Leray residue theory and Barlet forms on singular complete intersections

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We construct a complex $\Omega_S^\bullet(\log C)$ of sheaves of the so-called multi-logarithmic differential forms on a complex analytic manifold S with respect to a reduced complete intersection $C \subset S$, and define a residue map $\mathrm{res.}: \Omega_S^\bullet(\log C) \to \omega_C^\bullet$ from this complex onto the Barlet complex $\omega_C^\bullet$ of regular meromorphic differential forms on C. The residue map is proved to be a natural morphism between the two complexes; it follows then that sections of the complex $\omega_C^\bullet$ may be regarded as a generalization of the residue differential forms defined by Leray. Moreover, we show that the map res. can be given explicitly in terms of a certain integration current.

Content

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Introduction

In 1887, Henry Poincaré introduced the notion of a residue 1-form for rational 2-forms on $\mathbf{C}^2$; this notion was generalized by Georges de Rham (1954) and Jean Leray (1959) who defined and studied the Poincaré residue of a *d-closed* meromorphic q-form ω, having poles of the first order along a smooth divisor (see [17]). In 1972 J.-B. Poly [21] proved that the Leray residue is well defined for any (not necessarily d-closed) differential form ω, as soon as ω and $d\omega$ have a simple pole along a hypersurface.

In the present paper, we introduce a notion of the multi-logarithmic differential form. Such a meromorphic differential form has poles along a divisor $D = \bigcup_{i=1}^k D_i$ such that $C = \bigcap_{i=1}^k D_i$ is a reduced complete intersection. We then define the residue form of a multi-logarithmic differential form, as a generalization of corresponding definitions of J. Leray, J.-B. Poly and K. Saito. Our main theorem asserts that there is a natural morphism res. which maps the complex of the multi-logarithmic differential forms onto the complex of the regular meromorphic forms. The latter complex was defined and studied by D. Barlet [2], [3]. The first author has previously obtained a similar result for logarithmic differential forms along the divisor D. More precisely, it was proved [1] that the residue map defined by K. Saito [25] give a natural quasi-isomorphism between the complexes of sheaves of the logarithmic differential forms along the divisor D and of the regular meromorphic forms on D. In earlier works the second author and his collaborators [27], [28], [26] studied the properties of weakly holomorphic functions and residue currents on complete intersections. Building on these results, we give an explicit description of the natural morphism res. in terms of a certain integration current.

In Section 1, we give the basic notation and definitions and establish some properties of the multi-logarithmic differential forms with respect to a reduced complete intersection in a complex analytic manifold. In Section 2, we give the definition and discuss the properties of the regular meromorphic differential forms. In Section 3, our main theorem is proved. In Section 4, the above mentioned explicit description of this morphism res. is given. Finally, in the appendix we give a brief survey of the history of this subject, which does not seem to be widely known.

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1. Multi-logarithmic differential forms

Let U be an open subset of $S = \mathbf{C}^m$, and let $D_1, \ldots, D_k$ be a set of hypersurfaces defined by the equations $h_j(z) = 0,\ j = 1, \ldots, k,\ k \geq 1$, respectively. Here $h_j(z) = h_j(z_1, \ldots, z_m),\ j = 1, \ldots, k$, is a holomorphic function in U, and $z = (z_1, \ldots, z_m)$, $z_i \in \mathbf{C}$. In what follows, we shall suppose that $h_1, \ldots, h_k$ is a *regular sequence* in $\mathcal{O}_U$, and $I = (h_1, \ldots, h_k)\mathcal{O}_U$ is a *radical* ideal, that is $I = \sqrt{I}$. In other words, the complex analytic variety $C = D_1 \cap \ldots \cap D_k$ is a *reduced complete intersection.* In particular, $\dim C = m - k$ and the differential k-form $dh_1 \wedge \ldots \wedge dh_k$ vanishes identically on no irreducible component of C. Let $\Omega^\bullet_U = (\Omega^q_U, d)_{q=0,1,\ldots}$ be the de Rham complex of germs of holomorphic differential forms on U whose terms, locally at the point $0 \in U$, are defined as follows:

$$\Omega^q_{U,0} = \mathcal{O}_{U,0}\langle\ \ldots, dz_{i_1} \wedge \ldots \wedge dz_{i_q}, \ldots\rangle\ ,\ (i_1, \ldots, i_q) \in [1, m].$$

Let

$$\widehat{D_j} = D_1 \cup \ldots \cup D_{j-1} \cup D_{j+1} \cup \ldots \cup D_k, \quad j = 1, \ldots, k,$$

and let $\Omega^q_U(\star\widehat{D_j})$ be the $\mathcal{O}_U$-module of meromorphic differential forms of degree $q \geq 0$ consisting of all the differential q-forms with the polar divisor $\widehat{D_j}$. Write $\widehat{D_1} = \emptyset$ for $k = 1$, so that $\Omega^q_U(\star\widehat{D_1}) = \Omega^q_U$.

Proposition 1.1. *Let ω be a meromorphic differential q-form on U, $q \geq k$, with poles along the divisor $D = D_1 \cup \ldots \cup D_k$. The following conditions are equivalent:*

$$i)\ h_j\omega \in \sum_{i=1}^{k} \Omega^q_U(\star\widehat{D_i}),\ h_j d\omega \in \sum_{i=1}^{k} \Omega^{q+1}_U(\star\widehat{D_i}), \quad j = 1, \ldots, k\,;$$

$$ii)\ h_j\omega \in \sum_{i=1}^{k} \Omega^q_U(\star\widehat{D_i}),\ dh_j \wedge \omega \in \sum_{i=1}^{k} \Omega^{q+1}_U(\star\widehat{D_i}), \quad j = 1, \ldots, k\,;$$

iii) There is a holomorphic function g, not equal identically to zero on every irreducible component of C, a holomorphic $(q-k)$-form ξ and a meromorphic q-form $\eta \in \sum_{i=1}^{k} \Omega^q_U(\star\widehat{D_i})$ on U such that

$$g\omega = \frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k} \wedge \xi + \eta\,;$$

iv) There are analytic subsets $A_j \subset D_j$, $j = 1, \ldots, k$, of codimension at least 2 such that the germ of ω at any point $x \in \bigcup_{j=1}^{k}(D_j \setminus A_j)$ belongs to

$$\frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k} \wedge \Omega^{q-k}_{U,x} + \sum_{i=1}^{k} \Omega^q_{U,x}(\star\widehat{D_i}),$$

where $\Omega^{q-k}_{U,x}$ is the module of germs of holomorphic q-forms on U at x.

Proof. The equivalence of $i)$ and $ii)$ follows directly from the Leibnitz rule:

$$d(h_j\omega) = dh_j \wedge \omega + h_j d\omega.$$

Let us prove the implication $ii) \Rightarrow iii)$. Consider the following representation of the form ω :

$$\omega = \frac{\sum\limits_{|I|=q} a_I(z)\cdot dz_I}{h_1\cdots h_k},$$

where $I := I^q = (i_1,\ldots,i_q)$, $1 \le i_1,\ldots,i_q \le m$, is a multi-index, $dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_q}$, and $a_I(z)$ is a set of holomorphic functions on U, skew symmetric with respect to the index I. It is easy to see that the condition

$$dh_j \wedge \omega \in \sum_{i=1}^{k} \Omega_U^{q+1}(\star\widehat{D}_i)$$

is equivalent to the condition

$$dh_j \wedge \sum_I a_I(z)\cdot dz_I \in \sum_{\ell=1}^{k} h_\ell \Omega_U^{q+1}.$$

For an ordered multi-index $I := I^{q+1} = (i_1,\ldots,i_{q+1})$, the latter gives us the following system of relations between the coefficients a_I and the derivatives $\partial h_j/\partial z_i$:

$$\sum_{\ell=1}^{q+1}(-1)^{\ell-1}\frac{\partial h_j}{\partial z_{i_\ell}} a_{I\setminus i_\ell} = b_{jI}^1 h_1 + \ldots + b_{jI}^k h_k, \quad j=1,\ldots,k, \tag{1}$$

with holomorphic coefficients $b_{jI}^1,\ldots,b_{jI}^k \in \mathcal{O}_U$.

