THE FORMAL GROUP LAW AND TRANSFERRED CHERN CLASSES IN MORAVA K-THEORY

MALKHAZ BAKURADZE

ABSTRACT. Chern classes of transferred bundle for cyclic covering are evaluted in terms of transferred Chern classes by certain property of the formal group law in Morava K-theory

1. Introduction and Statements

This is joint project with S. Priddy in preparation. The author takes the opportunity as a visitor at Max-Planck Institute of Mathematics to give this material in more details with illustrated examples and tables.

A generalized cohomology theory $E^*(-)$ is a homotopy-invariant contravariant fuctor that assings for each integer n to each space X an abelian group $E^n(X)$ connected by natural suspension isomorphism $\Sigma : E^n(X) \simeq E^{n+1}(\Sigma X)$ of abelian groups and satisfies the axioms:
(i). $E^n(-)$ is half-exact: If $X = A \bigcup B$ and $y \in E^n(A)$ and $z \in E^n(B)$ agree in $E^n(A \bigcap B)$, there exists $x \in E^n(X)$ that lifts both y and z.
(ii). $E^n(-)$ is strongly additive: For any disjoint union $X = \coprod_\alpha X_\alpha$, the inclusions induce $E^n(X) \simeq \prod_\alpha E^n(X_\alpha)$.

The cohomology theory $E^*(-)$ is multiplicative if $E^*(X)$ is naturally a commutative graded ring with unit and suspension isomorphism of $E^*(X)$ modules.

A multiplicative cohomology theory E^* is called complex oriented of there exists a Thom class,that is, a class $u \in E^2(CP^\infty)$ that restricts to a generator of the free one-dimensional E^*-module $E^2(CP^1)$. Then by Dold theorem

$$E^*(CP^\infty) = E^*[[x]],$$

where x is the Chern class of canonical complex line bundle ξ over $CP^\infty = BU(1)$ and

$$E^*(BU(1)^n) = E^*[[x_1, ..., x_n]], \tag{1}$$

$x_i = c_1(\xi_i)$, and ξ_i is the pull back bundle over $BU(1)^n$ by the projection $BU(1)^n \to B(1)$ on the i-th factor.

If E has a line bundle Chern class then the splitting principle says that the diagonal inclusion of $U(1)^n$ in $U(n)$ induces the monomorphism and

$$E^*(BU(n)) = E^*[[c_1, ..., c_n]],$$

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where $c_i \in E^{2i}BU(n)$, the i-th Chern class of the universal $U(n)$ bundle over $BU(n)$, restricts to the i-th elementary symmetric function in $x_1, ..., x_n$. The following axioms determine the general Chern classes uniquely:
(i). Naturality with respect to pull backs;
(ii). Normalization, that is, $c_i(\xi) = 0$ for all $i > deg(\xi)$;
(iii). Hopf line bundle over CP^∞ has a Chern class;
(iv). The Whitney Formula:

$$c_k(\xi \oplus \eta) = c_k(\xi) + \sum_{1 \le i \le k-1} c_{k-i}(\xi)c_i(\eta) + c_k(\eta).$$

The formal group law. For computing $c_i(\xi \otimes \eta)$, the Chern class for tensor product of bundles, we need a formula for tensor product of line bundles and Whitney formula. In view of equation (1), there must be

$$c_1(\xi_1 \otimes \xi_2) = x_1 + x_2 + \sum_{i,j} a_{ij} x_1^i x_2^j = F(x_1, x_2)$$

in the universal case, and therefore generally, where

$$F(x, y) = x + y + \sum_{i,j} a_{ij} x^i y^j$$

in formal power series ring $E^*[[x, y]]$.

$F(x, y)$ is called formal group law in E^* and satisfies the identities which obviously follow from the corresponding bundle identities:
(i). $F(x, y) = F(y, x)$;
(ii). $F(F(x, y), z) = F(x, F(y, z))$;
(iii). $F(x, 0) = x$.

The universal example of complex-oriented cohomology theory is complex cobordism $MU^*(-)$, accosiated with Thom spectrum MU. By Quillen's theorem [5] for any complex oriented cohomology theory $E^*(-)$ there is a map of ring spectra $g : MU \to E$, which induces the formal group law, that is, $g(\alpha_{ij}^{MU}) = \alpha_{ij}^E$.
The Morava FGL. Let us turn to Morava K-theories. For fixed prime p there is complex oriented cohomology $K(s)^*(-)$ named after J.Morava with the coefficient ring $K^* = K^*(pt) = F_p[v_s, v_s^{-1}]$, with $| v_s |= 2(1 - p^s)$. The following lemma [3] plays central role in our discussion.

