QUADRATIC ENDOFUNCTORS OF THE CATEGORY OF GROUPS

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MPI/95-55

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Let $\underline{\underline{Gr}}$ be the category of groups. In this paper we study functors $F : \underline{\underline{Gr}} \to \underline{\underline{Gr}}$ which preserve cokernels and filtered colimits. The functor F is linear if the map

$$(Fr_1, Fr_2) : F(X \vee Y) \to F(X) \times F(Y)$$

is an isomorphism where $X \vee Y$ is the sum in the category of groups and $r_1 : X \vee Y \to X$, $r_2 : X \vee Y \to Y$ are the retractions. Moreover F is quadratic if $F(X \mid Y) = \text{kernel}(Fr_1, Fr_2)$ as a bifunctor is linear in X and Y.

Our main result shows that the monoidal category of such quadratic endofunctors of $\underline{\underline{Gr}}$ is equivalent to the monoidal category of square groups; see (3.10) and (8.9). Here a square group is a diagram

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$

where M_{ee} is an abelian group, M_e is a group of nilpotency degree 2, P is a homomoprhism and H is a quadratic function with properties as in (3.5). We show that the quadratic endofunctor F of $\underline{\underline{Gr}}$ can be described by a quadratic tensor product

$$F(G) = G \otimes M,\ G \in \underline{\underline{Gr}},$$

where M is a square group. A similar result for quadratic endofunctors of the category $\underline{\underline{Ab}}$ of abelian groups was obtained in [2]. In (3.10) we specify the square groups corresponding to quadratic functors $\underline{\underline{Ab}} \to \underline{\underline{Gr}}$ and $\underline{\underline{Gr}} \to \underline{\underline{Ab}}$ respectively.

The category of linear endofunctors of $\underline{\underline{Gr}}$ which preserve cokernels and filtered colimits is equivalent to the category of abelian groups. In fact, such linear endofunctors L of $\underline{\underline{Gr}}$ have a factorization

$$L : \underline{\underline{Gr}} \xrightarrow{ab} \underline{\underline{Ab}} \xrightarrow{\otimes A} \underline{\underline{Ab}} \subset \underline{\underline{Gr}}$$

where ab is the abelianization functor and $A = L(\mathbb{Z})$ is an abelian group. We show that the quadratic endofunctor F of $\underline{\underline{Gr}}$ has a similar factorization

$$F : \underline{\underline{Gr}} \xrightarrow{nil} \underline{\underline{Nil}} \xrightarrow{\otimes M} \underline{\underline{Nil}} \subset \underline{\underline{Gr}}$$

Here $\underline{\underline{Nil}}$ is the category of groups of nilpotency degree 2, nil is the nilization functor and $M = F\{\mathbb{Z}\}$ is the square group defined in (3.6) with $M_e = F(\mathbb{Z})$, $M_{ee} = F(\mathbb{Z}|\mathbb{Z})$.

The functor F is exact if F carries short exact sequences to short exact sequences. It is well known that exact linear endofunctors F of $\underline{\underline{Ab}}$ are given by torsion free (or flat) abelian groups E such that $F(A) = A \otimes E$. In (6.8) we classify the exact quadratic functors $\underline{\underline{Nil}} \to \underline{\underline{Nil}}$ by flat square groups. Such flat square groups are simply functions $H : E \to E$ on torsion free abelian groups E for which $H(a+b) - H(a) - H(b)$ is linear in a and b and $H(2a) - 4H(a) = a$ with $a, b \in E$. This result relies on the universal coefficient theorem for quadratic functors in [4]. For example

$$\mathbb{Z}_{nil} = (H : \mathbb{Z} \to \mathbb{Z},\, H(a) = a(a-1)/2)$$

is the flat square group corresponding to the identity functor of $\underline{\underline{Nil}}$. The quadratic tensor product satisfies

$$G \otimes \mathbb{Z}_{nil} = nil(G) = G/\Gamma_3 G$$

where $\Gamma_3 G$ is the subgroup of triple commutators in the group G.

Square groups may be considered as the quadratic analogue of abelian groups; they play a similar role in "quadratic algebra" as abelian groups in linear algebra. For example a square ring (as defined in [3]) is a monoid in the category of square groups (compare (8.10)); this generalizes the classical notion of ring being a monoid in the monoidal category of abelian groups. The many examples of square rings in [3] yield examples of square groups. Moreover $\mathbb{Z}_{nil}$ above is also a square ring which, in fact, is the initial object in the category of square rings.

In section § 7 we describe free square groups S which are sums of universal square groups $\mathbb{Z}^{\otimes}$ and $\mathbb{Z}^{Q}$. Each square group M admits a surjection $S \twoheadrightarrow M$. This implies that each quadratic endofunctor of $\underline{\underline{Gr}}$ is obtained as a natural quotient of a functor $\otimes S$. Moreover there is a square ring Q such that the category of Q-modules coincides with the category of square groups. In fact, Q is the endomorphism square ring of $\mathbb{Z}^{Q} \vee \mathbb{Z}^{\otimes}$ in the category of free square groups which is a quadratic category.

In the literature quadratic (and more generally polynomial) endofunctors were mainly studied for additive categories, in particular for the category of abelian groups [2], [5], [12], the category of rational vectorspaces [9], the category of $\mathbb{Z}/p$ -vector spaces [7]. On the other hand polynomial endofunctors of the category of topological spaces were already considered by Goodwillie [6]. The theory of polynomial endofunctors of the category of groups started in this paper is a necessary step to combine the algebraic and topological approach. In fact, applying polynomial endofunctors of $\underline{\underline{Gr}}$ to simplicial groups yields polynomial endofunctors for spaces.

§ 1 Linear and quadratic functors

Let $\underline{\underline{Ab}}$ be the category of abelian groups and let $\underline{\underline{A}}$ be an additive category, for example $\underline{\underline{A}} = \underline{\underline{Ab}}$. Recall that an additive category $\underline{\underline{A}}$ is a category for which the morphism sets $\underline{\underline{A}}(X,Y)$ are abelian groups and composition is bilinear and for which finite sums exist in $\underline{\underline{A}}$. Such sums $A \vee B$ are also products in $\underline{\underline{A}}$ and $A \vee B$ has the structure of a biproduct

(1.1) $$A \underset{r_1}{\overset{i_1}{\rightleftarrows}} A \vee B \underset{r_2}{\overset{i_2}{\leftrightarrows}} B$$

with $r_1 i_1 = 1_A$, $r_2 i_2 = 1_B$ and $i_1 r_1 + i_2 r_2 = 1_{A \vee B}$. The empty sum is the zero object $*$ in $\underline{\underline{A}}$. A functor

(1.2) $$F : \underline{\underline{A}} \to \underline{\underline{Ab}}$$

is <u>additive</u> if $F(f+g) = F(f) + F(g)$ for morphisms $f, g \in \underline{\underline{A}}(X,Y)$ and F is <u>quadratic</u> if Δ with

$$\Delta(f,g) = F(f+g) - F(g) - F(f)$$

is a bilinear function. Additive and quadratic functors satisfy $F(*) = 0$.

We now generalize the concept of additive, resp. quadratic functors as follows. For this we replace the additive category $\underline{\underline{A}}$ by a category $\underline{\underline{C}}$ with sums $X \vee Y$ and zero object $*$. Zero morphisms in $\underline{\underline{C}}$ are described by $0 : X \to * \to Y$. For each sum $X \vee Y$ in $\underline{\underline{C}}$ one has also a diagram

(1.3) $$X \underset{r_1}{\overset{i_1}{\rightleftarrows}} X \vee Y \underset{r_2}{\overset{i_2}{\leftrightarrows}} Y$$

with $r_1 i_1 = 1_X$, $r_2 i_2 = 1_Y$ where the retractions are given by $r_1 = (1,0)$ and $r_2 = (0,1)$. Moreover we replace the category $\underline{\underline{Ab}}$ in (1.2) by the category $\underline{\underline{Gr}}$ of groups and we consider a functor

(1.4) $$F : \underline{\underline{C}} \to \underline{\underline{Gr}} \qquad \text{with} \qquad F(*) = 0.$$

For objects X, Y in $\underline{\underline{C}}$ we define the <u>cross effect</u> $F(X \mid Y)$ by the kernel

$$F(X \mid Y) = \text{kernel}\,(F(X \vee Y) \overset{(r_{1*}, r_{2*})}{\longrightarrow} F(X) \times F(Y))$$

Hence one has a short exact sequence of groups

$$0 \to F(X \mid Y) \overset{i_{12}}{\longrightarrow} F(X \vee Y) \overset{(r_{1*}, r_{2*})}{\longrightarrow} F(X) \times F(X) \to 0$$

which is natural in $X, Y \in \underline{\underline{C}}$. We now say that F in (1.4) is <u>linear</u> if $F(X \mid Y) = 0$ for all $X, Y \in \underline{\underline{C}}$ and we say that F in (1.4) is <u>quadratic</u> if $F(X \mid Y)$ is linear in X and Y, that is, the functors $F(- \mid Y) : X \longmapsto F(X \mid Y)$ and $F(X \mid -) : Y \longmapsto F(X \mid Y)$ are linear in the sense above. This definition of linear and quadratic functors resembles the approach in [6]. The next lemma is well known.

(1.5) Lemma. *Let $F : \underline{\underline{A}} \to \underline{\underline{Ab}}$ be a functor as in (1.2). Then F is additive, resp. quadratic in the sense of (1.2) if and only if the composite $\underline{\underline{A}} \to \underline{\underline{Ab}} \subset \underline{\underline{Gr}}$ is linear, resp. quadratic in the sense of (1.4).*

Hence linear, resp. quadratic functors in (1.4) generalize additive, resp. quadratic functors in (1.2). We shall show that linear functors are actually often determined by additive functors; see § 2.

(1.6) Lemma. *Let $F : \underline{\underline{C}} \to \underline{\underline{Gr}}$ with $F(*) = 0$ be a functor and let $X \in \underline{\underline{C}}$. If F is linear then $F(X)$ is an abelian group. If F is quadratic then $F(X)$ is a group of nilpotency degree 2.*

Proof. Let $a, b \in F(X)$ and let $(a, b) = -a - b + a + b$ be the commutator in $F(X)$. Then $(a, b) = (1, 1)_* d$ where

$$d = (i_{1*}a,\, i_{2*}b) \in F(X \vee X)$$

and one has

$$(r_{1*},\, r_{2*})(d) = 0 \in F(X) \times F(X)$$

If F is linear then $(r_{1*},\, r_{2*})$ is an isomorphism and hence $d = 0$ which implies $(a, b) = 0$. If F is quadratic we consider for $x, y, z \in F(X)$ the triple commutator $c = ((x, y), z)$. Then

$$v = ((i_{1*}x,\, i_{2*}y),\, i_{3*}z) \in F(X \vee X \vee X)$$

satisfies $(1, 1, 1)_*(v) = c$. Moreover for $1 \leq i \leq j \leq 3$ the projection $r_{ij} : X \vee X \vee X \to X \vee X$ satisfies $(r_{ij})_*(v) = 0$. Hence the lemma (1.8) below implies $v = 0$ so that $c = 0$. q.e.d.

Let $F : \underline{\underline{C}} \to \underline{\underline{Gr}}$ be a functor with $F(*) = 0$ and let X_1, X_2, X_3 be objects in $\underline{\underline{C}}$. For $1 \leq i < j \leq 3$ let

$$r_{ij} : X_1 \vee X_2 \vee X_3 \to X_i \vee X_j$$

be the retractions which induce

$$(r_{12}, r_{13}, r_{23})_* : F(X \vee Y \vee Z) \to F(X \vee Y) \vee F(X \vee Z) \vee F(Y \vee Z)$$

Then

$$F(X \mid Y \mid Z) = \text{kernel}\,(r_{12}, r_{13}, r_{23})_* \tag{1.7}$$

is the <u>third cross effect</u> of F.

(1.8) Lemma. *The functor F is quadratic if and only if the third cross effect $F(X \mid Y \mid Z) = 0$ is trivial for all $X, Y, Z \in \underline{\underline{C}}$.*

Proof. One readily checks that

$$F(X \mid Y \mid Z) = \text{kernel}\,(F(X \mid Y \vee Z) \to F(X \mid Y) \times F(X \mid Z))$$

and the result follows. q.e.d.

We have inclusions of categories $\underline{\underline{Ab}} \subset \underline{\underline{Nil}} \subset \underline{\underline{Gr}}$ where $\underline{\underline{Nil}}$ is the full subcategory of groups of nilpotency degree 2. Hence by (1.6) any linear resp. quadratic functor $F : \underline{\underline{C}} \to \underline{\underline{Gr}}$ has the factorization

(1.9)
$$\begin{array}{ll} F : \underline{\underline{C}} \to \underline{\underline{Ab}} \subset \underline{\underline{Gr}} & \text{(linear)} \\ F : \underline{\underline{C}} \to \underline{\underline{Nil}} \subset \underline{\underline{Gr}} & \text{(quadratic)} \end{array}$$

For a quadratic functor F we see by (1.6) that $F(X \mid Y)$ is an abelian group. Moreover we get

(1.10) Lemma. *Let F be quadratic. Then*

$$0 \to F(X \mid Y) \to F(X \vee Y) \to F(X) \times F(Y) \to 0$$

is a central extension.

