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A Variant of the Proof of the Landweber Stong Conjecture

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0. Let G be a finite group, k a finite field of order q, V an n - dimensional $k[G]$ - module and let $S[V]^G$ be the graded ring of invariants with respect to the action of G on the symmetric algebra of V (where V is given grading 2 if q is odd and grading 1 otherwise). $S[V]^G$ is a module over the Dickson algebra $D(V) := S[V]^{\mathrm{Aut}\,V}$ which is known to be a polynomial algebra on the Dickson generators $c_1, \cdots, c_n$ of degree $\epsilon(q^n - q^{n-i})$ with $\epsilon = 2$ if q is odd and $\epsilon = 1$ otherwise. The reader is referred to [W] for more information on the Dickson algebra. In [LSt] Landweber and Stong stated the following conjecture.

Landweber Stong Conjecture. *The depth of $S[V]^G$ is the maximal k such that $c_1, \cdots, c_k$ is a regular sequence on $S[V]^G$.*

We recall that a sequence $x_1, \cdots, x_n$ of elements of positive degree in $S[V]^G$ is called regular if multiplication by x_1 is injective on $S[V]^G$, and for $i > 1$ multiplication by x_i is injective on $S[V]^G/(x_1, \cdots, x_{i-1})$; the depth of $S[V]^G$ is defined to be the supremum of the lengths of all possible regular sequences.

The conjecture has recently been proved by Bourguiba and Zarati [BZ] in case $k = \mathbb{F}_p$ is a prime field. Their proof uses deep results from the theory of unstable modules over the mod - p Steenrod algebra A_p, most notably results on Lannes' T - functor and the related functor Fix. The purpose of this note is to give a variant of their proof which we believe to be a bit simpler (but which also depends heavily on properties of Lannes' T - functor). For background information on unstable modules and unstable algebras over the Steenrod algebra the reader is referred to [S]. Both proofs can be generalized to arbitrary finite fields and we will also make a few remarks on this at the end of the paper. Details of this generalization (in the case of the proof given in [BZ]) are given in [B].

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1. We fix a prime p and an unstable algebra K over A_p which is finitely generated as a graded $\mathbb{F}_p$ - algebra (we will say that K is an unstable noetherian algebra) and we are concerned with the category $K_{fg} - \mathcal{U}$ of unstable finitely generated K - modules. We recall that the objects of this category are unstable A_p - modules together with a map of unstable modules $K \otimes M \longrightarrow M$ which defines on M a graded K - module structure; furthermore M is assumed to have a finite number of K - module generators. The morphisms in $K_{fg} - \mathcal{U}$ are maps of graded $\mathbb{F}_p$ - vector spaces which commute both with the action of K and of A_p. For more on this category the reader may consult [H2].

Let $d = d(K)$ denote the transcendence degree of K: by this we mean the Krull dimension of K as a graded $\mathbb{F}_p$ - algebra, or equivalently the maximum of the cardinalities of all sets of algebraically independent homogeneous elements. We assume that there are elements $c_{i,K}, i = 1, \dots, d$, in K with the following property: there exists a natural number e such that for every homomorphism φ of unstable algebras from K to the mod - p cohomology H^*W of an elementary p - group W (i.e. $W \cong (\mathbb{Z}/p)^n$ for some n, called the rank of W and also denoted by $\operatorname{rk} W$) which makes H^*W into a finitely generated K - module we have

$$\varphi(c_{i,K}) = \begin{cases} (c_{i,W})^{p^{e+d-\operatorname{rk} W}} & \text{if } i \leq \operatorname{rk} W, \\ 0 & \text{if } i > \operatorname{rk} W. \end{cases}$$

Here $c_{i,W}$ is the i - th Dickson invariant in H^*W (of degree $2(p^{\operatorname{rk} W} - p^{\operatorname{rk} W - i})$ if p is odd resp. of degree $2^{\operatorname{rk} W} - 2^{\operatorname{rk} W - i}$ if $p = 2$). We note that the finite generation assumption on H^*W assures that $\operatorname{rk} W \leq d$.