Lemma 1.1. *For the formal group law in Morava K-theory $K(s)^*$, $s > 1$ we have mod $y^{p^{2(s-1)}}$*

$$F(x, y) \equiv x + y - v_s \sum_{0<j<p} p^{-1} \binom{p}{j} (x^{p^{s-1}})^j (y^{p^{s-1}})^{p-j}.$$

Proof. As D. Ravenel explained, this result can be derived from the recursive formula for the FGL given in 4.3.8 [6]: For the FGL in Morava K-theory it reads

$$F(x, y) = F(x + y, v_s w_1(x, y)^{p^{s-1}}, v_s^{e_2} w_2(x, y)^{p^{2(s-1)}}, ...)$$

where w_i is a certain homogeneous polynomial of degree p^i defined by 4.3.5 and $e_i = (p^{is} - 1)/(p^s - 1)$. In particular $w_0 = x + y$ (the first term in the formula above),

$$w_1 = - \sum_{0<j<p} p^{-1} \binom{p}{j} x^j y^{p-j},$$

and $w_i \notin (x^p, y^p)$.

For $s > 1$ we can reduce modulo the ideal $v_s^{e_2}(x^{p^{2(s-1)}}, y^{p^{2(s-1)}})$ and get

$$F(x,y) \equiv F(x+y, v_s w_1(x,y)^{p^{s-1}})$$
$$= F(x+y+v_s w_1(x,y)^{p^{s-1}}, v_s w_1(x+y, v_s w_1(x,y)^{p^{s-1}})^{p^{s-1}}, \dots)$$
$$\equiv F(x+y+v_s w_1(x,y)^{p^{s-1}}, v_s w_1(x^{p^{s-1}}+y^{p^{s-1}}, v_s^{p^{s-1}} w_1(x,y)^{p^{2(s-1)}})),$$

and modulo $v_s^{1+p^{s-1}}(x^{p^{2(s-1)}}, y^{p^{2(s-1)}})$ we have $F(x,y) \equiv x+y+v_s w_1(x,y)^{p^{s-1}}$.

□

Let p be a prime. For a given action of cyclic group Z/p on a space X consider the regular covering

$$\rho : EZ/p \times X \to EZ/p \times_{Z/p} X.$$

Let us write for short $X_{hZ/p} = EZ/p \times_{Z/p} X$. For the permutation action of $\pi = Z/p$ on $X = Y^p$ we have $Y^p_{h\pi} = E\pi \times_\pi Y^p$. Let t be a generator of π and let $N_\pi = 1+t+\dots+t^{p-1}$ be the trace map. For a n-plane bundle η^n the corresponding classifying map

$$f : X \to BU(n)$$

gives the classifying map $(f, tf, \dots, t^{p-1}f)$ for the bundle $N_\pi \eta^n$ and the map of orbit spaces

$$f_{\eta^n} : X_{hZ/p} \to BU(n)^p_{h\pi}.$$

For the covering

$$\rho_\pi : E\pi \times BU(n)^p \to E\pi \times_\pi BU(n)^p$$

and the universal $U(n)$ bundle $\xi^n \to BU(n)$ consider the Atiyah transfer bundle [2]

$$\xi^n_\pi \to E\pi \times_\pi BU(n)^p,$$

the np-plane bundle

$$\xi^n_\pi = E\pi \times_\pi (\xi^n)^{\times p}.$$

Then the map f_{η^n} classifies the Atiyah transfer bundle for η^n and ρ. So by naturality of transfer [1] we can consider ρ_π as the universal example.

Recall that by the Künneth isomorphism $K(s)^*(BU(n)^p) = F \oplus T$ as a π module, where F is free and T is trivial.

Definition 1.2. Let $\omega_k^{(n)} \in F$ be defined modulo $kerN_\pi = Im(1-t^*)$ by

$$N_\pi(\omega_k^{(n)}) = c_k(N_\pi(\xi^n)),$$

where c_k are Chern classes, $k \in \{1, \dots, np\} \setminus \{p, 2p, \dots, np\}$.

By transfer property $Tr^*_\pi = Tr^*_\pi t^*$, where

$$Tr^*_\pi : K(s)^*(E\pi \times BU(n)^p)) \to K(s)^*(BU(n)^p_{h\pi}),$$

hence $Tr^*_\pi(\omega_k^{(n)})$ is well defined.

We write $\omega_k(\eta^n)$ for pullback by above map f_{η^n} of orbit spaces.

Recall $K(s)^*(BZ/p) = F_p[v_s, v_s^{-1}][z]/(z^{p^s})$, where $z = c_1(\theta)$ is the Chern class of the canonical line complex bundle over BZ/p.

Lemma 1.3. *We can define the polynomial in $1+n$ variables*

$$A_k^{(n)}(z, Z_1, ..., Z_n) \in K(s)^*[z, Z_1, ..., Z_n]$$

by the equation in $K(s)^*B(Z/p \times U(n))$:

$$C_k - v_s z^{p^s-1} p^{-1} \sum_{i_1+2i_2...+ni_n=k} \binom{p}{i_0, i_1, ..., i_n} c_1^{i_1} ... c_n^{i_n} = A_k^{(n)}(z, C_p, ..., C_{pn})$$

where $C_i = c_i(\xi^n \oplus \theta \otimes \xi^n \oplus ... \oplus \theta^{p-1} \otimes \xi^n)$, $c_j = c_j(\xi^n)$ are Chern classes, $k \in \{1, ...np\} \setminus \{p, ..., np\}$.