Proof. Since each element in $F(X \vee Y)$ has the form $b + i_1(x) + i_2(y)$ ($x \in F(X)$, $y \in F(Y)$, $b \in F(X \mid Y)$) it is enough to show that

$$c = -b - i_1(x) + b + i_1(x)$$

is zero. We consider the following element in $F(X \vee Y \vee X)$:

$$d = -(i_1, i_2)_*(b) - i_3(x) + (i_1, i_2)_*(b) + i_{3*}(x).$$

Here $i_3 : X \to X \vee Y \vee X$ and $(i_1, i_2) : X \vee Y \to X \vee Y \vee X$ are canonical inclusions. Then for $(i_1, i_2, i_1) : X \vee Y \vee X \to X \vee Y$ one has

$$(i_1, i_2, i_1)_* d = c$$

and $r_{ij*} d = 0$, $1 \leq i < j \leq 3$, and lemma (1.8) gives $d = 0$ so that $c = 0$. q.e.d.

(1.11) Definition. Let $F : \underline{\underline{C}} \to \underline{\underline{Gr}}$ be a functor with $F(*) = 0$. Then we define the natural transformation

$$P : F(X \mid X) \to F(X)$$

by the composition

$$F(X \mid X) \subset F(X \vee X) \xrightarrow{(1,1)_*} F(X)$$

Similarly we define

$$\bar{P} : F(X \mid X \mid X) \to F(X)$$

by the composition

$$F(X \mid X \mid X) \subset F(X \vee X \vee X) \xrightarrow{(1,1,1)_*} F(X)$$

(1.12) Lemma. *The image of P and $\bar{P}$ respectively is a normal subgroup. Moreover the image of P is central if F is quadratic.*

Proof. This is clear since $F(X \mid X)$ is normal in $F(X \vee X)$ and $(1,1)_*$ is epimorphic. The same argument holds for $\bar{P}$ and for the quadratic case. q.e.d.

(1.13) Lemma. *F is linear if and only if $P = 0$. Moreover F is quadratic if and only if $\bar{P} = 0$.*

Proof. Clearly $P = 0$ if F is linear. Conversely if $P = 0$ we obtain $F(X \mid Y) = 0$ since we have the commutative diagram

$$\begin{array}{ccc} F(X \vee Y \mid X \vee Y) & \xrightarrow{P} & F(X \vee Y) \\ & {\scriptstyle (i_1 \mid i_2)_*} \nwarrow \quad \nearrow {\scriptstyle i_{12}} & \\ & F(X \mid Y) & \end{array}$$

A similar argument holds in the quadratic case. q.e.d.

(1.14) Definition. Let $F : \underline{\underline{C}} \to \underline{\underline{Gr}}$ be a functor with $F(*) = 0$ as in (1.4). Then the additivization, resp. the quadratization

$$F^{ad} : \underline{\underline{C}} \to \underline{\underline{Ab}},$$
$$F^{quad} : \underline{\underline{C}} \to \underline{\underline{Nil}}$$

are the functors defined by $F^{ad}(X) = \text{cokernel}\,(P : F(X \mid X) \to F(X))$ and $F^{quad}(X) = \text{cokernel}\,(\bar{P} : F(X \mid X \mid X) \to F(X))$. Here F^{ad} is a linear functor and F^{quad} is a quadratic functor and we use (1.9). In fact, F^{ad} is linear by the following argument. Since P is natural and the diagram

$$\begin{array}{ccc} F(X \mid X) & \xrightarrow{P} & F(X) \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q} \\ F^{ad}(X \mid X) & \xrightarrow{P} & F^{ad}(X) \end{array}$$

commutes where q_2 is induced by the natural transformation q. Here q_2 is an epimorphism and $qP = 0$ so that P in the bottom row is zero. Hence F^{ad} is linear by (1.13). A similar argument shows that F^{quad} is quadratic.

Hence the inclusions of functor categories

$$\underline{\underline{linear\ functors}} \subset \underline{\underline{quadratic\ functors}} \subset \underline{\underline{functors}}\,(\underline{\underline{C}} \to \underline{\underline{Gr}})$$

have left adjoint functors given by additivization and quadratization respectively. We leave it to the reader to study more generally functors $\underline{\underline{C}} \to \underline{\underline{Gr}}$ of degree n (which are linear for $n = 1$ and quadratic for $n = 2$) and to prove similar results for such functors.

§ 2 Linear and quadratic functors on theories of cogroups

We generalize the notion of an additive category as follows. A theory of cogroups is a category $\underline{\underline{T}}$ with zero object $*$ and finite sums such that each object X has the structure of a cogroup given by the maps $\mu : X \to X \vee X$, $\nu : X \to X$ satisfying the usual identities. A morphism $f : X \to Y$ is linear if the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ X \vee X & \xrightarrow{f \vee f} & Y \vee Y \end{array}$$

commutes. Morphism sets $\underline{\underline{T}}(X, Y)$ in $\underline{\underline{T}}$ are groups (written additively) with $a + b = (a, b)\mu$, $-a = a\nu$, for $a, b \in \underline{\underline{T}}(X, Y)$. Moreover we assume that for all objects X, Y in $\underline{\underline{T}}$ there is given a diagram of linear morphisms

(2.1) $$X \underset{r_1}{\overset{i_1}{\rightleftarrows}} X \vee Y \underset{r_2}{\overset{i_2}{\leftrightarrows}} Y$$

with $r_1 = (1, 0)$, $r_2 = (0, 1)$ and $i_1 r_1 + i_2 r_2 = 1_{X \vee Y}$. Clearly by (1.1) any additive category is a theory of cogroups. We consider the covariant Hom functors

$$\underline{\underline{T}}(X, -) : \underline{\underline{T}} \to \underline{\underline{Gr}}$$

which carry $Y \in \underline{\underline{T}}$ to the group $\underline{\underline{T}}(X, Y)$.

(2.2) Lemma. *A theory $\underline{\underline{T}}$ of cogroups is an additive category if and only if all covariant Hom functors are linear.*

This is readily proved and leads to the next definition:

(2.3) Definition. A theory of cogroups $\underline{\underline{T}}$ is a quadratic category if all covariant Hom functors of $\underline{\underline{T}}$ are quadratic.

One can check that this notion of a quadratic category coincides with the one in [3] where quadratic categories are studied. For any theory of cogroups $\underline{\underline{T}}$ one obtains canonically an additive catgeory $\underline{\underline{T}}^{ad}$ and a quadratic category $\underline{\underline{T}}^{quad}$ together with quotient functors:

$$(2.4) \qquad \underline{\underline{T}} \to \underline{\underline{T}}^{quad} \to \underline{\underline{T}}^{ad}$$

Here the morphism sets in $\underline{\underline{T}}^{ad}$ and $\underline{\underline{T}}^{quad}$ respectively are given by the additivization and quadratization of the covariant Hom functors in $\underline{\underline{T}}$:

$$\underline{\underline{T}}^{ad}(X,Y) = \underline{\underline{T}}(X,-)^{ad}(Y),$$
$$\underline{\underline{T}}^{quad}(X,Y) = \underline{\underline{T}}(X,-)^{quad}(Y).$$

The quotient functors in (2.4) have the obvious universal properties with respect to functors $\underline{\underline{T}} \to \underline{\underline{C}}$ where $\underline{\underline{C}}$ is quadratic (resp. additive) category.

(2.5) __Example__. The category $\underline{\underline{gr}}$ of free groups is a theory of cogroups. Moreover the category $\underline{\underline{nil}}$ of free objects in the category $\underline{\underline{Nil}}$ is a quadratic category and the category $\underline{\underline{ab}}$ of free abelian groups is an additive category such that the obvious quotient functors

$$\begin{array}{ccccc}
\underline{\underline{gr}} & \longrightarrow & \underline{\underline{gr}}^{quad} & \longrightarrow & \underline{\underline{gr}}^{ad} \\
\| & & \| & & \| \\
\underline{\underline{gr}} & \longrightarrow & \underline{\underline{nil}} & \longrightarrow & \underline{\underline{ab}}
\end{array}$$

coincide with (2.4). Further examples are obtained by the categories of free algebras, free Lie algebras, etc. in a similar way.

(2.6) __Proposition.__ *Let $\underline{\underline{T}}$ be a theory of cogroups and let $F : \underline{\underline{T}} \to \underline{\underline{Gr}}$ be a functor with $F(*) = 0$. If F is linear, resp. quadratic, there is a unique factorization as follows where the vertical arrows denote the canonical functors.*

$$\begin{array}{ccc}
\underline{\underline{T}} & \xrightarrow{F} & \underline{\underline{Gr}} \\
\downarrow & & \uparrow \\
\underline{\underline{T}}^{ad} & \longrightarrow & \underline{\underline{Ab}}
\end{array} \qquad \textit{(linear)}$$

$$\begin{array}{ccc}
\underline{\underline{T}} & \xrightarrow{F} & \underline{\underline{Gr}} \\
\downarrow & & \uparrow \\
\underline{\underline{T}}^{quad} & \longrightarrow & \underline{\underline{Nil}}
\end{array} \qquad \textit{(quadratic)}$$

This shows that the category of linear functors $\underline{\underline{T}} \to \underline{\underline{Gr}}$ is equivalent to the category of additive functors $\underline{\underline{A}} \to \underline{\underline{Ab}}$ where $\underline{\underline{A}} = \underline{\underline{T}}^{ad}$ is an additive category. Moreover the category of quadratic functors $\underline{\underline{T}} \to \underline{\underline{Gr}}$ is equivalent to the category of quadratic functors $\underline{\underline{Q}} \to \underline{\underline{Nil}}$ where $\underline{\underline{Q}} = \underline{\underline{T}}^{quad}$ is a quadratic category. For the proof of (2.6) we will use the following lemma.

(2.7) Lemma. *Let $\underline{\underline{T}}$ be a theory of cogroups and $F : \underline{\underline{T}} \to \underline{\underline{Gr}}$ be a functor with $F(*) = 0$. We consider for $X, Y \in \underline{\underline{T}}$ and $a \in F(X)$ the evaluation function*

$$\underline{\underline{T}}(X,Y) \to F(Y),\ f \longmapsto f_*(a)$$

and its cross effect

$$(f \mid g)_*(a) = (f+g)_*(a) - g_*(a) - f_*(a)$$

Then F is linear (resp. quadratic) if and only if $(f \mid g)_*(a) = 0$ (resp. $(f \mid g)_*(a)$ is bilinear in f and g) for all X, Y and a.

We obtain a retraction of i_{12} in (1.4)

$$(2.8) \qquad \begin{aligned} &r_{12} : F(X \vee Y) \to F(X \mid Y) \qquad \text{by} \\ &i_{12} r_{12}(a) = a - i_2 r_2 a - i_1 r_1 a = (i_1 r_1 \mid i_2 r_2)_*(a) \end{aligned}$$

In fact as in (2.10) of [3] one shows that r_{12} is well defined and surjective. Here we use the assumptions that $i_1 r_1 + i_2 r_2 = 1_{X \vee Y}$. Using the retraction r_{12} we obtain the natural function

$$(2.9) \qquad H : F(X) \to F(X \mid X)$$

given by the composition

$$F(X) \xrightarrow{(i_1+i_2)} F(X \vee X) \xrightarrow{r_{12}} F(X \mid X)$$

The functions r_{12} and H need not to be homomorphisms. For $f, g : X \to Y$ and $a \in F(X)$ one readily gets the formula

$$(2.10) \qquad (f \mid g)_*(a) = P(f,g)_* H(a)$$

where $(f,g)_* : F(X \mid X) \to F(Y \mid Y)$ is induced by $f \vee g : F(X \vee X) \to F(Y \vee Y)$; compare (3.3) (6) in [3].

Proof of (2.7). If $(f \mid g)_*(a) = 0$ for all X, Y, a then $(i_1 r_1 \mid i_2 r_2)_*(a) = 0$ and hence $r_{12} = 0$. Therefore $F(X \mid Y) = 0$ and thus F is linear. On the other hand if F is linear then the inclusions $i_1, i_2 : X \to X \vee X$ satisfy $(i_1 + i_2)_* = i_{1*} + i_{2*} : F(X) \to F(X \vee X) = F(X) \times F(X)$. Hence for $f + g = (f,g)(i_1 + i_2)$ we get $(f+g)_* = f_* + g_*$. One gets the result in a similar way for quadratic functors.

q.e.d.

