathbb{C}$), we define L-functions

$$L^D(f, s) = \prod_{p \nmid D} \left(1 - \lambda_p p^{\kappa-2-s} + p^{2\kappa-3-2s}\right)^{-1},$$
$$L^{d_B}(f', s) = \prod_{p \nmid d_B} \left(1 - \lambda'_p p^{\kappa-2-s} + p^{2\kappa-3-2s}\right)^{-1}.$$

When $D = 1$, we write $L(f, s)$ for $L^D(f, s)$, which is the usual Hecke L-function of f.

5.5

We are now able to state the main result of the paper.

Theorem 5.1. *Let $f \in S_\kappa$ and $f' \in \mathcal{A}_\kappa$, and suppose that*

$$T_p\, f = \lambda_p\, f,$$
$$T'_p\, f' = \lambda'_p\, f'$$

for each $p < \infty$. Then $F(g) := \mathcal{L}(f, f')(g)$ is a Hecke eigenform and the Hecke eigenvalues are given as follows:

(i) If $p \nmid d_B$, we have

$$\mathcal{T}_p^0\, F = F,$$
$$\mathcal{T}_p^1\, F = \left(p\overline{\lambda_p} + p\lambda'_p\right) F,$$
$$\mathcal{T}_p^2\, F = \left(p\overline{\lambda_p}\lambda'_p + p^2 - 1\right) F.$$

(ii) If $p | d_B$, we have

$$\mathcal{T}_p^0\, F = \lambda'_p F,$$
$$\mathcal{T}_p^1\, F = \left(p\overline{\lambda_p} + (p-1)\lambda'_p\right) F.$$

Corollary 5.2. *Let f and f' be as in Theorem 5.1. Then we have*

$$L(\mathcal{L}(f,f'), spin, s) = L^D(\overline{f}, s)\, L^{d_B}(f', s) \prod_{p|D} \left(1 - \{\overline{\lambda_p} + (1-p)\lambda'_p\}p^{\kappa-2-s} + p^{2\kappa-3-2s}\right)^{-1}.$$

In particular, if $D = 1$, we have

$$L(\mathcal{L}(f,f'), spin, s) = L(\overline{f}, s)\, L^{d_B}(f', s).$$

Here we define $L^D(\bar{f}, s)$ or $L(\bar{f}, s)$ by replacing λ_p with $\overline{\lambda_p}$ in $L^D(f,s)$ or $L(f,s)$ respectively.

Remark 5.3. When $p \nmid d_B$ the formula for the Hecke eigenvalues is essentially the same as the corresponding result of Yoshida lifting (cf. [Y, Theorem 6.1]). For such p this leads to the following decomposition

$$L_p(\mathcal{L}(f,f'), spin, s) = (1 - \overline{\lambda_p}p^{\kappa-2-s} + p^{2\kappa-3-2s})^{-1}(1 - \lambda'_p p^{\kappa-2-s} + p^{2\kappa-3-2s})^{-1}.$$

6 Commutation relations

6.1

In this section, we state the commutation relations of Hecke operators, from which Theorem 5.1 immediately follows. For a function ϕ on H_p, we put $\widehat{\phi}(h) = \phi(h^{-1})\quad (h \in H_p)$. We define $\widehat{\phi'}$ for $\phi' \colon H'_p \to \mathbb{C}$ in a similar manner.

6.2

In this subsection, suppose that $p \nmid d_B$ and let the notations be the same as in §5.2. The metaplectic representation r in this case is given as follows:
Let $\Phi \in \mathcal{S}(M_{4,2}(\mathbb{Q}_p)) \otimes \mathcal{S}(\mathbb{Q}_p^\times)$, $X \in M_{4,2}(\mathbb{Q}_p)$ and $t \in \mathbb{Q}_p^\times$. Here $\mathcal{S}(M_{4,2}(\mathbb{Q}_p))$ (resp. $\mathcal{S}(\mathbb{Q}_p^\times)$) denote the space of locally constant and compactly supported functions on $M_{4,2}(\mathbb{Q}_p)$ (resp. $\mathbb{Q}_p^\times$). We have

$$\begin{aligned}
&r(g,1,1)\Phi(X,t) = |\nu(g)|_p^{-\frac{3}{2}}\Phi(g^{-1}X, \nu(g)t), \quad (g \in G_p),\\
&r(1,1,h')\Phi(X,t) = |\det h'|_p^{\frac{3}{2}}\Phi(Xh', (\det h')^{-1}t), \quad (h' \in H'_p),\\
&r(1, \left(\begin{smallmatrix} a & 0 \\ 0 & a' \end{smallmatrix}\right), 1)\Phi(X,t) = |a|_p^{\frac{7}{2}}|a'|_p^{-\frac{1}{2}}\Phi(aX, (aa')^{-1}t), \quad (a, a' \in \mathbb{Q}_p^\times),\\
&r(1, \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right), 1)\Phi(X,t) = \psi(\frac{bt}{2}\operatorname{Tr}({}^tXJXw^{-1}))\Phi(X,t), \quad (b \in \mathbb{Q}_p),\\
&r(1, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), 1)\Phi(X,t) = |t|_p^4 \int_{M_{4,2}(\mathbb{Q}_p)} \psi(t \cdot \operatorname{Tr}({}^tYJXw^{-1}))\Phi(Y,t)d_QY.
\end{aligned}$$

6.3

The commutation relations are stated as follows:

Proposition 6.1. *Suppose that* $p \nmid d_B$. *Then we have*
(i) $r(\Phi_p^1, 1, 1)\varphi_{0,p} = p \cdot r(1, \widehat{\phi}_p, 1)\varphi_{0,p} + p \cdot r(1, 1, \widehat{\phi}'_p)\varphi_{0,p}$,
(ii) $r(\Phi_p^2, 1, 1)\varphi_{0,p} + (1-p^2)r(\Phi_p^0, 1, 1)\varphi_{0,p} = p \cdot r(1, \widehat{\phi}_p, \widehat{\phi}'_p)\varphi_{0,p}$.