For example in $K(s)^*(B\pi \times BU(1))$ one has

$$C_1 = v_s \left(z^{p^s-1} c_1 + \sum_{i=1}^{s-1} z^{p^s-p^i} C_p^{p^{i-1}} \right),$$

thus

$$A_1^{(1)}(z, Z_1) = v_s \sum_{i=1}^{s-1} z^{p^s-p^i} Z_1^{p^{i-1}}.$$

Then using the polynomials $A_k^{(n)}$ we evaluate the transferred classes $\omega_k(\eta^n)$ for regular coverings:

Theorem 1.4. *Let $\rho : X \to X/_{Z/p}$ be the regular cyclic covering of prime index p defined by a free action of Z/p on X and $Tr^* = Tr^*_\rho$ be the associated transfer homomorphism. Let $\eta^n \to X$ be a complex n-plane bundle, $\eta^n_{Z/p} \to X/_{Z/p}$ the pn-plane bundle defined by Atyiah transfer, and $\psi \to X/_{Z/p}$ be the line complex bundle associated with ρ. Then*

$$c_k(\eta^n_{Z/p})) - Tr^*(\omega_k(\eta^n)) = A_k^{(n)}(c_1(\psi), c_p(\eta^n_{Z/p}), ..., c_{pn}(\eta^n_{Z/p})),$$

where $k \in \{1, ..., np\} \setminus \{p, ..., np\}$.

Example 1.5. $\omega_1 = c_1(\eta^n)$ hence if $k = 1$ we have

$$c_1(\eta^n_{Z/p})) - Tr^*(c_1(\eta^n)) = A_1^{(n)}(c_1(\psi), c_p(\eta^n_{Z/p}), ..., c_{pn}(\eta^n_{Z/p})).$$

If in addition $n = 1$ then by above example we have for the line bundle $\eta \to X$ and transferred Chern class

$$c_1(\eta_{Z/p}) - Tr^*(c_1(\eta)) = v_s \sum_{i=1}^{s-1} z^{p^s-p^i} c_p(\eta_{Z/p})^{p^{i-1}}.$$

2. Proofs

Proof of Lemma 1.3. It is clear that we have to prove the Lemma in case $n = 1$ and then apply the splitting principle. In case $n = 1$, let σ_k be the k-th symmetric function in p variables. Lemma 2.1 implies that in $K(s)^*B(\pi \times U(1)) = K(s)^*[[z, x]]/(z^{p^s})$ the difference

$$\sigma_k(x, F(x, z), ..., F(x, (p-1)z)) - p^{-1} \binom{p}{k} x^k v_s z^{p^s-1},$$

for $1 \leq k \leq p-1$ is a polynomial in z^{p-1} and $\sigma_p = \sigma_p(x, F(x,z), ..., F(x,(p-1)z))$. By definition the classes $C_k = \sigma_k$, and therefore define $A_k^{(1)}$, as polynomials in z^{p-1} and σ_p.

Next we consider the case $n = 2$ in detail. Let us apply the splitting principle. Consider $B\pi \times BU(1) \times BU(1)$ and write formally
$\xi^2 = \xi + \eta$ and

$$x = c_1(\xi),$$
$$X_i = c_i(\xi + \xi \otimes \theta + ... + \xi \otimes \theta^{p-1}),$$
$$y = c_1(\eta),$$
$$Y_i = c_i(\eta + \eta \otimes \theta + ... + \eta \otimes \theta^{p-1}),$$

$i = 1, ..., p.$

Then from the case $n = 1$ we have:

$$X_k = \sigma_k(x, F(x,z), ..., F(x,(p-1)z)) = p^{-1}\binom{p}{k} x^k v_s z^{p^{s-1}} + A_k^{(1)}(z^{p-1}, X_p) \qquad (*)$$

and

$$Y_k = \sigma_k(y, F(y,z), ..., F(y,(p-1)z)) = p^{-1}\binom{p}{k} y^k v_s z^{p^{s-1}} + A_k^{(1)}(z^{p-1}, Y_p) \qquad (**)$$

for $k = 1, ..., p-1$.

Then the classes $C_i = c_i(\xi^2 + \xi^2 \otimes \theta + ... + \xi^2 \otimes \theta^{p-1})$ we can express by X_i and Y_i using Whitney formula for the sum of bundles C_i is symmetric in classes X_i and Y_i and the above formulas (*), (**) for the elements X_i and Y_i defines the equation:

$$C_k = \sum_{i+2j=k} \delta_{ij} c_1^i c_2^j + A^{(2)}(z, C_p, C_2p)$$

We want to see that

$$\delta_{i,j} = v_s z^{p^s-1} p^{-1} \sum_{i+2j=k} \binom{p}{i_0, i, j}$$

For this we have to compute the contribution of this procedure to coefficients δ_{ij}.

