RELATION BETWEEN HOMOTOPY PROPERTIES OF CLOSED SYMPLECTIC MANIFOLDS

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1. Introduction

Homotopy properties of closed symplectic manifolds attract the attention of geometers since the classical papers of Sullivan [S] and Thurston [Th]. On one hand, "soft" homotopy techniques help in the solution of many "hard" problems in symplectic geometry, cf. [G1, McD, RT1, TO]. On the other hand, it is still unknown if there are specific homotopy properties of closed manifolds dependent on the existence of symplectic structures on them. It turns out that symplectic manifolds violate many specific homotopy conditions shared by the Kähler manifolds (which form a subclass of symplectic manifolds). In particular, if M is a closed Kähler manifold then the following holds:

(1) all the odd-degree Betti numbers $b_{2i+1}(M)$ are even;
(2) M has the Hard Lefschetz property;
(3) all Massey products (of all orders) in M vanish.

It is well known (and we shall see it below) that closed symplectic manifolds violate *all* the homotopy properties (1) – (3). However, it is not clear whether properties (1) – (3) are *independent* or not, in case of closed symplectic manifolds or certain classes of these. For example, can a combination of the type

(1) – (2) – not-(3)

be realized by a closed symplectic manifold (possibly, with prescribed properties)? The knowledge of answers to questions like this shed new light on the understanding of closed symplectic manifolds.

In Theorem 3.1 we have summarized our knowledge by writing down the corresponding tables. We have considered two classes of symplectic manifolds: the class of symplectically aspherical symplectic manifolds and the class of simply-connected symplectic manifolds. Recall that a symplectically aspherical manifold is a symplectic manifold (M,ω) such that $\omega|\pi_2(M)=0$, i.e.

$$\int_{S^2} f^{\#}\omega = 0$$

for every map $f: S^2\to M$. In view of the Hurewicz Theorem, a closed symplectically aspherical manifold always has a non-trivial fundamental group. It is well known that symplectically aspherical manifolds play an important role in geometry and topology of symplectic manifolds, [F, G2, H, RO, RT2].

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2. Preliminaries and notation

Given a topological space X, let $(\mathcal{M}_X, d)$ be the Sullivan model of X, that is, a certain natural commutative DGA algebra over the field of rational numbers $\mathbb{Q}$ which is a homotopy invariant of X, see [DGMS, TO, S] for details. Furthermore, if X is a nilpotent CW-space of finite type then $(\mathcal{M}_X, d)$ completely determines the rational homotopy type of X.

A space X is called *formal* if there exists a DGA-morphism

$$\rho : (\mathcal{M}_X, d) \to (H^*(X; \mathbb{Q}), 0)$$

inducing isomorphism on the cohomology level. Recall that every closed Kähler manifold in formal [DGMS].

We refer the reader to [K, Ma, RT1] for the definition of Massey products. It is well known and easy to see that Massey products yield an obstruction to formality [DGMS, RT1, TO]. In other words, if the space is formal then all Massey products must be trivial. Thus, all the Massey products in every Kähler manifold vanish.

We need also the following result of Miller [Mi]:

2.1. Theorem. *Every closed simply-connected manifold M of dimension ≤ 6 is formal. In particular, all Massey products in M vanish.* □

The next homotopy property related to symplectic (in particular, Kähler) structures is the *Hard Lefschetz property*. Given a symplectic manifold (M^{2m}, ω), we denote by $[\omega] \in H^2(M)$ the de Rham cohomology class of ω. Furthermore, we denote by $L_\omega : \Omega^k(M) \to \Omega^{k+2}(M)$ the multiplication by ω and by $L_{[\omega]} : H^k(M) \to H^{k+2}(M)$ the induced homomorphism in the de Rham cohomology $H^*(M)$. As usual we write L instead of L_ω or $L_{[\omega]}$ if there is no danger of confusion. We say that a symplectic manifold (M^{2m}, ω) has the *Hard Lefschetz property* if, for every k, the homomorphism

$$L^k : H^{m-k}(M) \to H^{m+k}(M)$$

is surjective. In view of the Poincaré duality, for closed manifolds M it means that every L^k is an isomorphism. We need also the following result of Gompf [G1, Theorem 7.1].