Proof of (2.6). Consider the diagram

$$\begin{array}{ccccc} \underline{\underline{T}}(X, Y \mid Y) & \xrightarrow{P} & \underline{\underline{T}}(X,Y) & \xrightarrow{F} & Hom(FX, FY) \\ & & \uparrow{\scriptstyle \bar{P}} & & \\ & & \underline{\underline{T}}(X, Y \mid Y \mid Y) & & \end{array}$$

If F is linear one has to show that $F(f+Px)=F(f)$ and if F is quadratic one has to show that $F(f+\bar{P}z)=F(f)$. Now let F be linear. For $a \in F(X)$ one gets

$$F(f+Px)(a) = F(f)(a) + F(P(x))(a) \qquad \text{by (2.7)}$$

where $F(P(x))(a)=0$ by the commutative diagram

$$\begin{array}{ccc} \underline{\underline{T}}(X,Y \mid Y) & \xrightarrow{ev} & F(Y \mid Y)=0 \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P} \\ \underline{\underline{T}}(X,Y) & \xrightarrow{ev} & F(Y) \end{array}$$

Here the evaluation ev with $ev(f)=f_*(a)$ is a natural transformation in Y. Actually this shows that for any linear functor F and $x \in \underline{\underline{T}}(X,Y \mid Y)$ with $Px : X \to Y$ the induced map $(Px)_* = 0 : F(X) \to F(Y)$ is trivial. Given a quadratic functor F this implies that also for $x \in \underline{\underline{T}}(X,Y \mid Y)$ the induced map $0=(1_Z,Px)_* : F(Z \mid X) \to F(Z \mid Y)$ is trivial. Thus also for any $f \in \underline{\underline{T}}(X,Y)$ we get $(f,Px)_*=0$ and by (2.10) this shows

$$(f \mid Px)_*(a) = 0 \quad \text{for any} \quad a \in F(X).$$

Now by (2.10) we get

$$(f+\bar{P}y)_*(a) = f_*(a) + (\bar{P}y)_*(a) + (f \mid \bar{P}y)_*(a)$$

where $(\bar{P}y)_*=0$ since $ev\,\bar{P} = \bar{P}\,ev$ as above for P. Moreover $\bar{P}y = Pz$ for some z and hence $(f \mid \bar{P}y)_*(1)=0$. This shows that $F(f+\bar{P}y)=F(f)$ and the proposition is proved.

q.e.d.

§ 3 Quadratic endofunctor of the category of groups

We classify certain linear and quadratic endofunctors of the category of groups. The linear case is easily described by the following result. We say that a functor $F : \underline{\underline{C}} \to \underline{\underline{K}}$ <u>preserves cokernels</u> if F carries each coequalizer in $\underline{\underline{C}}$ to a coequalizer in $\underline{\underline{K}}$.

(3.1) <u>Proposition</u>. *Let $F : \underline{\underline{Gr}} \to \underline{\underline{Gr}}$ be a linear functor which preserves cokernels. Then F has a unique factorization*

$$\begin{array}{ccc} \underline{\underline{Gr}} & \xrightarrow{F} & \underline{\underline{Gr}} \\ {\scriptstyle ab}\downarrow & & \uparrow \\ \underline{\underline{Ab}} & \xrightarrow{\tilde{F}} & \underline{\underline{Ab}} \end{array}$$

where the left side is the abelianization functor. Here $\tilde{F}$ again preserves cokernels. If F also preserves filtered colimits then $\tilde{F}$ is given by the tensor product

$$\tilde{F}(A) = A \otimes M$$

with $M = F(\mathbb{Z})$ and $A \in \underline{\underline{Ab}}$.

Filtered colimits are also termed direct limits.

(3.2) <u>Corollary</u>. *The category of linear endofunctors of $\underline{\underline{Gr}}$, resp. $\underline{\underline{Ab}}$, which preserve cokernels and filtered colimits is equivalent to the category $\underline{\underline{Ab}}$ of abelian groups.*

The quadratic analogue of these results is obtained below by the use of square groups which replace abelian groups.

<u>Proof of</u> (3.1). Since F preserves cokernels we see that F is determined by the restriction $F : \underline{\underline{gr}} \to \underline{\underline{Gr}}$ to the category $\underline{\underline{gr}}$ of free groups. Hence by (1.7) and (2.5), (2.6) we obatin $\tilde{F}$. Moreover if F preserves also filtered colimits then $\tilde{F}$ is determined by the restriction $\tilde{F} : fg - \underline{\underline{ab}} \to \underline{\underline{Ab}}$ where $fg - \underline{\underline{ab}}$ is the category of finitely generated free abelian groups. Hence $\tilde{F}$ is determined by the $\mathbb{Z}$ -module $M = \tilde{F}(\mathbb{Z}) = F(\mathbb{Z})$ with $\tilde{F}(A) = A \otimes F(\mathbb{Z})$.

q.e.d.

We now consider quadratic endofunctors of the category $\underline{\underline{Ab}}$ of abelian groups. For this we need the following notation from [2].

(3.3) <u>Definition</u>. A <u>quadratic $\mathbb{Z}$ -module</u>

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$

is a pair of abelian groups M_e, M_{ee} together with homomorphisms H, P satsfying $PHP = 2P$ and $HPH = 2H$. Given M we obtain a quadratic functor $\underline{\underline{Ab}} \to \underline{\underline{Ab}}$ which carries the abelian group A to the <u>quadratic tensor product</u> $A \otimes M$. Here $A \otimes M$ is the abelian group generated by symbols $a \otimes m$, $[a, b] \otimes n$ with $a, b \in A$ and $m \in M_e$, $n \in M_{ee}$ subject to the relations

$$(1) \qquad \begin{cases} (a+b) \otimes m = a \otimes m + b \otimes m + [a, b] \otimes H(m) \\ [a, a] \otimes n = a \otimes P(n) \end{cases}$$

where $a \otimes m$ is linear in m and $[a, b] \otimes n$ is linear in a, b and n. let $\underline{\underline{A}}$ be an additive category and $F : \underline{\underline{A}} \to \underline{\underline{Ab}}$ be a quadratic functor. Then each $X \in \underline{\underline{A}}$ determines the quadratic $\mathbb{Z}$-module

$$(2) \qquad F\{X\} = (F(X) \xrightarrow{H} F(X \mid X) \xrightarrow{P} F(X))$$

where H and P are defined as in (2.12), (1.11). With this notation we obtain the following quadratic analogue of (3.1) proved in [2].

(3.4) <u>Proposition</u>. *Let $F : \underline{\underline{Ab}} \to \underline{\underline{Ab}}$ be a quadratic functor which preserves cokernels and filtered colimits. Then one has the natural isomorphism*

$$F(A) = A \otimes M$$

where $M = F\{\mathbb{Z}\}$ is the quadratic $\mathbb{Z}$ -module given as in (3.3) (2). Moreover the category of such functors is equivalent to the category of quadratic $\mathbb{Z}$ -modules.

The next notion of a square group generalizes the notion of a quadratic $\mathbb{Z}$-module.

(3.5) Definition. A square group

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$

is given by a group M_e and an abelian group M_{ee}. Both groups are written additively. Moreover p is a homomorphism and H is a quadratic function, that is the cross effect

$$(a \mid b)_H = H(a+b) - H(b) - H(a)$$

is linear in $a, b \in Q_e$. In addition the following properties are satisfied ($x, y \in M_{ee}$)

(1) $\quad (Px \mid b)_H = 0 \quad \text{and} \quad (a \mid Py)_H = 0$

(2) $\quad P(a \mid b)_H = a + b - a - b$

(3) $\quad PHP(x) = P(x) + P(x)$

(4) $\quad \Delta(a) = HPH(a) + H(a+a) - 4H(a)$ is linear in a

By (1) and (2) P maps to the center of M_e and by (2) cokernel of P is abelian. Hence M_e is a group of nilpotency degree 2. Let *Square* be the category of square groups. Quadratic $\mathbb{Z}$ -modules are square groups for which $\Delta = 0$ is trivial and H is linear. One has the following additional formulas:

(5) $\quad H(a+b-a-b) = -(b \mid a)_H + (a \mid b)_H$

(6) $\quad \Delta P(x) = 0$

(7) $\quad P\Delta(a) = 0$

As an example we have the square group

$$\mathbb{Z}_{nil} = (\mathbb{Z} \xrightarrow{H} \mathbb{Z} \xrightarrow{0} \mathbb{Z})$$

with $H(r) = \binom{r}{2}$ and $P = 0$; compare [3]. Square groups arise naturally as follows.

(3.6) Proposition. *Let $\underline{\underline{T}}$ be a theory of cogroups and let $F : \underline{\underline{T}} \to \underline{\underline{Gr}}$ be a quadratic functor with $F(*) = 0$. Then each object $X \in \underline{\underline{T}}$ yields the square group*

$$F\{X\} = (F(X) \xrightarrow{H} F(X \mid X) \xrightarrow{P} F(X))$$

given by H and P in (2.12) and (2.11).

Proof. For $X, Y \in \underline{\underline{Q}}$ we have the following diagram in which the row is a central extension of groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(X \mid Y) & \xrightarrow{i_{12}} & F(X \vee Y) & \xrightarrow{(r_{1*}, r_{2*})} & F(X) \times F(Y) & \longrightarrow & 0 \\
 & & & & \downarrow r_{12} & & & & \\
 & & & & F(X \mid Y) & & & &
\end{array}$$

Here the function r_{12} with $r_{12}i_{12}(x) = x$ is defined by

$$r_{12}(u) = i_{12}^{-1}(u - (i_2 r_2)_* u - (i_1 r_1)_* u) \tag{1}$$

where $i_1 : X \to X \vee Y$, $i_2 : Y \to X \vee Y$ are the inclusions. By definition of H and P we have

$$H = r_{12}\mu_* : F(X) \to F(X \vee X) \to F(X \mid X) \tag{2}$$

$$P = \nabla_* i_{12} : F(X \mid X) \to F(X \vee X) \to F(X) \tag{3}$$

where $\mu = i_1 + i_2$ and $\nabla = (1, 1)$. We now check that $F\{X\} = (H, P)$ is a square group. In fact, P is central since ∇_* is surjective and since i_{12} is central. P is a homomorphism while H is quadratic since for $a, b \in F(X)$

$$H(a + b) - H(b) - H(a) = i_{12}^{-1}(i_{2*}a + i_{1*}b - i_{2*}a - i_{1*}b) \tag{4}$$

Compare the proof of (3.3) (2) [3]. The right hand side is a commutator in a group of nilpotency degree 2 and therefore linear in a and b. Moreover since P is central and natural we see that $i_{2*}(Px) = P(i_{2*}x)$ is central and therefore (3.5) (1) holds by (4). Also (3.5) (2) is a consequence of (4) and (3). We define

$$T : F(X \mid Y) \to F(Y \mid X) \tag{5}$$

by $i_{12}T = t_* i_{12}$ where $t : X \vee Y \to Y \vee X$ is the interchange map. Then $PT = P$ and $T = HP - 1$ as in the proof of (3.3)(3)[3]. Hence $PHP - P = P$ so that also (3.5) (3) holds. Finally we obtain (3.5) (4) by the following equations.

$$\begin{aligned}
i_{12}(\Delta(a)) &= i_{12}(HPH(a) + H(a + a) - 4H(a)) \\
&= i_{12}(TH(a) - H(a) + (a|a)_H) \\
i_{12}(TH(a) - H(a)) &= (i_2 + i_1)_* a - i_{1*}a - i_{2*}a - ((i_1 + i_2)_* a - i_{2*}a - i_{1*}a) \\
&= (i_2 + i_1)_* a - i_{1*}a + (-i_{2*}a + i_{1*}a + i_{2*}a - i_{1*}a) + i_{1*}a - (i_1 + i_2)_* a \\
&= (i_2 + i_1)_* a + i_{12}(-a|a)_H - (i_1 + i_2)_* a.
\end{aligned}$$

Hence we get

$$i_{12}\Delta(a) = (i_2 + i_1)_* a - (i_1 + i_2)_* a \in F(X \vee X)$$

and this function is linear in a since i_{12} is central and $(i_1 + i_2)_*$ and $(i_2 + i_1)_*$ are homomorphisms.

q.e.d.

The following non abelian version of the quadratic tensor product is obtained almost in the same way as in (3.3) (1).