Let us fix a multi-index $J^p = (j_1,\ldots,j_p) \subset [1,\ldots,m]$, $1 \le p \le k$, and denote the corresponding minor of the Jacobian matrix $\mathrm{Jac}(h_1,\ldots,h_k) = \|\partial h_i/\partial z_j\|$ as follows:

$$\Delta_{J^p} = \Delta_{j_1\ldots j_p} = \det\left\|\frac{\partial h_i}{\partial z_{j_r}}\right\|_{1\le i,r\le p}.$$

Let $\mathcal{I} \subseteq \mathcal{O}_U$ be the ideal generated by the sequence of germs $(h_1,\ldots,h_k)$. We shall prove the following relations:

$$\Delta_{J^p} a_{I^q} \equiv \sum_{\substack{K\subset I^q\\ |K|=p}} \mathrm{sgn}\begin{pmatrix} I^q \\ K, I^q\setminus K\end{pmatrix} \Delta_K\, a_{(J^p, I^q\setminus K)} \; (\mathrm{mod}\,(\mathcal{I})), \quad p=1,\ldots,k, \tag{2}$$

by induction on p. Suppose that $p=1$, $J^1 = j_1$, and set $I = (j, I^q) = (j_1, i_1,\ldots,i_q)$ in the formula (1). One gets

$$\frac{\partial h_1}{\partial z_j} a_I \equiv \sum_{\ell=1}^{q+1}(-1)^{\ell}\frac{\partial h_1}{\partial z_{i_\ell}} a_{I\setminus i_\ell} \; (\mathrm{mod}\,(\mathcal{I})),$$

and this coincides with the relation (2) for $p=1$.

Assuming the relation (2) to be true for $p-1$, one can prove it for p as follows. Making use of the cofactor expansion along the p-th row of the determinant Δ_{J^p}, one obtains:

$$\Delta_{J^p} a_I = \sum_{\ell=1}^{p} (-1)^{p-\ell} \frac{\partial h_p}{\partial z_{j_\ell}} \Delta^{p-1}_{j_1 \ldots \widehat{j_\ell} \ldots j_p} a_I.$$

We have

$$\Delta_{j_1 \ldots \hat{\ell} \ldots j_p} a_I \equiv \sum_{\substack{K' \subset I^q \\ |K'|=p-1}} \operatorname{sgn} \begin{pmatrix} I^q \\ K',\, I^q \setminus K' \end{pmatrix} \Delta_{K'} a_{(j_1 \ldots \widehat{j_\ell} \ldots j_p,\, I \setminus K')} \ (\mathrm{mod}\, (\mathcal{I})),$$

by the induction hypothesis. On substituting this relation into the previous equation and changing the order of summation, one obtains

$$\Delta_{J^p} a_I \equiv \sum_{\substack{K' \subset I^q \\ |K'|=p-1}} \operatorname{sgn} \begin{pmatrix} I^q \\ K',\, I^q \setminus K' \end{pmatrix} \Delta_{K'} \sum_{\ell=1}^{p} (-1)^{p-\ell} \frac{\partial h_p}{\partial z_\ell} a_{(j_1 \ldots \widehat{j_\ell} \ldots j_p,\, I \setminus K')} \ (\mathrm{mod}\, (\mathcal{I})).$$

The second sum consists of p summands contained in the formula (1) for $j = p,\ I = (i_1, \ldots, i_p,\ I \setminus K')$. Hence one can rewrite it as a sum of other $q-p+1$ summands with opposite signs and an element of the ideal $(h_1, \ldots, h_k)\mathcal{O}_U$. This yields the following relation moduli $\mathcal{I}$:

$$\Delta_{J^p} a_I \equiv \sum_{\substack{K' \subset I^q \\ |K'|=p-1}} \operatorname{sgn} \begin{pmatrix} I^q \\ K',\, I^q \setminus K' \end{pmatrix} \Delta_{K'} (-1)^{p-1} \sum_{i \in I \setminus K'} (-1)^{\#(i;\, I \setminus K')} \frac{\partial h_p}{\partial z_i} a_{(j_1 \ldots j_p,\, I \setminus K' \setminus i)}, \tag{3}$$

where $\#(i;\ I \setminus K')$ denotes the number of occurrences of the index i in the set $I \setminus K'$. Let us order all the pairs (K', i) in such a way that the multi-index $K' \cup \{i\}$ coincides with a given $K \subset I$. For any such pair, the corresponding coefficient $a_{(j_1 \ldots j_p,\, I \setminus (K' \setminus i))}$ is equal to $a_{(J,\, I \setminus K)}$. Thus the contribution of the set ordered as above to the relation (3) looks as follows:

$$a_{(J,\, I \setminus K)} (-1)^{p-1} \sum_{i \in K} \operatorname{sgn} \begin{pmatrix} I^q \\ K \setminus i,\ I^q \setminus K,\ i \end{pmatrix} (-1)^{\#(i;\, I \setminus (K \setminus i))} \frac{\partial h_p}{\partial z_i} \Delta_{K \setminus i} =$$

$$= \operatorname{sgn} \begin{pmatrix} I^q \\ K,\ I^q \setminus K \end{pmatrix} a_{J,\, I \setminus K}\, \Delta_K.$$

This completes the proof of the relation (2) for $p \geq 1$.

It is not difficult to see that the relations (2) imply the following equality

$$\Delta_{i_1 \ldots i_k} \cdot \sum_{|I|=q} a_I dz_I = dh_1 \wedge \ldots \wedge dh_k \wedge \Big(\sum_{|I'|=q-k} a_{i_1 \ldots i_k I'}\, dz_{I'} \Big) + \nu,$$

with $\nu \in \sum_{j=1}^{k} h_j \Omega_U^{q-k}$, so that $iii)$ holds, with $g = \Delta_{i_1 \ldots i_k}$, and with $\xi,\ \eta$ of the required shape. It remains to show that one can choose the function g in $iii)$

in such a way that $g \not\equiv 0$ on each of the irreducible components C_j of C. Let $\{\Delta_{i_1 \dots i_k}\}$, $(i_1, \dots . i_k) \in [1, \dots, m]$, be the set of all $k \times k$-minors, with elements $\Delta^1, \dots, \Delta^N$, where $N = \binom{m}{k}$. By assumption, for every non-singular point of C, there is a minor that does not vanish at this point. Let $z^{(j)} \in C_j$ and suppose that $z^{(j)}$ is non-singular, then the function

$$\Delta_t(z) = t\Delta^1 + t^2\Delta^2 + \dots + t^N\Delta^N$$

regarded as polynomial of t, does not vanish identically for every $z = z^{(j)}$. Hence, there is a value $t_0 \in \mathbf{C}$ such that $\Delta_{t_0}(z^{(j)}) \neq 0$ for all j; thus one can take $g(z) = \Delta_{t_0}(z)$.

Here is a purely algebraic proof of this result. Consider the ideal $\mathcal{G}$ of $\mathcal{O}_U$ generated by all the minors $\Delta_{j_1 \dots j_k}$ of maximal order of the Jacobian matrix $\mathrm{Jac}(h_1, \dots, h_k)$. The condition that $dh_1 \wedge \dots \wedge dh_k$ does not vanish identically on every irreducible component of C implies that the image $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ in the ring $\mathcal{O}_{C,0}$ is not equal to $\mathrm{Ann}\, \mathcal{O}_{C,0}$. Thus one can use [4], Th.2.4.(1) which implies that the $\mathcal{O}_{C,0}$-depth of the ideal $\widetilde{\mathcal{G}}$ is at least 1. Hence there is an element $g \in \mathcal{O}_C$ satisfying the condition in *iii)*.