2.2. Theorem. *For any even dimension $n \geq 6$, finitely presented group G and integer b there is a closed symplectic n-manifold M with $\pi_1(M) \cong G$ and $b_i(M) \geq b$ for $2 \leq i \leq n-2$, such that M does not satisfy the Hard Lefschetz condition. Furthermore, if $b_1(G)$ is even then all degree-odd Betti numbers of M are even.*

□

We denote such manifold M by $M(n, G, b)$.

2.3. Remark. Theorem 7.1 in [G1] is formulated in a slightly different way, but the *proof* is based on constructing of M by some "symplectic summation" in a way to violate the Hard Lefschetz property.

In our explicit constructions we will need some particular classes of manifolds, namely, *nilmanifolds*, resp. *solvmanifolds*. These are homogeneous spaces of the form G/Γ, were G is a simply connected nilpotent, resp. solvable Lie group and Γ

is a co-compact discrete subgroup (i.e. a *lattice*). The most important information for us is the following (see e.g. [TO] for the proofs):

2.4. Recollection. (i) Let $\mathfrak{g}$ be a nilpotent Lie algebra with structural constants c_k^{ij} with respect to some basis, and let $\{\alpha_1, ..., \alpha_n\}$ be the dual basis of $\mathfrak{g}^*$. Then the differential in the Chevalley–Eilenberg complex $(\Lambda^*\mathfrak{g}^*, d)$ is given by the formula

$$d\alpha_k = -\sum_{1\leq i<j<k} c_k^{ij}\alpha_i \wedge \alpha_j.$$

(ii) Let $\mathfrak{g}$ be the Lie algebra of a simply connected nilpotent Lie group G. Suppose that G admits a lattice Γ. By Nomizu's theorem, the Chevalley–Eilenberg complex $(\Lambda^*\mathfrak{g}^*, d)$ is quasi-isomorphic to the de Rham complex of G/Γ. Moreover, $(\Lambda^*\mathfrak{g}^*, d)$ is a minimal differential algebra, and hence it is isomorphic to the minimal model of G/Γ:

$$(\Lambda^*\mathfrak{g}^*, d) \cong (\mathcal{M}_{G/\Gamma}, d).$$

Let ω_0 be the standard symplectic form on $\mathbb{C}P^m$. Recall that every closed symplectic manifold (M^{2n}, ω) with integral form ω can be symplectically embedded into $\mathbb{C}P^m$ for m large enough, with the (known) smallest possible value of m equal to $n(n+1)$ [Gr, Ti]. We will use the *blow-up* construction with respect to such embedding [McD, RT1]. We need the following result.

2.5. Theorem. *Let (M^{2n}, ω) be a closed connected symplectic manifold, let i : $(M, \omega) \to (\mathbb{C}P^m, \omega_0)$ be a symplectic embedding, and let $\widetilde{\mathbb{C}P^m}$ be the blow-up along i. Then the following holds:*

(i) *$\widetilde{\mathbb{C}P^m}$ is a simply-connected symplectic manifold;*

(ii) *if there exists i such that $b_{2i+1}(M)$ is odd, then there exists k such that $b_{2k+1}(\widetilde{\mathbb{C}P^m})$ is odd;*

(iii) *if M possesses a non-trivial Massey triple product and $m-n \geq 4$, then $\widetilde{\mathbb{C}P^m}$ possesses a non-trivial Massey triple product. Moreover, if there is a non-trivial Massey product $\langle \alpha, \beta, [\omega]\rangle \in H^*(M)$, $\alpha, \beta \in H^*(M)$, then $\widetilde{\mathbb{C}P^m}$ possesses a non-trivial Massey triple product even for $m-n=3$.*

Proof. (i) and (ii) are proved in [McD], (i) and (iii) are proved in [RT1], cf. also [BT]. □

3. Two tables

3.1. Theorem. *The relations between the Hard Lefschetz property, evenness of odd-degree Betti numbers and vanishing of the Massey products for closed symplectic manifolds are given by the following tables:*

TABLE 1: *symplectically aspherical case;*

TABLE 2: *simply-connected case.*

The word **Impossible** in the table means that there is no closed symplectic manifold (aspherical or simply connected) that realizes the combination in the corresponding line.