Proposition 6.2. *Suppose that* $p|d_B$. *Then we have*

$$\begin{aligned} r(\Phi_p^0, 1, 1)\varphi_{0,p} &= r(1, 1, \widehat{\phi}'_p)\varphi_{0,p}, \\ r(\Phi_p^1, 1, 1)\varphi_{0,p} &= p \cdot r(1, \widehat{\phi}_p, 1)\varphi_{0,p} + (p-1)r(1, 1, \widehat{\phi}'_p)\varphi_{0,p}. \end{aligned}$$

7

Given a condition S, we set

$$\delta(S) := \begin{cases} 1 & \text{(if } S \text{ is satisfied)} \\ 0 & \text{(otherwise)} \end{cases}.$$

For the reamining part of this paper we often use this notation, and denote $\varphi_{0,p}$ simply by φ_0

7.1

In this section, we assume that $p \nmid d_B$ and prove Proposition 6.1. We keep the notations of §5.2 and §6.2. For $X \in M_2(\mathbb{Q}_p)$ with $wX + {}^tXw = 0$ and $Y \in GL_2(\mathbb{Q}_p)$, we put

$$u(X) := \begin{pmatrix} 1_2 & X \\ 0_2 & 1_2 \end{pmatrix}, \ \tau(Y) := \begin{pmatrix} Y & 0_2 \\ 0_2 & w^{-1}{}^tY^{-1}w \end{pmatrix} \in G_p.$$

The following is easily verified.

Lemma 7.1. *(i)*

$$\begin{aligned} K_p \operatorname{diag}(p,p,1,1)K_p = &\operatorname{diag}(1,1,p,p)K_p \cup \bigcup_{c \in \mathbb{Z}_p/p\mathbb{Z}_p} u(\left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right)) \operatorname{diag}(1,p,1,p)K_p \\ &\cup \bigcup_{b,d \in \mathbb{Z}_p/p\mathbb{Z}_p} \tau(\left(\begin{smallmatrix} 1 & d \\ 0 & 1 \end{smallmatrix}\right))u(\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)) \operatorname{diag}(p,1,p,1)K_p \\ &\cup \bigcup_{a,b,c \in \mathbb{Z}_p/p\mathbb{Z}_p} u(\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)) \operatorname{diag}(p,p,1,1)K_p. \end{aligned}$$

(ii)

$$K_p \operatorname{diag}(p^2,p,p,1)K_p = \operatorname{diag}(1,p,p,p^2)K_p \cup \bigcup_{d\in\mathbb{Z}_p/p\mathbb{Z}_p} u(\left(\begin{smallmatrix}1&d\\0&1\end{smallmatrix}\right)) \operatorname{diag}(p,1,p^2,p)K_p$$
$$\cup \bigcup_{a\in\mathbb{Z}_p/p\mathbb{Z}_p,\ c\in\mathbb{Z}_p/p^2\mathbb{Z}_p} u(\left(\begin{smallmatrix}a&0\\c&-a\end{smallmatrix}\right)) \operatorname{diag}(p,p^2,1,p)K_p$$
$$\cup \bigcup_{(a,b,c)\in\Lambda} u(p^{-1}\left(\begin{smallmatrix}a&b\\c&-a\end{smallmatrix}\right)) \operatorname{diag}(p,p,p,p)K_p$$
$$\cup \bigcup_{a,d\in\mathbb{Z}_p/p\mathbb{Z}_p,\ b\in\mathbb{Z}_p/p^2\mathbb{Z}_p} \tau(\left(\begin{smallmatrix}1&d\\0&1\end{smallmatrix}\right))u(\left(\begin{smallmatrix}a&b\\0&-a\end{smallmatrix}\right)) \operatorname{diag}(p^2,p,p,1)K_p,$$

where

$$\Lambda := \{(a,b,c)\in(\mathbb{Z}_p/p\mathbb{Z}_p)^3 \mid (a,b,c)\not\equiv(0,0,0) \mod p,\ a^2+bc\equiv 0 \mod p\}.$$

Here we note $\sharp\Lambda = p^2-1$.

7.2

Denote by $\sigma_{m,n}$ (resp. σ') the characteristic function of $M_{m,n}(\mathbb{Z}_p)$ (resp. $\mathbb{Z}_p^\times$). Then we have $\varphi_0(X,t) = \sigma_{4,2}(X)\sigma'(t)$. In the later discussion, we often write σ for $\sigma_{m,n}$ if there is no fear of confusion. The following facts are frequently used.

Lemma 7.2. *We write σ for $\sigma_{1,1}$. Then*

$$\sum_{a\in\mathbb{Z}_p/p\mathbb{Z}_p} \sigma(p^{-1}(t+a)) = \sigma(t) \quad (t\in\mathbb{Q}_p)$$

and

$$\sum_{a\in\mathbb{Z}_p/p\mathbb{Z}_p} \sigma(t)\sigma(p^{-1}(at+t')) = p\sigma(p^{-1}t)\sigma(p^{-1}t') + \sigma(t)\sigma(t') - \sigma(p^{-1}t)\sigma(t') \quad (t,t'\in\mathbb{Q}_p).$$

Lemma 7.3. *For $x\in M_{2,2}(\mathbb{Q}_p)$, set*

$$\lambda(x) := \sigma(\left(\begin{smallmatrix}1&0\\0&p^{-1}\end{smallmatrix}\right)x) + \sum_{a\in\mathbb{Z}_p/p\mathbb{Z}_p} \sigma(\left(\begin{smallmatrix}p^{-1}&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right)x)$$

and

$$\rho(x) := \sigma(x\left(\begin{smallmatrix}p^{-1}&0\\0&1\end{smallmatrix}\right)) + \sum_{a\in\mathbb{Z}_p/p\mathbb{Z}_p} \sigma(x\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\0&p^{-1}\end{smallmatrix}\right)).$$

Then

$$\lambda(x) = \rho(x).$$

For the rest of this section, we put

$$[i_1,i_2,i_3,i_4](x)=\sigma_{2,2}\left(\begin{pmatrix} p^{-i_1}x_1 & p^{-i_2}x_2 \\ p^{-i_3}x_3 & p^{-i_4}x_4 \end{pmatrix}\right)$$

for $x=\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)\in M_{2,2}(\mathbb{Q}_p)$ and $i_1,i_2,i_3,i_4\in\mathbb{Z}$.