The implication *iii)* $\Rightarrow$ *iv)* is obvious: take $A_j = D_j \cap \{g = 0\}$, $j = 1, \dots, k$.

Let us prove the implication *iv)* $\Rightarrow$ *ii)*. It is obvious that $h_j\omega$ can be expressed as a sum of meromorphic forms each of which has singularities along at most $k-1$ divisors, for example, on $\widehat{D_i}$, and on the subset $A_i \subset D_i$ of codimension 2. Hence by the Riemann extension theorem, such a meromorphic form has singularities *only* on $\widehat{D_i}$. As a result, $h_j\omega \in \sum_{i=1}^{k} \Omega_U^q(\star\widehat{D_i})$, $j = 1, \dots, k$. Similar considerations are valid for the differential form $dh_j \wedge \omega$. This completes the proof of the implication *iv)* $\Rightarrow$ *ii)*.

Remark 1.2. In fact, Proposition 1.1 is a generalization of the basic Lemma-Definition of K. Saito for the case $k = 1$ if $C = D$ is a reduced *hypersurface* (see [25], (1.1)).

Definition 1.3. A meromorphic differential q-form ω, $q \geq k$, on U is called *multi-logarithmic* with respect to the reduced complete intersection $C = D_1 \cap \dots \cap D_k$ if ω satisfies one of the conditions *i)* - *iv)* of Proposition 1.1.

Let C be a reduced complete intersection in S, and let $h_1 = 0, \dots, h_k = 0$ be a system of defining equations of C in the neighbourhood of $x \in C$. A meromorphic q-form ω, $q \geq k$, is logarithmic with respect to C at x, if for all $j = 1, \dots, k$ the differential forms $h_j\omega$ and $h_j d\omega$ satisfy either of conditions *i)* or *ii)* of Proposition 1.1. We shall denote the $\mathcal{O}_{S,x}$-module of germs of multi-logarithmic q-forms at x and the corresponding *sheaf* of multi-logarithmic differential q-forms on S by $\Omega_{S,x}^q(\log C)$ and $\Omega_S^q(\log C)$, $q \geq 0$, respectively. Thus the $\mathcal{O}_S$-module $\Omega_S^q(\log C)$ is a submodule of $\Omega_S^q(\star D)$, consisting of all the differential forms with polar singularities along the *divisor* D, where $D = D_1 \cup \dots \cup D_k$. Clearly, the sheaves $\Omega_S^q(\log C)$

and Ω^q_S coincide outside D, for all $q \geq k$. For the sake of convenience of notation we set

$$\Omega^q_S(\log C) = \sum_{i=1}^{k} \Omega^q_S(\star \widehat{D_i}), \quad q = 0, \ldots, k-1. \tag{4}$$

It should be also remarked that $\Omega^q_S(\log C)$, $q \geq 0$, is a *coherent* sheaf of $\mathcal{O}_S$-modules. Moreover, this sheaf is an $\mathcal{O}_{S,0}$-module of finite type, and the direct sum $\oplus_{q=0}^m \Omega^q_{S,0}(\log C)$ is an $\mathcal{O}_{S,0}$-exterior algebra.

In what follows, we let, for simplicity, $x = 0$ when considering the local problems in the neighbourhood of x. We shall also assume that U is an open subset of $S = \mathbf{C}^m$ containing the origin.

Definition 1.4. In the notation of *iii*), the restriction on $C = D_1 \cap \ldots \cap D_k$ of the form ξ/g is called the *residue form* of ω :

$$\mathrm{res.}\,\omega = \frac{\xi}{g}\Big|_C.$$

Proposition 1.5. *The residue morphism* res. *is well-defined.*

Proof. Let $\omega \in \Omega^q_S(\log C)$, $q \geq 0$. If ω has two different local presentations

$$g_\ell \omega = \frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k} \wedge \xi_\ell + \eta_\ell, \quad \ell = 1, 2,$$

then

$$dh_1 \wedge \ldots \wedge dh_k \wedge (g_1\xi_2 - g_2\xi_1) = h_1 \cdots h_k (g_1\eta_2 - g_2\eta_1) \in (h_1, \ldots, h_k)\Omega^q_S.$$

Hence

$$dh_1 \wedge \ldots \wedge dh_k \wedge (g_1\xi_2 - g_2\xi_1) \equiv 0 \ (\mathrm{mod}\,(h_1, \ldots, h_k)).$$

Now one can apply the part *i*) of the Theorem from [23] for $R = \mathcal{O}_{C,0}$, $M = \Omega^q_{S,0} \otimes \mathcal{O}_{C,0}$. It follows then that

$$\mathcal{G}^e(g_1\xi_2 - g_2\xi_1) \in dh_1 \wedge \Omega^{q-k}_{S,0} + \ldots + dh_k \wedge \Omega^{q-k}_{S,0}, \quad e \in \mathbf{Z},\ e \geq 1,$$

in $\Omega^q_{S,0} \otimes \mathcal{O}_{C,0}$, where the ideal $\mathcal{G}$ of $\mathcal{O}_{S,0}$ is generated by all minors of maximal order $\Delta_{j_1 \ldots j_k}$ of the Jacobian matrix $\mathrm{Jac}(h_1, \ldots, h_k)$. Since C is reduced, the image $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ in the ring $\mathcal{O}_{C,0}$ is not equal to $\mathrm{Ann}\,\mathcal{O}_{C,0}$. Thus one can use [4], Th.2.4.(1), which implies that the $\mathcal{O}_{C,0}$-depth of the ideal $\widetilde{\mathcal{G}}$ is at least 1. Hence there is a maximal minor $\Delta = \Delta_{i_1 \ldots i_k}$ which is not a zero-divisor in $\mathcal{O}_{C,0}$, and

$$\Delta^e(g_1\xi_2 - g_2\xi_1) \in dh_1 \wedge \Omega^{q-1}_{S,0} + \ldots + dh_k \wedge \Omega^{q-1}_{S,0}$$

in $\Omega^q_{S,0} \otimes \mathcal{O}_{C,0}$. Therefore the class of $\Delta^e(g_1\xi_2 - g_2\xi_1)$ in $\Omega^q_{C,0}$ is equal to zero. Thus the two elements $\dfrac{1}{g_1}\xi_1$ and $\dfrac{1}{g_2}\xi_2$ define the same class in $\mathcal{M}_{C,0} \otimes_{\mathcal{O}_{C,0}} \Omega^q_{C,0}$.

Lemma 1.6. *The map* res. *defines a homomorphism of* $\mathcal{O}_S$*-modules*

$$\mathrm{res.}: \ \Omega_S^q(\log C) \longrightarrow \mathcal{M}_C \otimes_{\mathcal{O}_C} \Omega_C^{q-k}, \quad q \geq k;$$

moreover, the following sequence of $\mathcal{O}_S$*-modules*

$$0 \longrightarrow \sum_{i=1}^{k} \Omega_S^q(\star\widehat{D}_i) \longrightarrow \Omega_S^q(\log C) \xrightarrow{\mathrm{res.}} \mathcal{M}_C \otimes_{\mathcal{O}_C} \Omega_C^{q-k}.$$

is exact, and the morphism res. *commutes with the exterior differentiation* d.

Proof. Writing $g\omega$ as in $iii)$ of Proposition 1.1, in view of the Definition 1.4 of the residue map res. , one obtains

$$\mathrm{res.}\,\omega = 0 \Leftrightarrow g\omega \in \sum_{i=1}^{k} \Omega_S^q(\star\widehat{D}_i) \Leftrightarrow \omega \in \sum_{i=1}^{k} \Omega_S^q(\star\widehat{D}_i),$$

and the assertion follows.