(3.7) Definition. Let G be a group and let M be a square group. We define the group $G \otimes M$ by the generators $g \otimes a$ and $[g,h] \otimes x$ with $g, h \in G, a \in M_e$ and $x \in M_{ee}$ subject to the relations

$$\begin{aligned}(g+h) \otimes a &= g \otimes a + h \otimes a + [g,h] \otimes H(a) \\ [g,g] \otimes x &= g \otimes P(x)\end{aligned}$$

where $g \otimes a$ is linear in a and where $[g,h] \otimes x$ is central and linear in each variable g, h and x. There are obvious induced maps for this tensor product so that one gets a bifunctor

$$\otimes : \underline{\underline{Gr}} \times \underline{\underline{\text{Square}}} \to \underline{\underline{Gr}}$$

This functor factors uniquely as follows

$$\begin{array}{ccc} \underline{\underline{Gr}} \times \underline{\underline{\text{Square}}} & \xrightarrow{\otimes} & Gr \\ \downarrow & & \uparrow \\ \underline{\underline{Nil}} \times \underline{\underline{\text{Square}}} & \xrightarrow{\otimes} & \underline{\underline{Nil}} \end{array}$$

where the bottom row is the restriction of $\otimes$ to the subcategory $\underline{\underline{Nil}} \subset \underline{\underline{Gr}}$ of groups of nilpotency degree 2. The left hand side is given by the nilization functor $nil : \underline{\underline{Gr}} \to \underline{\underline{Nil}}$ which carries G to G^{nil}. The commutativity of the diagram is a consequence of the next lemma.

(3.8)Lemma. *For any group G and square group M we have $G \otimes M \in \underline{\underline{Nil}}$ and $G \otimes M = G^{nil} \otimes M$. Moreover the functor $\underline{\underline{Gr}} \to \underline{\underline{Gr}}$, $G \mapsto G \otimes M$, is quadratic and preserves cokernels and colimits.*

We prove this lemma in §4. Next we state our result on the classification of quadratic endofunctors of the category of groups.

(3.9) Theorem. *Let $F : \underline{\underline{Gr}} \to \underline{\underline{Gr}}$ be a quadratic functor which preserves cokernels. Then F has a unique factorization*

$$\begin{array}{ccc} \underline{\underline{Gr}} & \xrightarrow{F} & \underline{\underline{Gr}} \\ {\scriptstyle nil}\downarrow & & \uparrow \\ \underline{\underline{Nil}} & \xrightarrow{\tilde{F}} & \underline{\underline{Nil}} \end{array}$$

where the left hand side to the nilization functor. Here $\tilde{F}$ preserves cokernels. If F also preserves filtered colimits then $\tilde{F}$ is given by the quadratic tensor product

$$\tilde{F}(G) = G \otimes M$$

with $M = F\{\mathbb{Z}\}$ and $G \in \underline{\underline{Nil}}$.

(3.10) Corollary. *The category of quadratic endofunctors of* $\underline{\underline{Gr}}$ *which preserve cokernels and filtered colimits is equivalent to the category* $\underline{\underline{Square}}$ *of square groups. Moreover this equivalence of categories yields equivalences of subcategories according to the following list.*

quadratic functors as in (3.10)	square groups
$\underline{\underline{Gr}} \longrightarrow \underline{\underline{Gr}}$	all square groups
$\underline{\underline{Ab}} \longrightarrow \underline{\underline{Gr}}$	square groups with $\Delta = 0$
$\underline{\underline{Gr}} \longrightarrow \underline{\underline{Ab}}$	square groups with H linear
$\underline{\underline{Ab}} \longrightarrow \underline{\underline{Ab}}$	quadratic $\mathbb{Z}$-module

Proof of (3.9). Since F preserves cokernels we see that F is determined by the restriction $F : \underline{\underline{gr}} \to \underline{\underline{Gr}}$. Hence by (1.7) and (2.5), (2.6) we obtain $\tilde{F}$. If $\tilde{F}$ preserves filtered limits then $\tilde{F}$ is determined by the restriction $\tilde{F} : fg - \underline{\underline{nil}} \to \underline{\underline{Nil}}$ where $fg - \underline{\underline{nil}}$ is the category of finitely generated free objects in $\underline{\underline{nil}}$. Hence (3.9) and (3.10) are consequences of (3.8) and the following result.

q.e.d.

(3.12) Proposition. *The category of quadratic functors* $fg - \underline{\underline{nil}} \to \underline{\underline{Nil}}$ *is equivalent to the category* $\underline{\underline{Square}}$ *of square groups. The equivalence carries* F *to* $F\{\mathbb{Z}\}$ *with the inverse carrying* M *to the functor* $G \mapsto G \otimes M$ *with* $G \in fg - \underline{\underline{nil}}$.

Proof of (3.12). By (3.8) we know that the functor

$$- \otimes M : fg - \underline{\underline{nil}} \longrightarrow \underline{\underline{Nil}}$$

is quadratic and one really checks that

$$(- \otimes M)\{\mathbb{Z}\} = M.$$

On the other hand for any quadratic functor $F : fg - \underline{\underline{nil}} \to \underline{\underline{Nil}}$ there exists a natural transformation

$$\begin{aligned} \tau_G = \tau &: G \otimes F\{\mathbb{Z}\} \to F(G) \\ \tau(g \otimes a) &= F(\tilde{g})(a) \\ \tau([g,h] \otimes x) &= p(F(\tilde{g},\tilde{h})(x)) \end{aligned}$$

Here $g, h \in G$, $a \in F\{\mathbb{Z}\}_e = F(\mathbb{Z})$, $x \in F\{\mathbb{Z}\}_{ee} = F(\mathbb{Z}|\mathbb{Z})$ and $\tilde{g}(1) = g$. Since $\tau_{\mathbb{Z}}$ of $\tau_{\mathbb{Z}\oplus\mathbb{Z}}(\mathbb{Z}|\mathbb{Z})$ are isomorphisms also τ is an isomorphism for any $G \in fg - \underline{\underline{nil}}$.

q.e.d.

§ 4 Properties of the quadratic tensor product

We describe some properties of the quadratic tensor product which in particular yield a proof of (3.8). Let G be a group and let M be a square group. Then the following equations hold for $g, h, k \in G$ and $a, b \in M_e$, $x, y \in M_{ee}$.

(4.1) $$[g,h] \otimes x = [h,g] \otimes Tx \quad \text{with } T = HP - 1$$

(1) $$(-g) \otimes a = -(g \otimes a) + g \otimes PHa$$

(2) $$(ng) \otimes a = g \otimes (na) + g \otimes \binom{n}{2} PHa$$

(3) $$-h \otimes a - g \otimes b + h \otimes a + g \otimes b = [g,h] \otimes (a|b)_H$$

For the commutator $c = -g - h + g + h \in G$ one has

(4) $$c \otimes a = [h,g] \otimes \Delta(c)$$

(5) $$[c,k] \otimes x = 0 = [k,c] \otimes x$$

We leave the proof of (4.1), (1) and (2) and (5) to reader.

Proof of (3). We have the following equations

$$\begin{aligned}
&(g+h) \otimes (a+b) = g \otimes (a+b) + h \otimes (a+b) + [g,h] \otimes H(a+b) = \\
&= g \otimes a + g \otimes b + h \otimes a + h \otimes b + [g,h] \otimes (H(a) + H(b) + (a|b)_H) \\
&(g+h) \otimes (a+b) = (g+h) \otimes a + (g+h) \otimes b = \\
&= g \otimes a + h \otimes a + [g,h] \otimes Ha + g \otimes b + h \otimes b + [g,h] \otimes Hb
\end{aligned}$$

Since $[g,h] \otimes Ha$ is central one therefore gets (3).

Proof of (4).

$$\begin{aligned}
&c \otimes a = (-g - h + g + h) \otimes a = \\
&(-g) \otimes a + (-h) \otimes a + g \otimes a + h \otimes a + [-g,-h] \otimes Ha + \\
&[-g,g] \otimes Ha + [-g,h] \otimes Ha + [-h,g] \otimes Ha + [-h,h] \otimes Ha + [g,h] \otimes Ha \\
&= -g \otimes a - h \otimes a + g \otimes a + h \otimes a - [h,g] \otimes Ha + [g,h] \otimes Ha
\end{aligned}$$

Here we use (1) and (3.7). Now (3) and (4.1) yield (4).

q.e.d.

(4.2) Theorem. *Let $G \vee H$ be the sum of groups G, H in the category $\underline{\underline{Gr}}$ with retractions r_1, r_2. Moreover let M be a square group. Then there is a natural short exact sequence*

$$0 \longrightarrow G^{ab} \otimes H^{ab} \otimes M_{ee} \xrightarrow{i} (G \vee H) \otimes M \xrightarrow{q} (G \otimes M) \times (H \otimes M) \longrightarrow 0$$

where $q = (r_1 \otimes M,\, r_2 \otimes M)$. The inclusion i carries $\{g\} \otimes \{h\} \otimes x$ to $[i_1 g,\, i_2 h] \otimes x$ for $g \in G$, $h \in H$, $x \in M_{ee}$.

Proof of (3.8). The theorem shows that the functor $G \longmapsto G \otimes M$ is quadratic. Clearly the functor preserves cokernels and colimits. Therefore (3.8) is a consequence of (2.6) and (2.5).

q.e.d.

(4.3) Addendum. For the product $G \times H$ of groups and the canonical map $G \vee H \to G \times H$ one obtains the following natural commutative diagram with short exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G^{ab} \otimes H^{ab} \otimes M_{ee} & \xrightarrow{i} & (G \vee H) \otimes M & \xrightarrow{q} & G \otimes M \times H \otimes M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & G^{ab} \otimes H^{ab} \otimes M_{\Delta} & \longrightarrow & (G \times H) \otimes M & \longrightarrow & G \otimes M \times H \otimes M & \longrightarrow & 0 \end{array}$$

where M_Δ is the cokernel of $\Delta : M_e \to M_{ee}$ and where $M_{ee} \to M_\Delta$ is the quotient map.

The proof of this addendum can be achieved by the same method which is used in the following proof of (4.2).

Proof of (4.2). One readily checks that i is well defined and that q is surjective with $qi = 0$. Since $(i_1 g + i_2 h) \otimes a \equiv (i_1 g) \otimes a + (i_2 h) \otimes a$ modulo image of i we see that the cokernel of i is defined by the same generators and relations as the group $(G \otimes M) \times (H \otimes M)$. Hence we get kernel$(q) =$ image(i). It remains to check that i is injective. For this it is enough to prove the theorem for free groups. In fact let X, Y be free simplicial groups with $\pi_0 X = G$, $\pi_0 Y = H$ and $\pi_i X = \pi_i Y = 0$ for $i > 0$. Then the free case of (4.2) yields the short exact sequence of simplicial groups

$$0 \to X^{ab} \otimes Y^{ab} \otimes M_{ee} \to (X \vee Y) \otimes M \to (X \otimes M) \times (Y \otimes M) \to 0$$

and hence the sequence in (4.2) is obtained as π_0 of this sequence. For this we use the homotopy exact sequence and the fact that

$$\pi_1(X \vee Y) \otimes M \xrightarrow{q_*} \pi_1((X \otimes M) \times (Y \otimes M)) = \pi_1(X \otimes M) \times \pi_1(Y \otimes M)$$

is surjective. A set theoretic splitting of q_* is given by $(\alpha, \beta) \longmapsto i_{1*}\alpha + i_{2*}\beta$. We now assume that G and H in (4.2) are free groups. The kernel of q is a quotient of the biadditive functor $(G, H) \longmapsto G^{ab} \otimes H^{ab} \otimes M_{ee}$ and hence kernel(q) is also a biadditive functor by naturality of the sequence. Moreover these functors are compatible with filtered colimits and therefore it suffices to prove the theorem for $G = H = \mathbb{Z}$. For this we introduce the following group $\tilde{M}$ given by the square group M. The elements of $\tilde{M}$ are the triples (a, b, x) with $a, b \in M_e$, $x \in M_{ee}$ with the group structure defined by

$$(a, b, x) + (a', b', x') = (a + a', b + b', x + x' - (b'|a)_H)$$

Then there exists a commutative diagram

$$\begin{array}{ccccccccc} & & M_{ee} & \xrightarrow{i} & (\mathbb{Z} \vee \mathbb{Z}) \otimes M & \xrightarrow{q} & \mathbb{Z} \otimes M \times \mathbb{Z} \otimes M & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \psi} & & \| & & \\ 0 & \longrightarrow & M_{ee} & \longrightarrow & \tilde{M} & \longrightarrow & M_e \times M_e & \longrightarrow & 0 \end{array}$$

in which the bottom row is the obvious short exact sequence; compare the first part of (4.4) for $M_e = \mathbb{Z} \otimes M$. We define ψ below such that

$$M_{ee} \xrightarrow{i} (\mathbb{Z} \vee \mathbb{Z}) \otimes M \xrightarrow{\psi} \tilde{M} \xrightarrow{\rho} M_{ee}$$

is the identity of M_{ee} where ρ is the projection (which is not a homomorphism). This shows that i is injective and the proof of the theorem is complete. Moreover ψ is actually an isomorphism of groups. The homomorphism ψ is the composition

$$\psi : (\mathbb{Z} \vee \mathbb{Z}) \otimes M \twoheadrightarrow (\mathbb{Z} \vee \mathbb{Z})^{nil} \otimes M \xrightarrow{\tilde{\psi}} \tilde{M}$$

where $(\mathbb{Z} \vee \mathbb{Z})^{nil}$ is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ as a set with the group structure

$$(r, s, t) + (r', s', t') = (r + r', t + t', s + s' + s \cdot r').$$

We define $\tilde{\psi}$ by commutativity of the diagram above and by the following formulas for the composite $\rho\tilde{\psi} = \tilde{\rho}$.

$$\tilde{\rho}((r, s, t) \otimes a) = t\Delta(a) + rs(H(a) - (a|a)_H)$$
$$\tilde{\rho}([r, s, t), (r', s', t')] \otimes x) = rs'x + sr\, T(x)$$

A somewhat tedious but straightforward calculation shows that $\tilde{\rho}$ yields a well defined homomorphism $\tilde{\psi}$. Moreover $\tilde{\rho}\, i$ is the identity of M_{ee} and hence $\rho\psi i$ is the identity of M_{ee}.

q.e.d.

