Symplectic topology of integrable Hamiltonian systems, II: Characteristic classes and integrable surgery

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Abstract

We give a topological and geometrical classification of integrable Hamiltonian systems with nondegenerate singularities in terms of their singularities, affine-structured orbit space, monodromies, and characteristic classes. In particular, we discover a new characteristic class, called the global Chern class, which lies in $H^2(O, \mathcal{R})$, where O is the base space of the system and $\mathcal{R}$ is some free Abelian (not locally constant in general) sheaf over O, called the affine monodromy sheaf. This characteristic class allows to classify systems topologically, and it coincides with the one found by Duistermaat for the case of regular Lagrangian torus foliations and by Boucetta and Molino for the case with only elliptic singularities. We discuss the obstructions to the construction of integrbalc systems from a given stratified integral affine manifold as the base space. As an application to symplectic geometry, we find a method, called integrable surgery, for constructing many known and unknown symplectic manifolds.

Keywords: *symplectic manifolds, integrable Hamiltonian systems, monodromy, characteristic classes, topological classification, integrable surgery*

AMS subject classification: 58F07, 70H05

1 Introduction

Integrable Hamiltonian systems (IHS's for short) are known to play a very important rôle in classical mechanics and physics, and it is a natural problem to study their topological properties. For such a study is important for the understanding of dynamical behavior of integrable systems and their perturbations, for finding obstructions to integrability, for detecting algebraically different or similar integrable systems, ect. On the other hand, according to a conjecture told to me by A.T. Fomenko, every symplectic manifold admits a nondegenerate IHS. Whether this conjecture is true or not, a topological study of IHS's will help us to understand more about the symplectic structures on manifolds.

In the previous paper [40] in this series, we studied the topological structure of nondegenerate singularities of IHS's. In this paper, we will give a topological and geometrical classification of IHS's with nondegenerate singularities. Before describing it, let us recall that the first significant result concerning topology of IHS's is what called Arnold-Liouville theorem, which gives the normal form, in terms of action-angle coordinates, for IHS's near a regular compact level set of a moment map (see e.g. [1]). The question about the existence of global action-angle coordinates for (the regular parts of) IHS's was studied, among other people, by Duistermaat [16], who found two important topological invariants, which are called monodromy and Duistermaat-Chern class respectively, and which together classify regular IHS's (i.e. without singulrities) over a given base space up to topological equivalence. The work of Duistermaat was made more precise (and extended to the case of complete isotropic foliations) by Dazord and Delzant [12], and was extended to the case of systems with only elliptic singularities by Boucetta and Molino [9]. In these papers [9, 12, 16], the Duistermaat-Chern class (or the Chern class as they call it) is defined with the help of local sections of the torus fibration, and it is an element in $H^2(O, \mathcal{R})$, where O is the base space of the foliation, and $\mathcal{R}$ is a locally constant free Abelian sheaf over O, which may be called the affine monodromy sheaf and defined as the sheaf of local $\mathbb{S}^1$ actions on the fibers of the Lagrangian fibration.

The main difficulty that we encountered while trying to generalize the above characteristic class to the case of integrable systems with more general nondegenerate singularities lies in the fact that we don't have local sections in the sense of fiber bundles: they do not exist near general hyperbolic singularities. There are two approaches to overcome this difficulty: The first approach is to avoid the use of local sections by dealing directly with local automorphism groups, and the second approach is to generalize the notion of local sections so that they always exist.

Using the first approach, we find a cohomological class, called *global Chern class* (cf. Subsection 4.1), which lies in $H^2(O, \mathcal{R})$, where O is the base space of the associated (singular) Lagrangian foliation of the system, and $\mathcal{R}$ is the sheaf of local $\mathbb{S}^1$-actions, just as before! Of course, our global Chern class coincides with the one studied by Duistermaat, Dazord, Delzant, Boucetta and Molino in the above-mentioned cases. The main differences from the regular case is that O is not a manifold but a stratified manifold, and $\mathcal{R}$ is not locally constant though still free Abelian. It is interesting to notice that the base space O has a natural affine structure which makes it into a *stratified integral affine manifold* (cf. Subsection 3.3).

The second approach is also carried out, for a large subset of possible base spaces (for which one can hope to have global generalized sections, cf. Subsection 4.3).

By generalizing the notion of monodromy from the regular case, we obtain the notion of *global monodromy*, which is the isomorphism class of the sheaf over the base space O, whose stalk at each point is the cohomology ring of the preimage of this point under the projection map, with integral coefficients (cf. Subsection 3.4). Two integrable systems are called *roughly topologically equivalent* if their base spaces can be identified via a homeomorphism, and under this identification, they have the same singularities topologically, and the same global monodromy. In order to define the global Chern class of an integrable system, we then have to compare it with another fixed system lying in the same rough topological equivalence class. (In the case of regular foliations, such a fixed system is the one which admits a global section. In general we don't have sections so the choice may be arbitrary).

Two integrable systems are called *topologically equivalent* if they have the same associated singular torus foliations topologically (cf. Subsection 3.1). Our main result is the following (cf. Subsection 4.1):

Theorem. Two integrable Hamiltonian systems with nondegenerate singularities are topologically equivalent if and only if their base spaces can be identified by a homeomorphism, and under this identification they have the same topological structure of singularities, the same global monodromy and the same global Chern class.

Two integrable systems are called *geometrically equivalent* if there is a smooth symplectomorphism between the two manifolds which preserves the associated singular foliations. By using the so called *Lagrangian global Chern class*, we also obtain a geometrical classification similar to the theorem stated above (cf. Subsection 4.1).

In the case of nondegenerate IHS's on 3-dimensional isoenergy submanifolds, studied by Fomenko and his school, the global Chern class vanishes, and the global monodromy may be characterized in terms of some numerical marks as in the so-called Fomenko-Zieschang invariant (cf. [19] and references therein). Let us mention here a nice recent application of topological invariants in this case: together with Maupertuis variational principle, they allowed Bolsinov and Fomenko to find some metrics on the 2-sphere, whose geodesic flows are integrable with the aid of first integrals of degree 3 or 4 in velocities, which cannot be reduced to linear or quadratic integrals (cf. [8] and references therein). Such integrable metrics have not been known before.

We want to advertise in this paper a simple idea, called *integrable surgery*, for constructing symplectic manifolds using integrable Hamiltonian systems. This method may complement other known methods for constructing symplectic structures (see e.g. [4, 30]). An integrable surgery is a symplectic surgery which respects some integrable systems on symplectic manifolds. Such a surgery is of course present implicitly in our definition of characteristic classes, and is important for integrable systems themselves. If the conjecture that every symplectic manifold admits a nondegenerate IHS is true then every symplectic manifold can be obtained from the simplest ones by integrable surgery. Anyway, symplectic manifolds constructed explicitly by symplectic surgery form a very large class, and may be used to check various conjectures in symplectic geometry, e.g. the conjecture about the rational homotopy type of simply-connected symplectic manifolds (cf. [35]). We will illustrate our idea by several simple examples throughout this paper.

The organization of this paper is as follows: Section 1 is this introduction. In Section 2 we recall briefly the theory of Duistermaat, Dazord and Delzant of regular torus Lagrangian foliations, for our work is a direct generalization of this theory to the case with singularities. The only original thing in this Section are two examples at the end: one is about the integrable point of view of Kodaira-Thurston example, the other one is an exotic symplectic structure on $\mathbb{R}^{2n}$ constructed using integrable surgery. Section 3 starts with a geometrical definition of nondegenerate IHS's, so that one can forget about the moment maps and deal only with singular Lagrangian foliations. Subsection 3.2 recalls the main results about nondegenerate singularities from [40], which are indespensable for the topological classification of integrable systems. In particular, these results allow us to study the integral affine structure of the base spaces in Subsection 3.3, and the sheaves of local automorphism groups in Subsection 3.5. In Subsection 3.4 we give the notion of affine and global monodromies, and rough topological / geometrical equivalences. Section 4 starts with the definition of characteristic classes, and classification theorems. Subsection 4.2 is devoted to the problem of constructing integrable systems from given stratified integral affine manifolds as the base spaces. We will find some homological obstructions for doing so. This Subsection also contains a few examples of integrable surgery. Subsection 4.3 is devoted to the study of generalized sections. Subsection 4.4, the last one, contains a theorem about the topology of the base space in the 2 degree of freedom case, which is an analog of Milnor's theorem about affine structures on 2-surfaces. It also contains some examples of integrable systems with two degrees of freedom, which are related to complex algebraic surfaces.

In this paper we will define the characteristic classes only for IHS's with nondegenerate singularities, but I think that they can be defined in the same way for IHS's with some degenerate singularities. To make it more precise, we would have to study in more detail degenerate singularities, and their automorphism groups in particular. It is desirable to have a theory similar to the Arnold theory of Lagrangian singularities, for degenerate singularities of IHS's. It is also a natural problem to study topological properties of IHS's not on symplectic manifolds, but on Poisson manifolds, i.e. allowing the systems to depend on some parameters....

2 Regular Lagrangian torus foliations

2.1 Local normal form

Let (M^{2n}, ω) be a smooth paracompact symplectic manifold, $H : (M^{2n}, \omega) \to \mathbb{R}$ a smooth Hamiltonian function. The Hamiltonian system $\dot{x} = X_H(x)$, defined by $i_{X_H}\omega = -dH$, is called *integrable* in the sense of Liouville (in this case we say that we have an IHS), if there exist n commuting first integrals $F_1 = H, F_2, \ldots, F_n$ which are functionally independent almost everywhere: $\{F_i, H\} = \{F_i, f_j\} = 0$, $dF_1 \wedge dF_2 \wedge \ldots \wedge dF_n \neq 0\, a.e.$. The map $\mathbf{F} = (F_1, ..., F_n) : M^{2n} \to \mathbb{R}^n$ is called the *moment map*. Of course, for a given Hamiltonian H, this moment map is not unique. However, under the nonresonance condition for H, the regular level sets of this moment map are uniquely determined by the original integrable Hamiltonian vector field X_H.

We will always assume that the level sets of the moment map $\mathbf{F} = (F_1, ..., F_n) : (M^{2n}, \omega) \to \mathbb{R}^n$ are compact (without this assumption Arnold-Liouville theorem may fail). Let $\Sigma = \{y \in \mathbb{R}^n | \exists x \in \mathbf{F}^{-1}(y), \operatorname{rank} d\mathbf{F}(x) < n\}$ be the *bifurcation diagram.* Put $M_0 = M \setminus \mathbf{F}^{-1}(\Sigma)$, and denote by O_0 the space of connected components of the regular level sets of $\mathbf{F}$ in M_0. Then we have a natural projection $\pi : M_0 \to O_0$, and the map $\mathbf{F} : M_0 \to \mathbb{R}^n$ can be factored through this projection to a map

$\tilde{\mathbf{F}} : O_0 \to \mathbb{R}^n$.

The following well-known theorem is an analog of Darboux's theorem in symplectic geometry, and it gives the normal form for an integrable system near a regular level set of the moment map.

Theorem 2.1 (Arnold-Liouville) *$\pi : M_0 \to O_0$ is a regular Lagrangian $\mathbb{T}^n$ (torus) fibration. Moreover, for each $y \in O_0$ there is a neighborhood $D^n = D(y)$ of y in O_0 such that $(\pi^{-1}(D^n), \omega)$ can be written as $(D^n \times \mathbb{T}^n, \sum_1^n dp_i \wedge dq_i)$ via a fibration-preserving sympletomorphism, where (p_i) is a system of coordinates in D^n, (q_i mod 1) is a system of periodic coordinates in $\mathbb{T}^n$, and the fibration of $(D^n \times \mathbb{T}^n, \sum_1^n dp_i \wedge dq_i)$ into Lagrangian tori is the projection $D^n \times \mathbb{T}^n \to D^n$.*

The functions p_i and q_i are called *action* and *angle* coordinates. Each torus of the fibration $M_0 \to O_0$ is called a *Liouville torus.* The fact that $M_0 \to O_0$ is a torus fibration and the system X_H is quasi-periodic on each torus was known to Liouville. The existence of action-angle coordinates was proved by Arnold under some additional assumptions, and then by Jost and others (see e.g. [1]).

Arnold-Liouville theorem implies that each Liouville torus has a natural flat structure given by angle coordinates, the Hamiltonian vector field in each Liouville torus is constant, and the Hamiltonian flow is quasi-periodic. It also implies that the (regular part of the) base space O_0 has a unique natural *integral affine structure* which makes it into an *integral affine manifold* (e.g., [16]): If (p_i, q_i) $(i = 1, \dots, n, q_i -$ mod 1) and (x_i, y_i) $(i = 1, \dots, n, y_i -$ mod 1) are two different systems of action-angle coordinates near a Liouville torus, then (x_i) and (p_i) are related by an integral affine transformation, that is $(x_i)^T = A\,(p_i)^T + (c_i)^T$, where A is an element of $GL(n, \mathbb{Z})$ and (c_i) are some real constants. Thus the integral affine structure of the base space is given locally by a system of action coordinates. We will call any first integral of the Hamiltonian system near a Liouville torus, whose Hamiltonian flow is periodic with minimal period equal to 1, a local *action function.* It is easy to see that locally near every Liouville torus, any action function is a coordinate function in some system of action-angle coordinates.

2.2 Global action-angle coordinates

Suppose now that we have a regular Lagrangian torus fibration $\pi : (M_0^{2n}, \omega) \to O_0^n$. It is a natural geometric setting of intebrable systems without singularities, because any two functions $f_1, f_2 : O_0 \to \mathbb{R}$ Poisson-commute if considered as functions on M_0 (i.e. $\{f_1 \circ \pi, f_2 \circ \pi\} = 0$), and any Hamiltonian function of the type $H = h \circ \pi, h : O_0 \to \mathbb{R}$, is integrable.

We can ask if there are global action-angle coordinates. That is, can $(M_0, \omega) \to O_0$ be written in the form

$$(O_0 \times \mathbb{T}^n, \sum_1^n dp_i \wedge dq_i) \to O_0$$

where $(p_i) : O_0 \to \mathbb{R}^n$ is an immersion, q_i mod 1 are periodic coordinates on $\mathbb{T}^n$.