Remark 1.7. In particular, for $q = k$ we have

$$0 \longrightarrow \sum_{i=1}^{k} \Omega_S^k(\star\widehat{D}_i) \longrightarrow \Omega_S^k(\log C) \xrightarrow{\mathrm{res.}} \mathcal{M}_C \cong \mathcal{M}_{\widetilde{C}},$$

where $\pi: \widetilde{C} \longrightarrow C$ is the normalization of C. K. Saito has observed that if $k = 1$ and $C = D$ is a hypersurface which is a normal crossing outside a subvariety of codimension at least 2, then the image Im (res.) *coincides* with $\pi_*(\mathcal{O}_{\widetilde{D}})$; it consists of the so-called *weakly holomorphic* functions on D, that is meromorphic functions, whose preimage becomes holomorphic on the normalization (see [25], (2.8)). Some interesting properties of the class of weakly holomorphic functions have been discussed in [27].

Remark 1.8. Thus the image $\mathrm{res.}\,\Omega_S^q(\log C)$, $q \geq k$, of the residue map is an $\mathcal{O}_C$-module. It follows from the definition of multi-logarithmic forms that

$$h_j \cdot \left(\Omega_{S,0}^q(\log C)/\sum_{i=1}^{k} \Omega_S^q(\star\widehat{D}_i)\right) = 0, \quad j = 1, \ldots, k.$$

Hence the ideal $(h_1, \ldots, h_k) \subset \mathcal{O}_S$ annihilates Im (res.).

Remark 1.9. The condition $iii)$ of Proposition 1.1 is equivalent to the following condition:

$$\Delta_{j_1 \ldots j_k} \cdot \omega \in \frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k} \wedge \Omega_U^{\bullet - k} + \Omega_U^{\bullet}, \quad (j_1, \ldots, j_k) \in [1, \ldots, m],$$

for any minor $\Delta_{j_1 \ldots j_k}$ of maximal order of the Jacobian matrix $\mathrm{Jac}(h)$.

2. Regular meromorphic differential forms

Let X be an analytic reduced subspace of a complex manifold S, let $\dim X = n$, and $\dim S = n+k$. The $\mathcal{O}_{X,0}$-module

$$\omega^n_{X,0} = \mathrm{Ext}^k_{\mathcal{O}_{S,0}}(\mathcal{O}_{X,0},\ \Omega^{n+k}_{S,0})$$

is said to be the Grothendieck dualizing module of the germ $(X,0)$.

Definition 2.1. The sheaf ω^q_X, $q \geq 0$, of regular meromorphic differential q-forms is locally given as follows:

$$\omega^q_{X,0} = \mathrm{Hom}_{\mathcal{O}_{X,0}}(\Omega^{n-q}_{X,0},\ \omega^n_{X,0});$$

equivalently, the sheaf $\omega^q_{X,0}$ consists of all the meromorphic forms ω of degree q on X such that $\omega \wedge \eta \in \omega^n_{X,0}$ for any $\eta \in \Omega^{n-q}_{X,0}$.

We shall study the case when X is taken to be the reduced germ of a complete intersection $C = D_1 \cap \ldots \cap D_k$, where $D_1, \ldots, D_k$ is a set of hypersurfaces defined by the equations $h_j(z) = 0,\ j = 1, \ldots, k,\ k \geq 1$, respectively (in the notation of Section 1). In this case,

$$\omega^n_{C,0} = \mathrm{Ext}^k_{\mathcal{O}_{S,0}}(\mathcal{O}_{C,0},\ \Omega^{n+k}_{S,0}) \cong \mathcal{O}_{C,0}\left(\frac{dz_1 \wedge \ldots \wedge dz_{n+k}}{dh_1 \wedge \ldots \wedge dh_k}\right)$$

is a *free* module of rank one, and

$$\omega^q_{C,0} \cong \mathrm{Hom}_{\mathcal{O}_{C,0}}(\Omega^{n-q}_{C,0},\ \mathcal{O}_{C,0}), \quad 0 \leq q \leq n.$$

Here are some properties of the regular meromorphic differential forms:

1) $\omega^q_C = 0$ for $q < 0$ and for $q > \dim C$;

2) The de Rham differential d is extended to the modules $\omega^q_C,\ 0 \leq q \leq n$, so that $(\omega^\bullet_C, d)$ is a complex;

3) In the case when C is *normal*, the sheaves ω^q_C have the following well-known interpretation:

$$\omega^q_C \cong j_* j^* \Omega^q_C,$$

where $j\colon C \setminus \mathrm{Sing}\, C \longrightarrow C$ is the canonical embedding. More precisely, in such a case, for any $q \geq 0$ there is an exact sequence of local cohomology groups with supports in $Z = \mathrm{Sing}\, C$:

$$0 \longrightarrow H^0_Z(\Omega^q_C) \longrightarrow \Omega^q_C \longrightarrow \omega^q_C \longrightarrow H^1_Z(\Omega^q_C) \longrightarrow 0.$$

Let $k = 1$, so that $C = D$ is a *hypersurface.* Then both the module $\omega^0_{D,0}$ and the $\mathcal{O}_D$-module res. $\Omega^1_{S,0}(\log D)$ contain all the germs of *locally bounded meromorphic* functions on D (cf. [3], §2, ex. i)). Moreover, these two sets coincide if the local fundamental group of the complement $S \setminus D$ is *abelian* (see [23]).

Theorem 2.2 ([3]). *The sequence of $\mathcal{O}_C$-modules*

$$0 \longrightarrow \omega_C^q \xrightarrow{\mathcal{C}} \mathrm{Ext}^k_{\mathcal{O}_S}(\mathcal{O}_C,\ \Omega_S^{q+k}) \xrightarrow{\alpha} \Big(\mathrm{Ext}^k_{\mathcal{O}_S}(\mathcal{O}_C,\ \Omega_S^{q+k+1})\Big)^k, \quad q \geq 0,$$

where $\omega_C^q \subset j_* j^* \Omega_C^q$, the map $\mathcal{C}$ is multiplication by the fundamental class of C in S, and $\alpha(e) = (e \wedge dh_1, \ldots, e \wedge dh_k)$, is an exact sequence of $\mathcal{O}_C$-modules.

Thus $\mathcal{C}(v)$ corresponds to a Čech cocycle w/h such that $w = v \wedge dh$. It is easy to see that

$$\omega_C^n \cong \mathrm{Ext}^k_{\mathcal{O}_S}(\mathcal{O}_C,\ \Omega_S^{n+k}).$$

In particular, ω_C^n is isomorphic to the Grothendieck dualizing module of the n-dimensional complete intersection $C \subset S$. In conclusion, we establish a remarkable property of regular meromorphic differential forms in terms of the trace map associated with the Noether normalization. In fact, this property in a slightly different context has been proved by D. Barlet (see [3]).

Theorem 2.3. *The $\mathcal{O}_{C,0}$-module ω_C^q, $q \geq 0$, consists of all the meromorphic forms ω of degree q on C satisfying the following condition: for any $\eta \in \Omega_{C,0}^{n-q}$, the trace*

$$\mathrm{Trace}_{\,C/\mathbf{C}^n}(\omega \wedge \eta)(z) = \sum_{x \in \pi^{-1}(z)} (\omega \wedge \eta)(x)$$

is the germ of a holomorphic form on C.

Proof. Taking the Noether normalization of the germ C, one can suppose, that the germ $(S, 0)$ is isomorphic to the direct product $(\mathbf{C}^n, 0) \times (\mathbf{C}^k, 0)$ with coordinates (z, w) so that the natural projection $\pi\colon (C, 0) \to (\mathbf{C}^n, 0)$ is a finite covering.