7.3 Proof of Proposition 6.1 (i)

Let $X=\begin{pmatrix} x \\ y \end{pmatrix}\in M_{4,2}(\mathbb{Q}_p)$ and $t\in\mathbb{Q}_p^\times$. In view of §6.2 and Lemma 7.1, we obtain

$$r(\Phi_p^1,1,1)\varphi_0(X,t)=p^{\frac{3}{2}}\sigma'(pt)I(X),$$

where

$$\begin{aligned} I(X)=&\sum_{a,b,c\in\mathbb{Z}_p/p\mathbb{Z}_p}\sigma\begin{pmatrix} p^{-1}\{x+\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)y\} \\ y \end{pmatrix} \\ &+\sum_{b,d\in\mathbb{Z}_p/p\mathbb{Z}_p}\sigma\begin{pmatrix} \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & d \\ 0 & 1 \end{smallmatrix}\right)\{x+\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)y\} \\ \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & d \\ 0 & 1 \end{smallmatrix}\right)y \end{pmatrix} \\ &+\sum_{c\in\mathbb{Z}_p/p\mathbb{Z}_p}\sigma\begin{pmatrix} \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right)\{x+\left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right)y\} \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right)y \end{pmatrix} \\ &+\sigma\begin{pmatrix} x \\ p^{-1}y \end{pmatrix}. \end{aligned}$$

On the other hand, since

$$U_p\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)U_p=\bigcup_{a\in\mathbb{Z}_p/p\mathbb{Z}_p}\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right)U_p\cup\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)U_p,$$

we obtain

$$p\{r(1,\widehat{\phi}_p,1)\varphi_0(X,t)+r(1,1,\widehat{\phi}_p')\varphi_0(X,t)\}=p^{\frac{3}{2}}\sigma'(pt)I'(X),$$

where

$$\begin{aligned} I'(X)=&\delta(\mathrm{Tr}(xw^{-1t}yw)\in p\mathbb{Z}_p)\sigma(X) \\ &+p^3\sigma(p^{-1}X) \\ &+p\sum_{a\in\mathbb{Z}_p/p\mathbb{Z}_p}\sigma(X\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right)) \\ &+p\sigma(X\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)). \end{aligned}$$

The proof of Proposition 6.1 (i) is reduced to that of the following formula:

$$I(X) = I'(X). \tag{7.1}$$

Without loss of generality, we may assume that $y = \begin{pmatrix} p^\lambda & 0 \\ 0 & p^\mu \end{pmatrix}$ with $\lambda \geq \mu \geq 0$ in view of the elementary divisor theorem. First suppose that $\mu > 0$. Then

$$I\begin{pmatrix} x \\ y \end{pmatrix} = p^3\sigma(p^{-1}x) + p\lambda(x) + \sigma(x)$$

and

$$I'\begin{pmatrix} x \\ y \end{pmatrix} = \sigma(x) + p^3\sigma(p^{-1}x) + p\rho(x).$$

The equality (7.1) immediately follows from Lemma 7.3.

Next suppose that $\lambda = \mu = 0$. Then

$$I(X) = \sum_{a,b,c \in \mathbb{Z}_p/p\mathbb{Z}_p} \sigma(p^{-1}(x + \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)))$$

and

$$I'(X) = \delta(\mathrm{Tr}(x) \in p\mathbb{Z}_p)\sigma(x).$$

Let $x = \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)$. If $x \notin M_2(\mathbb{Z}_p)$, we have $I(X) = I'(X) = 0$. Assume that $x \in M_2(\mathbb{Z}_p)$. If $\mathrm{Tr}(x) = x_1 + x_4 \in p\mathbb{Z}_p$ (resp. $\in \mathbb{Z}_p^\times$), we have $I(X) = I'(X) = 1$ (resp. $= 0$), which proves (7.1).

Finally suppose that $\lambda > 0$ and $\mu = 0$. Then

$$\begin{aligned} I(X) =& p \sum_{a,b \in \mathbb{Z}_p/p\mathbb{Z}_p} \sigma(p^{-1}x + \left(\begin{smallmatrix} 0 & p^{-1}b \\ 0 & p^{-1}a \end{smallmatrix}\right)) \\ &+ \sum_{b \in \mathbb{Z}_p/p\mathbb{Z}_p} \sigma(\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\{x + \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)\}) \\ =& p[1,0,1,0](x) + [1,0,0,0](x) \end{aligned}$$

and

$$I'(X) = \delta(x_1 \in p\mathbb{Z}_p)\sigma(x) + p[1,0,1,0](x),$$

hence we get $I(X) = I'(X)$. This complete the proof of Proposition 6.1 (i).

7.4 Proof of Proposition 6.1 (ii)

Fisrt observe

$$r(\Phi_p^2, 1, 1)\varphi_0(X, t) = p^3\sigma'(p^2t)J(X),$$

where

$$
\begin{aligned}
J(X) = {} & \\
& \sigma \begin{pmatrix} \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right) x \\ \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & p^{-2} \end{smallmatrix}\right) y \end{pmatrix} + \sum_{d \in \mathbb{Z}_p/p\mathbb{Z}_p} \sigma \begin{pmatrix} \left(\begin{smallmatrix} p^{-1} & p^{-1}d \\ 0 & 1 \end{smallmatrix}\right) x \\ \left(\begin{smallmatrix} p^{-2} & p^{-2}d \\ 0 & p^{-1} \end{smallmatrix}\right) y \end{pmatrix} + \sum_{\substack{a \in \mathbb{Z}_p/p\mathbb{Z}_p \\ c \in \mathbb{Z}_p/p^2\mathbb{Z}_p}} \sigma \begin{pmatrix} \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & p^{-2} \end{smallmatrix}\right) x + \left(\begin{smallmatrix} p^{-1}a & 0 \\ p^{-2}c & -p^{-2}a \end{smallmatrix}\right) y \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right) y \end{pmatrix} \\
& + \sum_{(a,b,c) \in \Lambda} \sigma \begin{pmatrix} p^{-1}x + p^{-2} \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) y \\ p^{-1} y \end{pmatrix} + \sum_{\substack{a,d \in \mathbb{Z}_p/p\mathbb{Z}_p \\ b \in \mathbb{Z}_p/p^2\mathbb{Z}_p}} \sigma \begin{pmatrix} \left(\begin{smallmatrix} p^{-2} & p^{-2}d \\ 0 & p^{-1} \end{smallmatrix}\right) x + \left(\begin{smallmatrix} p^{-2}a & p^{-2}(b+ad) \\ 0 & -p^{-1}a \end{smallmatrix}\right) y \\ \left(\begin{smallmatrix} p^{-1} & p^{-1}d \\ 0 & 1 \end{smallmatrix}\right) y \end{pmatrix}.
\end{aligned}
$$

We also have

$$r(\Phi_p^0, 1, 1)\varphi_0(X, t) = p^3 \sigma'(p^2 t) J'(X),$$

where

$$J'(X) := \sigma \begin{pmatrix} p^{-1} x \\ p^{-1} y \end{pmatrix}.$$

On the other hand, we obtain

$$r(1, \widehat{\phi}_p, \widehat{\phi}_p)\varphi_0(X, t) = p^2 \sigma'(p^2 t) J''(X),$$

where

$$
\begin{aligned}
J''(X) := {} & \delta(\mathrm{Tr}(x w^{-1} {}^t y w) \in p^2\mathbb{Z}_p) \sum_{b \in \mathbb{Z}_p/p\mathbb{Z}_p} \sigma\left(\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \right) \\
& + \delta(\mathrm{Tr}(x w^{-1} {}^t y w) \in p^2\mathbb{Z}_p) \sigma\left(\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) \\
& + p^3 \sum_{b \in \mathbb{Z}_p/p\mathbb{Z}_p} \sigma\left(\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-2} \end{pmatrix} \right) \\
& + p^3 \sigma\left(\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} p^{-2} & 0 \\ 0 & p^{-1} \end{pmatrix} \right).
\end{aligned}
$$