More generally, we can ask for a topological or geometrical classification of such fibrations $\pi : (M_0^{2n}, \omega) \to O_0^n$, assuming that the base sapce O_0^n is known. Here two regular Lagrangian torus fibrations $(M_a, \omega_a) \xrightarrow{\pi_a} O_a$ and $(M_b, \omega_b) \xrightarrow{\pi_b} O_b$ are called *topologically equivalent* if there are diffeomorphisms $\Phi : M_a \to M_b$, $\phi : O_a \to O_b$ which make the following diagram commutative:

$$\begin{array}{ccc} M_a & \xrightarrow{\Phi} & M_b \\ \downarrow \pi_a & & \downarrow \pi_b \\ O_a & \xrightarrow{\phi} & O_b \end{array}$$

They are called *geometrically equivalent* if Φ can be chosen to be a symplectomorphism.

A natural way to solve the above problem is via *obstruction theory*. If $\pi : (M_0^{2n}, \omega) \to O_0^n$ admits a global system of action-angle coordinates, then it has the following properties:

a) $\pi : M_0^{2n} \to O_0^n$ is a principal $\mathbb{T}^n$-bundle.

b) $\pi : M_0^{2n} \to O_0^n$ has a global section.

c) Moreover, it has a global *Lagrangian* section.

Conversely, if the above conditions are satisifed then one can show easily that $\pi : (M_0^{2n}, \omega) \to O_0^n$ admits global action-angle coordinates. The obstruction for the condition a) to be fulfilled is called the *(affine) monodromy*. It will be clear that the monodromy, besides of being a topological invariant of the foliation, can also be determined from the affine structure of the base space O_0 alone. The obstructions to b) and c) will be called *Duistermaat-Chern class* and *Lagrangian Duistermaat-Chern class*. It will also be clear that two regular Lagrangian torus fibrations over the same base space O_0 are topologically equivalent if they have the same monodromy and Duistermaat-Chern class, and geometrically equivalent if they induce the same affine structure on O_0 and have the same Lagrangian Duistermaat-Chern class.

In the next subsections we will discuss briefly affine monodromy and (Lagrangian) Duistermaat-Chern class (for more details see [12, 16]).

2.3 Affine monodromy

As in the previous subsection, consider a Lagrangian torus fibration $\pi : (M_0^{2n}, \omega) \to O_0^n$. One has an associated vector bundle of first homology groups

$$E \overset{H_1(\mathbb{T}^n, k)}{\longrightarrow} O_0^n$$

where k is a coefficient ring, say $\mathbb{R}$. On this vector bundle there is a unique natural locally flat connection, called the Gauss-Manin connection (e.g., [2]). The (affine) *monodromy* is defined as the holomomy of this connection, and is an element of $\hom(\pi_1(O_0^n), GL(n, \mathbb{R}))$ defined up to conjugacy. By choosing the coefficient ring k to be $\mathbb{Z}$, we see that it is actually an element of $\hom(\pi_1(O_0^n), GL(n, \mathbb{Z}))$.

From the definition it is clear that the monodromy is a topological invariant. We will now show that it is also an invariant of O_0^n as an integral affine manifold. Hence the adjective *affine*. Indeed, the vector bundle $E_{\mathbf{R}} \stackrel{H_1(\mathbb{T}^n,\mathbb{R})}{\longrightarrow} O_0^n$ can be identified with the bundle of constant vector fields on the fibers of $\pi : (M_0^{2n}, \omega) \to O_0^n$. If X is a constant vector field on $\mathbb{T}^n_y, y \in O_0^n$, then $\alpha(X) := -\omega(X, .)$ can be identified with an element of $T^*O_0^n$, and the map $X \mapsto \alpha(X)$ is an isomorphism. Hence $E_{\mathbb{R}} \stackrel{H_1(\mathbb{T}^n,\mathbb{R})}{\longrightarrow} O_0^n$ is isomorphic to the cotangent bundle T^*O_0 of O_0, and we have a natural flat connection on it. On the other hand, since O_0 is an integral affine manifold, the tangent bundle TO_0 has a natural flat connection, defined by the local trivializations given by the affine charts. The dual connection on the cotangent bundle T^*O_0 is therefore also flat. The holonomy of this connection is obviously an invariant of the affine structure. But it is easy to see, using Arnold-Liouville theorem, that this connection coincides with the flat connection defined before.

Notice also that $E_{\mathbf{Z}} \stackrel{H_1(\mathbb{T}^n,\mathbf{Z})}{\longrightarrow} O_0^n$ is a discrete subbundle of $E_{\mathbf{R}} \stackrel{H_1(\mathbb{T}^n,\mathbb{R})}{\longrightarrow} O_0^n$. Under the natural identification of $E_{\mathbf{R}}$ with T^*O_0, $E_{\mathbf{Z}}$ maps to a subbundle of T^*O_0, consisting of "integral" covectors. We will denoted this subbundle, or the discrete sheaf associated to it, by $\mathcal{R}$. It will be used in the definition of the Duistermaat-Chern class in Subsection 2.4.

First examples of integrable systems with nontrivial monodromy, namely the spherical pendulum and the Lagrange top, were observed by Cushman and others (e.g., [11, 16]). In these examples and in all other known examples arising from classical mechanics and physics, the nontriviality of the monodromy is due to the presence of the so-called *focus-focus singularities* (see [41] and below).

2.4 Duistermaat-Chern class

The Duistermaat-Chern class is defined as the obstruction for the Lagrangian torus fibration $M_0^{2n} \to O_0^n$ to admit a global section. This fibration is locally trivial. Let (U_i) be a trivializing open covering of O_0. Over each U_i there is a smooth section, denoted by s_i. The difference between two local sections, s_i and s_j, over $U_i \cap U_j$, can be written as

$$\mu_{ij} = s_j - s_i \in C^\infty(E_{\mathbf{R}}/E_{\mathbf{Z}})(U_i \cap U_j) = C^\infty(T^*O_0/\mathcal{R})(U_i \cap U_j)$$

Here $C^\infty(.)$ denotes the sheaf of smooth sections. It is immediate that (μ_{ij}) is an 1-cocycle, and it defines a Cech first cohomology class, not depending on the choice of sections:

$$\hat{\mu}_{DC} \in H^1(O_0, C^\infty(T^*O_0/\mathcal{R}))$$

Since $C^\infty(T^*O_0)$ is a fine sheaf, from the short exact sequence

$$0 \to \mathcal{R} \to C^\infty(T^*O_0) \to C^\infty(T^*O_0/\mathcal{R})) \to 0$$

we obtain that the coboundary map δ from $H^1(O_0, C^\infty(T^*O_0/\mathcal{R}))$ to $H^2(O_0, \mathcal{R})$ in the associated long exact sequence, is an isomorphism.

$\hat{\mu}_{DC}$, or its image μ_{DC} in $H^2(O_0, \mathcal{R})$ under the isomorphism δ, will be called the *Duistermaat-Chern class* [16]. In case the monodromy is trivial, i.e. $M_0 \to O_0$ is a principal $\mathbb{T}^n$ bundle, the Duistermaat-Chern class coincides with the usual Chern class (cf. [12]).

If one requires local sections s_i to be Lagrangian, then one has that

$$\mu_{ij} \in \mathcal{Z}(T^*O_0/\mathcal{R})(U_i \cap U_j)$$

($\mathcal{Z}$ means closed 1-forms), and it will define the *Lagrangian Duistermaat-Chern class*:

$$\mu_{LDC} \in H^1(O_0, \mathcal{Z}(T^*O_0/\mathcal{R}))$$

There is another short exact sequence

$$0 \to \mathcal{R} \to \mathcal{Z}(T^*O_0) \to \mathcal{Z}(T^*O_0/\mathcal{R}) \to 0$$

which leads to the following long exact sequence

$$\begin{aligned} \ldots \to H^1(O_0, \mathcal{R}) &\xrightarrow{\hat{d}} H^1(O_0, \mathcal{Z}(T^*O_0)) = H^2(O_0, \mathbb{R}) \to H^1(O_0, \mathcal{Z}(T^*O_0/\mathcal{R})) \\ \xrightarrow{\Delta} H^2(O_0, \mathcal{R}) &\xrightarrow{\hat{d}} H^2(O_0, \mathcal{Z}(T^*O_0)) = H^3(O_0, \mathbb{R}) \to H^2(O_0, \mathcal{Z}(T^*O_0/\mathcal{R})) \to \ldots \end{aligned}$$

Under the maps Δ and $\hat{d}$ we have $\mu_{LDC} \overset{\Delta}{\mapsto} \mu_{DC} \overset{\hat{d}}{\mapsto} 0$

Thus, if the integral affine manifold O_0 is given, then any element of the cohomology group $H^1(O_0, \mathcal{Z}(T^*O_0/\mathcal{R}))$ will be the Lagrangian Duistermaat-Chern class of some torus Lagrangian fibration over O_0, and the necessary and sufficient condition for an element μ in $H^2(O_0, \mathcal{R})$ to be the Duistermaat-Chern class of some Lagrangian torus fibration is that $\hat{d}(\mu) = 0$. To each element $\mu_{DC} \in H^2(O_0, \mathcal{R})$ such that $\hat{d}(\mu_{DC}) = 0$ there are $H^2(O_0, \mathbb{R})/\hat{d}H^1(O_0, \mathcal{R})$ choices of the element μ_{LDC} such that $\Delta(\mu_{LDC}) = \mu_{DC}$, and each choice corresponds to a geometrically different Lagrangian torus fibration with the same topological structure. If $\mu_{LDC} = 0$ then the corresponding fibration is geometrically equivalent to $T^*O_0/\mathcal{R} \longrightarrow O_0$ (cf. [12]).

One can write down the following natural theorem, which is a reformulation of the results due to Duistermaat [16], and Dazord and Delzant [12]:

Theorem 2.2 ([12, 16]) *Two regular Lagrangian torus fibrations $(M_a, \omega_a) \to O_0$ and $(M_b, \omega_b) \to O_b$ are topologically equivalent if and only if there is a diffeomorphism of the base spaces $\phi : O_a \to O_b$ which induces an identification of the affine monodromies and Duistermaat-Chern classes. They are geometrically equivalent if and only if ϕ can be chosen to preserve also the affine structure of the base spaces and the Lagrangian Duistermaat-Chern class.*

If O_0^n is 2-connected, then there is no room for the monodromy and Lagrangian Duistermaat-Chern class, and one obtains the following result due to Nekhoroshev:

Corollary 2.3 ([33]) *If O_0^n is an integral affine manifold with $\pi_1(O_0^n) = \pi_2(O_0^n) = 0$, then there is a unique Lagrangian torus fibration over O_0^n (which is compatible with the given affine structure of O_0^n), and it admits global action-angle coordinates.*

I don't know any example of a physically meaningful integrable system with nontrivial Duistermaat-Chern class (for the regular part of the system). However, it is not difficult to construct artificial examples:

Example 2.4 *Kodaira-Thurston example:*

Take O_0 to be the standard flat torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, with trivial monodromy. Then $H^3(O_0, \mathbb{R}) = 0$, and any element $\mu_{DC} \in H^2(O_0, \mathcal{R}) = \mathbb{Z}^2$ is realizable. The automorphism group of the base space acts on $H^2(O_0, \mathcal{R})$, and the quotient space is isomorphic to $\mathbb{Z}_+$ (nonnegative integers). Thus each integrable system with the base space $\mathbb{T}^2$ is characterized topologically by a nonnegative integer m, and its ambient symplectic manifold M^4_m has $H_1(M^4_m, \mathbb{Z}) = \mathbb{Z}^3 \oplus (\mathbb{Z}/m\mathbb{Z})$ as can be computed easily. For each m there are $H^2(O_0, \mathbb{R})/\hat{d}H^1(O_0, \mathcal{R}) = \mathbb{R}/\mathbb{Z}$ choices of the symplectic structure on the fibration $M^4_m \to O_0$, up to geometrical equivalence. Let us notice that the fibrations $M^4_m \to \mathbb{T}^2$ are topologically the same as a series of Kodaira primary complex surfaces (see e.g. [5]). In particular, when $m = 1$, M^4_1 is the well-known Kodaira-Thurston example [26, 36] of a manifold admitting both a conplex and a symplectic structure but not a Kähler structure.

The results of Duistermaat, Dazord and Delzant lead to the following fact in integrable surgery: Suppose O_1 and O_2 are base spaces of integrable systems $(M_1, \omega_1) \to O_1$ and $(M_2, \omega_2) \to O_2$, such that on the intersection $O_1 \cap O_2$ these two systems are regular and induce the same integral affine structure on $O_1 \cap O_2$. Then these two systems can be glued together into an integrable system with the base space $O = O_1 \cup O_2$ if and only if they have the same Lagrangian Duistermaat-Chern class when restricted to $O_1 \cap O_2$. In particular, if $O_1 \cap O_2$ is contractible, then the above two systems can always be glued together in a unique may. This very simple fact already has an interesting application given in the following example.

Example 2.5 *Exotic symplectic* $\mathbb{R}^{2n}$*s:*

Start with the following two integrable systems: The first one is given by the moment map $\mathbf{F} = (F_1, ..., F_n) = (\pi x_1^2 + \pi y_1^2, ..., \pi x_n^2 + \pi y_n^2)$ on the open ball of radius 1 of $\mathbb{R}^{2n}$ with coordinates x_i, y_i and with the standard metric and symplectic structure (i.e. a harmonic oscillator). On the base space O_1 of this system, the functions F_i are also integral affine coordinates of the induced affine structure outside the singularities. Let O_2 be an open n-disk, attached to O_1 in such a way that $O_1 \cup O_2$ is diffeomorphic to O_1 *rel.* singularities of O_1, and $O_1 \cap O_2$ is contractible. Extend the functions $F_1, ... F_n$ from O_1 to O_2 in such a way that $dF_1 \wedge ... \wedge dF_n \neq 0$ everywhere on O_2 and there is a point $y \in O_2$ with $F_1(y) = ... = F_n(y) = 0$. O_2 with the integral affine structure given by the functions F_i is the base space of a unique integrable system $(M_2, \omega_2) \to O_2$. This is our second system. By construction, our two systems can be glued in a unique natural way into an integrable system living on a symplectic manifold diffeomorphic to $\mathbb{R}^{2n}$. The preimage of y in this manifold is a Lagrangian torus, and in fact it is an *exact* Lagrangian torus (i.e. for any 1-form α such that $d\alpha$ is equal to the symplectic form, the restriction of α on this torus is cohomologous to 0). On the other hand, a famous result of Gromov [23] (see also

[34]) says that in the standard symplectic space there can be no smooth closed exact Lagrangian submanifold. Thus our symplectic space is exotic in the sense that it can not be symplectically embedded into the standard symplectic space of the same dimension. Let us notice here that the first explicit example of an exotic symplectic space was found by Bates and Peschke [6]. Our example is a kind of modification and simplification of their example. We have a conjecture that, by modifying our example (e.g. by creating more points y in O_2 with $F_1(y) = ... = F_n(y) = 0$), one can get an infinite series of exotic symplectic spaces, which are essentially different.