Let $\omega \in \omega_{C,0}^q$, so that for any $\eta \in \Omega_{C,0}^{n-q}$ one has

$$\omega \wedge \eta = \frac{\alpha}{dh_1 \wedge \cdots \wedge dh_k} \in \omega_{C,0}^n,$$

where $\alpha \in \Omega_{S,0}^{n+k}$. By definition, one can view the differential form $\omega \wedge \eta$ as an element of a Čech cochain in the open set $\{\Delta(z, w) \neq 0\}$:

$$\omega \wedge \eta = \frac{\varphi(z, w)}{\Delta} dz,$$

where φ is a holomorphic function and Δ is the determinant of the Jacobian matrix of the sequence of functions $h_1(z, w), \ldots, h_k(z, w)$ with respect to w, that is,

$$\Delta = \det \left\| \frac{\partial h_j}{\partial w_i} \right\|_{1 \leq i, j \leq k}.$$

Thus

$$\mathrm{Trace}_{\,C/\mathbf{C}^n}(\omega \wedge \eta)(z) = \sum_\ell \frac{\varphi}{\Delta}(z, w^\ell(z))\, dz.$$

It is well-known (see [28], Ch.1) that the trace of the meromorphic function φ/Δ, whose denominator is equal to the Jacobian of the map π, is holomorphic. The proof of this fact is based on the integral representation of the trace as

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^k \int_\Gamma \frac{\varphi(z,w)dw}{h_1(z,w)\cdots h_k(z,w)},$$

where the integrand is holomorphic on the parameters z and

$$\Gamma = \{w \in \mathbf{C}^k : |h_1(0,w)| = \ldots = |h_k(0,w)| = \varepsilon\}$$

is a compact cycle.

To prove the converse, for simplicity, we consider the case $n = 1$, so that C is a complete intersection curve. The general case can be studied in the same way. Let $\nu \in \Omega^1_{S,0}$, and suppose that

$$\text{Trace}\left(\varphi\,\frac{\nu}{g}\right)$$

is holomorphic, for any holomorphic function $\varphi \in \mathcal{O}_{C,0}$. Then one has to prove

$$\left.\frac{\nu}{g}\right|_C = \left.\frac{\psi}{dh_1 \wedge \ldots \wedge dh_k}\right|_C,$$

that is,

$$\nu \wedge dh = \psi g + h_1 f_1 + \ldots + h_k f_k, \tag{5}$$

where $f_1, \ldots, f_k \in \mathcal{O}_{S,0}$ are holomorphic functions. By assumption,

$$\text{Trace}\left(\varphi\,\frac{\nu}{g}\right)(z) = \sum_\ell \varphi\frac{\nu}{g}(z, w^\ell(z))$$

is holomorphic for any φ. Let us consider the Grothendieck residue

$$R_{h,g}[\varphi\nu \wedge dh_1 \wedge \ldots \wedge dh_k] = \int_{\Gamma_{\varepsilon,\delta}} \frac{\varphi\nu \wedge dh_1 \wedge \ldots \wedge dh_k}{gh_1 \ldots h_k},$$

where $\Gamma_{\varepsilon,\delta} = \{(z,w) \in (S,0) : |h_1| = \varepsilon, \ldots, |h_k| = \varepsilon,\ |g| = \delta\}$. The local variant of the Hilbert Nullstellensatz shows that for some $N > 1$, one can write

$$z^N = q_1 h_1 + \ldots + q_k h_k + Qg.$$

Therefore after applying the residue transformation formula one obtains

$$R_{h,g}[\varphi\nu \wedge dh_1 \wedge \ldots \wedge dh_k] = R_{h,r}[Q\varphi\nu \wedge dh_1 \wedge \ldots \wedge dh_k],$$

where $r = z^N$. The contour of integration in the latter integral

$$\gamma(\varepsilon, \delta) = \{(z, w) \in (S, 0) \colon |h_1| = \varepsilon, \ldots, |h_k| = \varepsilon,\ |z| = \delta\}$$

is fibered under the projection π over the circle $|z| = \delta$. Hence, integration with respect to w along the classical formula of the logarithmic residue yields

$$R_{h,g}[\varphi\nu \wedge dh_1 \wedge \ldots \wedge dh_k] =$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_{|z|=\delta} \operatorname{Trace}\left(\varphi\nu\,\frac{q}{r}\right)(z) = \frac{1}{2\pi\sqrt{-1}} \int_{|z|=\delta} \operatorname{Trace}\left(\varphi\,\frac{\nu}{g}\right)(z),$$

since the restrictions of q/r and $1/g$ to C are equal. By our assumption the last integral is equal to zero, hence the local duality theorem of Grothendieck implies (5). This completes the proof in the case $n = 1$.

3. The Poincaré residue map

We start with the definition of the classical Poincaré residue, following [10]. Let ω be a meromorphic differential m-form on an m-dimensional complex analytic *manifold* S with a polar divisor $D \subset S$. Thus, locally we have a representation:

$$\omega = \frac{f(z)dz_1 \wedge \ldots \wedge dz_m}{h(z)}.$$

By definition, the Poincaré residue $\text{rés}_D(\omega)$ is a meromorphic $(m-1)$-form on D whose singularities are contained in the singular locus $\operatorname{Sing} D \subset D$. To define this form explicitly, let us note that at each point $x \in D \setminus \operatorname{Sing} D$ one of the derivatives does not vanish:

$$\frac{\partial h}{\partial z_i}(x) \neq 0.$$

Hence, locally one may let

$$\text{rés}_D(\omega) = (-1)^{m-i} \left.\frac{f(x)dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_m}{\dfrac{\partial h}{\partial z_i}(x)}\right|_D.$$

It is not difficult to verify that this expression does not depend on the index i, on the local coordinates, and on the defining equation of D. Moreover, the Poincaré residue $\text{rés}_D(\omega)$ is holomorphic on the complement $S \setminus D$. When D is *smooth,* one can take $h(z) = z_m$, then

$$\text{rés}_D\left(\frac{f(x)dz_1 \wedge \ldots \wedge dz_m}{z_m}\right) = f(x)dz_1 \wedge \ldots \wedge dz_{m-1}$$

coincides with the usual residue.

For brevity, the Poincaré residue $\text{rés}_D(\omega)$ will be denoted by $\text{rés}(\omega)$. When $D \subset S$ is a *nonsingular* hypersurface, the Poincaré residue induces an exact sequence

$$0 \longrightarrow \Omega_S^m \longrightarrow \Omega_S^m(D) \xrightarrow{\text{rés}} \Omega_D^{m-1} \longrightarrow 0 \ ,$$

where $\Omega_S^m(D)$ denotes the set of meromorphic forms on S having a pole of the first order along the divisor D. In particular, it follows that the germ of every holomorphic $(m-1)$-form on D is a Poincaré residue. It is obvious that this is true globally when the first cohomology group vanishes: $H^1(S, \Omega_S^m) = 0$.

More generally, following J.Leray (see [12]) one can apply the above construction to meromorphic differential q-forms on S with a *smooth* polar divisor $D \subset S$ for any $1 \le q \le m$, and write locally for any *d-closed* differential q-form ω on $S \setminus D$, having poles of the first order along D, the following presentation:

$$\omega = \frac{dh}{h} \wedge \xi + \eta,$$

where ξ and η are holomorphic. The restriction of ξ to D is a holomorphic form independent of the choice of the local equation $h = 0$ of D; we denote it by $\text{rés}(\omega)$ as before. As a result, one gets (see [19]) an exact sequence

$$0 \longrightarrow \Omega_S^q \longrightarrow \Omega_S^q(D) \xrightarrow{\text{rés}} \Omega_D^{q-1} \longrightarrow 0, \quad 1 \le q \le m,$$

which is equivalent to the sequence

$$0 \longrightarrow \Omega_S^q \longrightarrow \Omega_S^q(\log D) \xrightarrow{\text{rés}} \Omega_D^{q-1} \longrightarrow 0 \ , \tag{6}$$

since the divisor D is a *smooth* hypersurface.

Remark 3.1. As it has been already remarked by de Rham and Leray (cf. [12]) that for a meromorphic q-form ω with poles of the first order along D the above definition gives us the *holomorphic* $(q-1)$-form $\text{rés}(\omega)$ in cases when the singular locus $\operatorname{Sing} D$ of the divisor D consists of isolated *double quadratic* points only and either $q < m$, or $q = m$, and the coefficient of m-form $h \cdot \omega$ vanishes on $\operatorname{Sing} D$. This phenomenon can be readily explained by considerations which we use in the next and previous sections.