(4.4) Lemma. $\mathbb{Z} \otimes M = M_e$, $(\mathbb{Z}|\mathbb{Z}) \otimes M = M_{ee}$ and $G \otimes \mathbb{Z}_{nil} = G^{nil}$.

Proof. By (4.1) (2) we have a homomorphism

$$\mathbb{Z} \otimes M \to M_e$$

given by $r \otimes a \longmapsto ra + \binom{r}{2}PHa$ and $[r,s] \otimes x \longmapsto rsP(x)$. This is an isomorphism with the inverse $a \longmapsto 1 \otimes a$. Moreover the cross effect $(\mathbb{Z}|\mathbb{Z}) \otimes M = M_{ee}$ is obtained by (4.2). The last statement is obtained by the fact that the functor $F(G) = G^{nil}$ satisfies $F\{\mathbb{Z}\} = \mathbb{Z}_{nil}$ and hence $F(G) = G \otimes \mathbb{Z}_{nil}$ by (3.9).

q.e.d.

(4.5) Example. Let $R \subset \mathbb{Q}$ be a subring of the rationals and let

$$R_{nil} = (R \xrightarrow{H} R \xrightarrow{0} R)$$

be the square group with $H(r) = \binom{r}{2}$ and $P = 0$. Then for any group G the group $G \otimes R_{nil}$ is the same as the classical R-localization of the group G^{nil} [14].

§ 5 Exactness properties of the quadratic tensor product and derived functors

We introduce derived functors of the quadratic tensor product which are used for the usual exactness properties.

(5.1) Definition. Simplicial objects in the category $\underline{\underline{Nil}}$ form a Quillen model category [Q]. For $G \in \underline{\underline{Nil}}$ let $K(G,0)^{Nil}$ be a free simplicial object in $\underline{\underline{Nil}}$ with

$$\pi_0\, K(G,0)^{Nil} = G,$$
$$\pi_i\, K(G,0)^{Nil} = 0 \quad \text{for} \quad i \geq 1.$$

Hence $K(G,0)^{Nil}$ is a cofibrant model of G. We define for any square group M the Nil-torsion groups by the homotopy groups

$$Tor_*^{Nil}(G,M) = \pi_*(K(G,0)^{Nil} \otimes M)$$

A short exact sequence of square groups

$$0 \to M_1 \to M \to M_2 \to 0 \qquad (5.2)$$

yields a short exact sequence of functors $\underline{\underline{nil}} \to \underline{\underline{Nil}}$ of the form

$$0 \to - \otimes M_1 \to - \otimes M \to - \otimes M_2 \to 0$$

Hence we obtain as usual the long exact sequence

$$\dots \to Tor_n^{Nil}(G,M_1) \to Tor_n^{Nil}(G,M) \to Tor_n^{Nil}(G,M_2) \to \dots$$

with $Tor_0^{Nil}(G,M) = G \otimes M$. Moreover for $i > 0$ we have $Tor_i^{Nil}(G,M) = 0$ if $G \in$ $\underline{\underline{nil}}$ is free. These Nil -torsion groups are related with the homology groups in the variety $\underline{\underline{Nil}}$ given by

$$H_n^{Nil}(G) = \pi_{n-1}(K(G,0)^{Nil})^{ab},\ n \geq 1. \tag{5.3}$$

Compare Leedham-Green [8] for the definition and [4] for the computation of these groups.

If N is a quadratic $\mathbb{Z}$-module then

$$G \otimes N = G^{ab} \otimes N.$$

We computed in this case the groups $Tor_i^{Nil}(G,N)$ by the universal coefficient theorem 3.5 and 3.6 of [4] which shows that the groups $Tor_i^{Nil}(G,N)$ are determined by the groups $H_i^{Nil}(G)$ and N. This implies that the groups $Tor_i^{Nil}(G,M)$ for any square group M can be computed up to extension problems by the following method. There is a functorial short exact sequence in $\underline{\underline{Square}}$

$$0 \to N_1 \to M \to N_2 \to 0 \tag{5.4}$$

where

$$N_1 = (Im\,(P) \xrightarrow{H} M_{ee} \xrightarrow{P} Im(P))$$
$$N_2 = (Coker\,(P) \to 0 \to Coker\,(P))$$

are quadratic $\mathbb{Z}$ -modules.

Next we deal with the exactness of $G \otimes M$ in the variable G. In this case we get for any central extension

$$0 \to A \xrightarrow{i} G \to E \to 0 \quad \text{in} \quad \underline{\underline{Nil}}$$

the exact sequence

$$Tor_1^{Nil}(G,M) \to Tor_1^{Nil}(E,M) \to i \otimes M \to G \otimes M \to E \otimes M \to 0 \tag{5.5}$$

where $i \otimes M$ is the cokernel of the map

$$\begin{cases} w : A \otimes A \otimes Coker\,(\Delta) \to (A \otimes M) \times (A \otimes G^{ab} \otimes Coker\,\Delta) \\ w(a \otimes b \otimes \bar{x}) = (-[a,b] \otimes x,\ a \otimes i(b) \otimes \bar{x}) \end{cases}$$

Proof of (5.5). Let X_* be the simplicial object in $\underline{\underline{Nil}}$ with $X_n = G \times A^n = G \times A \times \ldots \times A$ and

$$d_i(a_0, \dots, a_n) = \begin{cases} (a_0, \dots, a_i + a_{i+1}, \dots, a_n), & i < n \\ (a_0, \dots, a_{n-1}), & i = n \end{cases}$$

$$s_i(a_0, \dots, a_n) = (a_0, \dots, a_i, 0, a_{i+1}, \dots, a_n)$$

Then $\pi_0(X_*) = E$ and $\pi_i(X_*) = 0$, $i > 0$. As in $[P_1]$, $[P_2]$ one has the spectral sequence with

$$E^1_{pq} = Tor^{Nil}_q(X_p, M) \Longrightarrow Tor^{Nil}_{p+q}(Q, M)$$

which in low degrees yields the exact sequence (5.5).

q.e.d.

If M is a quadratic $\mathbb{Z}$ -module one has a similar exact sequence as in (5.5) by B.9 in [2]. For the square group $M = (\mathbb{Z} \to 0 \to \mathbb{Z})$ the sequence (5.5) coincides with the corresponding five-term exact sequence of Stammbach for the homology in varieties of groups; see III.2 of [14].

§ 6 Exactness and flat square groups

A functor $T : \underline{\underline{Nil}} \to \underline{\underline{Nil}}$ is <u>right exact</u> if for any short exact sequence

(6.1) $$0 \to G_1 \to G \to G_2 \to 0$$

in $\underline{\underline{Nil}}$ the <u>induced sequence</u>

(6.2) $$TG_1 \to TG \to TG_2 \to 0$$

is exact. Moreover T is <u>exact</u> if the induced sequence of any short sequence is short exact.

(6.3) <u>Lemma</u>. *T is right exact, resp. exact, if and only if (6.2) is right exact, resp. short exact, whenever (6.1) is a central extension in $\underline{\underline{Nil}}$.*

<u>Proof</u>. Let $C = \{c \in G_1,\ c + x = x + c,\ x \in G\}$ be the centralizer of G_1 in G. Then C is central in G and G_1 and G_1/C is central in G/C. Using the corresponding exact sequences (6.1) follows.

q.e.d.

(6.4) <u>Lemma</u>. *If T is right exact then T carries a product in $\underline{\underline{Nil}}$ to a product.*

<u>Proof</u>. Consider the short exact sequence

$$0 \to G_1 \to G_1 \times G_2 \to G_2 \to 0.$$

q.e.d.

(6.5) Proposition. *Any right exact functor $T : \underline{\underline{Nil}} \to \underline{\underline{Nil}}$ is quadratic. Moreover the category of right exact functors $\underline{\underline{Nil}} \to \underline{\underline{Nil}}$ which preserve filtered colimits is equivalent the category of square groups for which Δ is surjective.*

By (3.5) (7) the surjectivity of Δ implies that $P = 0$ and that M_e is abelian. In fact we get the following equivalent description:

(6.6) Lemma. *A square group M for which Δ is surjective is the same as a pair of abelian groups M_e, M_{ee} together with a function $H : M_e \to M_{ee}$ such that the cross effect $(a|b)_H$ is linear in $a, b \in M_e$ and such that $\Delta : M_e \to M_{ee}$ with $\Delta(a) = H(2a) - 4H(a)$ is surjective.*

One readily checks that Δ in (6.6) is always a homomorphism.

Proof of (6.5). By (6.4) and (1.6) T is linear on $\underline{\underline{Ab}}$. Hence T carries the central extension

$$0 \to X^{ab} \otimes Y^{ab} \to X \vee Y \to X \times Y \to 0$$

in $\underline{\underline{Nil}}$ with $X, Y \in \underline{\underline{Nil}}$ to an exact sequence

$$T(X^{ab} \otimes Y^{ab}) \to T(X \vee Y) \to T(X) \times T(Y) \to 0$$

where $T(X^{ab} \otimes Y^{ab})$ is additive in X and Y since T is linear on $\underline{\underline{Ab}}$. Hence also the image of $T(X^{ab} \otimes Y^{ab}) \to T(X \vee Y)$ is biadditive and therefore T is quadratic. Now the equivalence of categories in (6.5) follows from (3.10) and (4.3).

q.e.d.

(6.7) Definition. A flat square group is a torsion free abelian group F together with a function $H : F \to F$ for which the cross effect $(a, b)_H$ is linear in $a, b \in F$ and for which

$$H(2a) - 4H(a) = a.$$

We point out that the category of flat square groups is equivalent to the category of square groups M for which Δ is an isomorphism and M_e is torsion free. The equivalence carries M to $\Delta^{-1}H : M_e \to M_e$; compare (6.6).

(6.8) Theorem. *The category of exact functors $\underline{\underline{Nil}} \to \underline{\underline{Nil}}$ which preserve filtered colimits is equivalent to the category of flat square groups.*

(6.9) Addendum. *For a square group M the following statements (a) and (b) are equivalent.*

(a) $\quad -\otimes M : \underline{\underline{Nil}} \to \underline{\underline{Nil}}$ *is exact*

(b) $\quad \Delta : M_e \to M_{ee}$ *is an isomorphism of torsion free groups.*

The addendum describes in (a) the classical flatness condition for M and (b) shows that M corresponds to a flat square group as defined in (6.7). Clearly (6.8) is a consequence of (6.9); compare (6.5). The proof of (6.9) is based on the following lemma.

(6.10) Lemma. *Let $T : \underline{\underline{Nil}} \to \underline{\underline{Nil}}$ be a right exact functor. Then T is exact if and only if one of the following condition holds.*

(a) *For any central inclusion $i : A \to G$ with abelian cokernel the induced map $T(i) : T(A) \to T(G)$ is injective.*
(b) *For any inclusion $j : B \to C$ of abelian groups the induced map $T(j)$ is injective and condition (a) is satisfied whenever the cokernel G/A is free abelian.*

Proof of (6.10). Obviously the exactness of T implies (a) and (b). Now assume (a) holds. Consider first (6.1) with G_2 abelian. Then (a) implies that (6.2) is short exact. In fact, let C be the centre of G and $A = G/C$. Then A is abelian since $G \in \underline{\underline{Nil}}$. Let $C_1 = G_1 \cap C$, $G_2 = C/C_1$ and $A_1 = G_1/C_1$. Then rows and columns in the following diagram are short exact

$$\begin{array}{ccccc} A_1 & \longrightarrow & A & \longrightarrow & A_2 \\ \uparrow & & \uparrow & & \uparrow \\ G_1 & \longrightarrow & G & \longrightarrow & G_2 \\ \uparrow & & \uparrow & & \uparrow \\ C_1 & \longrightarrow & C & \longrightarrow & C_2 \end{array}$$

We can apply (a) to all short exact sequences of the diagram except the row in the middle. Hence T carries this row also to a short exact sequence.