In fact, for each unstable noetherian algebra K one can find such elements $c_{i,K}$ and they are unique up to the process of taking iterated p - th powers. This can be deduced from the work of Rector [R] and Lam [Lm1, Lm2] and was apparently known to Landweber/Stong and Wilkerson in the 1980's; it is also implicit in [HLS1]. For a detailed proof we refer to the appendix by J. Lannes in [BZ]. If $K = H^*W$ or $K = D(W^*)$ (with W^* denoting the dual of W) then the elements $c_{i,W}$ are well-known to qualify as elements c_{i,H^*W} and also as elements $c_{i,D(W^*)}$ (in both cases we have $e = 0$!).

After these preparations we can now state a theorem which in the case of prime fields generalizes the conjecture of Landweber and Stong. (Take $K = S[V]^{\mathrm{Aut}(V)}$ and $M = S[V]^G$!) As before we assume that all elements in regular sequences are homogeneous of positive degree and depth is defined in terms of such sequences.

THEOREM 1. *Let K be an unstable noetherian algebra of transcendence degree d and let M be an unstable finitely generated K- module. Then*

$$\textit{depth } M = \textit{max}\, \{0 \leq r \leq d \mid c_{1,K}, \dots, c_{rK} \;\; \textit{is a regular sequence on } M\}\,.$$

This result was proved recently by Bourguiba and Zarati [BZ]. Their proof as well as ours depends crucially on the progress in the theory of unstable A_p - modules within the last 10 years; in particular they depend heavily on the work of Lannes [Ln] and the earlier work of Lannes/Schwartz [LS] and Lannes/Zarati [LZ]. The first chapter of the recent book of Schwartz [S] provides more than adequate background on these prerequisites.

Beyond this we believe that the proof given here is more direct and represents a simplification over the original proof by Bourguiba and Zarati. It may therefore be of some interest.

2. We will follow Bourguiba and Zarati and call an unstable finitely generated K - module M for which the conclusion of the theorem holds an LS - module. The following result is the key-observation to the proof of [BZ] and also to our proof.

PROPOSITION 2. *Let K and M be as in the theorem. Then M can be embedded into an LS - module M' with* $\operatorname{depth} \ M' \geq \operatorname{depth} \ M$.

PROOF OF THEOREM 1. (assuming Proposition 2) It is enough to show that if depth $M \geq r$ then $c_{1,K}, \ldots, c_{r,K}$ is a regular sequence on M and we will do this by induction on r. The case $r = 0$ is trivial.

So assume depth $M = r > 0$ and consider the short exact sequence

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$$

where the embedding of M into M' is as in Proposition 2. Because depth can be characterized in terms of non-vanishing of certain Ext - groups ([M, Thm. 28, p.100f]) we obtain depth$M'' \geq r-1$, so $c_{1,K}, ..., c_{r-1,K}$ is a regular sequence for M'' by induction hypothesis, and hence M'' is free as a module over the polynomial subalgebra $\mathbb{F}_p[c_{1,K}, ..., c_{r-1,K}] \subset K$ ([E, Prop. 10.3.4]). Therefore the exact sequence splits if it is considered as a sequence of modules over this polynomial algebra and consequently we still have an injection after dividing out the sequence $c_{1,K}, ..., c_{r-1,K}$, i.e. $M/(c_{1,K}, ..., c_{r-1,K})$ injects into $M'/(c_{1,K}, ..., c_{r-1,K})$. Because M' is an LS - module with depth $\geq r$ the latter is torsion free with respect to $c_{r,K}$ and therefore the former is as well and we are done. □

3. We now turn towards the proof of Proposition 2 and we start by describing certain objects of $K_{fg} - \mathcal{U}$ which are easily seen to be LS - modules.

So let $\varphi : K \longrightarrow H^*W$ be a homomorphism of unstable algebras which makes H^*W into a finitely generated module over K and consider the corresponding "connected component" $T_W(K;\varphi)$ of the unstable algebra $T_W(K)$. We recall that T_W is a functor from the category $\mathcal{U}_p$ of unstable modules to itself which is left adjoint to tensoring with H^*W. In fact, T_W even lifts to a functor from unstable algebras to itself and continues to have the adjointness property in this context. In particular the adjoint of φ is a map $T_W K \longrightarrow \mathbb{F}_p$ of unstable algebras. Then $T_W(K;\varphi)$ is defined as the "component" $T_W K \otimes_{T_W^0 K} \mathbb{F}_p(\varphi)$ where T_W^0 denotes the homogeneous part of degree 0 of T_W and $\mathbb{F}_p(\varphi)$ denotes the $T_W^0 K$ - module $\mathbb{F}_p$ with module structure given by the adjoint of φ.