An extension of the exact sequence (6) to the case when the divisor D has arbitrary singularities was described in [1]

$$0 \longrightarrow \Omega_S^q \longrightarrow \Omega_S^q(\log D) \xrightarrow{\text{res.}} \omega_D^{q-1} \longrightarrow 0, \quad q \ge 1. \tag{7}$$

The following theorem is devoted to the further generalization of this construction to the case of complete intersection.

Theorem 3.2. *In the notation of Section* 1 *let* $C \subset S$ *be a reduced complete intersection. Then there is an exact sequence*

$$0 \longrightarrow \sum_{i=1}^{k} \Omega_S^q(\star\widehat{D_i}) \longrightarrow \Omega_S^q(\log C) \xrightarrow{\text{res.}} \omega_C^{q-k} \longrightarrow 0, \quad q \geq k,$$

and a natural isomorphism of $\mathcal{O}_C$*-modules*

$$\text{res.}\, \Omega_S^q(\log C) \cong \omega_C^{q-k}, \quad q \geq k.$$

Proof. The proof goes similarly to [1], §4. Thus, it is sufficient to verify our statements locally. The remark 1.9 implies

$$\Delta_{j_1 \ldots j_k} \cdot \text{res.}\, \Omega_S^q(\log C)|_U \subset \Omega_C^{q-k}|_{C\cap U},$$

for all maximal minors $\Delta_{j_1 \ldots j_k}$, $(j_1, \ldots, j_k) \in [1, \ldots, m]$ of the Jacobian matrix $\text{Jac}(\underline{h})$. Since $\omega_{C,0}^n \cong \mathcal{O}_{C,0}(dz_1 \wedge \ldots \wedge dz_{n+k}/dh_1 \wedge \ldots \wedge dh_k)$ the definition of the modules of regular meromorphic differential forms implies immediately that

$$\text{res.}\, \Omega_S^q(\log C) \subset \omega_C^{q-k}.$$

Let now $\mathcal{K}_\bullet(\underline{h})$ be the usual Koszul complex associated with the sequence $\underline{h} = (h_1, \ldots, h_k)$:

$$0 \to \mathcal{O}_{S,0}\langle e_0 \wedge \ldots \wedge e_{k-1}\rangle \xrightarrow{d_{k-1}} \ldots \longrightarrow \mathcal{O}_{S,0}\langle e_0\rangle + \ldots + \mathcal{O}_{S,0}\langle e_{k-1}\rangle \to$$
$$\xrightarrow{d_0} \mathcal{O}_{S,0} \xrightarrow{d_{-1}} \mathcal{O}_{C,0} \to 0\ ,$$

where

$$\mathcal{K}_k(\underline{h}) = \mathcal{O}_{S,0}\langle e_0 \wedge \ldots \wedge e_{k-1}\rangle,\ \ldots,\ \mathcal{K}_1(\underline{h}) = \mathcal{O}_{S,0}\langle e_0\rangle + \ldots + \mathcal{O}_{S,0}\langle e_{k-1}\rangle,$$

and $\mathcal{K}_0(\underline{h}) = \mathcal{O}_{S,0},\ d_0(e_i) = h_{i+1},\ i = 0, \ldots, k-1,\ d_{-1}(1) = 1$. From the dual exact sequence one obtains, by definition,

$$\text{Ext}^k_{\mathcal{O}_{S,0}}(\mathcal{O}_{C,0},\ \Omega^{q+1}_{S,0}) \cong \text{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_k(\underline{h}),\ \Omega^{q+1}_{S,0})/d^{k-1}(\text{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_{k-1}(\underline{h}),\ \Omega^{q+1}_{S,0})).$$

This means that any element of $\text{Ext}^k_{\mathcal{O}_{S,0}}(\mathcal{O}_{C,0},\ \Omega^{q+1}_{S,0})$ can be represented as a Čech $(k-1)$-cochain (more explicitly, a $(k-1)$-cocycle)

$$\frac{\nu}{h_1 \cdots h_k} \in \text{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_k(\underline{h}),\ \Omega^{q+1}_{S,0}) \cong C^{k-1}_{(k)}(\Omega^{q+1}_{S,0}),$$

where $\nu \in \Omega^{q+1}_{S,0}$. Choose now an element $\nu \in \Omega^{q+1}_{S,0}$ such that

$$\frac{\nu}{h_1 \cdots h_k} \wedge dh_j \in \text{Ext}^k_{\mathcal{O}_{S,0}}(\mathcal{O}_{C,0},\ \Omega^{q+2}_{S,0}), \quad j = 1, \ldots, k,$$

corresponds to the *trivial* element. This means that for any $j = 1, \dots, k$ the form $\nu \wedge dh_j/h_1 \cdots h_k$, is defined by an element of $d^{k-1}(\mathrm{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_{k-1}(\underline{h}), \Omega^{q+2}_{S,0}))$. This yields that

$$\nu \wedge dh_j \in \sum_{i=1}^{k} h_i \Omega_S^{q+2}, \quad j = 1, \dots, k,$$

or, equivalently,

$$\omega \wedge dh_j \in \sum_{i=1}^{k} \Omega_S^{q+1}(\star \widehat{D}_i), \quad j = 1, \dots, k, \quad \text{where} \quad \omega = \frac{\nu}{h_1 \cdots h_k}.$$

In view of Proposition 1.1 this means that $\omega \in \Omega^q_{S,0}(\log C)$. Now we shall use the notation and exact sequence of Theorem 2.2. Set $\tilde{\nu} = \mathcal{C}^{-1}(\nu/h_1 \cdots h_k)$. Then $\mathcal{C}(\tilde{\nu})$ corresponds to the Čech cocycle $\nu/h_1 \cdots h_k$ such that $\nu = \tilde{\nu} \wedge dh_1 \wedge \cdots \wedge dh_k$ (take $v = \tilde{\nu}$, $w = \nu$ in the description of ω^q_C in terms of multiplication by the fundamental class of C in S). This implies

$$\frac{\nu}{h_1 \cdots h_k} = \tilde{\nu} \wedge \frac{dh_1}{h_1} \wedge \cdots \wedge \frac{dh_k}{h_k}, \quad \text{and} \quad \mathrm{res.}\left(\frac{\nu}{h_1 \cdots h_k}\right) = \tilde{\nu}.$$

Thus, for any element $\tilde{\nu} \in \omega_C^{q-k}$ there is a preimage under the multi-logarithmic residue map represented by $\nu/h_1 \cdots h_k$.

4. On residues of multi-logarithmic forms and $\bar{\partial}$-closed currents

We first remark, that in view of Theorem 3.2, the sections of $\omega^\bullet_C$ can be considered as "regular meromorphic forms." More precisely, in complex analysis "regular" or "holomorphic" differential forms are usually described as $\bar{\partial}$-closed forms (cf. [13]).

The key idea and further considerations in this section are closely related to the methods developed in [27], where the theory of weakly holomorphic functions is studied. Moreover, they lead to another proof of the statement of Theorem 3.2 concerning the representation of sections of Barlet sheaves $\omega^\bullet_C$ as residues of multi-logarithmic differential forms. The main result of this section is the following theorem.

Theorem 4.1. *There is a natural one to one correspondence between the sets of residues of multi-logarithmic forms and meromorphic $\bar{\partial}$-closed currents on the complete intersection $C = D_1 \cap \ldots \cap D_k$.*

The proof will be given below. We start with the definition of the integration current associated with a meromorphic differential form, and recall the basic properties of residue currents.

