Immersed curves, knots and singularities

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A divide is a relative generic immersion of a 1-manifold $(M, \partial M)$ in the unit 2-disk $(D, \partial D)$. In this talk a divide will be an immersion of a finite union of copies of $[0,1]$ and will be called a picture. To a picture $P \subset D$ is associated a classical link in the 3-sphere. Consider the tangent space $T\mathbb{R}^2$ as $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^2 + i\mathbb{R}^2 = \mathbb{C}^2$. The link $L(P)$ of the picture P is the following subset in the unit sphere S^3 of $\mathbb{C}^2$:

$$L(P) := \{(x,u) \in T\mathbb{R}^2 = \mathbb{R}^4 \mid x \in P, u \in T_xP, ||x||^2 + ||u||^2 = 1\}$$

The link $L(P)$ does not change its type if we move P with an isotopy in the space of generic relative immersions. So we may assume that in the collar $D - \frac{1}{\sqrt{2}}D$ the picture P consists of radial line segments, and we see that the link $L(P)$ is transversal to the contact structure of complex planes in TS^3.It follows that the link $L(P)$ has natural orientation. Links $L(P)$ of pictures P have special properties. For instance, the involution $(x,u) \to (x,-u)$ induces a rigid symmetry of the link $L(P)$, which induces for each component a strong inversion.

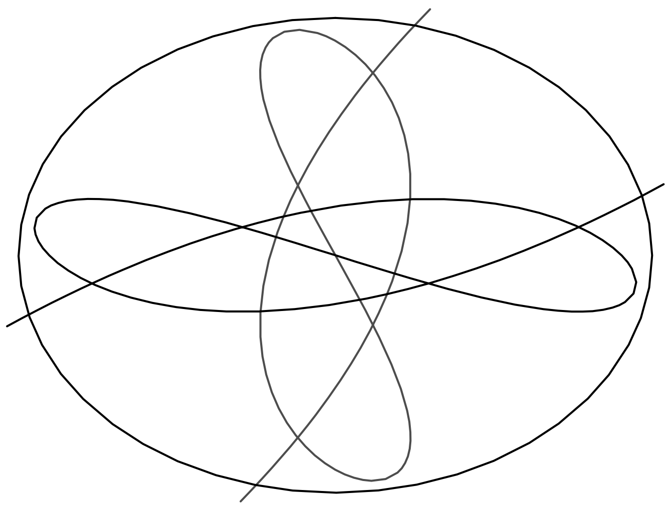

Figure 1: A picture with 2 branches

The unknotting number of the link of a picture is computed by:

THEOREM. *The unknotting number and the 4-ball genus of the link $L(P)$ of the picture P is the number of double points of the immersion P.*

The proof relies on the positive answer of Kronheimer and Mrowka to the Thom conjecture.

For given natural numbers $r, \delta > 0$ there are infinitely many links with r components and with unknotting number δ, but only a finite number of transversal isotopy classes of pictures with r branches and δ double points. So we see that links $L(P)$ of pictures are as rare among links as orchids among flowers. It is a surprise that the knots $10_{139}, 10_{145}, 10_{152}$ of the table of Rolfsen, for which the unknotting number has been determined only recently by Tomomi Kawamura [K], are knots of pictures. The knot 10_{139} is very rare since it is a knot of two pictures which can not be related by an transversal isotopy (see Fig. 2).

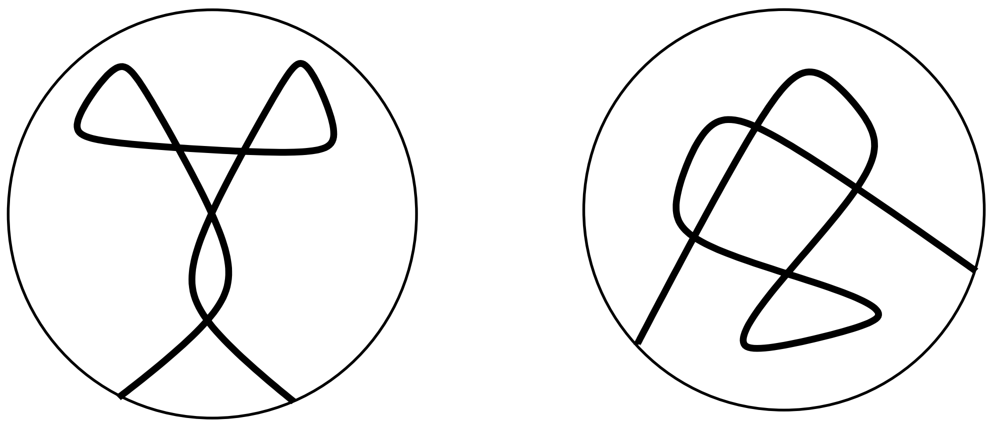

Figure 1: Two pictures for the knot 10_{139}.

Links of pictures tend to be fibered.

THEOREM. *The link of a connected picture is a fibered link.*

The proof uses a complex function f_P on $TD \subset \mathbb{C}^2$ which satisfies the Cauchy-Riemann equations along the zero section D in TD, such that the 0-level of f_P intersects D along P.

The geometric monodromy can be read from the picture in a similar way as the monodromy of a plane curve singularity from a picture provided by a small local real deformation of the singularity with the maximal possible number of double points in $\mathbb{R}^2$.