Now assume (6.1) is a central extension and now let C be the centre of G_2, $B = G_2/C$ and form the following diagram with short exact rows and columns.

$$\begin{array}{ccccc} G_1 & \longrightarrow & \tilde{C} & \longrightarrow & C \\ \| & & \downarrow & & \downarrow \\ G_1 & \longrightarrow & G & \longrightarrow & G_2 \\ & & \downarrow & & \downarrow \\ & & B & = & B \end{array}$$

Here B and C are abelian so that T carries the corresponding rows and columns to short exact sequences. Thus by (6.3) T is exact.

Next assume that condition (6) is satisfied. One shows with similar arguments that (b) implies (a) and hence the proof of (6.10) is complete. In fact, choose a free resolution $R \to F$ of G_2 in $\underline{\underline{Ab}}$ and consider

$$\begin{array}{ccccc} & & R & = & R \\ & & \downarrow & & \downarrow \\ G_1 & \longrightarrow & \tilde{G} & \longrightarrow & F \\ \| & & \downarrow & & \downarrow \\ G_1 & \longrightarrow & G & \longrightarrow & G_2 \end{array}$$

q.e.d.

Proof of (6.9). We show that (a) or (b) are also equivalent to one of the following conditions

(c) Δ is surjective and $\mathrm{Tor}_1^{Nil}(-, M) = 0$. In this case also $\mathrm{Tor}_i^{Nil}(-, M) = 0$ for $i \geq 1$.

(d) Δ is surjective, M_e is torsion free and $\mathrm{Tor}_1^{Nil}(F, M) = 0$ for any free abelian group F.

We now show $(a) \iff (c) \iff (d) \iff (b)$.

$(a) \Rightarrow (c)$: We know by (6.5) that Δ is surjective. Moreover any exact functor preserves the Moore normalisation of a simplical object in $\underline{\underline{Nil}}$ and hence $\mathrm{Tor}_i^{Nil}(-, M) = 0$ for $i \geq 1$.
$(c) \Rightarrow (a)$: By (6.5) we know that $- \otimes M$ is right exact. Then (5.5) and (6.3) imply that $- \otimes M$ is exact.
$(a), (c) \Rightarrow (d)$: We only have to show that M_e is torsion free. But this follows from the fact that for abelian A we have $A \otimes M = A \otimes M_e$. This equation clearly holds for $A = \mathbb{Z}$ and by (4.3) we see that $A \mapsto A \otimes M$ is an additive right exact functor preserving colimits so that the equation holds for any abelian group. Now it is well known that the exactness of the functor $A \mapsto A \otimes M_e$ is equivalent to the torsion freeness of M_e.
$(d) \Rightarrow (a)$: This is a consequence of lemma (6.10) and (5.5).
$(d) \Leftrightarrow (b)$: In (b) and (d) the function Δ is surjective and M_e is torsion free. Hence $P = 0$ and we have the short exact sequence of square groups in (5.4) with

$$N_1 = (0 \to M_{ee} \to 0)$$
$$N_2 = (M_e \to 0 \to M_e)$$

This yields the exact sequence

$$Tor_2^{Nil}(F, N_2) \xrightarrow{\partial} Tor_1^{Nil}(F, N_1) \to Tor_1^{Nil}(F, M) \to Tor_1^{Nil}(F, N_2) \xrightarrow{\partial} F \otimes N_1$$

where $F \in \underline{\underline{Nil}}$. If F is free abelian we have

$$\begin{aligned} F \otimes N_1 &= (\Lambda^2 F) \otimes M_{ee} \\ Tor_1^{Nil}(F, N_2) &= (\Lambda^2 F) \otimes M_e \\ Tor_1^{Nil}(F, N_1) &= F \otimes \Lambda^2 F \otimes M_{ee} \\ Tor_2^{Nil}(F, N_2) &= F \otimes \Lambda^2 F \otimes M_e \end{aligned}$$

where ∂ is in both cases induced by $\Delta : M_e \to M_{ee}$. Hence Δ is an isomoprhism if and only if $Tor_1^{Nil}(F, M) = 0$ and hence the proof of (d) $\Leftrightarrow$ (b) is complete. The computation of the torsion groups above is a consequence of [4] and the remarks in (5.3) and (5.4) above on the computation of such groups.

q.e.d.

§7 Universal quadratic functors and free square groups

The universal quadratic endofunctors of the category of groups are the functors

$$\otimes^2,\ Q^2 : \underline{\underline{Gr}} \to \underline{\underline{Gr}}$$

defined as follows. Here $\otimes^2$ is the <u>tensor square of the abelianization</u>

(7.1) $$\otimes^2(G) = G^{ab} \otimes G^{ab}.$$

Moreover Q^2 is defined by a <u>universal quadratic map</u>

(7.2) $$\gamma : G \to Q^2(G)$$

obtained as follows. Let G, G' be groups written additively (but not necessarily abelian). A function $H : G \to G'$ is termed <u>quadratic</u> if the cross effect $(a|b)_H = H(a+b) - H(b) - H(a)$ is a central element in G' and is linear in $a, b \in G$. Then (7.2) is the quadratic map with the property that for each quadratic map H there is a unique homomoprhism $\bar{H}$ such that

$$\begin{array}{ccc} Q^2(G) & \xrightarrow{\ \bar{H}\ } & G' \\ {\scriptstyle\gamma}\nwarrow & & \nearrow{\scriptstyle H} \\ & G & \end{array}$$

commutes. Next we define square groups $\mathbb{Z}^{\otimes}$ and $\mathbb{Z}^{Q}$ corresponding to the functors $\otimes^2$, Q^2 above. For this let $\mathbb{Z}(x_1, \dots, x_n)$ be the free abelian group generated by the elements $x_1, \dots, x_n$. Let

(7.3) $$\mathbb{Z}^{\otimes} = (\mathbb{Z}(Px) \xrightarrow{H} \mathbb{Z}(x, HPx - x) \xrightarrow{P} \mathbb{Z}(Px))$$

where H and P are homomorphisms with

$$P(x) = Px,\ P(HPx - x) = Px\,,$$
$$H(Px) = x + (HPx - x)\,.$$

(7.4) $$\mathbb{Z}^{Q} = (\mathbb{Z}(e, PHe) \xrightarrow{H} \mathbb{Z}(He, HPHe, (e|e)_H) \xrightarrow{P} \mathbb{Z}(e, PHe))$$

Here P is a homomorphism with

$$P(He) = PHe,\ P(HPHe) = 2PHe,\ P(e|e)_H = 0$$

and H is quadratic with $H(e) = He$, $H(PHe) = HPHe$ and

$$H(a+b) - H(b) - H(a) = \begin{cases} 0 & \text{for} \quad a = PHe \quad \text{or} \quad b = PHe \\ (e|e)_H & \text{for} \quad a = b = e \end{cases}$$

(7.5) Lemma. *$\otimes^2$ and Q^2 above are quadratic functors which preserve cokernels and filtered colimits and $\otimes^2\{\mathbb{Z}\} = \mathbb{Z}^{\otimes}$, $Q^2\{\mathbb{Z}\} = \mathbb{Z}^{Q}$ are the corresponding square groups. Hence one has natural isomorphisms ($G \in \underline{\underline{Gr}}$)*

$$\otimes^2(G) = G \otimes \mathbb{Z}^{\otimes},\ Q^2(G) = G \otimes \mathbb{Z}^{Q}$$

The lemma implies that $Q^2(G) \in \underline{\underline{Nil}}$ and $Q^2(G) = Q^2(G^{nil})$, moreover $Q^2(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ and $Q^2(\mathbb{Z}|\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Proof. For $\otimes^2$ compare [2]. For Q^2 we observe that the function

$$\gamma : G \to G \otimes \mathbb{Z}^{Q},\ \gamma(g) = g \otimes e$$

is quadratic. Moreover γ is in fact universally quadratic since a quadratic function $H : G \to G'$ induces the unique homomorphism $\bar{H} : G \otimes \mathbb{Z}^{Q} \to G'$ with $\bar{H}\gamma = H$. Here $\bar{H}$ is given on generators by

$$g \otimes e \longmapsto Hg$$
$$g \otimes PHe \longmapsto (g|g)_H$$
$$[a,b] \otimes He \longmapsto (a|b)_H$$
$$[a,b] \otimes HPHe \longmapsto (a|b)_H + (b|a)_H$$
$$[a,b] \otimes (e|e)_H \longmapsto -Hb - Ha + Hb + Ha$$

Uniqueness of $\bar{H}$ is readily checked by the relations in (3.7).

q.e.d.

Looking at the definition of $\mathbb{Z}^{\otimes}$ and $\mathbb{Z}^{Q}$ one readily obtains the following freeness property. Let $\underline{\underline{Square}}(M, N)$ be the set of morphisms $M \to N$ in the category of square groups.

(7.6) Lemma. *For any $M \in \underline{\underline{Square}}$ one has the natural bijections*

$$\underline{\underline{Square}}(\mathbb{Z}^Q, M) = M_e,\ f \longmapsto f(e)$$
$$\underline{\underline{Square}}(\mathbb{Z}^{\otimes}, M) = M_{ee}, g \longmapsto g(x)$$

The category $\underline{\underline{Square}}$ has limits and colimits. For example limits are obtained 'pointwise', in particular the product $M \times N$ is given by $(M \times N)_e = M_e \times N_e$ and $(M \times N)_{ee} = M_{ee} \times N_{ee}$ with $H = H_M \times H_N$, $P = P_M \times P_N$. Below we also give an explicit description of coproducts $M \vee N$. By (7.6) we see that the forgetful functors

$$\phi_e, \phi_{ee} : \underline{\underline{Square}} \to \underline{\underline{Set}} \tag{7.7}$$

which carry M to $\phi_e M = M_e$ and $\phi_{ee} M = M_{ee}$ respectively have left adjoints $\bar{\phi}_e$ and $\bar{\phi}_{ee}$ given by the coproduct over a set S,

$$\bar{\phi}_e(S) = \bigvee_S \mathbb{Z}^Q$$
$$\bar{\phi}_{ee}(S) = \bigvee_S \mathbb{Z}^{\otimes}$$

This shows the freeness property of $\mathbb{Z}^Q$ and $\mathbb{Z}^{\otimes}$. It also shows that there is a canonical surjection of square groups

$$\bigvee_{M_e} \mathbb{Z}^Q \vee \bigvee_{M_{ee}} \mathbb{Z}^{\otimes} \twoheadrightarrow M \tag{7.8}$$

for any square group M. Limits and colimits in $\underline{\underline{Square}}$ correspond via the equivalence (3.10) to limits and colimits in the category of quadratic endofunctors of $\underline{\underline{Nil}}$ which preserve cokernels and filtered colimits. The sum of such endofunctors F, F' is the quadratization $F \vee_{quad} F'$ of the functor $F \vee F'$ which is given by the sum of groups in $\underline{\underline{Nil}}$

$$(F \vee F')(G) = F(G) \vee F'(G).$$

Moreover by (7.8) and (7.5) we get the following result:

(7.9) Proposition. *For each quadratic endofunctor F of $\underline{\underline{Gr}}$ (or $\underline{\underline{Nil}}$) which preserves cokernels and filtered colimits there exists a set S and a natural surjective homomorphism*

$$\bigvee_S (Q^2(G) \vee \otimes^2(G)) \twoheadrightarrow F(G),\ G \in \underline{\underline{Gr}}.$$

Here the left hand side denotes a sum in the category Nil of groups of nilpotency degree 2.

Let G_1, G_2 be groups in *Nil*. Then the sum $G_1 \vee G_2$ in *Nil* is given by the nilization of the sums of G_1 and G_2 in *Gr*. There is a well known exact central extension in *Nil*

$$0 \to G_1^{ab} \otimes G_2^{ab} \xrightarrow{w} G_1 \vee G_2 \to G_1 \times G_2 \to 0 \tag{7.10}$$

where w is the commutator map. This shows that any element in $G_1 \vee G_2$ can be written as a sum of elements $a + b + w$ where $a \in G_1$, $b \in G_2$ and w is a sum of commutators of the form $a + b - a - b$. We obtain the sum $M \vee N$ of square groups by the quadratization of the functor $G \longmapsto G \otimes M \vee G \otimes N = F_{M,N}(G)$, that is

$$G \otimes (M \vee N) = F_{M,N}^{quad}(G) \tag{1}$$

Using (7.10) this leads to the following explicit description.