To show the second part of Proposition 6.1, it remains to prove

$$J(X) + (1 - p^2) J'(X) = J''(X). \tag{7.2}$$

As in §7.3, we may assume that

$$y = \begin{pmatrix} p^{\alpha} & 0 \\ 0 & p^{\beta} \end{pmatrix} \quad (\alpha \geq \beta \geq 0).$$

We divide the proof into the five cases as follows:

$$(a)\ \beta \geq 2\ \ (b)\ \alpha \geq 2,\ \beta = 1\ \ (c)\ \alpha = \beta = 1\ \ (d)\ \alpha \geq 1,\ \beta = 0\ \ (e)\ \alpha = \beta = 0.$$

(a) Under the setting of this case, we have

$$\begin{aligned}
J(X)+(1-p^2)J'(X) =&\sigma\left(\begin{pmatrix} 1 & 0 \\ 0 & p^{-1}\end{pmatrix}x\right)+\sum_{d\in\mathbb{Z}_p/p\mathbb{Z}_p}\sigma\left(\begin{pmatrix} p^{-1} & p^{-1}d \\ 0 & 1\end{pmatrix}x\right) \\
&+p^3\sigma\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-2}\end{pmatrix}x\right) \\
&+p^3\sum_{d\in\mathbb{Z}_p/p\mathbb{Z}_p}\sigma\left(\begin{pmatrix} p^{-2} & p^{-2}d \\ 0 & p^{-1}\end{pmatrix}x\right) \\
=&\lambda(x)+p^3\lambda(p^{-1}x)
\end{aligned}$$

and

$$J\text{''}(X)=\rho(x)+p^3\rho(p^{-1}x).$$

The equality (7.2) follows from Lemma 7.3.
(b) For this case, we have

$$\begin{aligned}
J(X)=&[1,1,0,0](x)+p^2\sum_{a\in\mathbb{Z}_p/p\mathbb{Z}_p}\sigma\left(\begin{pmatrix} p^{-1}x_1 & p^{-1}x_2 \\ p^{-2}x_3 & p^{-2}x_4-p^{-1}a\end{pmatrix}\right)+\sum_{(a,b,c)\in\Lambda}\sigma\left(\begin{pmatrix} p^{-1}x_1 & p^{-1}(x_2+b) \\ p^{-1}x_3 & p^{-1}(x_4-a)\end{pmatrix}\right) \\
&+p\sum_{\substack{d\in\mathbb{Z}_p/p\mathbb{Z}_p \\ b\in\mathbb{Z}_p/p^2\mathbb{Z}_p}}\sigma\left(\begin{pmatrix} p^{-2}(x_1+dx_3) & p^{-2}(x_2+dx_4)+p^{-1}b \\ p^{-1}x_3 & p^{-1}x_4\end{pmatrix}\right) \\
=&[1,1,0,0](x)+p^2[1,1,2,1](x) \\
&+\sum_{\substack{a\equiv 0 \mod p \\ bc\equiv 0 \mod p,\ (b,c)\not\equiv(0,0) \mod p}}\sigma\left(\begin{pmatrix} p^{-1}x_1 & p^{-1}(x_2+b) \\ p^{-1}x_3 & p^{-1}x_4\end{pmatrix}\right)+\sum_{\substack{a\not\equiv 0 \mod p \\ b\not\equiv 0 \mod p}}\sigma\left(\begin{pmatrix} p^{-1}x_1 & p^{-1}(x_2+b) \\ p^{-1}x_3 & p^{-1}(x_4-a)\end{pmatrix}\right) \\
&+p^2\sum_{d\in\mathbb{Z}_p/p\mathbb{Z}_p}\sigma\left(\begin{pmatrix} p^{-2}(x_1+dx_3) & p^{-1}(x_2+dx_4) \\ p^{-1}x_3 & p^{-1}x_4\end{pmatrix}\right) \\
=&[1,1,0,0](x)+p^2[1,1,2,1](x)+(p-1)[1,1,1,1](x)+[1,0,1,0](x)-[1,1,1,0](x) \\
&+p^2\sigma(p^{-1}x_2)\sigma(p^{-1}x_4)\sum_{d\in\mathbb{Z}_p/p\mathbb{Z}_p}\sigma(p^{-1}x_3)\sigma(p^{-1}(dp^{-1}x_3+p^{-1}x_1)) \\
=&[1,1,0,0](x)+p^2[1,1,2,1](x)+(p-1)[1,1,1,1](x)+[1,0,1,0](x)-[1,1,1,0](x) \\
&+p^2\{p[2,1,2,1](x)+[1,1,1,1](x)-[1,1,2,1](x)\}
\end{aligned}$$

and hence

$$\begin{aligned}
J(X)+(1-p^2)J'(X)=&[1,1,0,0](x)+p[1,1,1,1](x)+[1,0,1,0](x) \\
&-[1,1,1,0](x)+p^3[2,1,2,1](x).
\end{aligned}$$

On the other hand,

$$\begin{aligned}J''(X) =&\delta(x_1 \in p\mathbb{Z}_p) \sum_{b\in\mathbb{Z}_p/p\mathbb{Z}_p} \sigma\left(\left(\begin{smallmatrix} x_1 & p^{-1}(bx_1+x_2) \\ x_3 & p^{-1}(bx_3+x_4)\end{smallmatrix}\right)\right) \\ &+ \delta(x_1 \in p\mathbb{Z}_p)\sigma\left(\left(\begin{smallmatrix} p^{-1}x_1 & x_2 \\ p^{-1}x_3 & x_4\end{smallmatrix}\right)\right) + p^3\sigma\left(\left(\begin{smallmatrix} p^{-2}x_1 & p^{-1}x_2 \\ p^{-2}x_3 & p^{-1}x_4\end{smallmatrix}\right)\right) \\ =&\sigma(p^{-1}x_1)\sigma(p^{-1}x_2) \sum_{b\in\mathbb{Z}_p/p\mathbb{Z}_p} \sigma(x_3)\sigma(p^{-1}(bx_3+x_4)) \\ &+ [1,0,1,0](x) + p^3[2,1,2,1](x) \\ =&p[1,1,1,1](x) + [1,1,0,0](x) - [1,1,1,0](x) \\ &+ [1,0,1,0](x) + p^3[2,1,2,1](x).\end{aligned}$$

The equality (7.2) in this case immediately follows.