Let $\psi = \xi/g$ be a meromorphic differential form on a reduced analytic subset C of a domain $U \subset \mathbf{C}^m$; here ξ is a holomorphic differential form and g is a holomorphic function. Such a form defines the following current

$$\Big\langle [\psi],\, \varphi \Big\rangle = \Big\langle \left[\frac{\xi}{g}\right]\Big|_C,\, \varphi \Big\rangle = \lim_{\varepsilon\to 0} \int_{C\cap\{|g|>\varepsilon\}} \frac{\xi}{g} \wedge \varphi, \quad \varphi \in \mathcal{D}^\bullet(U), \tag{8}$$

where φ is a test C^∞-differential form with compact support on U, and the real dimension of ξ is equal to $\deg\varphi + \deg\psi$. This is an integration current on C in the principal value sense.

A residue current R_h associated with a holomorphic map $h = (h_1, \ldots, h_k) : U \to \mathbf{C}^k$ is is a functional on the space $\mathcal{D}^\bullet(U)$ of C^∞-differential form with compact support on U. It is defined as follows (see [5] or [28]):

$$R_h(\varphi) = \lim_{\varepsilon\to 0} \int_{T_\varepsilon(h)} \frac{\varphi}{h_1 \cdots h_k}, \quad \varphi \in \mathcal{D}^{2m-k}(U),$$

where the integral is taken over the tube

$$T_\varepsilon(h) = \{z \in S\colon |h_j(z)| = \varepsilon_j,\ j = 1, \ldots, k\}$$

with a fixed orientation. Here the approach of the radii $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ to zero is required to be along a so-called admissible path: a path $\varepsilon = \varepsilon(\tau)$ is called *admissible,* if $\varepsilon_k \to 0$ as $\tau \to 0$ and for each $j = 1, \ldots, k-1$, the coordinate $\varepsilon_j(\tau)$ approaches zero faster than any power of the next coordinate $\varepsilon_{j+1}(\tau)$.

The *principal value* of the residue current R_h with respect to a function $g\colon U \to \mathbf{C}$ is defined as the limit

$$P_g R_h(\varphi) = \lim_{\tau\to 0} \int_{T_\varepsilon(h)\cap\{|g|>\delta\}} \frac{\varphi}{g h_1 \cdots h_k}$$

along the admissible path $(\varepsilon(\tau), \delta(\tau))\colon [0,1] \to \mathbf{R}^{k+1}$.

When the set $h^{-1}(0) = \{h_1 = \ldots = h_k = 0\}$ is a complete intersection of codimension k then

$$R_h = \bar\partial\left[\frac{1}{h_1}\right] \wedge \ldots \wedge \bar\partial\left[\frac{1}{h_k}\right].$$

Respectively, when $g \not\equiv 0$ on every irreducible component of the analytic set $h^{-1}(0)$ then

$$P_g R_h = \left[\frac{1}{g}\right] \bar\partial\left[\frac{1}{h_1}\right] \wedge \ldots \wedge \bar\partial\left[\frac{1}{h_k}\right].$$

When $\omega = \alpha/h_1 \cdots h_k$ is a meromorphic differential form of degree ℓ then there is the current

$$R_h[\omega](\varphi) := R_h(\alpha \wedge \varphi), \quad \varphi \in \mathcal{D}^{2m-k-\ell}(U).$$

Analogously for $\omega = \alpha/g{\cdot}h_1 \cdots h_k$ we set

$$P_g R_h[\omega](\varphi) = P_g R_h(\alpha \wedge \varphi).$$

Let us assume that the zero set defined by the system of equations

$$\{g = h_1 = \ldots = h_k = 0\}$$

is a *complete intersection.* Then the residue currents R_h and their principal values P_g with respect to g have the following properties:

1) $R_h[\omega] = P_g R_h[g\omega]$;

2) $P_g R_h\left[\frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k} \wedge \frac{\xi}{g}\right] = \left[\frac{\xi}{g}\right]\Big|_C = \left[\frac{\xi}{g}\right] \wedge [C]$;

where $[C]$ is the integration current (a holomorphic chain) on C. In particular, when ξ and g are identically equal to the constant 1, then

$$R_h\left[\frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k}\right] = dh_1 \wedge \ldots dh_k \wedge \bar\partial\left[\frac{1}{h_1}\right] \wedge \ldots \wedge \bar\partial\left[\frac{1}{h_k}\right] = [C];$$

3) $h_j P_g R_h = 0, \quad j = 1, \ldots, k$;

4) If U is a domain of holomorphy (or a Stein manifold) then for $v \in \mathcal{O}_U$ one has

$$vR_h = 0 \Longleftrightarrow v \in (h_1, \ldots, h_k)\mathcal{O}_U.$$

In particular, this property is valid in the local rings $\mathcal{O}_{U,x},\ x \in U$.

The properties 1) and 3) are proved in [5], Theorem 4.4.1 and Theorem 1.7.6, respectively, the property 2) is obtained in [26], Theorem 2.1; the property 4) is obtained in [18], Corollary 6.1.2.

Proof of Theorem 4.1. Let $\omega \in \Omega^{q+k}_{S,0}(\log C)$, so that

$$\omega = \frac{\alpha}{h_1 \cdots h_k}, \quad \text{and} \quad g\omega = \frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k} \wedge \xi + \eta,$$

where α, ξ are holomorphic and η satisfies the condition *iii*) of Proposition 1.1. In view of property 1) one has

$$R_h[\omega] = P_g R_h[g\omega] = P_g R_h\left[\frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k} \wedge \frac{\xi}{g}\right] + P_g R_h\left[\frac{\eta}{g}\right].$$

The second summand corresponds to the current which is trivial by property 3). The property 2) implies

$$R_h[\omega] = \left[\frac{\xi}{g}\right]\Big|_C.$$

Hence the current $\left[\frac{\xi}{g}\right]\Big|_C$ defined by the residue res. $\omega = \frac{\xi}{g}\Big|_C$ of the multi-logarithmic form ω coincides with the residue current $R_h[\omega]$ which is $\bar\partial$-closed [5].

Conversely, under the same assumptions, let ξ/g be a meromorphic form on C, and the corresponding current (8)

$$\Big\langle [\psi],\, \varphi \Big\rangle = \Big\langle \left[\frac{\xi}{g}\right]\Big|_C,\, \varphi \Big\rangle = \lim_{\varepsilon\to 0}\int_{C\cap\{|g|>\varepsilon\}} \frac{\xi}{g}\wedge\varphi, \quad \varphi\in\mathcal{D}^{\bullet}(U)$$

is $\bar\partial$-closed. Then in virtue of property 2)

$$0 = \bar\partial\left[\frac{\xi}{g}\right]\Big|_C = \bar\partial\Big(\left[\frac{\xi}{g}\right]\wedge[C]\Big).$$

Since $\bar\partial[C]=0$, applying the Leibniz rule and the property 2), one obtains

$$0 = \bar\partial\left[\frac{\xi}{g}\right]\wedge[C] = \bar\partial\left[\frac{\xi}{g}\right]\wedge dh_1\wedge\ldots\wedge dh_k\wedge\bar\partial\left[\frac{1}{h_1}\right]\wedge\ldots\wedge\bar\partial\left[\frac{1}{h_k}\right].$$

The property 4) yields

$$dh_1\wedge\ldots\wedge dh_k\wedge\xi\in(g,h_1,\ldots,h_k)\Omega^{q+k}_{U,0},$$

that is,

$$dh_1\wedge\ldots\wedge dh_k\wedge\xi = \alpha g - h_1\eta_1 - \ldots - h_k\eta_k, \tag{9}$$

where $\alpha,\ \eta_1,\ldots,\eta_k\in\Omega^{q+k}_U$ are holomorphic forms on the domain $U\subset S$. Here the property 4) is used in computing the coefficients of the form $\xi\wedge dh$. Let us consider the meromorphic form $\omega=\alpha/h_1\cdots h_k$. We have

$$g\cdot\omega = g\cdot\frac{\alpha}{h_1\cdots h_k} = \frac{dh\wedge\xi + h_1\eta_1+\ldots+h_k\eta_k}{h_1\cdots h_k} = \frac{dh_1}{h_1}\wedge\ldots\wedge\frac{dh_k}{h_k}\wedge\xi+\eta,$$

where

$$\eta = \sum_{i=1}^{k}\frac{\eta_i}{h_1\cdots h_k}\in\sum_{i=1}^{k}\Omega^{q+k}_{U,0}(\star\widehat{D}_i).$$

This completes the proof of Theorem 4.1.