Let $k = 1, ..., p-1$. By the Whitney formila

$$C_k = X_k + Y_k + ...$$

but it follows from (*), (**) that the other terms $X_i Y_j$ are irrelevant since z is nilpotent and these terms have no contribution to δ_{ij}.(Of course they are relevant to $A^{(2)}(z, C_p, C_{2p})$ but we are looking at deltas only)

Then as above the contribution of X_k is

$$p^{-1}\binom{p}{k} x^k v_s z^{p^{s-1}}$$

and the total contribution will

$$p^{-1}\binom{p}{k}(x^k + y^k) v_s z^{p^{s-1}}$$

hence for $1 \leq k \leq p-1$ the result follows from the following :
Lemma(***). Let

$$x^k + y^k = \sum_{i+2j=k} \beta_{ij} c_1^i c_2^j$$

where $c_1 = x + y$, $c_2 = xy$ then modulo p

$$p^{-1}\binom{p}{k}\beta_{ij} = p^{-1}\sum_{i+2j=k}\binom{p}{i_0, i, j}.$$

To complete the proof in case $n = 2$ we have to consider C_k for $k = p+1, ...2p-1$. Again

$$C_{p+l} = X_l Y_p + X_p Y_l + ...$$

$l = 1, ..., p-1$, the other terms are irrelevant. Also since $Y_p = y^p +$ terms divisible by z the contribution of $X_l Y_p$ is

$$p^{-1} v_s z^{p^s-1}\binom{p}{l} x^l y^p$$

hence we have the total contribution for C_{p+l}

$$p^{-1} v_s z^{p^s-1}\binom{p}{l}(x^l y^p + x^p y^l). \quad (IV*)$$

Then

$$x^l y^p + x^p y^l = x^l y^l (x^{p-l} + y^{p-l}) = c_2^l (x^{p-l} + y^{p-l}).$$

By above Lemma(***) we have that if

$$c_2^l (x^{p-l} + y^{p-l}) = c_2^l \sum_{i+2j=p-l} \gamma_{ij} c_1^i c_2^j$$

then modulo p

$$p^{-1}\binom{p}{p-l}\gamma_{ij} = p^{-1}\sum_{i_1+2j=p-l}\binom{p}{i_0, i, j}.$$

Here in binomial coefficient we can replase $p - l$ by l. Also since in multinomial coefficient $l = p - i - 2j$ we can replace j by $j + l = p - i - j = i_0$. Thus for above (IV*) we have

$$p^{-1} v_s z^{p^s-1}\binom{p}{l}(x^l y^p + x^p y^l) = p^{-1} v_s z^{p^s-1}\sum_{i_1+2j=p-l}\binom{p}{i_0, i, j} c_1^i c_2^i c_2^l$$

$$= p^{-1} v_s z^{p^s-1}\sum_{i_1+2(j+l)=p+l}\binom{p}{i_0, i, j} c_1^i c_2^{j+l}.$$

This completes the froof in case $n = 2$.

The general case is completely analogous.

Initially the guess was another. Namely the formula from Lemma 1.3 should follow from similar but less tractable formula in BP^*. In $BP*$ should be

$$C_k - Tr^*(1)p^{-1}\sum_{i_1+2i_2...+ni_n=k}\binom{p}{i_0, i_1, ..., i_n} c_1^{i_1} ... c_n^{i_n}$$

on the left side since C_k pulls back to $c_k(\xi^n + ... + \xi^n)$, (p times.) in $BP^*(BZ/p \times BU(n))$ and the total Chern class $1 + C_1 + ... + C_{np}$ pulls back to $(1 + c_1 + ... + c_n)^p$ So C_k pulls back to

$$pp^{-1} \sum_{i_1 + 2i_2 ... + ni_n = k} \binom{p}{i_0, i_1, ..., i_n} c_1^{i_1} ... c_n^{i_n}$$

so is

$$Tr^*(1)p^{-1} \sum_{i_1 + 2i_2 ... + ni_n = k} \binom{p}{i_0, i_1, ..., i_n} c_1^{i_1} ... c_n^{i_n}.$$

Proof of Theorem 1.4. Consider the homotopy orbit space $BU(n)^p_{h\pi} = E\pi \times_\pi BU(n)^p$ as the universal example. The diagonal map $BU(n) \to BU(n)^p$ induces the inclusion

$$i : B\pi \times BU(n) \to E\pi \times_\pi BU(n)^p.$$

We use a result from [4] (Proposition 4.2) which implies that since $BU(n)$ is a unitary-like space (i.e. $K(s)^*BU(n)$ has no nilpotent elements) the map $(i \vee \rho_\pi)^*$ is a monomorphism. Since $\rho_\pi^* Tr^* = N_\pi$, the difference

$$c_k(\eta_\pi^n) - Tr^*(\omega_k^{(n)})$$

belongs to $ker \rho_\pi^*$ and hence is detected by i^*. The result now follows from Lemma 1.3.