(c) For this case, we have

$$\begin{aligned}J(X) =& \sum_{\substack{a\in\mathbb{Z}_p/p\mathbb{Z}_p \\ c\in\mathbb{Z}_p/p^2\mathbb{Z}_p}} \sigma\left(\left(\begin{smallmatrix} p^{-1}x_1 & p^{-1}x_2 \\ p^{-2}x_2+p^{-1}c & p^{-2}x_4-p^{-1}a\end{smallmatrix}\right)\right) \\ &+ \sum_{(a,b,c)\in\Lambda} \sigma\left(p^{-1}\left(x + \left(\begin{smallmatrix} a & b \\ c & -a\end{smallmatrix}\right)\right)\right) \\ &+ \sum_{\substack{a,d\in\mathbb{Z}_p/p\mathbb{Z}_p \\ b\in\mathbb{Z}_p/p^2\mathbb{Z}_p}} \sigma\left(\left(\begin{smallmatrix} p^{-2}(x_1+dx_3)+p^{-1}a & p^{-2}(x_2+dx_4)+p^{-1}(b+ad) \\ p^{-1}x_3 & p^{-1}x_4-a\end{smallmatrix}\right)\right) \\ =&p\sigma(p^{-1}x) + \delta(x_1+x_4 \in p\mathbb{Z}_p,\ x_1^2+x_2x_3 \in p\mathbb{Z}_p)(\sigma(x)-\sigma(p^{-1}x)) + p^2\sigma(p^{-1}x) \\ =&(p^2+p-1)\sigma(p^{-1}x) + \delta(x_1+x_4 \in p\mathbb{Z}_p,\ x_1^2+x_2x_4 \in p\mathbb{Z}_p)\sigma(x).\end{aligned}$$

On the other hand, we immediately have

$$J'(X) = \sigma(p^{-1}x)$$

and

$$\begin{aligned}J''(X) =&\delta(x_1+x_4 \in p\mathbb{Z}_p)\left\{\sigma\left(x\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1\end{smallmatrix}\right)\right) + \sum_{b\in\mathbb{Z}_p/p\mathbb{Z}_p} \sigma\left(x\left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{-1}\end{smallmatrix}\right)\right)\right\} \\ =&\delta(x_1+x_4 \in p\mathbb{Z}_p)\left\{\sigma\left(\left(\begin{smallmatrix} p^{-1}x_1 & x_2 \\ p^{-1}x_3 & x_4\end{smallmatrix}\right)\right) + \sum_{b\in\mathbb{Z}_p/p\mathbb{Z}_p} \sigma\left(\left(\begin{smallmatrix} x_1 & p^{-1}(bx_1+x_2) \\ x_3 & p^{-1}(bx_3+x_4)\end{smallmatrix}\right)\right)\right\}.\end{aligned}$$

To prove (7.2), it is sufficient to show that

$$\begin{aligned}&p\sigma(p^{-1}x) \\ &+ \delta(x_1+x_4 \in p\mathbb{Z}_p)\left\{\delta(x_1+x_2x_3 \in p\mathbb{Z}_p)\sigma(x) - \sum_{b\in\mathbb{Z}_p/p\mathbb{Z}_p} \sigma\left(\left(\begin{smallmatrix} x_1 & p^{-1}(bx_1+x_2) \\ x_3 & p^{-1}(bx_3+x_4)\end{smallmatrix}\right)\right) - [1,0,1,0](x)\right\}\end{aligned}$$

vanishes. This is proved by a tedious but straightforward calculation and we omit its proof.
(d) In this case, we have

$$\begin{aligned}
J(X) =& \sum_{\substack{a\in\mathbb{Z}_p/p\mathbb{Z}_p\\ b\in\mathbb{Z}_p/p^2\mathbb{Z}_p}} \sigma\left(\left(\begin{smallmatrix} p^{-2} & 0 \\ 0 & p^{-1}\end{smallmatrix}\right) x + \left(\begin{smallmatrix} p^{\alpha-2}a & p^{-2}b \\ 0 & -p^{-1}a\end{smallmatrix}\right)\right), \\
J'(X) =& 0, \\
J''(X) =& \delta(x_1 + p^\alpha x_4 \in p^2\mathbb{Z}_p)\sigma\left(\left(\begin{smallmatrix} p^{-1}x_1 & x_2 \\ p^{-1}x_3 & x_4\end{smallmatrix}\right)\right)
\end{aligned}$$

for $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$. Since

$$\begin{aligned}
J(X) =& \sum_{a\in\mathbb{Z}_p/p\mathbb{Z}_p} \sigma\left(\left(\begin{smallmatrix} p^{-2}x_1+p^{\alpha-2}a & x_2 \\ p^{-1}x_3 & p^{-1}(x_4-a)\end{smallmatrix}\right)\right) \\
=& J''(X),
\end{aligned}$$

we are done.
(e) For this remaining case, we have

$$J(X) = J'(X) = J''(X) = 0$$

and the proof of (7.2) has been completed.

8

8.1

In this section, we assume that $p|d_B$ and prove Proposition 6.2. The proof of the first formula of the proposition is straightforward. To prove the second formula, we need some preparation. By $\sigma_{m,n}$, we denote the characteristic function of $M_{m,n}(\mathcal{O}_p)$. As in §7, we often omit the subscripts of $\sigma_{m,n}$'s. For a subset A of B_p, we put $A^- := \{a \in A \mid \mathrm{tr}(a) = 0\}$. Recall that $\mathcal{O}_p$ is the maximal order of B_p, Π is a (fixed) prime element of B_p and $\pi = n(\Pi)$. We put $\mathfrak{P}_p = \Pi\mathcal{O}_p$.

8.2

We now collect several facts on the arithmetic of B_p used in the later discussion.