The sign **?** means that we (the authors) do not know whether a manifold with corresponding properties exists.

Table 1: Symplectically Aspherical Symplectic Manifolds

Triviality of Massey Products	Hard Lefschetz Property	Evenness of b_{2i+1}	
yes	yes	yes	Kähler ($\mathbb{T}^{2n}$)
yes	yes	no	Impossible
yes	no	yes	?
yes	no	no	?
no	yes	yes	?
no	yes	no	Impossible
no	no	yes	$K \times K$
no	no	no	K

Table 2: Simply-Connected Symplectic Manifolds

Triviality of Massey Products	Hard Lefschetz Property	Evenness of b_{2i+1}	
yes	yes	yes	Kähler ($\mathbb{C}P^n$)
yes	yes	no	Impossible
yes	no	yes	M(6,{e},0)
yes	no	no	?
no	yes	yes	?
no	yes	no	Impossible
no	no	yes	$\widetilde{\mathbb{C}P^5} \times \widetilde{\mathbb{C}P^5}$
no	no	no	$\widetilde{\mathbb{C}P^5}$

Proof. We prove the theorem via line-by-line analysis of Tables 1 and 2.

Line 1 in Tables 1 and 2. For closed Kähler manifolds, the Hard Lefschetz property is proved in [GH], the evenness of b_{2i+1} follows from the Hodge theory [W], the triviality of Massey products follows from the formality of any closed Kähler manifold [DGMS].

One can ask whether there are non-Kähler manifolds having the properties from line 1. In the symplectically aspherical case the answer is affirmative. Let $G = \mathbb{R} \times_\phi \mathbb{R}^2$ be the semidirect product determined by the one-parameter subgroup $\phi(t) = \operatorname{diag}(e^{kt}, e^{-kt}), t \in \mathbb{R}, e^k + e^{-k} \neq 2$. One can check that G contains a lattice, say Γ. Then the compact solvmanifold

$$M = G/\Gamma \times S^1$$

is symplectic and has the same minimal model as the Kähler manifold $S^2 \times T^2$. Hence such manifold fits into line 1. It cannot be Kähler, since it admits no complex structure. The latter follows from the Kodaira–Yau classification of compact complex surfaces (see [TO] for details).

Line 2 in Tables 1 and 2. Any manifold satisfying the Hard Lefschetz property must have *even* b_{2i+1}. Indeed, consider the usual non-singular pairing $p : H^{2k+1}(M) \otimes H^{2m-2k-1}(M) \to \mathbb{R}$ of the form

$$p([\alpha], [\beta]) = \int_M \alpha \wedge \beta.$$

Define a skew-symmetric bilinear form $\langle -, - \rangle : H^{2k+1}(M) \otimes H^{2k+1}(M) \to \mathbb{R}$ via the formula

$$\langle [\alpha], [\gamma] \rangle = p\left([\alpha], L^{m-2k-1}[\gamma]\right),$$

for $[\alpha], [\gamma] \in H^{2k+1}(M)$. Since this form is non-degenerate and skew-symmetric, its domain $H^{2k+1}(M)$ must be even-dimensional, i.e. b_{2k+1} is even.

Line 3 in Table 1. We do not know any non-simply-connected (and, in particular, symplectically aspherical) examples to fill in this line.