We will need to consider the map $\widetilde{\varphi} : K \longrightarrow H^*W \otimes T_W(K;\varphi)$ which is adjoint to the projection $T_W K \longrightarrow T_W(K;\varphi)$. This map has the form $\widetilde{\varphi} k = \varphi k \otimes 1 + y$ where y is a sume of terms which have positive degree in the second tensor variable. Now let F be an unstable $T_W(K;\varphi)$ - module which is finite as a graded vector space and consider $H^*W \otimes F$ as an unstable K - module via the K - module structure which is pulled back from the obvious $H^*W \otimes T_W(K;\varphi)$ - module structure via $\widetilde{\varphi}$. The proof of the following result is straightforward, say by filtering $H^*W \otimes F$ by the degree in the second tensor factor so that the associated graded object has its K - module structure given via φ only; we leave this to the reader.

Lemma 3. *The unstable K - module $H^*W \otimes F$ described above has the following properties.*

a) $H^*W \otimes F$ is finitely generated as K - module.

b) The sequence $c_{1,K}, \dots c_{\mathrm{rk}W,K}$ is a regular sequence on $H^*W \otimes F$. Furthermore $\mathrm{depth}(H^*W \otimes F) = \mathrm{rk}\ W$*; in particular $H^*W \otimes F$ is an LS - module.*

The following result is now the crucial input from the theory of unstable modules. A geodesic to its proof for somebody familiar with chapter I of [S] should be given by section 1 of [H1] and section 1 of [H2].

Theorem 4 ([HLS2, Thm I.4.9], [H2, Thm 1.9]). *For any unstable finitely generated K - module M there exists an embedding into a finite direct sum $\bigoplus_\alpha H^*W_\alpha \otimes F_\alpha$ where each summand is a module as in Lemma 3.*

Proof of Proposition 2. Assume that $\mathrm{depth}M = r$. We will take an embedding $M \hookrightarrow \bigoplus_\alpha H^*W_\alpha \otimes F_\alpha$ as provided by Theorem 4 and we will assume that this embedding is *reduced* in the sense that no summand can be deleted without losing the embedding property. It suffices to show that in such a reduced embedding all summands will have $\mathrm{rk}\ W_\alpha \geq r$.

Otherwise we would find $x \in M$ and some α with $\mathrm{rk}\ W_\alpha < r$ such that the unstable K - submodule M_x generated by x embedds into $H^*W_\alpha \otimes F_\alpha$. In particular, the order of the pole at $t = 1$ of the Poincaré series $\chi(M_x) := \Sigma_n \dim_{\mathbb{F}_p}(M_x)^n t^n$ is equal to $\mathrm{rk}W_\alpha < r$. On the other hand M has depth r, hence ([E, Prop. 10.3.4]) it is a free module over a polynomial algebra $\mathbb{F}_p[x_1, \dots x_r]$ whose generators x_i form a regular sequence on M and are of positive degree. M_x is a non - trivial submodule in M and hence $\chi(M_x)$ dominates from above $\chi(J)$ where J is a non-trivial principal ideal inside this polynomial algebra, namely the ideal generated by a non-trivial component of x if x is expressed as a linear combination of the basis elements of the free module M. Now a principal ideal over an integral domain is always free on its generator, hence the order of the pole at $t = 1$ of $\chi(J)$ is equal to r, i.e. the order of the pole at $t = 1$ of $\chi(M_x)$ is at least r and we arrive at a contradiction.$\square$

4. The proof of the Landweber-Stong-Conjecture in the case of an arbitrary field follows the same strategy as in the prime field case. In particular this involves finding appropriate definitions of the Steenrod algebra, the categories $K - \mathcal{U}$, the T - functor etc. in the case of an arbitrary finite field. Most of these concepts have been already developed by N. Kuhn in [K1, K2]. Once the basic definitions have been set up one can verify that the proof given here resp. in [BZ] can be carried over. More details on this can be found in [B].

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