Lemma 8.1. *We have*

$$\begin{aligned}
\sharp\mathcal{O}_p/\pi\mathcal{O}_p &= p^4,\\
\sharp\mathcal{O}_p/\Pi\mathcal{O}_p &= p^2,\\
\sharp\mathcal{O}_p^-/\pi\mathcal{O}_p^- &= p^3,\\
\sharp(\Pi^{-1}\mathcal{O}_p)^-/\mathcal{O}_p^- &= p^2.
\end{aligned}$$

Lemma 8.2. *Let $\sigma = \sigma_{1,1}$ be the characteristic function of $\mathcal{O}_p$.*

$$\begin{aligned}
&(i) \quad \sum_{b\in\mathcal{O}_p^-/\pi\mathcal{O}_p^-} \sigma(\pi^{-1}(x+b)) = \delta(\mathrm{tr}(x)\in p\mathbb{Z}_p)\sigma(x),\\
&(ii) \quad \sum_{b\in\mathcal{O}_p^-/\pi\mathcal{O}_p^-} \sigma(\Pi^{-1}(x+b)) = p^2\delta(\mathrm{tr}(x)\in p\mathbb{Z}_p)\sigma(x),\\
&(iii) \quad \sum_{b\in(\Pi^{-1}\mathcal{O}_p)^-/\mathcal{O}_p^-} \sigma(x+b) = \sigma(\Pi x),\\
&(iv) \quad \sum_{b\in\mathfrak{P}_p^-/\pi\mathfrak{P}_p^-} \sigma(\pi^{-1}(x+b)) = p\sigma(\Pi^{-1}x),\\
&(v) \quad \sum_{b\in\mathcal{O}_p^-/\mathfrak{P}_p^-} \sigma(\Pi^{-1}(x+b)) = \delta(\mathrm{tr}(x)\in p\mathbb{Z}_p)\sigma(x),\\
&(vi) \quad \sum_{b\in\mathfrak{P}_p^-/\pi\mathfrak{P}_p^-} \sigma(\pi^{-1}\Pi^{-1}x+b) = \delta(\mathrm{tr}(x)\in p^2\mathbb{Z}_p)\sigma(\Pi^{-1}x).
\end{aligned}$$

8.3

We first consider the case $p|D$. Let ϕ_p^+ (resp. ϕ_p^-) be the characteristic function of $U_p \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{-1}\end{smallmatrix}\right) U_p$ (resp. $U_p \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1\end{smallmatrix}\right) U_p$). Note that $\widehat{\phi}_p = \phi_p^+ + \phi_p^-$.

Lemma 8.3. *We have*

$$r(\Phi_p^1, 1, 1)\varphi_0(X,t) = p^{\frac{3}{2}}\sigma'(\pi t)I(X),$$

where

$$\begin{aligned}
I\begin{pmatrix} x \\ y \end{pmatrix} = &\sum_{b\in\mathcal{O}_p^-/\pi\mathcal{O}_p^-} \sigma\begin{pmatrix} \pi^{-1}(x+by) \\ y \end{pmatrix}\\
&+ \sum_{c\in(\Pi^{-1}\mathcal{O}_p-\mathcal{O}_p)^-/\mathcal{O}_p^-} \sigma\begin{pmatrix} \Pi^{-1}(x+cy) \\ \Pi^{-1}y \end{pmatrix}\\
&+ \sigma\begin{pmatrix} x \\ \pi^{-1}y \end{pmatrix}.
\end{aligned}$$

Proof. This follows from the definition of r and the coset decomposition

$$
\begin{aligned}
K_p \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} K_p = & \bigcup_{b \in \mathcal{O}_p^-/\pi\mathcal{O}_p^-} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} K_p \\
& \cup \bigcup_{c \in (\Pi^{-1}\mathcal{O}_p - \mathcal{O}_p)^-/\mathcal{O}_p^-} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Pi & 0 \\ 0 & \Pi \end{pmatrix} K_p \\
& \cup \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} K_p.
\end{aligned}
$$

□

Lemma 8.4. *(i) If $y \in \mathcal{O}_p^\times$, we have*

$$
I \begin{pmatrix} x \\ y \end{pmatrix} = \delta(\mathrm{tr}(xy^{-1}) \in p\mathbb{Z}_p)\sigma(x).
$$

(ii) If $y \in \Pi\mathcal{O}_p^\times$, we have

$$
I \begin{pmatrix} x \\ y \end{pmatrix} = p^2\delta(\mathrm{tr}(xy^{-1}) \in p\mathbb{Z}_p)\sigma(\Pi^{-1}x) - \sigma(\Pi^{-1}x) + \sigma(x).
$$

(iii) If $y \in \pi\mathcal{O}_p$, we have

$$
I \begin{pmatrix} x \\ y \end{pmatrix} = p^3\sigma(\pi^{-1}x) + (p^2-1)\sigma(\Pi^{-1}x) + \sigma(x).
$$

Proof. When $y \in \mathcal{O}_p^\times$,

$$
\begin{aligned}
I \begin{pmatrix} x \\ y \end{pmatrix} = & \sum_{b \in \mathcal{O}_p^-/\pi\mathcal{O}_p^-} \sigma(\pi^{-1}(xy^{-1}+b)) \\
= & \delta(\mathrm{tr}(xy^{-1}) \in p\mathbb{Z}_p)\sigma(xy^{-1}) \\
= & \delta(\mathrm{tr}(xy^{-1}) \in p\mathbb{Z}_p)\sigma(x).
\end{aligned}
$$

When $y \in \Pi\mathcal{O}_p^\times$,

$$
\begin{aligned}
I \begin{pmatrix} x \\ y \end{pmatrix} = & \sum_{b \in \mathcal{O}_p^-/\pi\mathcal{O}_p^-} \sigma(\Pi^{-1}(xy^{-1}+b)) + \sum_{c \in (\Pi^{-1}\mathcal{O}_p)^-/\mathcal{O}_p^-} \sigma(xy^{-1}+c) - \sigma(xy^{-1}) \\
= & p^2\delta(\mathrm{tr}(xy^{-1}) \in p\mathbb{Z}_p)\sigma(xy^{-1}) + \sigma(\Pi xy^{-1}) - \sigma(xy^{-1}) \\
= & p^2\delta(\mathrm{tr}(xy^{-1}) \in p\mathbb{Z}_p)\sigma(\Pi^{-1}x) + \sigma(x) - \sigma(\Pi^{-1}x).
\end{aligned}
$$

When $y \in \pi\mathcal{O}_p$,

$$I\begin{pmatrix} x \\ y \end{pmatrix} = \sharp(\mathcal{O}_p^-/\pi\mathcal{O}_p^-)\sigma(\pi^{-1}x) + \sharp((\Pi^{-1}\mathcal{O}_p - \mathcal{O}_p)^-/\mathcal{O}_p^-)\sigma(\Pi^- x) + \sigma(x)$$
$$= p^3\sigma(\pi^{-1}x) + (p^2-1)\sigma(\Pi^{-1}x) + \sigma(x).$$

□

Lemma 8.5. *We have*

$$r(1,\phi_p^+,1)\varphi_0(X,t) = p^{\frac{1}{2}}\sigma'(pt)J^+(X),$$

where

$$J^+\begin{pmatrix} x \\ y \end{pmatrix} = \sigma(y) \times \begin{cases} \delta(\mathrm{tr}(xy^{-1}) \in p\mathbb{Z}_p)\sigma(x) & (y \in \mathcal{O}_p^\times) \\ \sigma(x) & (y \in \Pi\mathcal{O}_p) \end{cases}.$$