3 Strongly nondegenerate IHS's

3.1 Geometric definition of nondegenerate IHS's

Since we will consider IHS's only up to topological or geometrical equivalence, throughout this paper we will adopt the following definition of IHS's which is a little bit different from the usual one: An *integrable Hamiltonian system* (with n degrees of freedom) on a symplectic manifold (M^{2n}, ω) is a *triple* $(M, \omega, \mathcal{L})$, where $\mathcal{L}$ is an *admissible singular Lagrangian foliation*, that is, a decomposition of M into disjoint connected compact subsets (called *leaves*), satisfying the following condition: For every leaf N of $\mathcal{L}$ there is a neighborhood $U(N)$ of N in M saturated by the leaves of $\mathcal{L}$, and a smooth map (called the *moment map*) $\mathbf{F} = (F_1, ..., F_n) : U(N) \to \mathbb{R}^n$, which is constant on each leaf of $\mathcal{L}$ in $U(N)$, nondegenerate almost everywhere, and such that its components Poisson-commute.

It follows from the definition and Arnold-Liouville theorem that most leaves of $\mathcal{L}$ are Lagrangian tori. $\mathcal{L}$ will be called, as usual, the (associated) singular Lagrangian foliation of the system.

The space O of leaves of $\mathcal{L}$ of an IHS $(M, \omega, \mathcal{L})$, equipped with the induced topology from M, will be called the *base space* (or the *orbit space*) of the integrable system. Under some nondegeneracy conditions (see below), O will turn out to be a stratified manifold.

Notice that we don't require the global existence of a moment map, but only its existence near each leaf of the Lagrangian foliation. Thus, the moment map is something which is not fixed, but can be changed if necessary (for the sake of regularity).

A *fixed point* of an IHS with n degrees of freedom $(M^{2n}, \omega, \mathcal{L})$ is a point in the symplectic manifold at which the differential of any moment map of the system is equal to 0. If the moment map can be chosen so that its quadratic part near a fixed point generates a Cartan subalgebra of the symplectic algebra of quadratic functions under the standard Poisson bracket, then this fixed point is called *nondegenerate*. In general, a point x is called *nondegenerate singular of corank k*, if under a local Marsden-Weinstein reduction with respect to the first $(n - k)$ components of a moment map, it becomes a nondegenerate fixed point of an IHS with k degrees of freedom.

A *singularity* of an IHS is by definition the germ of the associated Lagrangian foliation near a singular leaf, and is denoted by $(\mathcal{U}(N), \mathcal{L})$. Here N is the singular leaf, $\mathcal{U}(N)$ means a saturated sufficiently small tubular neighborhood, and $\mathcal{L}$ the Lagrangian foliation. $(\mathcal{U}(N), \mathcal{L})$ is called *nondegenerate* if every singular point of the system in it is nondegenerate.

A nondegenerate singularity $(\mathcal{U}(N), \mathcal{L})$ is called *topologically stable* if the singular value set of the moment map restricted to $\mathcal{U}(N)$ coincides with the singular value set of the moment map restricted to a small neighborhood of a singular point of maximal corank in N. This topological stability is a rather natural condition, and is satisfied by all nondegenerate singularities of all known algebraically integrable systems. Hereafter, we will assume all nondegenerate singularities to be topologically stable, though often we will not mention it explicitly.

A singular point is called *clean* if it becomes an isolated fixed point after a local Marsden-Weinstein reduction. Clearly, all nondegenerate singular points are clean.

Definition 3.1 An IHS is called *strongly nondegenerate* if all of its singularities are topologically stable nondegenerate. An IHS is called *nondegenerate* if all of its singular points are clean, a dense subset of these points lies in nondegenerate singular leaves, and all nondegenerate singularities are topologically stable.

Each nondegenerate IHS gives rise to other nondegenerate IHS's with fewer degrees of freedom, in the following way: Consider the set of singular points of corank at least k in an IHS with n degrees of freedom $(M^{2n}, \omega, \mathcal{L})$. Then, according to Eliasson-Vey theorem about the local structure of nondegenerate singular points [18, 38], this set is an immersed symplectic manifold of dimension $2(n-k)$ in M^{2n}, which may be empty. We will call it the *center submanifold* of dimension $2(n-k)$ of the system. Each center submanifold has a natural orientation given by the symplectic structure. Moreover, it also has an induced nondegenerate IHS with $(n-k)$ degrees of freedom. It is a standard way to obtain "small" IHS's from the "big" ones. These big IHS's may be even infinite-dimensional.

Definition 3.2 Two strongly nondegenerate IHS's $(M_1, \omega_1, \mathcal{L}_1)$, $(M_2, \omega_2, \mathcal{L}_2)$ are called *topologically equivalent* if there is a diffeomorphism $\Phi : M_1 \to M_2$ which sends $\mathcal{L}_1$ to $\mathcal{L}_2$ and which preserves the orientation of M_1 and its center submanifolds. They are called *geometrically equivalent* if the above diffeomorphism can be chosen to be a sympectomorphism.

Clearly, this definition is compatible with the one given in Subsection 2.2 for the case of regular Lagrangian foliations. It may seem more natural to assume $\Phi : M_1 \to M_2$ to be only a homeomorphism in the definition of topological equivalence. However, from the theory of characteristic classes to be discussed below, it will be clear that if such a foliation-preserving homeomorphism exists, it can always be chosen to be a diffeomorphism.

3.2 Arnold-Liouville with singularities

A classical theorem of Williamson [39] asserts that the quadratic part of the moment map near a nondegenerate fixed point x can be decomposed, after a linear transformation, into components of three types: *elliptic* ($F_i = p_i^2 + q_i^2$), *hyperbolic* ($F_i = p_i q_i$) and *focus-focus* ($F_i = p_i q_i + p_{i+1} q_{i+1}, F_{i+1} = p_i q_{i+1} - p_{i+1} q_i$) ($\omega(x) = \sum_i dp_i \wedge dq_i$). If a nondegenerate fixed point has k_e elliptic, k_h hyperbolic and k_f focus-focus components ($k_e + k_h + 2k_f = n$), then it is called *of Williamson type* (k_e, k_h, k_f). To each nondegenerate singular point of corank k there is also a corresponding uniquely determined Williamson type (k_e, k_h, k_f), with $k_e + k_h + 2k_f = k$.

The *codimension* of a nondegenerate singularity $(\mathcal{U}(N), \mathcal{L})$ is by definition the maximal corank of singular points in N. Each (singular) leaf N of a strongly nondegenerate IHS has a natural stratification (by the orbits of the Poisson $\mathbb{R}^n$ action of an appropriate local moment map). If the codimension of $(\mathcal{U}(N), \mathcal{L})$ is k then strata of minimal dimension in N are $(n-k)$-dimensional tori and they are the only closed strata in N. To each nondegenerate singularity there is also a corresponding uniquely determined *Williamson type*, which is the Williamson type of a singular point of maximal corank in it.

Let $(\mathcal{U}(N), \mathcal{L})$ be a nondegenerate singularity of Williamson type (k_e, k_h, k_f) and codimension k, of an IHS with n degrees of freedom. The following three theorems, which together form an analog of Arnold-Liouville theorem, hold [40]:

Theorem 3.3 (Torus action) *There is a natural Hamiltonian torus $\mathbb{T}^{n-k_h-k_f}$ action in $(\mathcal{U}(N), \mathcal{L})$ which preserves the moment map of the IHS and which is free almost everywhere. This action is unique, up to automorphisms of $\mathbb{T}^{n-k_h-k_f}$ which preserves two special torus subgroups: the subgroup $\mathbb{T}_e^{k_e} \subset \mathbb{T}^{n-k_h-k_f}$ which is due to elliptic components and which is the maximal subgroup acting trivially on N, and the subgroup $\mathbb{T}_f^{k_f} \subset \mathbb{T}^{n-k_h-k_f}$ which is due to focus-focus components and which acts trivially on closed (minimal) strata of N.*

Theorem 3.4 (Action-angle coordinates) *There is a unique natural normal finite covering $\overline{\mathcal{U}(N)}_{can}$ of $\mathcal{U}(N)$ with the following properties:*
i) $\overline{\mathcal{U}(N)}_{can}$ is symplectomorphic to the direct product $\mathrm{D}^{n-k} \times \mathbb{T}^{n-k} \times P^{2k}$ with the symplectic form

$$\omega = \sum_1^{n-k} dx_i \wedge dy_i + \pi^*(\omega_1)$$

where x_i are Euclidean coordinates on D^{n-k}, y_i $(\mathrm{mod}\ 1)$ are coordinates on $\mathbb{T}^{n-k}$, ω_1 is a symplectic form on a $2k$-dimensional symplectic manifold P^{2k}, and π means the projection. Under this symplectomorphism, the moment map (lifted from $\mathcal{U}(N)$ to $\overline{\mathcal{U}(N)}_{can}$) does not depend on y_i.
ii) $\mathcal{U}(N) = \overline{\mathcal{U}(N)}_{can}/\Gamma_{can} = (\mathrm{D}^{n-k} \times \mathbb{T}^{n-k} \times P^{2k})/\Gamma_{can}$, where the finite group Γ_{can} acts on this product freely, symplectically, and component-wise, i.e. it commutes with the projections. Γ_{can} is a subgroup of $\mathbb{T}^{n-k}$, and all of its nontrivial elements are of order 2.

A system of coordinates (x_i, y_i) as in the above theorem is called an *equivariant non-complete system of action-angle coordinates* for the singularity $(\mathcal{U}(L), \mathcal{L})$. The above theorem implies in particular that every singularity, up to a normal finite covering, can be reduced to singularities which contain fixed points, by a Marsden-Weinstein reduction [28].

Theorem 3.5 (Canonical model) *Diffeomorphically, $(\mathcal{U}(N), \mathcal{L})$ can be written in a unique canonical way as a quotient of a direct product singularity by a free component-wise action of a finite group Γ_{CAN}:*

$$(\mathcal{U}(N), \mathcal{L}) = (\overline{\mathcal{U}(N)}_{CAN}, \mathcal{L}_{CAN})/\Gamma_{CAN} = \{(\mathcal{U}(\mathbb{T}^{n-k}), \mathcal{L}_r) \times (P^2(N_1), \mathcal{L}_1) \times \ldots \times \\ \times (P^2(N_{k_e+k_h}), \mathcal{L}_{k_e+k_h}) \times (P^4(N'_1), \mathcal{L}'_1) \times \ldots \times (P^4(N'_{k_f}), \mathcal{L}'_{k_f})\}/\Gamma_{CAN}$$

Here $(\mathcal{U}(\mathbb{T}^{n-k}), \mathcal{L}_r)$ denotes the Lagrangian foliation in a tubular neighborhood of a regular Lagrangian $(n-k)$-torus of an IHS with $n-k$ degrees of freedom; $(P^2(N_i), \mathcal{L}_i)$ for $1 \leq i \leq k_e + k_h$ denotes a codimension 1 nondegenerate surface singularity (= singularity of an IHS with one degree of freedom); $(P^4(N'_i), \mathcal{L}'_i)$ for $1 \leq i \leq k_f$ denotes a focus-focus singularity of an IHS with two degrees of freedom. Γ_{CAN} acts on the above product component-wise, and moreover, it acts trivially on all possible elliptic components of the product.

By definition, $(\overline{\mathcal{U}(N)}_{CAN}$, action of $\Gamma_{CAN})$ is called the *canonical model* of the singularity $(\mathcal{U}(N), \mathcal{L})$, and Γ_{CAN} the *Galois group* of $(\mathcal{U}(N), \mathcal{L})$. Its relation with $(\overline{\mathcal{U}(N)}_{can}$, action of $\Gamma_{can})$ is as follows: In case of codimension 1 they coincide. In general $\overline{\mathcal{U}(N)}_{CAN}$ is a normal finite covering of $\overline{\mathcal{U}(N)}_{can}$, and Γ_{can} is a quotient group of Γ_{CAN}.

Elliptic and hyperbolic components of nondegenerate singularities of IHS's lie on 2-dimensional surfaces and are rather simple. Focus-focus components lie on 4-dimensional manifolds and deserve a special mention:

Proposition 3.6 ([41]) *Assume that $(\mathcal{U}(N), \mathcal{L})$ has Williamson type (0,0,1), i.e. it is a focus-focus singularity of a system with 2 degree of freedom, and assume that N contains exactly $m \geq 1$ fixed (focus-focus) points. Then N has a natural stratification into m points and m cylinders. The base space of $(\mathcal{U}(N), \mathcal{L})$ is a disk with a "removable" singular point in the center which is the image of N. Affinely, it can be obtained from a small disk near the origin in the standard affine space $\mathbb{R}^2$ by cutting out the angle between two directions $(0,0) \to (m,1)$ and $(0,0) \to (0,1)$, and gluing the remaining edges by the unimodular map $(p_1, p_2) \mapsto (p_1 - mp_2, p_2)$. The affine monodromy around the image of N in the base space is generated by the matrix $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$.*

In fact, $(\mathcal{U}(N), \mathcal{L})$ in the focus-focus case is a "good torus fibration" in the sense of Matsumoto, and the monodromy can be given by the classical Picard-Lefschetz theorem (see [29] and references therin).

In a nondegenerate IHS, singular leaves which are not nondegenerate are called *simply-degenerate*. By continuation, one sees easily that simply-nondegenerate singularities also admit torus actions of appropriate dimensions, which much them not much worse than nondegenerate singularities. In fact, we think that a theory similar to Arnold's Lagrangian singularities may be developed for degenerate singularities of IHS's. In this paper we will deal mainly with strongly nondegenerate IHS's. However, we suspect that our main results are still valid for systems with "reasonable" degenerate singularities.