One can deduce the main statement of Theorem 3.2 from the following theorem.

Theorem 4.2. *A meromorphic differential q-form on the complete intersection $C=D_1\cap\ldots\cap D_k$ defines a meromorphic $\bar\partial$-closed current on C if and only if it is a section of the Barlet sheaf ω^q_C of regular meromorphic forms.*

Proof. Let $\omega=\xi/g$ be a meromorphic q-form C. It is not difficult to see that

$$\bar\partial(\omega)=0 \Longleftrightarrow \bar\partial(\omega\wedge\nu)=0 \quad \text{for any } \nu\in\Omega^{n-q}_C.$$

Making use of the property 4) and of similar considerations of the converse part of the proof of Theorem 4.1 we obtain that the property of $\bar\partial$-closedness of ω is equivalent to the existence of the relation

$$dh\wedge(\xi\wedge\nu) = \alpha g - h_1\eta_1 - \ldots - h_k\eta_k,$$

where $\alpha, \eta_1, \ldots, \eta_k \in \Omega_U^m$ are holomorphic forms on the domain $U \subset S$. This is equivalent to the condition

$$\frac{\xi}{g} \wedge \nu \Big|_C = \frac{\alpha}{dh_1 \wedge \cdots \wedge dh_k}\Big|_C \in \omega_C^n.$$

Hence, by definition,

$$\omega = \frac{\xi}{g} \in \omega_C^q$$

as required.

Remark 4.3. The idea of this proof is the same as that of the proof of Theorem 2 in [13].

Remark 4.4. Thus, any section of the Barlet sheaf ω_C^q is the residue of a multi-logarithmic form. Such meromorphic forms on the complete intersection C are characterized by the following two conditions:

a) they are holomorphic on the set of non-singular points of C (cf. [12]),

b) they have no residues with respect to the singular locus of C (this is the so-called $\bar{\partial}$-closedness property).

A differential form ψ which satisfies condition *a*) is not *a priori* meromorphic. Nevertheless it should define the following integration current (see [26]):

$$\Big\langle\, [\psi],\, \varphi \Big\rangle = \lim_{\delta \to 0} \int_{C \cap \{|dh| > \delta\}} \psi \wedge \varphi, \tag{10}$$

where $dh = dh_1 \wedge \cdots \wedge dh_k$. This integration current differs from (8) by the choice of the "principal value" of the integral. Recall that under our assumptions the singular locus $\operatorname{Sing} C$ of C is defined by the condition $dh_1 \wedge \cdots \wedge dh_k|_C = 0$. Then a natural question arises: *Whether the class of differential q-forms ψ which are holomorphic on the non-singular part of C and define the $\bar{\partial}$-closed currents by means of formula* (10), *and the class of sections of Barlet sheaf ω_C^q coincide?*

In connection with this problem it should be noted that there are *non-exact* $\bar{\partial}$-closed currents corresponding to non-holomorphic differential forms (see [8]).

If one supposes that ψ is *meromorphic* then Proposition 1 of [26] asserts that it is true. Thus one gets the following elegant description of the Grothendieck dualizing module on a reduced complete intersection in terms of $\bar{\partial}$-closed currents. We remark that *a priori* all sections of the dualizing module are meromorphic on C and holomorphic on the non-singular part of C.

Corollary 4.5. *Any element ψ of the Grothendieck dualizing module of the reduced complete intersection $C = D_1 \cap \ldots \cap D_k$ defines a meromorphic $\bar{\partial}$-closed current on C by formula* (10) *and vice versa.*

In conclusion we pose a problem which is a stronger version of the above question. Let Ψ^q be the set of differential q-forms ψ on U such that they are holomorphic on the non-singular part of C and satisfy the following condition (cf. formula (10)) :

$$\Big\langle\, [\psi],\, \varphi \Big\rangle = \lim_{\delta\to 0} \int_{C\cap\{|dh|=\delta\}} \psi\wedge\varphi = 0. \tag{11}$$

Conjecture 1. The classes of differential q-forms defined by the set Ψ^q and by the sections of the Barlet sheaf ω_C^q coincide.

Remark 4.6. It is well-known that any pure dimensional analytic set can be regarded locally as a component of some reduced complete intersection (see [9], p.52). In fact, this leads to a generalization of the notion of the multi-logarithmic differential form with respect to an arbitrary pure dimensional analytic set.

5. Appendix. Historical remarks and comments

In 1887, Henry Poincaré [20] introduced the notion of a residue 1-form for rational 2-forms on $\mathbf{C}^2$. In the papers of G. de Rham, [22], and J. Leray, [17], this notion was extended to *d-closed* meromorphic q-form ω having poles of the first order along a smooth divisor. They proved that the residue of such a form is holomorphic. Influenced by the papers of Leray and P. Dolbeault, M. Herrera, P. Liberman, N. Coleff and others have developed theory of the integral representation and residue currents (see [6], [7], [5]).

The next stage the development of the theory starts at the beginning of the seventies. It is closely related to the general theory of logarithmic differential forms. In fact, such forms were first considered in 1972 in a work of J.-B. Poly [21] who defined and studied the Leray residue for *semi-meromorphic* differential forms. Poly had proved that the Leray residue is well defined for any (not necessarily d-closed) semi-meromorphic differential form ω, as soon as ω and $d\omega$ have a simple pole along a hypersurface [loc. cite, Remarque 4]. Later, K. Saito further investigated the class of *meromorphic* differential forms ω satisfying these conditions; in his series of works, these forms were called *logarithmic* along a hypersurface (see [23], [24]). Saito established the basic properties of meromorphic logarithmic forms, and of their residues, and discussed some important and interesting applications of these objects. The same notion has occurred also in the book by Ph. Griffiths and J. Harris [11], with no references to previous works; in this book, the authors use this notion only in the case of divisors which are normal crossings of smooth hypersurfaces. This case has been also treated in earlier works by P. Deligne, J. Steenbrink, F. Pham, and many others.

Almost at the same time, in 1976, in a work of D. Barlet [2] a special complex $\omega_X^\bullet$ of subsheaves of the sheaves of meromorphic differential forms on an analytic subvariety X of a complex manifold was introduced and investigated (see also [3]).

If $\dim X = n$, the sheaf ω_X^n is naturally isomorphic to the Grothendieck dualizing module. In a purely algebraic context, a similar notion appeared also in the works of E. Kunz [15], [16], M. Kersken [14], and others on general duality theory. They call ω_X^q the sheaf of *regular meromorphic* differential forms of degree q. In a joint paper [13], G. Henkin and M. Passare give an analytic interpretation of regular meromorphic forms. Some of their results are essentially deduced from the basic properties of weakly holomorphic functions, established in a series of earlier works of the second author and his collaborators (see [27], [28], [26]).

In 1990 it was proved (see [1]) that the residue map defined by K. Saito [25] gives a natural quasi-isomorphism between the complexes of sheaves of logarithmic and of regular meromorphic differential forms on a reduced hypersurface in a complex analytic manifold. In our paper, this approach is further developed to treat reduced complete intersections. In fact, the notions of the multi-logarithmic differential form and its residue generalize the corresponding definitions of J. Leray, J.-B. Poly and K. Saito; while a natural morphism between the complexes of multi-logarithmic and of regular meromorphic differential forms on a complete intersection is a generalization of an earlier argument of [1] applicable to the case of hypersurfaces. An explicit description of this morphism in terms of integration currents is given in our paper, based on results obtained in the papers [27], [28], [26], [5] and [18] cited above.

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