Proof. Since $U_p \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right) U_p = \bigcup_{b \in \mathbb{Z}_p/p\mathbb{Z}_p} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} U_p$, we have

$$r(1,\phi_p^+,1)\varphi_0(X,t) = p^{-\frac{1}{2}}\sigma'(pt)\sum_{b \in \mathbb{Z}_p/p\mathbb{Z}_p}\psi\left(\frac{bt}{2}\mathrm{tr}(X^*QX)\right)\sigma(X)$$
$$= p^{-\frac{1}{2}}\sigma'(pt)\cdot p\cdot\delta(\mathrm{tr}(x^\sigma y) \in \mathbb{Z}_p)\sigma_{2,1}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$$
$$= p^{\frac{1}{2}}\sigma'(pt)\sigma(x) \times \begin{cases} \delta(\mathrm{tr}(xy^{-1}) \in p\mathbb{Z}_p) & (y \in \mathcal{O}_p^\times) \\ 1 & (y \in \Pi\mathcal{O}_p) \end{cases},$$

which proves the lemma. □

Lemma 8.6. *We have*

$$r(1,\phi_p^-,1)\varphi_0(X,t) = p^{\frac{5}{2}}\sigma'(pt)J^-(X),$$

where

$$J^-\begin{pmatrix} x \\ y \end{pmatrix} = \sigma(y) \times \begin{cases} 0 & (y \in \mathcal{O}_p^\times) \\ (\delta(\mathrm{tr}(xy^{-1}) \in p\mathbb{Z}_p) - p^{-1})\sigma(\Pi^{-1}x) & (y \in \Pi\mathcal{O}_p^\times) \\ p\sigma(\pi^{-1}x) + (1-p^{-1})\sigma(\Pi^{-1}x) & (y \in \pi\mathcal{O}_p) \end{cases}.$$

Proof. To prove the lemma, we recall that, for $h \in GL_2(\mathbb{Q}_p)$,

$$(\mathcal{I}\cdot r(1,h,1)\varphi)(X,t) = |\det h|^{-\frac{1}{2}}\mathcal{I}\varphi(\det(h)\cdot h^{-1}X, \det(h^{-1})\cdot t),$$

where

$$\mathcal{I}\varphi\left(\left(\begin{array}{c} x \\ y \end{array}\right), t\right) = \int_{B_p} \psi(-t\,\mathrm{tr}(u^\sigma x))\varphi\left(\left(\begin{array}{c} u \\ y \end{array}\right), t\right) du$$

(cf. Lemma 3.2). Here the measure du on B_p is normalized by $\mathrm{vol}(\mathcal{O}_p) = p^{-1}$.

It is easily verified that

$$\mathcal{I}\varphi_0\left(\left(\begin{array}{c} x \\ y \end{array}\right), t\right) = p^{-1}\sigma(\Pi x)\sigma(y)\sigma'(t)$$

and

$$\mathcal{I}^{-1}\varphi\left(\left(\begin{array}{c} x \\ y \end{array}\right), t\right) = |t|^4 \int_{B_p} \psi(t\,\mathrm{tr}(u^\sigma x))\varphi\left(\left(\begin{array}{c} u \\ y \end{array}\right), t\right) du.$$

It follows that

$$\begin{aligned}(\mathcal{I}\cdot r(1,\phi_p^-,1)\varphi_0)(X,t) =& p^{-\frac{1}{2}} \sum_{b\in\mathbb{Z}_p/p\mathbb{Z}_p} \mathcal{I}\varphi_0\left(\begin{pmatrix}1 & 0\\ b & p^{-1}\end{pmatrix}\left(\begin{array}{c} x \\ y \end{array}\right), pt\right)\\ =& p^{-\frac{3}{2}}\sigma(\Pi x)\sigma'(pt)\sum_{b\in\mathbb{Z}_p/p\mathbb{Z}_p}\sigma(bx+p^{-1}y).\end{aligned}$$

We thus have

$$\begin{aligned}r(1,\phi_p^-,1)\varphi_0(X,t) =& |t|^4\int_{B_p}\psi(t\,\mathrm{tr}(u^\sigma x))p^{-\frac{3}{2}}\sigma(\Pi u)\sigma'(pt)\sum_{b\in\mathbb{Z}_p/p\mathbb{Z}_p}\sigma(bu+p^{-1}y)du\\ =& p^{\frac{5}{2}}\sigma'(pt)K\left(\begin{array}{c} x \\ y \end{array}\right),\end{aligned}$$

where

$$K\left(\begin{array}{c} x \\ y \end{array}\right) = \int_{B_p}\psi(p^{-1}\,\mathrm{tr}(u^\sigma x))\sigma(\Pi x)\sum_{b\in\mathbb{Z}_p/p\mathbb{Z}_p}\sigma(bu+p^{-1}y)du.$$

First observe that $K\left(\begin{array}{c} x \\ y \end{array}\right) = 0$ if $y \notin \Pi\mathcal{O}_p$. Assume that $y \in \Pi\mathcal{O}_p$. Then

$$\begin{aligned}K\left(\begin{array}{c} x \\ y \end{array}\right) =& \sigma(p^{-1}y)\int_{\Pi^{-1}\mathcal{O}_p}\psi(p^{-1}\,\mathrm{tr}(u^\sigma x))du + \sum_{b\in(\mathbb{Z}_p-p\mathbb{Z}_p)/p\mathbb{Z}_p}\int_{\Pi^{-1}\mathcal{O}_p}\psi(p^{-1}\,\mathrm{tr}(u^\sigma x))\sigma(b^{-1}u+p^{-1}y)du\\ =& \mathrm{vol}(\Pi^{-1}\mathcal{O}_p)\sigma(p^{-1}y)\sigma(p^{-1}x) + \sum_{b\in(\mathbb{Z}_p-p\mathbb{Z}_p)/p\mathbb{Z}_p}\int_{\Pi^{-1}\mathcal{O}_p}\psi(p^{-1}\,\mathrm{tr}((u-bp^{-1}y)^\sigma x))\sigma(b^{-1}u)du\\ =& p\sigma(\pi^{-1}x)\sigma(\pi^{-1}y) + \sum_{b\in(\mathbb{Z}_p-p\mathbb{Z}_p)/p\mathbb{Z}_p}\psi(p^{-2}b\,\mathrm{tr}(y^\sigma x))\int_{\Pi^{-1}\mathcal{O}_p}\psi(p^{-1}\,\mathrm{tr}(u^\sigma x))\sigma(u)du\\ =& p\sigma(p^{-1}x)\sigma(\pi^{-1}y) + \{p\delta(\mathrm{tr}(y^\sigma x)\in p^2\mathbb{Z}_p) - 1\}\,\mathrm{vol}(\mathcal{O}_p)\sigma(\Pi^{-1}x).\end{aligned}$$

The last term is equal to

$$\{\delta(\mathrm{tr}(xy^{-1}) \in p\mathbb{Z}_p) - p^{-1}\}\sigma(\Pi^{-1}x)$$

if $y \in \Pi\mathcal{O}_p^\times$, and

$$(1 - p^{-1})\sigma(\Pi^{-1}x)$$

if $y \in \pi\mathcal{O}_p$. This proves the lemma. □

The following lemma is clear.