3.3 Affine structure of the base space

Let $(M^{2n}, \omega, \mathcal{L})$ be a strongly nondegenerate IHS, and O the associated base space. Since M^{2n} is paracompact, it follows easily that the base space O is also paracompact, so we can use Cech cohomologies for the sheaves over O (see e.g. [21]). Theorem 3.5 shows that O is a stratified manifold in a most natural sense. Indeed, for a codimension 1 surface singularity, the associated local base space is a local graph which is stratified into edges and one vertex. For a codimension 2 focus-focus singularity of an IHS with two degrees of freedom, the associated local base space is homeomorphic to a disk with a marked point inside it, and it is stratified into this point and the two-stratum around this point. Locally near every singular point, O is homeomorphic to a quotient of a direct product of $\mathbf{D}^{n-k}$ with some 2-dimensional disks with marked points (focus-focus components) and graphs (elliptic and hyperbolic components) by a finite group Γ_{CAN}. This direct product has a natural stratification, and the action of Γ_{CAN} preserves this stratification.

Naturally, the image of each singular leaf of codimension k of $(M^{2n}, \omega, \mathcal{L})$ is a point lying in an $(n-k)$-dimensional stratum of O. In particular, points of n-dimensional strata of O correspond to nonsingular Liouville tori of the system.

A real function $f : O \to \mathbb{R}$ is called a *smooth function on O* if its pull-back to M^{2n} is smooth. Similarly, a differential form on O corresponds to a basic differential form on M^{2n} in the sense of foliation theory. The algebra of differential forms on O gives us the algebra of de Rham cohomologies $H^*_{DR}(O, \mathbb{R})$. Of particular interest is the second cohomology group $H^2_{DR}(O, \mathbb{R})$: if $\tilde{\Omega}$ is a closed 2-form on O and Ω its pull-back on M then $\omega + \Omega$ (ω plus a "magnetic term") will be a new symplectic form of M for which $\mathcal{L}$ remains a Lagrangian foliation. It follows from our description of singularities that the sheaf of differential forms of each degree on O is a fine sheaf, the Poincaré lemma holds for standard neighborhoods of points in O (which are balls in case of regular points). Hence the de Rham cohomologies on O are the same as the Cech cohomologies of the constant sheaf $\mathbb{R}$ over O.

For IHS's which are nondegenerate but not strongly nondegenerate, stratification of the base space may be a delicate problem. However, if the system is algebraic or analytic, then the base space itself has an analytic structure, and a natural stratification always exists. Only now a stratum of codimension k of the base space may correspond either to nondegenerate singularities of codimension k, or simply-degenerate singularities of codimension smaller than k. We will not enter this problem, and will assume for simplicity that base spaces of all nondegenerate IHS's under investigation have a natural structure of a stratified manifold.

The theorems in the previous subsection give us something more. Namely, O has a natural stratified integral affine structure. Indeed, n-strata of O have a natural integral affine structure, which is given by local systems of action coordinates, as discussed in Subsection 2.1. Each m-dimensional stratum C^m of O ($m < n$) corresponds to an m-dimensional family of nondegenerate codimension $(n-m)$ singular leaves. Then action functions $(x_1, ..., x_m)$ given by Theorem 3.4 can be projected to O and restricted to C^m to become a local system of integral affine coordinates in C^m. It is easy to see that different systems of action functions when restricted C^m will differ from each other by an affine transformation with integral linear part. Thus on C^m we have a well defined integral affine structure. We will show that the integral affine structures on different strata of O are compatible in an natural way. First let us recall some definitions.

A manifold C^m is said to have an *affine structure* if there is a family of open charts $U^{(k)} \subset C^m$ with coordinates $(x_1^{(k)}, \ldots, x_m^{(k)})$ respectively, such that $\bigcup U^{(k)} = C^m$, and on each intersection $U^{(k)} \cap U^{(l)}$ we have $(x_i^{(k)})^T = A^{(kl)}\,(x_i^{(l)})^T + (b_i^{(kl)})^T$, where $A^{(kl)}$ is a constant invertible matrix and $b_i^{(kl)}$ are real constants, $(.)^T$ means transpose. This affine structure is called *integral* if $A^{(kl)}$ belong to the discrete group $GL(m, \mathbb{Z})$. In this case, $(x_i^{(k)})$ are called local *systems of integral affine coordinates.* Two different integral affine structure on a manifold C^m, given by two different local systems of coordinates $(x_i^{(k)})$ and $(y_i^{(l)})$, are called *commensurable*, if $\partial x_i^{(k)}/\partial y_j^{(l)}$ are rational numbers.

A real function on an affine manifold is called an *affine function* if it is affine in every affine chart of the manifold. An affine function f on an integral affine manifold C^m is called an *integral affine function* if in every local integral affine system of coordinates (x_i), $\partial f/\partial x_i$ are integers. Locally, this function belongs to a system of integral affine coordinates if and only if the greatest common divisor of $\partial f/\partial x_i$, $i = 1, ..., m$, is 1.

Definition 3.7 A stratified manifold C is called a *stratified integral affine manifold* if: i) Each stratum $C^m \subset C$ is equipped with an integral affine structure.
ii) These affine structures are compatible in the following sense: If C^m is a stratum of C of dimension m, then for every point $x \in C^m$, in a sufficiently small neighborhood $U(x)$ of x in C there are m functions $f_1, ..., f_m$, whose restriction to each stratum in $U(x)$ can be completed to a local system of integral affine coordinates

For every point $x \in O$, the action functions (x_i) near the preimage N_x in M^{2n}, given by Theorem 3.4, can also play the role of functions (f_i) in Definition 3.7 (due to the fact that the corresponding torus action is free almost everywhere by Theorem 3.3). Thus we have:

Theorem 3.8 *For every strongly nondegenerate IHS, the associated base space has a unique natural structure of a stratified integral affine manifold.*

Remark. Though M^{2n} has a natural orientation given by the symplectic form, the base space O does not have to be orientable at all, even in the regular case.

For every stratum $C^m \subset O$, $m < n$, denote by $Q^{2m} = Q^{2m}_{C^m}$ the set of singular points of corank $(n-m)$ in the preimage of C^m in M^{2n}. As we noted before, Q^{2m} is a symplectic submanifold of M^{2n}, and the projection $Q^{2m} \to C^m$ gives rise to a regular Lagrangian fibration in Q^{2m}. In other words, Q^{2m} is provided with an induced IHS in terms of Lagrangian foliations. However, the fiber of $Q^{2m} \to C^m$ need not be connected in general, and the base space $O(Q^{2m})$ of Q^{2m} is a finite covering of C^m. On $O(Q^{2m})$ we have two integral affine structures: one is given by Arnold-Liouville theorem, the other one is lifted from C^m. These affine structures may be different, but they are commensurable, as easily seen from Theorem 3.4.

An *integral geodesic interval* on a n-dimensional integral affine manifold is a curve given by the equation $\{f_1 = ... = f_{n-1} = 0\}$ in some local system of integral affine coordinates $(f_1, ..., f_n)$. We define the *affine length* of this interval to be $|f_n(x) - f_n(y)|$ where x, y are the end points of the interval. It is clear that the affine length is well-defined, i.e. it is independent of the choice of local integral affine coordinates. For the case of stratified integral affine manifolds, the analog of an integral geodesic interval is an *integral geodesic graph* (1-dimensional stratified submanifold), which near each codimension-1 stratum is given by $\{f_1 = ... = f_{n-1} = 0\}$ where $(f_1, ..., f_{n-1})$ are the integral affine functions in Definition 3.7. We define the affine length of a geodesic graph to be the sum of the affine legnths of its 1-dimensional strata. For the orbit spaces of strongly nondegenerate IHS's, the affine length function has the following remarkable local linear variation property: Consider an integral geodesic graph γ with end-points $x_1, ..., x_h$ in the base space O of a strongly nondegenerate IHS, and assume that $\gamma \subset \{f_1 = ... = f_{n-1} = 0\}$ where $f_1, ..., f_{n-1}$ are integral affine functions as before. For each $i = 1, ..., h$ let f_n^i be a local integral affine function near x_i such that $f_i(x_i) = 0$ and $(f_1, ..., f_{n-1}, f_n^i)$ is a local system of integral affine coordinates near x_i. Then we have a $(n-1)$-dimensional family of integral geodesic graphs $\gamma_{\epsilon_1...\epsilon_{n-1}}$ ($|\epsilon_1|, ..., |\epsilon_{n-1}|$ small), $\gamma_{\epsilon_1...\epsilon_{n-1}} \subset \{f_1 = \epsilon_1, ..., f_{n-1} = \epsilon_{n-1}\}$ with the end-points lying on $\{f_n^1 = 0\}, ..., \{f_n^h = 0\}$. Denote the affine length of $\gamma_{\epsilon_1,...,\epsilon_{n-1}}$ by $l(\epsilon_1, ..., \epsilon_{n-1})$. Then l is a linear function: $l = l(0, .., 0) + \sum l_i \epsilon_i$, where l_i are integers or half-integers. (For singularities with $\Gamma_{can} = 0$, i.e. with free torus actions, l_i are integers). For the proof, use the description of codimension 1 hyperbolic singularities given in [40] or the previous section, the fact that $l(\epsilon_1, ..., \epsilon_n)$ is, up to a factor of 1 or $1/2$, the symplectic area of an appropriate surface in the symplectic manifold whose image under the projection to the orbit space is $\gamma_{\epsilon_1,...,\epsilon_{n-1}}$, and of course the closeness of the symplectic form. The above linear variation property of the affine length is analogous to the Duistermaat-Heckman theorem [17] about the linear variation of the cohomology calss of the symplectic form on the reduced phase spaces of a Hamiltonian $\mathbb{S}^1$-action on a symplectic manifold. In our case, we don't have a global $\mathbb{S}^1$-action in general, but instead we have many local $\mathbb{S}^1$ actions.

The integral affine structure provides the base space O with a natural volume element, which is equal to $dx_1...dx_n$ in any local system (x_i) of integral affine coordinates of its n-dimensional strata. This volume element is, up to orientation, the image of the volume element $\omega^n/n!$ of (M^{2n}, ω) under the Gysin homomorphism of the foliation $M^{2n} \to O$:

$$\int_{\mathbb{T}^n} \omega^n/n! = \int_{\mathbb{T}^n} dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n = \pm dx_1 \wedge \ldots \wedge dx_n$$

Here (x_i, y_i) is a local system of action-angle coordinates. Thus we have:

Proposition 3.9 *The volume of the symplectic manifold M^{2n} is equal to the volume of the base space O.*

One can give to each stratum of the base space a triple (k_e, k_h, k_f) called the Williamson type, which is the Williamson type of the (singular) Lagrangian leaves in the preimage. For example, a stratum C^m of dimension m of the base space O of $(M^{2n}, \omega, \mathcal{L})$ is called *elliptic* if it corresponds to elliptic singularities, i.e. to singularities of Williamson type $(n-m, 0, 0)$. The base space behaves in a very simple way near elliptic singularities. In fact, due to a result by Eliasson [18] and Dufour and Molino [14] about the normal form of elliptic singularities, near every point of an elliptic stratum C^m the base space is isomorphic as an stratified integral affine manifold to a local $1/2^{n-m}$-space of the Euclidean space $\mathbb{R}^n$. A stratum C^m of dimension m of the base space O of $(M^{2n}, \omega, \mathcal{L})$ is called *hyperbolic* if it corresponds to (purely) hyperbolic singularities, i.e. to singularities of Williamson type $(0, n-m, 0)$. It is called *focus-focus* if the corresponding Williamson type is $(0, 0, (n-m)/2)$. Recall [40] that for hyperbolic singularities and only for them, the action functions (x_i) given in Theorem 3.4 are uniquely determined up to constants and integral linear transformations. It corresponds to an interesting property of the base space: near hyperbolic strata, the affine functions f_i entering in Definition 3.7 are uniquely determined up to affine transformations.

3.4 Monodromies and rough equivalence

Let $(M, \omega, \mathcal{L})$ be a strongly nondegenerate IHS with the base space O and the projection map $\pi : M \to O$. We can associate to it two discrete sheaves over O as follows.

The sheaf $\mathcal{R}$ of *local $\mathbb{S}^1$-actions*: for each open subset $U \subset O$, the group $R(U)$ of the sheaf $\mathcal{R}$ over U consists of $\mathcal{S}^1$-actions on $\pi^{-1}(U)$ which preserve the leaves of the singular foliation $\mathcal{L}$ and the affine structure on each leaf. Clearly, this is an Abelian sheaf. Theorem 3.3 implies that the stalk of $\mathcal{R}$ over a point $x \in O$ is isomorphic to $\mathbb{Z}^{n-k_h-k_f}$, where (k_e, k_h, k_f) is the Williamson type of the (singular) leaf $\pi^{-1}(x)$. In case of regular Lagrangian torus fibrations, this sheaf coincides with the sheaf $\mathcal{R}$ defined in Subsection 2.3.

The sheaf $\mathcal{H}$ of *cohomologies with integral coefficients*: The stalk of $\mathcal{H}$ at each point $x \in O$ is $H_x := H^*(\pi^{-1}(x), \mathbb{Z}) = H^*(\pi^{-1}(U(x)), \mathbb{Z})$ where $U(x)$ is a standard neighborhood of $x \in O$ (so that Theorem 3.5 holds for $(\pi^{-1}(U(x)), \mathcal{L})$). The induced homomorphisms in cohomologies give us the restriction maps, hence $\mathcal{H}$ is a well-defined sheaf. In case of regular foliations, $\mathcal{H}$ is isomorphic to the exterior (Grassmann) algebra of the dual of $\mathcal{R}$.