3. Some exapmles and tables

First recall from [6] that generators for

$$\begin{array}{ccc} \pi_* BP & \subset & H_* BP \\ \updownarrow & & \updownarrow \\ Z_{(p)}[v_1, v_2, ...] & \subset & Z_{(p)}[m_1, m_2, ...] \end{array}$$

$$|v_n| = 2(p^n - 1) = |m_n|$$

are given by

$$v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^{p^i}.$$

We use the above formula with the following well known formulas

$$F(x, y) = exp(logx + logy),$$

$$logx = \sum_{n \geq 0} m_n x^{n+1}$$

for computing $\alpha_{ij} = \alpha_{ij}^{BP}$.

The coefficients of formal group law in BP theory

$\alpha_{11} = -v_1;$
$\alpha_{12} = v_1^2;$
$\alpha_{13} = -2v_1^3 - 2v_2;$
$\alpha_{22} = -4v_1^3 - 3v_2;$
$\alpha_{14} = 3v_1^4 + 4v_1 v_2;$
$\alpha_{23} = 10v_1^4 + 11v_1 v_2;$
$\alpha_{15} = -4v_1^5 + 6v_1^2 v_2;$
$\alpha_{24} = -21v_1^5 - 28v_1^2 v_2;$

$\alpha_{33} = -34v_1^5 - 43v_1^2v_2;$
$\alpha_{16} = 6v_1^6 + 12v_1^3v_2 + 4v_2^2;$
$\alpha_{25} = 43v_1^6 + 75v_1^3v_2 + 18v_2^2;$
$\alpha_{34} = 101v_1^6 + 164v_1^3v_2 + 34v_2^2;$
$\alpha_{17} = -10v_1^7 - 24v_1^4v_2 - 14v_1v_2^2 - 4v_3;$
$\alpha_{26} = -88v_1^7 - 190v_1^4v_2 - 89v_1v_2^2 - 14v_3;$
$\alpha_{35} = -275v_1^7 - 551v_1^4v_2 - 226v_1v_2^2 - 28v_3;$
$\alpha_{44} = -394v_1^7 - 765v_1^4v_2 - 302v_1v_2^2 - 35v_3;$
$\alpha_{18} = 15v_1^8 + 40v_1^5v_2 + 28v_1^2v_2^2 + 8v_1v_3;$
$\alpha_{27} = 169v_1^8 + 420v_1^5v_2 + 257v_1^2v_2^2 + 46v_1v_3;$
$\alpha_{36} = 680v_1^8 + 1586v_1^5v_2 + 879v_1^2v_2^2 + 126v_1v_3;$
$\alpha_{45} = 1303v_1^8 + 2933v_1^5v_2 + 1543v_1^2v_2^2 + 203v_1v_3;$
$\alpha_{19} = -22v_1^9 - 66v_1^6v_2 - 58v_1^3v_2^2 - 12v_1^2v_3 - v_2^3;$
$\alpha_{28} = -312v_1^9 - 880v_1^6v_2 - 688v_1^3v_2^2 - 104v_1^2v_3 - 72v_2^3;$
$\alpha_{37} = -1573v_1^9 - 4192v_1^6v_2 - 3001v_1^3v_2^2 - 382v_1^2v_3 - 260v_2^3;$
$\alpha_{46} = -3861v_1^9 - 9900v_1^6v_2 - 6707v_1^3v_2^2 - 791v_1^2v_3 - 523v_2^3;$
$\alpha_{55} = -5156v_1^9 - 13042v_1^6v_2 - 8671v_1^3v_2^2 - 1001v_1^2v_3 - 654v_2^3.$

Example $n = 2$

Let $\alpha_{ij} = \alpha_{ij}^{G(2)}$ then using the map $BP \to G(n)$, $BP^* = Z_{(p)}[v_1, v_2, ...]$, $G(n)^* = Z_{(p)}[v_n.v_n^{-1}]$, which satisfies $v_r \to v_r$ if $r = n$, and $v_r \to 0$, otherwise
$\alpha_{13} = -2v_2;$
$\alpha_{22} = -3v_2;$
$\alpha_{16} = 4v_2^2;$
$\alpha_{25} = 18v_2^2;$
$\alpha_{34} = 34v_2^2;$
and otherwise $\alpha_{ij} = 0$, for $i + j = 8$.

Below are the initial segments of FGL of Morava K-theory $K(s)^*$.

$p = 2$, $s = 2$:

$x + y + v_2x^2y^2$
$+{v_2}^3\left(x^6y^4 + x^4y^6\right)$
$+{v_2}^5\left(y^{12}x^4 + y^4x^{12}\right)$
$+{v_2}^7\left(y^{10}x^{12} + y^{12}x^{10} + y^{14}x^8 + y^8x^{14}\right)$
$+{v_2}^9\left(y^{20}x^8 + y^8x^{20}\right)$
$+{v_2}^{11}\left(y^8x^{26} + y^{10}x^{24} + y^{24}x^{10} + y^{26}x^8\right)$
$+{v_2}^{13}\left(y^{12}x^{28} + y^{28}x^{12}\right)$
$+{v_2}^{15}\left(y^{30}x^{16} + y^{16}x^{30} + y^{18}x^{28} + y^{20}x^{26} + y^{22}x^{24} + y^{24}x^{22} + y^{26}x^{20} + y^{28}x^{18}\right)$
$+{v_2}^{17}\left(y^{20}x^{32} + y^{32}x^{20}\right)$
$+{v_2}^{19}\left(y^{16}x^{42} + y^{18}x^{40} + y^{40}x^{18} + y^{42}x^{16}\right)$
$+{v_2}^{21}\left(y^8x^{56} + y^{20}x^{44} + y^{40}x^{24} + y^{56}x^8 + y^{24}x^{40} + y^{44}x^{20}\right)$
$+{v_2}^{23}\left(y^{16}x^{54} + y^{18}x^{52} + y^{20}x^{50} + y^{22}x^{48} + y^{48}x^{22} + y^{50}x^{20} + y^{52}x^{18} + y^{54}x^{16}\right)$
$+{v_2}^{25}\left(y^{28}x^{48} + y^{16}x^{60} + y^{48}x^{28} + y^{60}x^{16}\right)$
$+{v_2}^{27}\left(y^{24}x^{58} + y^{26}x^{56} + y^{56}x^{26} + y^{58}x^{24}\right)$
$+{v_2}^{29}\left(y^{16}x^{72} + y^{28}x^{60} + y^{32}x^{56} + y^{40}x^{48} + y^{48}x^{40} + y^{60}x^{28} + y^{56}x^{32} + y^{72}x^{16}\right)$

$+{v_2}^{31}\left(y^{32}x^{62}+y^{34}x^{60}+y^{38}x^{56}+y^{36}x^{58}+y^{40}x^{54}+y^{42}x^{52}+y^{44}x^{50}+y^{48}x^{46}\right.$
$\left.+y^{46}x^{48}+y^{50}x^{44}+y^{52}x^{42}+y^{54}x^{40}+y^{56}x^{38}+y^{58}x^{36}+y^{60}x^{34}+y^{62}x^{32}\right)$
$+\ldots$

$p=2,\ s=3$:

$x+y+v_3x^4y^4$
$+{v_3}^{5}\left(y^{20}x^{16}+y^{16}x^{20}\right)$
$+{v_3}^{9}\left(y^{16}x^{48}+y^{48}x^{16}\right)$
$+{v_3}^{21}\left(y^{80}x^{68}+y^{84}x^{64}+y^{64}x^{84}+y^{68}x^{80}\right)$
$+{v_3}^{25}\left(y^{112}x^{64}+y^{64}x^{112}\right)$
$+\ldots$

$p=2,\ s=4$:

$x+y+v_4y^8x^8$
$+{v_4}^{9}\left(y^{72}x^{64}+y^{64}x^{72}\right)$
$+{v_4}^{17}\left(y^{64}x^{192}+y^{192}x^{64}\right)$
$+{v_4}^{73}\left(y^{520}x^{576}+y^{512}x^{584}+y^{584}x^{512}+y^{576}x^{520}\right)$
$+\ldots$

$p=2,\ s=5$:

$x+y+v_5y^{16}x^{16}$
$+{v_5}^{17}\left(y^{272}x^{256}+y^{256}x^{272}\right)$
$+{v_5}^{33}\left(y^{768}x^{256}+y^{256}x^{768}\right)$
$+\ldots$

$p=3,\ s=2$:

$x+y+v_2\left(2\,x^6y^3+2\,x^3y^6\right)$
$+{v_2}^{4}\left(2\,y^{12}x^{21}+y^9x^{24}+y^{15}x^{18}+y^{18}x^{15}+y^{24}x^9+2\,y^{21}x^{12}\right)$
$+{v_2}^{7}\left(2\,y^{21}x^{36}+2\,y^{18}x^{39}+2\,y^{36}x^{21}+y^{30}x^{27}+y^{27}x^{30}+2\,y^{39}x^{18}\right)$
$+{v_2}^{10}\left(2\,y^{45}x^{36}+2\,y^{36}x^{45}+2\,y^{18}x^{63}+2\,y^{9}x^{72}+2\,y^{72}x^{9}+2\,y^{63}x^{18}\right)$
$+{v_2}^{13}\left(y^{48}x^{57}+2\,y^{27}x^{78}+y^{57}x^{48}+2\,y^{33}x^{72}+y^{30}x^{75}+2\,y^{60}x^{45}+y^{36}x^{69}\right.$
$+2\,y^{39}x^{66}+2\,y^{66}x^{39}+2\,y^{45}x^{60}+y^{42}x^{63}+2\,y^{51}x^{54}+2\,y^{54}x^{51}+y^{75}x^{30}$
$+2\,y^{78}x^{27}+y^{63}x^{42}+y^{69}x^{36}+2\,y^{72}x^{33}$
$+{v_2}^{16}\left(2\,y^{90}x^{39}+2\,y^{93}x^{36}+2\,y^{39}x^{90}+2\,y^{36}x^{93}+y^{63}x^{66}+y^{66}x^{63}\right)$
$+{v_2}^{19}\left(2\,y^{36}x^{117}+y^{27}x^{126}+y^{81}x^{72}+y^{72}x^{81}+2\,y^{54}x^{99}+2\,y^{99}x^{54}+y^{126}x^{27}+2\,y^{117}x^{36}\right)$
$+{v_2}^{22}\left(y^{69}x^{108}+y^{63}x^{114}+2\,y^{57}x^{120}+y^{60}x^{117}+2\,y^{66}x^{111}+y^{54}x^{123}+y^{123}x^{54}\right.$