(7.11) The sum of square groups. Let

$$M = (M_e \xrightarrow{H_M} M_{ee} \xrightarrow{P_M} M_e),\ N = (N_e \xrightarrow{H_N} N_{ee} \xrightarrow{P_N} N_e)$$

be square groups. Then the sum

$$M \vee N = ((M \vee N_e \xrightarrow{H} (M \vee N)_{ee} \xrightarrow{P} (M \vee N)_e) \tag{1}$$

in the category *Square* is given by

$$(M \vee N)_{ee} = M_{ee} \oplus N_{ee} \oplus \operatorname{cok}(P_M) \otimes \operatorname{cok}(P_N) \oplus \operatorname{cok}(P_N) \otimes \operatorname{cok}(P_M) \tag{2}$$

$$(M \vee N)_e = (M_e \vee N_e)/\sim \tag{3}$$

Here $M_e \vee N_e$ is the sum in *Nil* and the equivalence relation is generated by

$$P_M(x) + c \sim c + P_M(x),$$
$$a + P_N(u) \sim P_N(u) + a$$

for $x \in M_{ee}$, $c \in N_e$, $u \in N_{ee}$, $a \in M_e$. Let $\bar{x} \in \operatorname{cok}(P_M)$, $\bar{u} \in \operatorname{cok}(P_N)$ be the elements in cokernels represented by x and u respectively. The operators H and P for $M \vee N$ are defined by

$$H(a + c + (a_1 + c_1 - a_1 - c_1)) = H_M(a) + H_N(c) + \bar{a} \otimes \bar{c} + \bar{a}_1 \otimes \bar{c}_1 - \bar{c}_1 \otimes \bar{a}_1$$

Compare (7.10). Moreover

$$P(x+u+\bar{a}_1\otimes\bar{c}_1+\bar{c}_2\otimes\bar{a}_2)=P_M(x)+P_N(u)+(a_1+c_1-a_1-c_1)+(c_2+a_2-c_2-a_2)$$

This completes the construction of $M \vee N$. Let

$$(7.12) \qquad \underline{\underline{square}} \subset \underline{\underline{Square}}$$

be the full subcategory consisting of sums

$$L = \bigvee_S \mathbb{Z}^{\otimes} \vee \bigvee_R \mathbb{Z}^{Q}$$

where S and R are sets. Here L is termed a free square group. For a square group M one has the canonical bijection

$$(7.13) \qquad \underline{\underline{Square}}(L,M) = \underline{\underline{Set}}(S,M_{ee}) \times \underline{\underline{Set}}(R,M_e)$$

so that $\underline{\underline{Square}}(L,M)$ has the structure of a group in $\underline{\underline{Nil}}$ provided L is free. We use the next lemma to show that $\underline{\underline{square}}$ is a quadratic category; see (2.3). The lemma describes an analogue of the central extension in (7.10).

(7.14) Lemma. *For $X,Y \in \underline{\underline{Square}}$ one has a short exact sequence in $\underline{\underline{Square}}$*

$$0 \to X \sharp Y \to X \vee Y \to X \times Y \to 0$$

where $X\sharp Y$ is the quadratic $\mathbb{Z}$ -module

$$X\sharp Y = (A\otimes B \xrightarrow{H} (A\otimes B)\oplus(B\otimes A) \xrightarrow{P} A\otimes B)$$

with $A = cokernel(P : X_{ee} \to X_e)$, $B = coker(P : Y_{ee} \to Y_e)$ and $H(a\otimes b) = a\otimes b - b\otimes a$ and $P(a\otimes b + b_1\otimes a_1) = a\otimes b - a_1\otimes b_1$.

This is readily proved by (7.10) and (7.10) (1). Since P in $\mathbb{Z}^{\otimes}$ is surjective the lemma implies that

$$(7.15) \qquad \mathbb{Z}^{\otimes} \vee \mathbb{Z}^{Q} \cong \mathbb{Z}^{\otimes} \times \mathbb{Z}^{Q}$$

Moreover using the fact that $X\sharp Y$ is linear in X and Y we obtain by (7.13) and (7.14):

(7.16) Proposition. *$\underline{\underline{square}}$ is a quadratic category.*

This leads to the following alternative description of the category of square groups.

(7.17) Proposition. *Let $Q = End(\mathbb{Z}^{\otimes} \vee \mathbb{Z}^{Q})$ be the endomoprhism square ring of the object $\mathbb{Z}^{\otimes} \vee \mathbb{Z}^{Q}$ in the quadratic category $\underline{\underline{square}}$. Then there is an isomorphism of categories*

$$\underline{\underline{Square}} = \underline{\underline{Mod}}(Q)$$

where $\underline{\underline{Mod}}(Q)$ is the category of Q-modules in the sense of 7.9 [3].

This proposition is the generalization of a result for quadratic $\mathbb{Z}$ -modules in 2.2 [2]; hence the square ring Q in (7.17) is the non abelian analogue of the ring $Q(2)$ defined in [12].

(7.18) Remark. An explicit computation of the square ring $Q = \text{End}(\mathbb{Z}^{\otimes} \vee \mathbb{Z}^{Q})$ is obtained as follows. As a square group Q is given by

$$Q = (Q_e \xrightarrow{H} Q_{ee} \xrightarrow{P} Q_e)$$

(1) $$Q_e = \mathbb{Z}^8,\ Q_{ee} = \mathbb{Z}^3$$

where the basis of Q_e is denoted by $a,\ b,\ c,\ p,\ h,\ ph,\ hp,\ hph$ and the basis of Q_{ee} is x, y, z. We define P by

(2) $$P(x) = c = P(y) \quad \text{and} \quad P(z) = 0.$$

In the multiplicative monoid Q_e the following equations hold:

(3) $$a^2 = a,\ b^2 = b,\ ab = ba = 0,\ a + b = 1,$$

(4) $$ha = 0,\ hb = 0,\ ap = 0,\ pb = 0,\ php = 2p,$$

(5) $$(p) \cdot h = ph,\ (hp) \cdot h = hph,\ (h) \cdot p = hp.$$

The equations (3), (4), (5) do not yet determine the structure of the monoid Q_e completely since the distributivity laws do not hold; but they yield a unique ring structure on

(6) $$R = \text{cokernel}(P)$$

Clearly $R = \mathbb{Z}^7$ has the basis $a,\ b,\ p,\ h,\ ph,\ hp,\ hph$ and the multiplication law is given by (3), (4), (5). We define the ring homomorphism

(7) $$\begin{cases} \epsilon : R \to \mathbb{Z}, \\ \epsilon(a) = 1,\ \epsilon(b) = \epsilon(p) = \epsilon(h) = 0\,. \end{cases}$$

Then the structure of Q_{ee} as an $R \otimes R \otimes R^{op}$ -module is defined by

$$(8) \qquad \begin{aligned} &(r \otimes s) \cdot u = \epsilon(r) \cdot \epsilon(s) \cdot u \quad \text{for} \quad r, s \in R,\ u \in Q_{ee}\,, \\ &0 = x \cdot a = x \cdot h = y \cdot a = y \cdot h\,, \\ &x \cdot p = z,\ y \cdot p = -z,\ z \cdot h = x - y\,. \end{aligned}$$

The quadratic map $H : Q_e \to Q_{ee}$ is uniquely defined by

$$(9) \qquad \begin{aligned} &H(a) = H(b) = H(p) = 0\,, \\ &H(h) = x,\ H(c) = x - y\,, \\ &(r|s)_H = -\epsilon(r) \cdot \epsilon(s) \cdot z\,. \end{aligned}$$

One can check that these data (1) ... (9) determine the square ring Q completely. The identification of a Q-module M and a square group $(M_e \to M_{ee} \to M_e)$ in 7.13 is given as follows; compare (7.9) [3]. We have

$$(10) \qquad M = M_e \times M_{ee} \quad \text{with} \quad M_e = M \cdot a,\ M_{ee} = M \cdot b\,.$$

Moreover mutliplication by h and p yield maps $H = \cdot h : M_e \to M_{ee}$, $P = \cdot p : M_{ee} \to M_e$. Moreover $\cdot c : M_e \to M_{ee}$ coincides with the function $v \longmapsto (v|v)_H$, $v \in M_e$. Finally the bracket Q-operations on M are given by $(v, w \in M_e)$

$$(11) \qquad \begin{aligned} &[v, w] \cdot x = (v|w)_H \in M_{ee}\,, \\ &[v, w] \cdot y = (w|v)_H \in M_{ee}\,, \\ &[v, w] \cdot z = w + v - w - v \in M_e\,. \end{aligned}$$

We obtain (1) by (7.15) and

$$\begin{aligned} Q_e &= \underline{\underline{Square}}\,(\mathbb{Z}^{\otimes} \vee \mathbb{Z}^{Q},\ \mathbb{Z}^{\otimes} \vee \mathbb{Z}^{Q}) \\ &= \underline{\underline{Square}}\,(\mathbb{Z}^{\otimes},\ \mathbb{Z}^{\otimes} \times \mathbb{Z}^{Q}) \times \underline{\underline{Square}}\,(\mathbb{Z}^{Q},\ \mathbb{Z}^{\otimes} \times \mathbb{Z}^{Q}) \\ &= \mathbb{Z}^{\otimes}_e \times \mathbb{Z}^{Q}_e \times \mathbb{Z}^{\otimes}_{ee} \times \mathbb{Z}^{Q}_{ee} = \mathbb{Z}^8 \end{aligned}$$

See (7.3) and (7.4). Similarly one gets by (7.14)

$$Q_{ee} = \underline{\underline{Square}}\,(\mathbb{Z}^{\otimes} \vee \mathbb{Z}^{Q},\ X \sharp X)$$

where $X = \mathbb{Z}^{\otimes} \times \mathbb{Z}^{Q}$. By (7.14) we have

$$X \sharp X = (\mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z}) = \mathbb{Z}^{\otimes}$$

so that $Q_{ee} = \mathbb{Z}^3$. Here we use the fact that P in $\mathbb{Z}^{\otimes} \times \mathbb{Z}^{Q}$ satisfies cokernel $(P) = \mathbb{Z}$.

The category *Square* has similar properties as the category *Nil*. Recall that for $G \in$ *Nil* one has an exact sequence

$$\Lambda^2 G^{ab} \xrightarrow{w} G \to G^{ab} \to 0 \tag{7.19}$$

where w is the commutator map which is injective if G is a free object in *Nil*. We now describe the analogue of (7.19) for the category *Square*. We define the abelianization M^{ab} of a square group M by the property

$$(G \otimes M)^{ab} = G \otimes (M^{ab}),\ G \in \underline{Gr}\,. \tag{7.20}$$

One can check that

$$M^{ab} = (M_e^{ab} \xrightarrow{H} M_{ee}/\sim \xrightarrow{P} M_e^{ab})$$

is given by M via the equivalence realization $(a|b)_H \sim 0$ for $a, b \in M_{ee}$. Here H and P in M^{ab} are induced by H and P in M. We can define the functor $M \longmapsto M^{ab}$ alternatively as follows. Let *Absquare* be the category consisting of the objects

$$A = (A_e \xrightarrow{H} A_{ee} \xrightarrow{P} A_e)$$

which are diagrams in *Ab* with $PHP = 2P$. We have the full embedding

$$\underline{Absquare} \subset \underline{Square} \tag{7.21}$$

Here the objects in *Absquare* are exactly all square groups M for which H is linear and hence they correspond by (3.10) to quadratic functors $\underline{Gr} \to \underline{Ab}$. The left adjoint of the inclusion (7.21) is the functor which carries M to M^{ab}. We observe that *Absquare* is an abelian category for which

$$(\mathbb{Z}^{\otimes} \vee \mathbb{Z}^{Q})^{ab} = \mathbb{Z}^{\otimes} \oplus (\mathbb{Z}^{Q})^{ab} \tag{7.22}$$

is a projective generator. Here we get by (7.20) and (7.4)

$$(\mathbb{Z}^{Q})^{ab} = (\mathbb{Z}(e, PHe) \xrightarrow{H} \mathbb{Z}(He, HPHe) \xrightarrow{P} \mathbb{Z}(e, PHe))$$

with $H(e) = He$, $H(PHe) = HPHe$, $P(He) = PHe$, $P(HPHe) = ePHe$. The projective generator (7.22) shows that the category *Absquare* is the same as the category of R-modules where R is the endomorphism ring of the object (7.22) in the abelian category *Absquare*. Here R coincides with the ring $R = \text{cokernel}\,(P)$ associated to the square ring Q in (7.18) (6). In particular $R = \mathbb{Z}^7$ as an abelian group. The ring considered in 2.2 [2] is the quotient $R^{op}/hph = 2h$.