Clearly, the discrete sheaves $\mathcal{R}$ and $\mathcal{H}$ are topological invariants of the system, which are closely related. In case of regular foliations, the isomorphism class of $\mathcal{R}$ is determined uniquely by the affine monodromy (holonomy representation). In the general case, the isomorphism class of the restriction of $\mathcal{R}$ and $\mathcal{H}$ to each stratum of O is also determined uniquely by the holonomy representations. By analogy, for a strongly nondegenerate IHS, we will call the isomorphism class of $\mathcal{R}$ (resp., $\mathcal{H}$) the *affine monodromy* (resp., *global monodromy*) of the system. Notice that, for general strongly nondegenerate IHS's, $\mathcal{R}$ is also determined uniquely by the affine structure of the base space. We will call $\mathcal{R}$ the *affine monodromy sheaf* and $\mathcal{H}$ the *global monodromy sheaf.*

It is an interesting problem to characterize isomorphism classes of $\mathcal{R}$ and $\mathcal{H}$ in terms of some numerical invariants, representations, ect. It seems to be a delicate problem because $\mathcal{R}$ and $\mathcal{H}$ are not locally constant in general. In a simplest case of IHS's with two degrees of freedom near an isoenergy hypersurface, the affine and global monodromies can be classified in terms of some rational numbers, called *marks* in the so-called Fomenko-Zieschang invariant (see e.g. [19]).

Two strongly nondegenerate IHS's over the same base space will be called roughly equivalent if and only if they have the same singularities and global monodromy. More precisely:

Definition 3.10 Two nondegenerate IHS's $(M_a, \omega_a, \mathcal{L}_a) \xrightarrow{\pi_a} O_a$ and $(M_a, \omega_a, \mathcal{L}_a) \xrightarrow{\pi_a} O_a$ are called *roughly topologically equivalent* (or have the same *rough topological type*) if there is a homeomorphism $\phi : O_a \to O_b$, a covering of O_a by open subsets U_i, a diffeomorphism $\Phi_i : \pi_a^{-1}(U_i) \to \pi_b^{-1}(\phi(U_i))$ for each i, such that $\pi_b \circ \Phi_i = \phi \circ \pi_a|_{\pi_a^{-1}(U_i)}$, and $\Phi_i^{-1}\Phi_j$ induces the identity map on the cohomology algebra $H^*(\pi_a^{-1}(x), \mathbb{Z})$ for each $x \in U_i \cap U_j$. The two systems are called *roughly geometrically equivalent* (or have the same *rough geometrical type*) if, in addition, Φ_i are symplectomorphisms.

Two systems without singularities or with only elliptic singularities are roughly equivalent if and only if they have the same base space and affine monodromy. The topological and geometrical type of an IHS $(M, \omega, \mathcal{L}) \to O$ will be denoted by $\hat{O}_{top}$ and $\hat{O}_{geom}$ respectively. It may be considered as a *framed base space*, the framing being given by singularities and global monodromy. The following lemma follows immediately from the above defitition:

Lemma 3.11 *If two strongly nondegenerate IHS's are roughly topologically equivalent, then they have the same affine monodromy. If they are roughly geometrically equivalent, then the map ϕ between the base spaces in the definition preserves the affine structure.*

Remark. The inverse to the first assertion of the above lemma is not true: one can use some codimension 1 hyperbolic singularities to construct some simple systems with the same singularities, affine monodromies but different global monodromies. Of course, the inverse to the second assertion of the above lemma is not true either.

It is an interesting problem to find in every class of roughly (topologically / geometrically) equivalent systems a particular, most remarkable one. If such a particular system exists, then other systems in the same rough equivalence class may be compared to this one, and their "difference" will be a characteristic class in some cohomology group. For example, in the case of regular fibrations, such a particular system is the one with a (Lagrangian) section. More generally, systems admitting so-called generalized sections (see Subsection 4.3) are also particular. If there is no such an apparently particular system, then we just fix an arbitrary system in the class and take it as the "point of reference".

Let us mention that, for 2 roughly topologically equivalent systems to be roughly geometrically equivalent, a necessary (and sufficient) condition is that the affine structure of the base spaces are the same and the singularities are not only topologically equivalent but also geometrically equivalent. Geometrical invariants of simplest singularities (of Williamson types (0,1,0) and (0,0,1)) were studied by Dufour, Molino, Toulet [15, 37] and Grossi [24], who found as the invariants some formal power series with arbitrary real coefficients. For example, consider the case of a simplest hyperbolic singularity (which contains only one singular point) of a Hamiltonian system with one degree of freedom. The singular leaf is a number 8 figure. According to a result by Colin de Verdier and Vey [10], near the singular point there are local coordinates (x, y) such that the symplectic form is $\omega = dx \wedge dy$ and the foliation is given by $xy = constant$. Take $H = xy$ (and extend it to a neighborhood of the singular leaf). We can assume that the set $H = \epsilon$ consists of two circles for $\epsilon > 0$ small. Consider the 2 cylinders lying between $H = 0$ and $H = \epsilon$ and denote their symplectic areas by $a_1(\epsilon)$ and $a_2(\epsilon)$. Then $a_1(\epsilon) + \epsilon \lg \epsilon$ and $a_1(\epsilon) + \epsilon \lg \epsilon$ are smooth functions, and the Taylor series of these functions at $\epsilon = 0$ form a complete set of geometrical invariants (modulo the topological structure) of this simplest hyperbolic singularity (see [37] for details).

3.5 Sheaves of local automorphisms

Besides the discrete monodmomy sheaves discussed in the previous subsection, over the base space O of a system $(M, \omega, \mathcal{L}) \xrightarrow{\pi} O$ we also have the following sheaves of groups of local automorphisms:

The sheaf $\mathcal{A}_{top}$ of *local topological automorphisms*: The group of $\mathcal{A}_{top}$ over an open subset $U \subset O$, denoted by $A_{top}(U)$, consists of diffeomorphisms from $\pi^{-1}(U)$ to itself which leave the leaves of $\mathcal{L}$ invariant and induce the identity homomorphism on the cohomology ring with integral coefficients of each leaf.

The sheaf $\mathcal{A}_{geom}$ of *local geometrical automorphisms*: The group of $\mathcal{A}_{geom}$ over an open subset $U \subset O$, denoted by $A_{geom}(U)$, is the subgroup of $A_{top}(U)$ consisting of the elements which preserve the symplectic structure ω.

It is clear that $\mathcal{A}_{top}$ and $\mathcal{A}_{geom}$ satisfy the axioms of sheaves. We can make $A_{top}(U)$ and $A_{geom}(U)$ into topological groups by giving them, say, C^∞ topology.

The aim of this subsection is to study an important natural extension of $\mathcal{A}_{top}$ and $\mathcal{A}_{geom}$ by the affine monodromy sheaf $\mathcal{R}$.

An important property of the elements of $A_{top}(U(x))$ which will be used is that they preserve each stratum of $N = \pi^{-1}(x)$:

Lemma 3.12 *Assume that $\psi : \pi^{-1}(U(x)) \to \pi^{-1}(U(x))$ is a homeomorphism commuting with the projection π and inducing the identity homomorphism on each cohomology algebra $H^*(\pi^{-1}(y), \mathbb{Z}_2)$, $y \in U(x)$. Then ψ preserves every stratum of every leaf $\pi^{-1}(y)$, $y \in U(x)$.*

Proof. By continuity, it is enough to prove that ψ preserves every n-dimensional stratum, because smaller-dimensional strata lie in their boundary. Again by continuity, it suffices to deal only with leaves of Williamson type $(0, 1, 0)$ or $(0, 0, 1)$. (If an n-dimensional stratum S lies in a leaf of Williamson type say $(0, 2, 0)$, then there are two families of n-dimensional strata S_{1t}, S_{2t}, $t \in \mathbb{R}_+$, which lie in leaves of Williamson type $(0, 1, 0)$, and such that $S = \lim_{t\to 0} S_{1t} \cap \lim_{t\to 0} S_{2t}$). Thus we can reduce the above lemma to the following two cases: 1) $\pi^{-1}(x)$ is of Williamson type $(0, 1, 0)$ and 2) $\pi^{-1}(x)$ is of Williamson type $(0, 0, 1)$. In the second case, the Galois group is trivial, and the statment follows easily from Künneth formular and by considering second cohomology groups $H^2(\pi^{-1}(x), \mathbb{Z}_2)$. In the first case, the statement also follows easily by considering the group $H^n(\pi^{-1}(x), \mathbb{Z}_2)$. □

Recall from Theorem 3.3 that we have the action of the torus group $\mathbb{T}^{n-k_h-k_f}$ on $(\pi^{-1}(U), \omega, \mathcal{L})$, if $U = U(x)$ is a standard neighborhood of a point $x \in O$ and (k_e, k_h, k_f) is the Williamson type of the leaf $\pi^{-1}(x)$, and this action gives a natural embedding of $\mathbb{T}^{n-k_h-k_f}$ into $A_{geom}(U)$ and $A_{top}(U)$. The fundamental group of this torus, $\mathbb{Z}^{n-k_h-k_f}$ is naturally isomorphic to the group $R(U)$ of the affine monodromy sheaf.

We now define a topologically trivial fibration $A_{top}(U(x)) \to \mathbb{T}^{n-k_h-k_f}$ as follows: Take a point p of maximal corank in $N = \pi^{-1}(x)$. In other words, p lies in a stratum of N isomorphic to a torus $\mathbb{T}^m$, $m = n - k_e - k_h - 2k_f$. Consider the normal vector space at p to this stratum in $\pi^{-1}(U)$. Since the Williamson type of N is (k_e, k_h, k_f), this normal vector space decomposes in a unique natural way into a direct sum of $k_e + k_h$ 2-dimensional subspaces (elliptic and hyperbolic components) and k_f 4-dimensional subspaces (focus-focus components). On each of these components we have a nondegenerate linear Poisson action of $\mathbb{R}^1$ (elliptic and hyperbolic case) or $\mathbb{R}^2$ (focus-focus case). In each of these components take a ray (a nonzero vector considered only up to positive scalars), which is tangent to an invariant subspace (of dimension 1 or 2 respectively) of the Poisson action in the hyperbolic and focus-focus case. The set of these $k_e + k_h + k_f$ rays is called a *framing* of p. p with such a framing will be called a *framed point* and denoted by $< p >$. An important observation is that the torus group $\mathbb{T}^{n-k_h-k_f}$ acts freely on the set of framed points of each closed stratum of $N = \pi^{-1}(x)$ (though it may act not freely on the stratum itself). Moreover, the condition of preserving cohomologies and Lemma 3.12 implies that, for any element ψ in $A_{top}(U(x))$ there is a unique element χ in $\mathbb{T}^{n-k_h-k_f}$ such that the diffeomorphism $\chi^{-1} \circ \psi$ preserves the framed point p. Denote by $A_{top}^{<p>}(U(x))$ the subgroup of $A_{top}(U(x))$ consisting of elements which preserve $< p >$. Then it follows that the map $\psi \mapsto (\chi^{-1} \circ \psi, \chi)$ is a homeomorphism between $A_{top}(U(x))$ and $A_{top}^{<p>}(U(x)) \times \mathbb{T}^{n-k_h-k_f}$.

As a consequence, we have

$$\begin{aligned} \pi_1(A_{top}(U(x)), Id) &= \pi_1(A_{top}^{<p>}(U(x)), Id) \times \pi_1(\mathbb{T}^{n-k_h-k_f}, 0) \\ &= \pi_1(A_{top}^{<p>}(U(x)), Id) \times R(U(x), 0) \end{aligned}$$

where Id denotes the identity map, 0 denotes the zero element in $\mathbb{T}^{n-k_h-k_f}$. (Conjecture: all homotopy groups of $A_{top}^{<p>}(U(x))$ are trivial). In particular, $\pi_1(A_{top}^{<p>}(U(x)))$ is a normal subgroup of $\pi_1(A_{top}(U(x)))$ with the quotient group equal to $R(U(x))$, and we have the following extension of $A_{top}(U(x))$ with respect to that subgroup of the fundamental group:

$$0 \to R(U(x)) \to \hat{A}_{top}(U(x)) \to A_{top}(U(x)) \to 0$$

Moreover, $R(U(x))$ lies in the center of $\hat{A}_{top}(U(x))$. Apparently, the above definition of $\hat{A}_{top}(U(x))$ depends on the choice of the framed point $< p >$. However, the following lemma shows that it is in fact canonical and does not depend on $< p >$:

Lemma 3.13 *The subgroup $\pi_1(A_{top}^{<p>}(U(x)))$ of $\pi_1(A_{top}(U(x)))$ does not depend on the choice of $< p >$.*

Proof. Let $< p >$ and $< q >$ be two framed points (of maximal corank) in $N = \pi^{-1}(x)$. Then the lemma is reduced to the following two cases:

Case 1: There is an element $\chi \in \mathbb{T}^{n-k_h-k_f}$ such that $\chi(< p >) =< q >$. Take any curve $\chi_t \in \mathbb{T}^{n-k_h k_f}$, $t \in [0,1]$, such that $\chi_0 = 0, \chi_1 = \chi$. Then $A_{top}^{<p>}(U(x))$ is homotopic to $A_{top}^{<q>}(U(x))$ in $A_{top}(U(x))$ via $\chi_t A_{top}^{<p>}(U(x))\chi_t^{-1}$.