Let the polynomial function $f : \mathbb{C}^2 \to \mathbb{C}$ has at $0 \in \mathbb{C}^2$ an isolated singularity. We may assume without restricting the local topology of the singularity that the factorization of f into irreducible local branches has real factors. Let $B(f) \subset S^3$ be the local link of the singularity of f at 0. Let $P(\bar{f})$ be the picture of some small real deformation $\bar{f}$ with the maximal number of local double points of the singularity.

The following theorem gives new insight for the local topology of plane curve singularities.

THEOREM. *The links $B(f)$ and $L(P(\bar{f}))$ are equivalent.*

An embedded tree B in the unit disk D, such that the intersection $B \cap \partial D$ consists of one terminal vertex r of B, is called a rooted planar tree. For a rooted planar tree B there exists an immersed copy $P_B \subset D$ of the interval $[0, 1]$ with the following properties:
(i) The immersion is relative, i.e. the endpoints are embedded in ∂D.
(ii) The immersion is generic, i.e. there are only transversal crossing points, only the endpoints lie on ∂D and the immersion is transversal to ∂D.
(iii) The double points of P_B lie in the interior of the edges of B, such that the local branches are transversal to the edge of B.
(iv) Each connected component of $D \setminus P_B$ contains exactly one vertex of B.
(v) The only intersection points of P_B with B are the double points of P_B.

The immersed curve P_B is well-defined up to regular relative isotopy and is called the slalom curve of the rooted planar tree B, see Fig. 3. The left picture of Fig. 2 is a slalom curve.

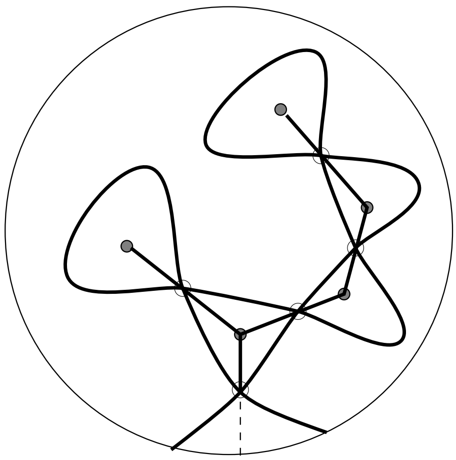

Figure 3: Rooted planar tree, its Dynkin diagram E_{10} and slalom.

The Dynkin diagram Δ_B of the slalom curve P_B is deduced from the rooted tree B as follows: First make a new tree B' by subdividing each edge of B with a new vertex, which is placed at the crossing point of P_B on the edge; next, remove from B' the root vertex r and the terminal edge of B' pointing to r. In Fig. 3 the tree B has the shape of the classical Dynkin diagram D_6 but the Dynkin diagram Δ_B of P_B has 10 vertices and we can denote it by E_{10}. The Dynkin diagram Δ_B of a rooted tree B is a bicolored rooted tree with an embedding in the plane. The root is the new vertex which lies on the edge of B originating from the root point of B and the bicoloring is such that the new vertices are of the same color. Moreover, the Dynkin diagram Δ_B has the property that the terminal vertices of Δ_B different from the root, are never new.

THEOREM. *Let B be a rooted tree. The complement of the slalom knot K_B admits a complete hyperbolic metric of finite volume if and only if the Dynkin diagram Δ_B is neither the diagram A_{2k}, $1 \leq k$, nor the diagrams E_6 or E_8.*

Using in addition to rooted trees also "disk-wide-webs" the theorem has a formulation which includes also extended Dynkin diagrams.

If the Dynkin diagram Δ_B is among $A_{2k},\ 1 \leq k,\ E_6,\ E_8$, the knot K_B is the torus knot $(2, 2k+1)$, $(3,4)$ or $(3,5)$ and appears as the local knot of a simple plane curve singularity; the monodromy diffeomorphism (with free boundary) of the knot K_B can be chosen to be of finite order in those cases and its complement does not carry a complete hyperbolic metric.

From the above theorem we get many examples of hyperbolic fibered knots, whose monodromy diffeomorphism and gordian number are known explicitly. The monodromy diffeomorphism of a slalom knot can be realized as a product of right Dehn twists of a system of simple closed curves on the fiber surface, such that the union of the curves is a spline in the fiber surface and the dual graph of the system is the Dynkin diagram of the rooted tree; the gordian number of a slalom knot equals the number of crossings of the slalom divide. We call the isotopy class of the monodromy diffeomorphism of the slalom knot of a rooted tree the Coxeter diffeomorphism of the Dynkin diagram of the rooted tree. It follows from the theorem and a theorem of Thurston that a Coxeter diffeomorphism of the Dynkin diagram of a rooted tree is pseudo-anosov if and only if the Dynkin diagram is not a classical Dynkin diagram.

Slalom knots can be characterized by the following properties:
- the knot is fibered,
- the monodromy has minimal complexity $4g-1$, where g is the genus of the knot,
- the monodromy is a product of Dehn twists, which all belong to the same conjugacy class in the mapping class group.

The complexity of an element T of the mapping class group, which we use here, is the minimum of the quantity $a+b$ over all the product decompositions of T as product of Dehn twists, where a is the length of the product and where b is the number of mutual intersection points of the core curves of the twists involved in the product decomposition.

References

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