We define the exterior square functor

(7.23) $$\Lambda^2 : \underline{\underline{Absquare}} \to \underline{\underline{Absquare}}$$

by the property that for $A \in \underline{\underline{Absquare}}$ the functor $G \longmapsto G \otimes \Lambda^2 A$ is the quadrization of the functor $G \longmapsto \Lambda^2(G \otimes A)$. Then (7.19) yields the exact sequence

(7.24) $$\Lambda^2(M^{ab}) \xrightarrow{w} M \to M^{ab} \to 0$$

of square groups which is natural in $M \in \underline{\underline{Square}}$. Moreover w is injective if M is a free square group. The map w is central in M. More explicitly one obtains via the cokernel $\bar{A} = \operatorname{cok}(P : A_{ee} \to A_e)$ the exterior square (7.23) by

$$\Lambda^2(A) = (\Lambda^2 \bar{A} \xrightarrow{H} \bar{A} \otimes \bar{A} \xrightarrow{P} \Lambda^2 \bar{A})$$

with $H(x \wedge y) = x \otimes y - y \otimes x$ and $P(x \otimes y) = x \wedge y$. The commutator map w in (7.24) is given by

$$\begin{cases} w_{ee}(x \otimes y) = (x|y)_H \\ w_e(x \otimes y) = x + y - x - y \end{cases}$$

(7.25) Remark. Let $\underline{\underline{nil}}$ be the subcategory of free objects in $\underline{\underline{Nil}}$. Then (7.19) yields the linear extension of categories

$$\operatorname{Hom}(-, \Lambda^2) \to \underline{\underline{nil}} \to \underline{\underline{ab}}$$

where $\underline{\underline{ab}}$ is the category of free abelian groups. This extension is for example studied in chapter VI [1]. The bifunctor $\operatorname{Hom}(-, \Lambda^2)$ on $\underline{\underline{ab}}$ carries A, B to the group of homomorphisms $\operatorname{Hom}(A, \Lambda^2 B)$ in $\underline{\underline{Ab}}$. Similarly we get by (7.24) the linear extension of categories

$$\operatorname{Hom}(-, \Lambda^2) \to \underline{\underline{square}} \to \underline{\underline{absquare}}$$

where $\underline{\underline{absquare}}$ is the category of free objects in the category $\underline{\underline{Absquare}}$. Such free objects are sums of the universal objects $\mathbb{Z}^{\otimes}$ and $(\mathbb{Z}^Q)^{ab}$. Here the bifunctor $\operatorname{Hom}(-, \Lambda^2)$ on $\underline{\underline{absquare}}$ carries A, B to the abelian group of morphisms $\operatorname{Hom}(A, \Lambda^2 B)$ in $\underline{\underline{Absquare}}$ where $\Lambda^2 B$ is defined by (7.23).

§ 8 The composition product for square groups

Endofunctors of the category of groups form a monoidal category with the monoidal structure given by the composition of endofunctors. The composition of quadratic endofunctors F_1 and F_2 however needs not to be quadratic. We still obtain a composition product in the category of quadratic functors by the quadratization

(8.1) $$F_1 \square F_2 = (F_1 \circ F_2)^{quad}$$

of the composition $F_1 \circ F_2$. If F_1 and F_2 are quadratic endofunctors which preserve filtered colimits and cokernels then also $F_1 \square F_2$ is a quadratic endofunctor which preserves filtered colimits and cokernels. Hence the equivalence of categories in (3.10) yields a composition product $M \square N$ for square groups M, N with the defining property

$$(8.2) \qquad (F_1 \square F_2)(G) = G \otimes (M \square N)$$

for $F_1(G) = G \otimes M$, $F_2(G) = G \otimes N$ and $G \in \underline{\underline{Gr}}$.

(8.3) **Theorem.** *Let $Quad(\underline{\underline{Gr}})$ be the category of quadratic endofunctors of $\underline{\underline{Gr}}$ which preserve cokernels. Then the composition product (8.1) yields the structure of a monoidal category for $Quad(\underline{\underline{Gr}})$. The unit object is the nilization functor $nil : \underline{\underline{Gr}} \to \underline{\underline{Nil}} \subset \underline{\underline{Gr}}$.*

For the proof of the theorem we need the following lemmata.

(8.4) **Lemma.** *Let $F \in Quad(\underline{\underline{Gr}})$ and let $A \xrightarrow{d} B \xrightarrow{q} C \to 0$ be an exact sequence in $\underline{\underline{Gr}}$. Then F induces the exact sequence in $\underline{\underline{Gr}}$*

$$F(A|B) \times F(A) \xrightarrow{\partial} F(B) \xrightarrow{F(q)} F(C) \to 0$$

with $\partial(x,y) = PF(d|1_B)(x) + F(d)(y)$.

Proof. The exact sequence (q,d) yields the coequalizer in $\underline{\underline{Gr}}$

$$A \vee B \underset{(0,1)}{\overset{(d,1)}{\rightrightarrows}} B \xrightarrow{q} C$$

where $A \vee B$ is the sum of groups. Since F preserves coequalizers we obtain the exact sequence $(F(q), \partial)$. Here ∂ is the restriction of $F(d,1)$ to kernel $F(0,1)$. Since F is quadratic we have kernel $F(0,1) = F(A|B) \times F(A)$ where $F(A|B)$ is central in $F(A \vee B)$.

q.e.d.

(8.5) **Lemma.** *Let F_1, F_2 be endofunctors of $\underline{\underline{Gr}}$. Then the quadratization satisfies*

$$(F_1^{quad} F_2)^{quad} = (F_1 F_2)^{quad}$$

Proof. We obviously have the natural transformation $(F_1F_2)^{quad} \to (F_1^{quad}F_2)^{quad}$ induced by the identity of F_1F_2 via the universal property. We construct the inverse by the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
F_1(F_2G|F_2G|F_2G) & \xrightarrow{\tilde{P}} & F_1F_2(G) & \longrightarrow & F_1^{quad}F_2(G) & \longrightarrow & 0 \\
\downarrow \psi & & \| & & \downarrow & & \\
F_1F_2(G|G|G) & \xrightarrow{\tilde{P}} & F_1F_2(G) & \longrightarrow & (F_1F_2)^{quad}(G) & \longrightarrow & 0
\end{array}$$

Here ψ is induced by the canonical map

$$F_2G \vee F_2G \vee F_2G \to F_2(G \vee G \vee G)$$

.

q.e.d.

(8.6) **Lemma.** *Let F_1, F_2 be endofunctors of $\underline{\underline{Gr}}$ where F_1 is quadratic and preserves cokernels. Then quadratization satisfies*

$$(F_1F_2^{quad})^{quad} = (F_1F_2)^{quad}$$

Proof. Again we have the natural transformation $(F_1F_2)^{quad} \to (F_1F_2^{quad})^{quad}$ induced by the identity of F_1F_2. The inverse is constructed by the following commutative diagram with exact rows where the top row is given as in (8.4) by the exact sequence

$$K \xrightarrow{P} F_2(G) \to F_2^{quad}(G) \to 0$$

with $K = F_2(G|G|G)$.

$$\begin{array}{ccccccc}
F_1(K|F_2G) \times F_1(K) & \xrightarrow{\partial} & F_1F_2(G) & \longrightarrow & F_1F_2^{quad}(G) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \psi=(\psi_1,\psi_2)} & & \| & & \downarrow & & \\
K' & \xrightarrow{\bar{P}} & F_1F_2(G) & \longrightarrow & (F_1F_2)^{quad}(G) & \longrightarrow & 0
\end{array}$$

with $K' = (F_1F_2)(G|G|G)$. Since K' is abelian we define ψ by the components ψ_1 and ψ_2. Here $\psi_2 : F_1(K) \to K'$ is induced by the identity of $F_1F_2(G \vee G \vee G)$. Moreover $\psi_2 : F_1(K|F_2G) \to K'$ is given by the following commutative diagram

$$\begin{array}{ccc}
F_1(K|F_2G) & \xrightarrow{\psi_2} & K' \\
\downarrow{\scriptstyle F_1(i_{123}|1)} & & \\
F_1(F_2(G \vee G \vee G)|F_2G) & & \downarrow{\scriptstyle i_{123}} \\
\downarrow{\scriptstyle i_{12}} & & \\
F_1(F_2(G \vee G \vee G) \vee F_2G) & \xrightarrow{\psi_3} & F_1F_2(G \vee G \vee G)
\end{array}$$

Here i_{12}, i_{123} are the inclusions and ψ_3 is induced by the map $F_2(G \vee G \vee G) \vee F_2G \to F_2(G \vee G \vee G)$ which is the identity on $F_2(G \vee G \vee G)$ and is Fj on F_2G. Here $j : G \to G \vee G \vee G$ is one of the inclusions. We have to show that the factorization ψ_2 exists. For this we use the definition of K' by the projections $r_{ij} : G \vee G \vee G \to G \vee G$. Since $F_1(i_{123}|1)F_1(F_2(r_{ij})|1) = F_1(0|1) = 0$ we see that $(F_1F_2(r_{ij}))\psi_3 i_{12}(F_1(i_{123}|1)) = 0$. Here we know that $F_1(0|1) = 0$ since F_1 is assumed to be quadratic.

q.e.d.

Proof of (8.3). We have by (8.5) and (8.6)

$$\begin{aligned} F_1\square(F_2\square F_3) &= (F_1 \circ (F_2 \circ F_3)^{quad})^{quad} \\ &= (F_1 \circ F_2 \circ F_3)^{quad} \\ &= ((F_1\square F_2)^{quad} \circ F_3)^{quad} \\ &= (F_1\square F_2)\square F_3 \end{aligned}$$

Moreover by (3.9) we see that *nil* is the unit.

q.e.d.

We now compute the composition product $M\square N$ for square groups defined in (8.2) which by (8.3) yields a monoidal structure of the category Square. For a square group

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$

we define the abelian group

$$(8.7) \qquad M^{ad} = cokernel\,(P : M_{ee} \to M_e)$$

and for $a \in M_e$ let $\bar{a} \in M^{ad}$ be the class of a in the cokernel. We observe by the axioms of a square group in (3.5) that the functions $(\ \ |\ \ )_H$ and Δ induce homomorphisms

$$\begin{aligned} (\ \ |\ \ )_H &: M^{ad} \otimes M^{ad} \to M_{ee} \\ \Delta &: M^{ad} \to M_{ee} \end{aligned}$$

(8.8) Definition. We define the composition product $M\square N$ of square groups M, N explicitly as follows:

$$M\square N = ((M\square N)_e \xrightarrow{H} (M\square N)_{ee} \xrightarrow{P} (M\square N)_e).$$

The group $(M\square N)_e$ is defined via the tensorproduct $M_e \otimes N$ in (3.7) by the quotient

$$(1) \qquad (M\square N)_e = M_e \otimes N/[a, Px] \otimes u \sim 0$$

Here and in the following we use the elements $a, b \in M_e$, $x, y \in M_{ee}$, $c, d \in N_e$, $u, v \in N_{ee}$. The abelian group $(M\square N)_{ee}$ is the quotient

$$(2) \qquad (M\square N)_{ee} = (M_{ee} \otimes N^{ad} \oplus M^{ad} \otimes M^{ad} \otimes N_{ee}/ \approx$$

where we use the equivalence relation

$$(a|b)_H \otimes c \approx \bar{b} \otimes \bar{z} \otimes \Delta(c).$$

Next the homomorphism P for $M\square N$ is given by

$$\begin{cases} P(x \otimes \bar{c}) = (Px) \otimes c, \\ P(\bar{a} \otimes \bar{b} \otimes u) = [a,b] \otimes u. \end{cases}$$

This implies that

$$(4) \qquad (M\square N)^{ad} = M^{ad} \otimes N^{ad}.$$

We now define the homomorphism

$$(5) \qquad (\ \ |\ \ )_H : (M\square N)^{ad} \otimes (M\square N)^{ad} \to (M\square N)_{ee}$$

by

$$(\bar{a} \otimes \bar{c} | \bar{b} \otimes \bar{d})_H = \bar{b} \otimes \bar{a} \otimes (c|d)_H$$

Then H in $M\square N$ is the unique quadratic function with this crosseffect $(\ \ |\ \ )_H$ which is defined on generators by

$$(6) \qquad \begin{cases} H(a \otimes c) = (Ha) \otimes \bar{c} + \bar{a} \otimes \bar{a} \otimes Hc \\ H([a,b] \otimes u) = \bar{a} \otimes \bar{b} \otimes u + \bar{b} \otimes \bar{a} \otimes Tu \end{cases}$$

(8.9) Theorem. *The composition products $M\square N$ for square groups in (8.2) and (8.8) coincide and hence yield a monoidal structure on the category Square with the square group $\mathbb{Z}_{nil}$ as unit object.*

(8.10) Addendum. *A square ring as defined in 7.4 of [3] is the same as a monoid in the monoided category (Square, $\square$, $\mathbb{Z}_{nil}$).*

The proof of (8.9) is unfortunately fairly long and technical. Since this proof is only based on ideas and facts described in this paper we omit the details. It is clear by theorem (8.3) that the monoidal structure of the category *Square* is well defined; only the explicit formula for the composition product of square groups in (8.8) has to be checked. This formula is compatible with the properties of square rings described in [3] and therefore we obtain (8.10).

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