Case 2. There is no such χ as above, but there is a curve $\gamma : [0,1] \to N$ such that $\gamma_0 = p, \gamma_1 = q$ and γ_t for all other t lie in a same stratum of N and have Williamson type $(k_e, k_h - 1.k_f)$ or $(k_e, k_h, k_f - 1)$ where (k_e, k_h, k_f) is the Williamson type of N. We can assume in addition that if p and q lie in the same stratum of N then $p = q$. Denote by $A_{top}^{<p,q>}(U(x))$ the set of elements of $A_{top}(U(x))$ which preserve both $< p >$ and $< q >$. Then it is easy to construct bundles $A_{top}^{<p>}(U(x)) \to A_{top}^{<p,q>}(U(x))$, $A_{top}^{<q>}(U(x)) \to A_{top}^{<p,q>}(U(x))$ with contractible fibers and inclusion maps $A_{top}^{<p,q>}(U(x)) \to A_{top}^{<p>}(U(x))$, $A_{top}^{<p,q>}(U(x)) \to A_{top}^{<q>}(U(x))$ as sections, and it will follow that $A_{top}^{<p>}(U(x))$ is homotopic to $A_{top}^{<p,q>}(U(x))$ and $A_{top}^{<q>}(U(x))$ in $A_{top}(U(x))$. For example, consider the case when N has Williamson type $(0,1,0)$, the $\mathbb{T}^{n-1}$ action is free, and p and q lie in different $(n-1)$-strata. take a fucntion f in $\pi^{-1}(U(x))$ which is invariant under the $\mathbb{T}^{n-1}$-action, such that $f(p) = 0$, $f(q) = 1$. Denote the projection $\mathbb{R}^{n-1} \to \mathbb{T}^{n-1}$ by ρ. For each $\psi \in A_{top}^{<p>}(U(x))$ there is a unique $\bar{\chi} \in \mathbb{R}^{n-1}$ such that the curve $t \mapsto \rho(f(\psi(\gamma(t)))\bar{\chi})\psi(\gamma(t))$ is homotopic to $\gamma(t)$ rel. p, q. In particular, $\rho(f(\psi(q))\bar{\chi})\psi(q) = q$. We have a family of elements $\psi_s \in A_{top}^{<p>}(U(x))$, $s \in [0,1]$, with $\psi_0 = \psi$ and $\psi_1 \in A_{top}^{<p,q>}(U(x))$, which is defined by $\psi_s(w) = \rho(sf(\psi(w))\bar{\chi})\psi(w)$. The map $\psi_0 \mapsto \psi_1$ defines the bundle that we are looking for. □

Using the above lemma, we can define a natural extension of the sheaf $\mathcal{A}_{top}$ by the sheaf $\mathcal{R}$:

$$0 \to \mathcal{R} \to \hat{\mathcal{A}}_{top} \to \mathcal{A}_{top} \to 0$$

Here the sheaf $\hat{\mathcal{A}}_{top}$ is defined by its stalk at each point $x \in O$ to be $\lim_{U(x)\to x} \hat{A}_{top}(U(x))$.

In an absolutely similar way, the group $A_{geom}(U(x))$ admits a natural extension

$$0 \to R(U(x)) \to \hat{A}_{geom}(U(x)) \to A_{geom}(U(x)) \to 0,$$

and so does the sheaf $\mathcal{A}_{geom}$:

$$0 \to \mathcal{R} \to \hat{\mathcal{A}}_{geom} \to \mathcal{A}_{geom} \to 0$$

Lemma 3.14 *$\hat{A}_{geom}(U(x))$ is naturally isomorphic to the Abelian group, denoted by $Z^1(U(x))$, of closed 1-form on $U(x)$ (i.e. basic closed 1-forms on $(\pi^{-1}(U(x)), \mathcal{L})$). Thus the sheaf $\hat{\mathcal{A}}_{geom}$ is isomorphic to the sheaf $\mathcal{Z}^1$ of local closed differential 1-forms on O, and $\mathcal{A}_{geom}$ is isomorphic to $\mathcal{Z}^1/\mathcal{R}$.*

Proof. By definition, there is a map $\mathbf{F} = (F_1, ..., F_n) : U(x) \to \mathbb{R}^n$ such that $\mathbf{F} \circ \pi : (\pi^{-1}(U(x)), \mathcal{L}) \to \mathbb{R}^n$ is a "good" moment map, that is, singular points of $\mathbf{F} \circ \pi$ are nondegenerate and coincide with singular points of $\mathcal{L}$ in $\pi^{-1}(U(x))$. Let ψ be an element of $A_{geom}(U(x))$. Then ψ preserves $\mathbf{F}$ and the symplectic form ω, therefor it commutes with the Poisson $\mathbb{R}^n$-action generated by $\mathbf{F}$. Denote the Hamiltonian vector fields of $F_1, ...F_n$ which generates the above Poisson action by $X_1, ...X_n$ respectively. Since ψ also preserves each orbit of this Poisson action, there exist (locally well-defined) n functions $a_1, ..., a_n : \pi^{-1}(U(x)) \to \mathbb{R}^n$, which invariant on each orbit of the Poisson action, such that ψ is equal to the 1-map of the flow of the vector field $\sum a_i X_i$ (which is smooth on each orbit of the Poisson action). Considering only regular points in $\pi^{-1}(U(x))$, one obtains that a_i are smooth functions. Considering ψ near a singular point of maximal corank in $\pi^{-1}(x)$, one obtains that a_i can be made single-valued. Thus a_i can be viewed as smooth functions on U. Since ψ is symplectic, the 1-form $\sum a_i dF_i$ is closed, just like in the regular case. Conversely, if $\sum a_i dF_i$ is a closed 1-form on U then the 1-map of the flow of the vector field $\sum a_i X_i$ will be an element in $A_{geom}(U(x))$. Just we have a surjective map $Z^1(U(x)) \to A_{geom}(U(x))$, whose kernel can be easily seen to be $R(U(x))$. In particular, $Z^1(U(x))$ is the universal covering of $A_{geom}(U(x))$, $\pi_1(A_{geom}(U(x))) = R(U(x))$, and it follows that $\hat{A}_{geom}(U(x)) = Z^1(U(x))$. □

4 Charateristic classes and integrable surgery

4.1 Characteristic classes and classification

In each class of roughly topologically or geometrically equivalent strongly nondegenerate IHS's, we choose an element $(M_a, \omega_a, \mathcal{L}_a) \xrightarrow{\pi_a} O_a$ of it, and consider this element as a "point of reference" for defining characteristic classes of the other elements in the same class. Over O_a we have the sheaves $\mathcal{R}$, $\mathcal{H}$, $\mathcal{A}_{top}$, $\hat{\mathcal{A}}_{top}$, $\mathcal{A}_{geom} = \mathcal{Z}^1/\mathcal{R}$, $\hat{\mathcal{A}}_{geom} = \mathcal{Z}^1$ as defined in Subsection 3.4 and 3.5.

Assume that another system $(M, \omega, \mathcal{L}) \xrightarrow{\pi} O$ has the same rough (topological or geometrical) equivalence type as $(M_a, \omega_a, \mathcal{L}_a)$. Here for convenience we assume that the base spaces are already identified by an appropriate homeomorphism. Let U_i be a fine enough open covering of $O = O_a$. Recall from the definition of rough equivalence that we have local isomorphisms $\Phi_i : \pi_a^{-1}(U_i) \to \pi^{-1}(U_i)$, which give us local "sections" $\mu_{ij} := \Phi_j \circ \Phi_i^{-1}|_{\pi_a^{-1}(U_i \cap U_j)} \in \mathcal{A}_{type}(U_i \cap U_j)$ of the sheaf $\mathcal{A}_{type}$, where *type* = *top* or *geom*. Thus we have an 1-cocycle in $\mathcal{A}_{top}$ or $\mathcal{A}_{geom}$. Its cohomology class will be denoted by $\hat{\mu}_{gC}$ or μ_{LgC} respectively. By analogy with the regular case, we have:

Definition 4.1 The first cohomology class $\hat{\mu}_{gC}$ (resp., μ_{LgC}) will be called the *global Chern class* (resp., *Lagrangian global Chern class*) of the system $(M, \omega, \mathcal{L})$ with repect to the system $(M_a, \omega_a, \mathcal{L}_a)$.

From the above definition we immediately obtain the following:

Theorem 4.2 *Two roughly topologically (resp., geometrically) equivalent strongly nondegenerate IHS's are topologically (resp., geometrically) equivalent if and only if, after appropriate homeomorphisms between the base spaces of them with the base space of a reference system in the same rough equivalence class are chosen, they have the same global Chern class (resp., Lagrangian global Chern class). In other words, two strongly nondegenerate IHS's are topologically (geometrically) equivalent if and only if they have the same structure of singularities, global monodromy and (Lagrangian) global Chern class.*

The short exact sequences

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{R} & \to & \hat{\mathcal{A}}_{top} & \to & \mathcal{A}_{top} & \to & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{R} & \to & \mathcal{Z}^1 & \to & \mathcal{A}_{geom} & \to & 0
\end{array}$$

give us the associated long exact sequences of cohomologies over the base space O_a, and the following commutative diagram:

$$\begin{array}{ccccccccc}
\ldots & \to & H^1(\hat{\mathcal{A}}_{top}) & \to & H^1(\mathcal{A}_{top}) & \xrightarrow{\delta} & H^2(\mathcal{R}) & \to & H^2(\hat{\mathcal{A}}_{top}) & \to & \ldots \\
 & & \uparrow & & \uparrow & & \| & & \uparrow & & \\
\ldots & \xrightarrow{\hat{d}} & H^1(\mathcal{Z}^1) & \to & H^1(\mathcal{A}_{geom}) & \xrightarrow{\Delta} & H^2(\mathcal{R}) & \xrightarrow{\hat{d}} & H^2(\mathcal{Z}^1) & \to & \ldots \\
 & & \| & & & & & & \| & & \\
 & & H^2(O_a, \mathbb{R}) & & & & & & H^3(O_a, \mathbb{R}) & &
\end{array}$$

Definition 4.3 The image of $\hat{\mu}_{gC}$ in $H^2(O_a, \mathcal{R})$ under the coboundary map δ in the above sequences will be denoted by μ_{gC}, and will also be called the (second) *global Chern class* of the system.

It follows from the above commutative diagram that if the system $(M, \omega, \mathcal{L})$ is roughly geometrically equivalent to the reference system $(M_a, \omega_a, \mathcal{L}_a)$, then under the maps Δ and $\hat{d}$ we have $\mu_{LgC} \xrightarrow{\Delta} \mu_{gC} \xrightarrow{\hat{d}} 0$. It follows from the construction of characteristic classes that any element in $H^1(\mathcal{A}_{geom})$ is the Lagrangian global Chern class of some IHS which is roughly geometrically equivalent to $(M_a, \omega_a, \mathcal{L}_a)$. Thus we have the following proposition which is similar to a result of Dazord and Delzant [12] for the regular case:

Proposition 4.4 *An element $\mu \in H^2(O_a, \mathcal{R})$ is the second global Chern class of some IHS roughly geometrically equivalent to $(M_a, \omega_a, \mathcal{L}_a)$ if and only if $\hat{d}(\mu) = 0$. Under this condition, the space of IHS's roughly geometrically equivalent to $(M_a, \omega_a, \mathcal{L}_a)$ (considered together with appropriate homeomorphisms between their base spaces and O_a), which have the same second global Chern class μ, is naturally isomorphic to $H^2(O_a, \mathbb{R})/\hat{d}(H^1(\mathcal{R}))$. (The systems in this space differ only by a "magnetic term" in the symplectic form).*

Open question: Does the condition $\hat{d}\mu_{gC} = 0$ still hold if $(M, \omega, \mathcal{L})$ is roughly equivalent to $(M_a, \omega_a, \mathcal{L}_a)$ only topologically, but not geometrically?. In other words, can ω always be changed so that $(M, \omega, \mathcal{L})$ becomes roughly geometrically to $(M_a, \omega_a, \mathcal{L}_a)$? For systems with two degrees of freedom, the condition $\hat{d}\mu_{gC} = 0$ is empty because $H^3(O, \mathbb{R}) = 0$, so in this case the answer is YES. For systems which are regular or have only elliptic singularities, the answer is also YES. The answer seems to be YES in some other simple cases as well.

Of course, if two systems are topologically equivalent, then they have the same second cohomology class μ_{gC}. The inverse is also true, which justifies our definition:

Proposition 4.5 *If two roughly topologically equivalent systems have the same second global Chern class μ_{gC}, then they are topologically equivalent.*

For the case of IHS's which have only elliptic singularities, the sheaf $\mathcal{R}$ is locally isomorphic to $\mathbb{Z}^n$ everywhere ($n - k_h - k_f = n$). In this case, the global Chern class μ_{gC} was defined and studied by Boucetta and Molino [9]. The definition of μ_{gC} given by Boucetta and Molino is different from ours: they use the notion of local (Lagrangian) sections, which for the case with only elliptic singularities is absolutely similar to the regular case, and proceeds like Duistermaat. (It should be clear that their definition and that of ours give the same result). In particular, it turns out that in each class of roughly topologically (geometrically) equivalent systems with only elliptic singularities, there is a unique particular element up to topological (geometrical) equivalence, which admits a global (Lagrangian) section [9].

Example 4.6 *Toric manifolds:*

Consider a Hamiltonian $\mathbb{T}^n$ action on a closed symplectic $2n$-dimensional manifold (M, ω), which is free somewhere. (M, ω) together with this torus action may be called a *Hamiltonian toric manifold.* The regular (singular) orbits of this $\mathbb{T}^n$ action are Lagrangian (isotropic) tori, and they are leaves of an IHS with only elliptic singularities. The base space of this system is integral-affinely equivalent to a polytope in the Euclidean space $\mathbb{R}^n$, whose each vertex has exactly n edges and these edges can be moved to the principal axis of $\mathbb{R}^n$ by an integral affine transformation. (This fact follows easily from the normal form of elliptic singulartities given by Eliasson [18] and Dufour and Molino [14]). A famous theorem of Delzant [13] says that each polytope satisfying the above condition on vertices is the base space of a Hamiltonian toric manifold which is unique up to geometrical equivalence. (These Hamiltonian toric manifolds admit a Kähler structure and a complex torus action which make them toric manifolds in the sense of complex algebraic geometry, see [13, 3]). The uniqueness in Delzant's theorem is evident from our point of view: Since $\mathcal{R}$ in this case is isomorphic to the constant sheaf $\mathbb{Z}^n$, and the base space is contractible, there is no room for characteristic classes. The existence is also simple: one starts from a Lagrangian section, and reconstructs the system (and the ambient manifold) in a unique way (cf. [9]).

Example 4.7 *Twisted products:*

We may call a *twisted product* of two IHS's $(M_1, \omega_1, \mathcal{L}_1) \xrightarrow{\pi_1} O_1$ and $(M_2, \omega_2, \mathcal{L}_2) \xrightarrow{\pi_2} O_2$ an integrable system over $O_1 \times O_2$, which is not topologically equivalent but roughly geometrically equivalent to the direct product of the two systems, and with the following property: the Marsden-Weinstein reduction of this system to $\{y\} \times O_2$ (resp., $O_1 \times \{y\}$) is geometrically equivalent to $(M_2, \omega_2, \mathcal{L}_2)$ (resp., $(M_1, \omega_1, \mathcal{L}_1)$) for any $y \in O_1$ (resp., $y \in O_2$). For example, if M_i are symplectic 2-tori, with the systems given by Morse functions, then $H^2(O_1 \times O_2, \mathcal{R}) = \mathbb{Z}^4$ (here $\mathcal{R}$ is the corresponding affine monodromy sheaf, and the formula is obtained easily via Meyer-Vietoris sequences), and non-zero elements of this group correspond to twisted products.