Line 3 in Table 2. We use Theorem 2.2 with $n = 6$ and $G = \{e\}$. Then, for every b, the corresponding manifold $M(6, \{e\}, b)$ has even odd-degree Betti numbers and does not have the Hard Lefschetz property. Furthermore, all the Massey products in M vanish by 2.1.

Line 4 and 5 in Tables 1 and 2. We do not know any examples to fill in these lines.

Line 6 in Tables 1 and 2. This is impossible, see the argument concerning line 2.

Lines 7 and 8 in Table 1. Consider the Kodaira-Thurston manifold K [T]. Recall that this manifold is defined as a nilmanifold

$$K = N_3/\Gamma \times S^1,$$

where N_3 denotes the 3-dimensional nilpotent Lie group of triangular unipotent matrices and Γ denotes the lattice of such matrices with integer entries. One can check that the Chevalley–Eilenberg complex of the Lie algebra $\mathfrak{n}_3$ is of the form

$$(\Lambda(e_1, e_2, e_3), d), \ de_1 = de_2 = 0, \ de_3 = e_1e_2.$$

with $|e_i| = 1$. By 2.4(ii), the minimal model of any nilmanifold N/Γ is isomorphic to the Chevalley–Eilenberg complex of the Lie algebra $\mathfrak{n}$. In particular, one can get the minimal model of the Kodaira–Thurston manifold in the form

$$(\Lambda(x, e_1, e_2, e_3), d), \quad dx = de_1 = de_2 = 0, \ de_3 = e_1e_2$$

with degrees of all generators equal 1. One can check that the vector space $H^1(K)$ has the basis $\{[x], [e_1], [e_2]\}$. Hence, $b_1(K) = 3$, which also shows that K does not have the Hard Lefschetz property. Furthermore, K possesses a symplectic form ω with $[\omega] = [e_1e_3 + e_2x]$, and one can prove that the Massey triple product $\langle [e_1], [e_1], [\omega] \rangle$ is non-trivial, see e.g. [RT1]. Thus, K realizes Line 8 of Table 1.

Finally, $K \times K$ realizes Line 7 of Table 1.

Lines 7 and 8 in Table 2. We use Theorem 2.5. Consider a symplectic embedding $i : K \to \mathbb{C}P^m, m \geq 5$, and perform the blow-up along i. Then, by 2.5(i), $\widetilde{\mathbb{C}P^m}$ is simply-connected. Furthermore, it realizes Line 8 of Table 2 by 2.5(ii) and 2.5(iii).

Finally, $\widetilde{\mathbb{C}P^m} \times \widetilde{\mathbb{C}P^m}$ realizes Line 7 of Table 2. □

3.2. Remark. The result of Lupton [L] shows that the problem of constructing of a non-formal manifold with the Hard Lefschetz property turns our to be very delicate. In [L] there is an example of a DGA, whose cohomology has the Hard Lefschetz property, but which is not *intrinsically* formal. This means that there is also a *non-formal* minimal algebra with the same cohomology ring. Sometimes, using Browder–Novikov theory, one can construct a smooth closed manifold M with such non-formal Sullivan minimal model. However, there is no way in sight to get a symplectic structure on M.

3.3. Remark. It is interesting to consider a corresponding table for 4-dimensional manifolds. (Notice that the dimension 4 is the lowest interesting dimension, because every 2-dimensional closed symplectic manifold is Kähler by trivial reasons.) Moreover, in this case we have the following interesting question. For all 4-dimensional closed Kähler manifolds the following Bogomolov–Miyaoka–Yau inequality holds:

$$c_1^2(M) \leq 3c_2(M) \text{ provided } c_2(M) > 0,$$

see e.g. [BPV, Ch. VII, Theorem (4.1)]. It is well known that every symplectic manifold admits an almost complex structure which is unique up to homotopy. In particular, the Chern classes of symplectic manifolds are well-defined, and one can ask whether the Bogomolov–Miyaoka–Yau inequality holds for closed symplectic 4-manifolds, or how is it related with Hard Lefschetz property or formality, etc.

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