$$
\begin{aligned}
&+2\,y^{81}x^{96}+y^{108}x^{69}+2\,y^{96}x^{81}+2\,y^{111}x^{66}+y^{114}x^{63}+y^{117}x^{60}+2\,y^{120}x^{57}\\
&+y^{93}x^{84}+y^{84}x^{93}+2\,y^{87}x^{90}+2\,y^{90}x^{87}\big)\\
&+{v_2}^{25}\left(2\,y^{57}x^{144}+2\,y^{54}x^{147}+2\,y^{75}x^{126}+2\,y^{72}x^{129}+y^{84}x^{117}+y^{81}x^{120}\right.\\
&+y^{99}x^{102}+2\,y^{93}x^{108}+2\,y^{90}x^{111}+y^{102}x^{99}+2\,y^{111}x^{90}+2\,y^{108}x^{93}+y^{117}x^{84}\\
&\left.+y^{120}x^{81}+2\,y^{129}x^{72}+2\,y^{126}x^{75}+2\,y^{144}x^{57}+2\,y^{147}x^{54}\right)\\
&+{v_2}^{28}\left(2\,y^{54}x^{171}+2\,y^{90}x^{135}+2\,y^{135}x^{90}+2\,y^{171}x^{54}\right)\\
&+\ldots
\end{aligned}
$$

$p=3,\ s=3$:

$$
\begin{aligned}
&x+y+v_3\left(2\,x^{9}y^{18}+2\,x^{18}y^{9}\right)\\
&+{v_3}^{10}\left(y^{81}x^{180}+2\,y^{90}x^{171}+y^{99}x^{162}+y^{162}x^{99}+2\,y^{171}x^{90}+y^{180}x^{81}\right)\\
&+{v_3}^{19}\left(2\,y^{171}x^{324}+2\,y^{162}x^{333}+y^{243}x^{252}+2\,y^{333}x^{162}+y^{252}x^{243}+2\,y^{324}x^{171}\right)\\
&+{v_3}^{28}\left(2\,y^{567}x^{162}+2\,y^{162}x^{567}+2\,y^{648}x^{81}+2\,y^{81}x^{648}+2\,y^{324}x^{405}+2\,y^{405}x^{324}\right)\\
&+\ldots
\end{aligned}
$$

$p=7,\ s=2$:

$$
\begin{aligned}
&x+y+v_2\left(4\,y^{35}x^{14}+6\,y^{42}x^{7}+4\,y^{14}x^{35}+2\,y^{28}x^{21}+2\,y^{21}x^{28}+6\,y^{7}x^{42}\right)\\
&+{v_2}^{8}\left(4\,y^{105}x^{280}+3\,y^{98}x^{287}+3\,y^{126}x^{259}+4\,y^{119}x^{266}+2\,y^{154}x^{231}+3\,y^{140}x^{245}\right.\\
&+5\,y^{147}x^{238}+4\,y^{133}x^{252}+y^{91}x^{294}+2\,y^{182}x^{203}+5\,y^{175}x^{210}+2\,y^{168}x^{217}\\
&+5\,y^{161}x^{224}+2\,y^{217}x^{168}+5\,y^{210}x^{175}+2\,y^{203}x^{182}+5\,y^{196}x^{189}+5\,y^{189}x^{196}\\
&+4\,y^{252}x^{133}+3\,y^{245}x^{140}+5\,y^{238}x^{147}+2\,y^{231}x^{154}+5\,y^{224}x^{161}\\
&+3\,y^{273}x^{112}+4\,y^{266}x^{119}+3\,y^{259}x^{126}+6\,y^{315}x^{70}+y^{308}x^{77}+6\,y^{301}x^{84}\\
&+y^{294}x^{91}+3\,y^{287}x^{98}+4\,y^{280}x^{105}+y^{336}x^{49}+6\,y^{329}x^{56}+y^{322}x^{63}\\
&\left.+y^{49}x^{336}+y^{63}x^{322}+6\,y^{56}x^{329}+6\,y^{70}x^{315}+y^{77}x^{308}+6\,y^{84}x^{301}+3\,y^{112}x^{273}\right)\\
&+{v_2}^{15}\left(4\,y^{413}x^{308}+6\,y^{427}x^{294}+2\,y^{420}x^{301}+2\,y^{448}x^{273}+6\,y^{441}x^{280}+4\,y^{98}x^{623}\right.\\
&+4\,y^{462}x^{259}+4\,y^{455}x^{266}+4\,y^{308}x^{413}+6\,y^{490}x^{231}+5\,y^{119}x^{602}+6\,y^{476}x^{245}\\
&+4\,y^{315}x^{406}+6\,y^{126}x^{595}+2\,y^{469}x^{252}+2\,y^{322}x^{399}+2\,y^{518}x^{203}+4\,y^{511}x^{210}\\
&+4\,y^{504}x^{217}+2\,y^{497}x^{224}+2\,y^{553}x^{168}+y^{546}x^{175}+3\,y^{539}x^{182}+6\,y^{525}x^{196}\\
&+5\,y^{609}x^{112}+5\,y^{602}x^{119}+6\,y^{595}x^{126}+4\,y^{588}x^{133}+3\,y^{574}x^{147}+4\,y^{133}x^{588}\\
&+y^{567}x^{154}+2\,y^{560}x^{161}+3\,y^{147}x^{574}+4\,y^{623}x^{98}+6\,y^{616}x^{105}\\
&+6\,y^{329}x^{392}+y^{154}x^{567}+2\,y^{168}x^{553}+2\,y^{161}x^{560}+3\,y^{182}x^{539}+y^{175}x^{546}\\
&+6\,y^{196}x^{525}+4\,y^{210}x^{511}+2\,y^{203}x^{518}+4\,y^{217}x^{504}+6\,y^{231}x^{490}+2\,y^{224}x^{497}\\
&+6\,y^{392}x^{329}+2\,y^{252}x^{469}+4\,y^{259}x^{462}+6\,y^{245}x^{476}+4\,y^{406}x^{315}+2\,y^{399}x^{322}\\
&+2\,y^{273}x^{448}+2\,y^{301}x^{420}+4\,y^{266}x^{455}+6\,y^{280}x^{441}+6\,y^{294}x^{427}+6\,y^{105}x^{616}\\
&+5\,y^{112}x^{609}+\ldots
\end{aligned}
$$