4.2 Realization problem and integrable surgery

Given a stratified integral affine manifold O, we can ask wether it can be realized as the base space of some nondegenerate IHS. If it is the case, we say that O is *realizable*. Of course, if O is to be realizable, it has to be *locally realizable*: each singular point y in O corresponds to some singularity of some integrable system, that is a singular Lagrangian foliation with the base space $U(y)$ where $U(y)$ is a neighborhood of y in O, in such a way that the following compatibility conditions are satisfied: If $U(y_1) \cap U(y_2) \neq \emptyset$ then there is a foliation-preserving symplectomorphism $\Phi_{y_1 y_2}$ between the two foliations over $U(y_1)$ and $U(y_2)$ restricted to $U(y_1) \cap U(y_2)$; If $U(y_1) \cap U(y_2) \cap U(y_3) \neq \emptyset$ then for the restriction of the corresponding 3 foliations over $U(y_1) \cap U(y_2) \cap U(y_3)$, the map $\Phi_{y_1 y_2} \circ \Phi_{y_2 y_3} \circ \Phi_{y_3 y_1}$ is isotopic to identity. A stratified integral manifold O equipped with such singularities will be called a *formal rough geometrical type* and denoted by $\hat{O}_{geom}$ before. The problem now is: given a formal rough geometrical type $\hat{O}_{geom}$, is there any integrable system roughly geometrically equivalent to it? A natural way to solve this problem is via integrable surgery: one tries to glue (a finite number of) integrable systems over subsets of O to obtain an integrable system over O. At each step, we are in the following situation: Assume given two IHS $(M_1, \omega_1, \mathcal{L}_1) \to O_1$ and $(M_2, \omega_2, \mathcal{L}_2) \to O_2$, with $O_1 \cup O_2 = O$, such that they are roughly geometrically equivalent when restricted to the common base space $O_1 \cap O_2$. Is there exists an integrable system over O, which is roughly geometrically equivalent to the above two systems when restricted to O_1 and O_2? The answer to this question may be given in terms of characteristic classes:

Proposition 4.8 *Denote the difference between the Lagrangian global Chern classes of the systems $(M_1, \omega_1, \mathcal{L}_1) \to O_1$ and $(M_2, \omega_2, \mathcal{L}_2) \to O_2$ restricted to $O_1 \cap O_2$ by*

$\mu_{LgC} \in H^1(O_1 \cap O_2, \mathcal{Z}^1/\mathcal{R})$. *Then there is an integrable system with the base space O and roughly geometrically equivalent to the above two systems when restricted to O_1 and O_2 if and only if μ_{LgC} lies in the sum of the images of $H^1(O_1, \mathcal{Z}^1/\mathcal{R})$ and $H^1(O_2, \mathcal{Z}^1/\mathcal{R})$ in $H^1(O_1 \cap O_2, \mathcal{Z}^1/\mathcal{R})$ under restriction maps. In particular, if $(M_1, \omega_1, \mathcal{L}_1) \to O_1$ and $(M_2, \omega_2, \mathcal{L}_2) \to O_2$ are topologically equivalent when restricted to $O_1 \cap O_2$, then the obstruction to such a system over O lies in the quotient of $H^2(O_1 \cap O_2, \mathbb{R})/\hat{d}H^1(O_1 \cap O_2, \mathcal{R})$ by the sum of the images of $H^2(O_1, \mathbb{R})/\hat{d}H^1(O_1, \mathcal{R})$ and $H^2(O_2, \mathbb{R})/\hat{d}H^1(O_2, \mathcal{R})$ in $H^2(O_1 \cap O_2, \mathbb{R})/\hat{d}H^1(O_1 \cap O_2, \mathcal{R})$ under the restriction maps.*

Proof. It is a direct consequence of the results of the previous subsection. □

If O is 2-dimensional, then we have:

Proposition 4.9 *Any rough geometrical type $\hat{O}_{geom}$ with O two-dimensional is realizable.*

Proof. If O is 2-dimensional, then we can always choose $O_1 \cap O_2$ in Proposition 4.8 to be a tubular neighborhood of something 1-dimensional, so all the obstructions vanish. □

Proposition 4.8 suggests that there may be obstructions for a given stratified integral affine manifold of dimension greater or equal to 3 to be the base space of some IHS. It is really the case, as the following example shows.

Example 4.10 *A fake base space:*

Let (S^2, ω) be a symplectic 2-sphere, $f : (S^2, \omega) \to \mathbb{R}$ a Morse function with 2 maximal points of the same value ($= 1$), 2 minimal points of the same value ($= -1$), two saddle points of different values ($= \pm 1/2$), such that f is invariant under an involution of S^2 which preserves the symplectic form and two saddle points. Denote the base space of this integrable system with one degree of freedom by $G = G_+ \cup G_-$, where G_+ (resp., G_-) correponds to the part of the sphere with $f \geq 0$ (resp., $f \leq 0$). G is a tree with 5 edges: 2 upper, 1 middle, and 2 lower. Denote by σ the involution of G which preserves f and lower edges but interchanges two upper edges (so σ cannot be lifted to an involution on S^2). Denote by K^2 the Klein bottle with a standard integral affine structure. We have $\pi_1(K^2) =< a, b | abab^{-1} = 1 >$. Denote by $\bar{K}$ the double covering of K^2 corresponding to the subgroup of $\pi_1(K^2)$ which is generated by a^2 and b (so $\bar{K}$ is also a Klein bottle), and denote the involution on $\bar{K}$ corresponding to that double covering also by σ. Put $O = \bar{K} \times_\sigma G = (\bar{K} \times G)/\mathbb{Z}_2$, with the integral affine structure induced from the product of the integral affine structures of $\bar{K}$ and G. We have $O = O_+ \cup O_-$ with $O_- = \bar{K} \times_\sigma G_- = K^2 \times G_-$ and $O_+ = \bar{K} \times_\sigma G_+$ a twisted product. O_- and O_+ are base spaces of integrable systems induced from the direct product of the subsystems over G_- and G_+ with a system over $\bar{K}$. These two systems are roughly equivalent over $O_0 = O_+ \cap O_- = K^2$, but they are not equivalent, so that they cannot be glued together to obtain a system over O. More precisely, the affine monodromy sheaf over $O_0 = K^2$ in O

is $\mathcal{R} = \mathcal{R}_{K^2} \oplus \mathbb{Z}$ where $\mathcal{R}_{K^2}$ is the affine monodromy of K^2 as an affine manifold itself; $H^2(O_0, \mathcal{R}) = H^2(K^2, \mathcal{R}_{K^2}) \oplus H^2(K^2, \mathbb{Z}) = H^2(K^2, \mathcal{R}_{K^2}) \oplus \mathbb{Z}_2$, and we have a natural projection map to the second component $\rho : H^2(O_0, \mathcal{R}) \to \mathbb{Z}_2$. Any system over O_- will have a global Chern class, which when restricted to O_0 will map to 0 under the map ρ; but any system over O_+ will have a global Chern class, which when restricted to O_0 will map to the nontrivial element of $\mathbb{Z}_2$. Thus those systems can never be glued together to a system with the base space O. In other words, O is not realizable.

Besides gluing, integrable surgery may be used also for cutting, for changing a system over a small piece of the base space, ect. In the rest of this subsection we will discuss some simple examples of integrable surgery.

Example 4.11 *Blowing-up:*

Blowing up and down, one of the main tools in algebraic geometry, is also a natural and useful process in symplectic geometry, see e.g. [4, 25, 30]. (By the way, it is also useful for the study of elliptic singularities of integrable systems, cf. [14]). In symplectic category, it consists of cutting away a symplectic $2n$-dimensional ball and collapsing the boundary of this ball to $\mathbb{C}P^n$ by collapsing each of the characteristic curves on this boundary to a point. Since a symplectic ball admits a simple natural integrable system, namely the harmonic oscillator, blowing up can be done also by integrable surgery: Start with a purely elliptic singularity of rank 0 and corank n of an IHS with n degrees of freedom. The corresponding local base space is locally equivalent to the corner $\{x_1 \geq 0, ..., x_n \geq 0\}$ of the Euclidean space $\mathbb{R}^n$. Cut out from this corner a small simplex by the cutting hyperplane $\{x_1 + ... + x_n = \epsilon > 0\}$. The new base space admits an integrable system which is different from the former one only near that corner, and the new ambient symplectic manifold is the result of a blowing up (in the symplectic sense) at the elliptic fixed point from the former one. If instead of an elliptic fixed point, we consider a symplectic submanifold in M consisting of elliptic singular points of some constant positive rank, and cut the base space by an appropriate "hyperplane" near the image of that elliptic symplectic submanifold, then the result will be a blowing up along a symplectic submanifold.

Example 4.12 *Dehn surgery:*

Consider an IHS over a 2-dimensional base space O with the projection map π, and let $D^2 \in O$ a closed disk lying in the regular part of O. Cut out the piece $\pi^{-1}(D^2)$ from the system, and then glue it back after some twisting along the fiber. This operation may be called a *Dehn surgery*, in complete analogy with the well known Dehn surgery in low-dimensional topology. It is easy to see that any two 2-degree-of-freedom integrable systems over the same base space can be transformed to each other by Dehn surgeries and adding of a magnatic term.

Example 4.13 *Hopf bifurcation:*

The bifurcation from elliptic (codimension 2) to focus-focus singularities (from linearly stable to linearly unstable critical orbits) under some parameter change (e.g. energy) happens in many Hamiltonian systems, e.g. the Lagrange and Kirkhoff tops, and it is usually called a Hamiltonian *Hopf bifurcation* (see e.g. [31]). Integrable surgery allows us to do the same thing, i.e. changing elliptic codimension 2 to focus-focus singularities, in a way that does not affect the system outside a small neighborhood of the (center manifold containing the) codimension 2 elliptic point. It may be described locally as follows: Consider a parallelopiped P, which is obtained from a right triangle T (base space of $\mathbb{C}P^2$ under torus action) by a blowing-up. This blowing up consists of cutting from T a small homothetic T' triangle at one of its vertices. Now push T' inside a little bit along an edge of T to get a new small triangle T'' which still lies on this edge but does not contain any vertice of T. Cutting T'' away from T and gluing the edges of the angle that have been cut cut together, we obtain a new triangle Q with one focus-focus inside it. The process of going from P to Q (which can be made without the use of T) is our Hopf bifurcation: in it, a vertice of P goes inside and become a focus-focus point; two eadges of P meeting at that elliptic point becomes one edge of Q, the other edges being untouched. Of course, Hopf bifurcation can also be performed in higher dimensions: it consists of changing elliptic codimension 2 strata into focus-focus codimension 2 strata. by repeating this process, one can kill all elliptic singularities of codimension 2 or higher, from any strongly nondegenerate IHS on a compact symplectic manifold.

4.3 Generalized sections

In topology, a cross section of a bundle map $\pi : M \to O$ is usually defined to be a continuous map $\phi : O \to M$ such that $\pi \circ \phi = id$ on O. If π is the projection map of an integrable system, then in general near hyperbolic singularities cross sections do not exist even locally, except for the simplest cases. However, we can generalize the notion of cross sections as follows, to assure that they always exist locally:

Definition 4.14 Let $(M, \omega, \mathcal{L}) \xrightarrow{\pi} O$ be a strongly nondegenerate IHS. A *generalized smooth section* of this system is a subset $S \subset M$ with the following properties:
i) The induced projection map $\pi : S \to O$ is surjective, and it is injective outside the singularities of O.
ii) S is a subset of some $S_1 \subset M$, where S_1 is an n-dimensional submanifold of M transversal to the foliation $\mathcal{L}$ in a natural sense. (So each point of S_1 is either nonsingular or purely elliptic).
iii) If $C \subset O$ is a stratum which cooresponds to singularities having some hyperbolic components, then the preimage of C in S^1 is a union of a finite number of topological sections of the topological locally trivial fibration $\pi^{-1}(C) \to C$. (The number of sections is equal to the number of local n-strata adjacent to C). We require these sections to be homotopic.
S is called a *generalized Lagrangian section* if S_1 can be chosen to be a Lagrangian submanifold.

If in the above definition we replace O by a subset of it, then we get the definition of a local generalized section. Clearly, generalized sections always exist locally.

Moreover, they are sections in the classical sense at elliptic and focus-focus singularities. That is, they are really generalized only at singularities which have some hyperbolic components. It is also clear that the existence of a generalized section implies some conditions on the global monodromy.

Definition 4.15 A rough topological type $\hat{O}_{top}$ is called *sectionable* if there is an integrable Hamiltonian system whgich is roughly topologically equivalent to $\hat{O}_{top}$ and which admits a generalized section. A rough geometrical type $\hat{O}_{geom}$ is called *sectionable* if there is an integrable system roughly geometrically equivalent to $\hat{O}_{geom}$ which admits a generalized Lagrangian section.

For example, a direct product of a finite number of IHS's with one degree of freedom admits a generalized section, so its rough topological type is sectionable. An example of a non-sectionable rough topological type is already present in the construction of a fake base space (Example 4.10).

If we have two local generalized sections in the neighborhood of a hyperbolic singularity, then it may happen that they cannot be deformed from one to another by smooth isotopy. However, they can be deformed from one to another after a finite number of "jumping over singular points" (more precisely, singular points for which $k_h = 1, k_f = 0$ or $k_h = 0, k_f = 1$ in the Williamson type). It is important to notice that such jumpings can be made in a homotopically caninical way. Hence, such jumpings can also be performed globally if we have global generalized sections. By using these jumpings one can overcome difficulties caused by hyperpolic and focus-focus singularities in the study of the global Chern class.

Proposition 4.16 *If two IHS's $(M_1, \omega_1, \mathcal{L}_1)$ and $(M_2, \omega_2, \mathcal{L}_2)$ are roughly topologically equivalent, and if both of them admit a generalized section, then they are topologically equivalent.*

Proof. It follows directly from proposition 4.5. □

The above proposition shows that if $\hat{O}_{top}$ is sectionable then in the class of all systems roughly topologically equivalent to $\hat{O}_{top}$ there is a natural distinguished element, and the global Chern class may be defined as the obstruction to the existence of a generalized section.