Lemma 8.7. *We have*

$$r(1, 1, \phi_p')\varphi_0(X, t) = p^{\frac{3}{2}}\sigma'(pt)J'(X),$$

where

$$J'\begin{pmatrix} x \\ y \end{pmatrix} = \sigma(\Pi^{-1}x)\sigma(\Pi^{-1}y).$$

A straightforward calculation shows the following, which completes the proof of Proposition 6.2 in the case where $p|D$.

Proposition 8.8. *We have*

$$J(X) - J^+(X) - p^2J^-(X) + (1 - p)J'(X) = 0.$$

8.4

In this subsection, we suppose that $p \nmid D$. We only give a sketch of the proof of Proposition 6.2 in this case, since the proof is similar to that in §8.3. First we have the following coset decomposition:

$$\begin{aligned} K_p \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} K_p = & \bigcup_{b \in \mathfrak{P}_p^- / \pi\mathfrak{P}_p^-} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} K_p \\ & \cup \bigcup_{c \in (\mathcal{O}_p - \mathfrak{P}_p)^- / \mathfrak{P}_p^-} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Pi & 0 \\ 0 & \Pi \end{pmatrix} K_p \\ & \cup \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} K_p. \end{aligned}$$

Lemma 8.9. *We have*

$$\begin{aligned} r(\Phi_p^1, 1, 1)\varphi_0(X, t) &= p^{3/2}\delta'(\pi t)I(X), \\ r(1, \widehat{\phi}_p, 1)\varphi_0(X, t) &= p^{1/2}\delta'(\pi t)J(X), \\ r(1, 1, \widehat{\phi}_p)\varphi_0(X, t) &= p^{3/2}\delta'(\pi t)J'(X), \end{aligned}$$

where

$$I(X) = \sum_{b\in\mathfrak{P}_p^-/\pi\mathfrak{P}_p^-} \sigma\begin{pmatrix} \pi^{-1}(x+by) \\ \Pi y \end{pmatrix} + \sum_{c\in(\mathcal{O}_p-\mathfrak{P}_p)^-/\mathfrak{P}_p^-} \sigma\begin{pmatrix} \Pi^{-1}(x+cy) \\ y \end{pmatrix} + \sigma\begin{pmatrix} x \\ \Pi^{-1}y \end{pmatrix},$$

$$J(X) = \delta(\mathrm{tr}(x^\sigma y)\in p\mathbb{Z}_p)\sigma\begin{pmatrix} x \\ \Pi y \end{pmatrix} + p^3\sigma\begin{pmatrix} \pi^{-1} \\ \Pi^{-1}y \end{pmatrix},$$

$$J'(X) = \sigma\begin{pmatrix} \Pi^{-1}x \\ y \end{pmatrix}.$$

Using Lemma 8.2 and Lemma 8.9, we obtain the following formula, from which Proposition 6.2 immediately follows.

Proposition 8.10.

$$I(X) = J(X) + (p-1)J'(X).$$

References

[Ar-1] T. Arakawa, unpublished note.

[Ar-2] T. Arakawa, On certain automorphic forms of $Sp(1,q)$, Automorphic forms of several variables, Taniguchi Symposium, Katata (1983), 1-48.

[E] M. Eichler, Quadratische Formen und Modulfunktionen, Acta Arith., 4 (1958), 217-239.

[Ge] S. Gelbart, Weil's representation and the spectrum of the metaplectic group, Lecture Notes in Math. 530, Springer-Verlag (1976).

[G-W] B. Gross and N. Wallach, On quaternionic discrete series representations, and their continuations, J. Reine. Angew. Math., 481 (1996), 73-123.

[H-K] M. Harris and S. Kudla, Arithmetic automorphic forms for the nonholomorphic discrete series of $GSp(2)$, Duke. Math. J., 66 (1992), 59-121.

[H-S] T. Hina and T. Sugano, On the local Hecke series of some classical groups over $\mathfrak{p}$-adic fields, J. Math. Soc. Japan, 35 (1983), 133-152.

[Ku] S. Kudla, On certain Euler products for $SU(2,1)$, Compositio Math., 42 (1981), 321-344.

[N-1] H. Narita, On certain automorphic forms of $Sp(1,q)$ (Arakawa's results and recent progress), Proceedings of the conference in memory of Tsuneo Arakawa, Automorphic forms and zeta functions, World Scientific (2006), 314-333.

[N-2] H. Narita, Fourier-Jacobi expansion of automorphic forms on $Sp(1, q)$ generating quaternionic discrete series, to appear in Journal of functional analysis.

[N-3] H. Narita, Theta lifting from elliptic cusp forms to automorphic forms on $Sp(1, q)$, preprint, (2006).

[Ra] S. Rallis, Langlands functoriality and the Weil representation, Amer. J. Math., 104 (1982), 469-515.

[Ro] B. Roberts, The theta correspondence for similitudes, Israel J. Math., 94 (1996), 285-317.

[Shim-1] G. Shimura, On modular correspondence for $Sp(N, \mathbb{Z})$ and their congruence relations, Proc. Nat. Acad. Sci. U.S.A., 49 (1963), 824-828.

[Shim-2] G. Shimura, Arithmetic of unitary groups, Ann. Math., 79 (1964), 369–409.

[Su] T. Sugano, On holomorphic cusp forms on quaternion unitary groups of degree 2, J. Fac. Sci. Univ. Tokyo Sect. IA, Math., 31 (1984), 521-568.

[Sz] H. Shimizu, Theta series and automorphic forms on GL_2, J. Math. Soc. Japan, 24 (1972), 638-683.

[Y] H. Yoshida, Siegel's modular forms and the arithmetic of quadratic forms, Invent. Math., 60 (1980), 193-248.

[W] J. L. Waldspurger, Sur les valeurs de certaines fonctions L automorphes en leur centre de symmetrie, Compositio Math., 54 (1985), 173–242.

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