Examples of the polynomials A^1_k from the Theorem 1.4

Below are some $A^1_k(z,Z)$ from Lemma 1.3, $y=z^{p-1}$, $\sigma_p=Z$

$p=3, s=2$

$$
\begin{aligned}
&\sigma_1=v_2y^3\sigma_3+v_2y^4x\\
&\sigma_2=2\,{v_2}^2y^3{\sigma_3}^4+2\,v_2y^2{\sigma_3}^2+x^2v_2y^4+2\,y
\end{aligned}
$$

$p = 5, s = 3$

$\sigma_1 = v_3 y^{25} \sigma_5{}^5 + v_3 y^{30} \sigma_5 + v_3 y^{31} x$

$\sigma_2 = 4\, v_3{}^2 y^{25} \sigma_5{}^{30} + 4\, v_3{}^2 y^{30} \sigma_5{}^{26} + 3\, v_3 y^{19} \sigma_5{}^{10} + v_3 y^{24} \sigma_5{}^6 + 3\, v_3 y^{29} \sigma_5{}^2 + 2\, v_3 y^{31} x^2$

$\sigma_3 = 2\, v_3{}^3 y^{25} \sigma_5{}^{55} + 2\, v_3{}^3 y^{30} \sigma_5{}^{51} + v_3{}^2 y^{19} \sigma_5{}^{35} + 2\, v_3{}^2 y^{24} \sigma_5{}^{31} + v_3{}^2 y^{29} \sigma_5{}^{27} + 2\, v_3 y^{13} \sigma_5{}^{15} + v_3 y^{18} \sigma_5{}^{11} + v_3 y^{23} \sigma_5{}^7 + 2\, v_3 y^{28} \sigma_5{}^3 + 2\, x^3 v_3 y^{31}$

$\sigma_4 = 4\, v_3{}^4 y^{25} \sigma_5{}^{80} + 4\, v_3{}^4 y^{30} \sigma_5{}^{76} + 4\, v_3{}^3 y^{19} \sigma_5{}^{60} + 3\, v_3{}^3 y^{24} \sigma_5{}^{56} + 4\, v_3{}^3 y^{29} \sigma_5{}^{52} + 4\, v_3{}^2 y^{13} \sigma_5{}^{40} + 2\, v_3{}^2 y^{18} \sigma_5{}^{36} + 2\, v_3{}^2 y^{23} \sigma_5{}^{32} + 4\, v_3{}^2 y^{28} \sigma_5{}^{28} + 4\, y^7 v_3 \sigma_5{}^{20} + y^{12} v_3 \sigma_5{}^{16} + 4\, y^{17} v_3 \sigma_5{}^{12} + y^{22} v_3 \sigma_5{}^8 + 4\, y^{27} v_3 \sigma_5{}^4 + x^4 v_3 y^{31} + 4\, y$

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Razmadze Mathematical Institute, Tibilsi 380093, Republic of Georgia

Current address: Max-Planck-Institut Für Mathematik, Bonn

E-mail address: `bakuradz@mpim-bonn.mpg.de`