Proposition 4.17 *If two IHS's $(M_1, \omega_1, \mathcal{L}_1)$ and $(M_2, \omega_2, \mathcal{L}_2)$ are roughly geometrically equivalent, and if both of them admit a generalized Lagrangian section, then they are geometrically equivalent.*

Proof. The proof is similar to the regular case. First we try to map the Lagrangian generalized section of the first system to that of the second system. To do this, we can change the first Lagrangian generalized section by a finite number of jumpings which preserve the property of being Lagrangian, and then by a Lagrangian isotopy, so that it becomes locally isomorphic to the second Lagrangian generalized section in a natural way. Then we identify the two Lagrangian generalized sections are, and after that there is a unique way to extend this identification to a foliation preserving symplectomorphism between the two systems. □

Proposition 4.18 *If* $(M, \omega, \mathcal{L}) \xrightarrow{\pi} O$ *admits a generalized section, then we can change* ω *by a magnetic term, i.e. a closed 2-form* ω_1 *on* O*, such that* $(M, \omega + \pi^*\omega_1, \mathcal{L})$ *admits a generalized Lagrangian section.*

Proof. The proof follows from the previous proposition, by using integrable surgery. □

Proposition 4.19 *If a rough geometrical type* $\hat{O}_{geom}$ *is roughly topologically equivalent to a rough topological type* $\hat{O}_{top}$ *and* $\hat{O}_{top}$ *is sectionable, then* $\hat{O}_{geom}$ *is also sectionable. In particular, for any IHS* $(M, \omega, \mathcal{L})$ *of the rough topological type* $\hat{O}_{top}$*, we have that* $\hat{d}\mu_{gC}(M, \omega, \mathcal{L}) = 0$ *in* $H^3(O, \mathbb{R})$*, if* $\hat{O}_{top}$ *is sectionable.*

Proof. It follows easily from the previous propositions. □

Example 4.20 *Systems on compact coadjoint orbits:*

On coadjoint orbits of compact Lie algebras, one can construct integrable systems using argument shift method (see e.g. [20, 7]), and if the shift is generated by an element lying in the compact Lie algebra itself, then the obtained systems have no hyperbolic singularities (cf. the last section of [7]). Since there are no hyperbolic singularities, one can speak of sections instead of generalized sections, and I suspect that all sush systems admit sections. If the generator of the shift lies outside the compact Lie algebra, then the obtained systems may have hyperbolic singularities, and it is also an interesting question wether all sush systems admit generalized sections.

4.4 When the dimension is 4

In this subsection we will first prove an analog of Milnor's theorem [32] for the case of stratified affine manifolds which are base spaces of nondegenerate (but not necessarily strongly nondegenerate) integrable Hamiltonian systems. Then we will discuss some interesting examples of symplectic 4-manifolds admitting nondegenerate IHS's.

Let O^2 be the base space of a nondegenerate IHS with two degrees of freedom. Then besides the usual stratification as an affine manifold, O^2 has another, topological stratification, which is cruder than the affine stratification. Namely, Proposition 3.6 allows us to forget focus-focus points in O^2 as 0-dimensional strata, and consider them as ordinary points in 2-dimensional strata. In other words, if $\tilde{C}$ is a 2-stratum in O^2 then we will add to $\tilde{C}$ all focus-focus points in its boundary. As a result, we will get a 2-dimensional topological stratum, denoted by C, of the new stratification of O^2. We will call each such C a (topological) *2-domain* of O^2. Of course, if the IHS contains no focus-focus singularities, then 2-domains of O^2 coincide with affine 2-strata.

Theorem 4.21 *Let* O^2 *be the base space of a nondegenerate integrable Hamiltonian system on a compact (may be with boundary) symplectic manifold* M^4 *and* C

a topological 2-domain of it. Assume that the image of the boundary of M does not intersect with the closure of C (if M^4 is closed then this condition is satisfied automatically). Then C is homeomorphic to either an annulus, a Mobius band, a Klein bottle, a torus, a disk, a projective space, or a sphere (in case of sphere or projective space, C must contain focus-focus points).

Proof. We will prove for the case C is orientable. Then the non-orientable case can be treated by taking a double covering. Suppose that C contains exactly k handles ($k \geq 0$). Take a simple smooth oriented loop $\gamma = \gamma(t), x_0 = \gamma(0) = \gamma(1)$ a regular point in C, such that it divides C into 2 parts, one of which contains no handle and the other one contains all the handles but no focus-focus point and has γ as the only boundary component.

Assume that $k > 0$. Provide γ with such an orientation that the handles of C are inside of γ. Then homotopically $\gamma = a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_k^{-1} b_k^{-1}$, where a_i, b_i are generators of the fundamental group of the corresponding handles. Fix a Riemannian metric on C. Cut C along smooth loops a_i, b_i (with the common point $x_0 = \gamma(0)$) to obtain a polygone with $4k$ edges, every angle of which equal to $\pi/2k$, as usual.

Take a non-zero vector f at x_0. Transport it parallelly along γ with respect to the affine structure, we get a family $f_\gamma(t)$ of vectors at $\gamma(t)$. Consider the angle function $A_\gamma(t) := \angle(f_\gamma(t), \dot{\gamma}(t))$ which is the algebraic angle spanned from $f_\gamma(t)$ to the curve γ. Of course, this angle function can be chosen to be continuous, and we will make so.

We want to evaluate the difference of the value of this angle function between the end points. First note that, by construction, the value $A_{\gamma,f}(1) - A_{\gamma,f}(0)$ does not change when γ changes smoothly leaving x_0 fixed. Moreover, the vector $f_\gamma(1)$ depends only on the homotopy type of γ and on the initial vector $f = f_\gamma(0)$. For any smooth closed curve c with the end point at x_0 and any vector g at x_0, redenote $g_c(1)$, the result of parallel transporting of g along c with respect to the affine structure, simply by g_c, and the difference $A_{c,g}(1) - A_{c,g}(0)$ by $D(c, g)$. Note that for any two non-zero vectors g and g' we have $|D(c,g) - D(c,g')| < \pi$, since if, for example, $A_{c,g}(0) < A_{c,g'}(0) < A_{c,g}(0) + \pi$, then also $A_{c,g}(1) < A_{c,g'}(1) < A_{c,g}(1) + \pi$. It follows that $D(c,g) + D(c^{-1}1, g') < \pi$ for any closed curve c and vectors g, g'.

By construction, we can decompose $D(\gamma, f) = A_{\gamma,f}(1) - A_{\gamma,f}(0)$ as follows:
$D(\gamma, f) =$
$(D(a_1, f) + \pi/2k - \pi) + (D(b_1, f_{a_1}) + \pi/2k - \pi) +$
$(D(a_1^{-1}, f_{a_1 b_1}) + \pi/2k - \pi) + (D(b_1^{-1}, f_{a_1 b_1 a_1^{1-}}) + \pi/2k - \pi) +$
$\ldots +$
$(D(a_n^{-1}, f_{a_1 b_1 a_1^{-1} b_1^{-1} \ldots b_n}) + \pi/2k - \pi) + (D(b_n^{-1}, f_{a_1 b_1 a_1^{1-} b_1^{-1} \ldots b_n a_n^{-1}}) + \pi/2k - \pi) =$
$\sum[D(a_i, f_i) + D(a_i^{-1}, f_{i'})] + \sum[D(b_i, f_{i''}) + D(b_i^{-1}, f_{i'''})] - (4k-2)\pi$
where $f_i, f_{i'}, \ldots$ are short notes for $f_{a_1 b_1 a_1^{-1} \ldots}$.

Using inequalities of the type $D(c,g) + D(c^{-1}, g') < \pi$, we obtain that $D(\gamma, f) < k\pi + k\pi - (4k-2)\pi = -(2k-2)\pi \leq 0$. Thus we have proved:

Lemma 4.22 *If $k \geq 1$ then for any initial non-zero vector f we have $D(\gamma, f) := A_{\gamma,f}(1) - A_{\gamma,f}(0) < 0$.*

Furthermore, we have:

Lemma 4.23 *Assume that a simple smooth loop Γ with the end point x_0 goes around some boundary component E of C in the positive direction (i.e. so that the boundary component is outside of Γ), so that the domain between Γ and this boundary component is an annulus without any focus-focus point. Then a non-zero vector f at x_0 can be chosen in such a way that $D(\Gamma, f) > 0$.*

Proof of Lemma 4.23 . We divide the lemma into 3 cases.

Case 1: The boundary component E is a smooth closed curve corresponding to codimension 1 elliptic singularities. Then an action function, which is zero on this boundary component and positive elsewhere, is well defined near it. In this case, take the vector f to be tangent to a level set of this action function, and we will have $D(\Gamma, f) = 0$.

Case 2: Points of E correspond to only codimension 1 singularities, most of which are hyperbolic and a finite number of which can be simply-degenerate. Then near this boundary component there is a well-defined affine geodesic direction transversal to it, called the *preferred geodesic direction* (which is given by the parallel level sets of a unique action function given by Theorem 3.4). Then take the vector f to be parallel to this transversal direction. If E has a cusp due to some simply degenerate points then $D(\Gamma, f)$ will be a positive multiple of π. Otherwise we will have $D(\Gamma, f) = 0$.

Case 3: There are nondegenerate fixed points (of saddle-saddle, center-saddle or center-center type) on E. For simplicity, we will assume that all fixed points are of saddle-saddle type. The other cases can be treated similarly. Then near every saddle-saddle point there are two preferred geodesic directions, which are transversal to two local parts of E at this point respectively. Note that every preferred geodesic direction is a preferred direction for simultaneously 2 saddle-saddle points (which are connected by a path of codimension 1 singular points). Let f be parallel to one of the two preferred directions near one of the saddle-saddle points. Then simple comparisons show that $D(\Gamma, f) > 0$. □

Lemma 4.24 *Let $\Delta, \Delta(0) = x_0$, be a simple closed curve going around a focus-focus point (i.e. it divides C into 2 parts, one of which is a disk containing a focus-focus point and no more singularities), which is oriented in negative way (so that the focus-focus point is outside of Δ). Then for any initial non-zero vector f we have $D(\Delta, f) \geq -2\pi$.*

The proof of the above lemma follows directly from Proposition 3.6.

Now suppose C has k handles, m boundary components and n focus-focus points ($k, m, n \geq 0$). In case $m = n = 0$, C is a sphere, and it must contain singular focus-focus points because $\mathbb{S}^2$ has no regular affine structure. Suppose now that $m+n > 0$. Let γ a simple closed curve as in Lemma 4.22, $\Gamma_1, \ldots, \Gamma_m$ be simple closed curves corresponding to boundary components as in Lemma 4.23, and $\Delta_1, \ldots, \Delta_n$ be simple closed curves going around focus-focus points as in Lemma 4.24. Then we can choose them so that the cycle $\Gamma_1 \ldots \Gamma_m \gamma^{-1} \Delta_1 \ldots \Delta_n$ is homotopically trivial. It follows that

$D(\Gamma_1, f) + \ldots + D(\Gamma_m, f_{\Gamma_1 \ldots \Gamma_{m-1}}) + D(\gamma^{-1}, f_{\Gamma_1 \ldots \Gamma_m}) + D(\Delta_1, f_{\ldots}) + \ldots + D(\Delta_n, f_{\ldots}) = -2(m+n)\pi$ for any initial non-zero vector f at x_0. Here each $f_{\ldots}$ denotes some appropriate vector.

Using Lemma 4.24 we obtain that

$$D(\Gamma_1, f) + \ldots + D(\Gamma_n, f_{\ldots}) + D(\gamma^{-1}, f_{\ldots}) \leq -2m\pi.$$

But by Lemmas 4.22 and 4.23, $D(\gamma^{-1}, f_{\ldots}) > 0$ if $k > 0$, $D(\Gamma_i, f_{\ldots}) > -\pi$ and if $m > 0$ we can choose f so that $D(\Gamma_1, f) \geq 0$. It follows that there are only three possible cases: $(m = 0, n = 1)$ (disk), $(m = 0, n = 2)$ (annulus), $(m = 1, n = 0)$ (torus). □

Example 4.25 *K3, ruled manifolds, ect.:*

It is easy to construct IHS's for which a 2-domain C of the orbit space is any of the allowed cases listed in Theorem 4.21. The most interesting case is S^2. S^2 admits an integral affine structure with 24 singular points of focus-focus type, which may be constructed as follows: Start from an integral affine triangle (base space of $\mathbb{C}P^2$ under torus action). Cut out from this triangle 3 small homothetic triangles, each lying on one edge. Gluing together the edges of each of the 3 angles that have been cut out, we obtain a triangle with an integral affine structure with 3 singular points of focus-focus type. We can glue 8 copies of this new triangle together to obtain a 2-sphere with an integral affine structure with 24 focus-focus points. Proposition 4.9 shows that this S^2 is the base space of some IHS with 24 (simple) focus-focus singularities. Topologically, it is a torus fibration over S^2 with 24 singular fibers of type I^+, in the sense of Matsumoto, and the ambient manifold is diffeomorphic to a K3 surface (see [29] and references therein). We can also go the other way around (less explicitly): Start with a holomorphic integrable system on an K3 surface (cf. [27]). Forgetting about the complex structure and taking the real part of the holomorphic symplectic form, we get an integrable system with 2 degrees of freedom whose base space is homeomorphic to S^2.

Assume now that the base space has no focus-focus singular point and is homeomorphic to the direct product of a graph or a circle with a closed interval. (The affine structure on O needs not be a direct product). The ambient manifolds of IHS's with such an orbit space O are rational and ruled symplectic 4-manifolds in the sense of McDuff (see e.g. [4, 30]). They are symplectic analogs of complex ruled surfaces (see e.g. [5]). It can be shown easily that in this case, as in the case of S^2 with 24 focus-focus points, we have $H^2(O, \mathcal{R}) = 0$ (for any realizable affine structure on O). If we take as O a product of 2 graghs which are not trees, then it will correspond to many topologically different IHS's, like in Example 4.7. Using integrable surgery, one can create more complicated 2-dimensional base spaces. It is an interesting problem to study such 2-dimensional base spaces, and their corresponding systems and 4-manifolds, in more detail.

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