Combinations of rational double points on the deformation of quadrilateral singularities III

Tohsuke Urabe

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
5300 Bonn 3

MPI/89-1

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§0. Introduction

In this Part III we would like to study the hypersurface quadrilateral singularity $W_{1,0}$. In addition to two kinds of transformations of Dynkin graphs – an elementary transformation and a tie transformation– (Urabe [2], [3], [4]), the notion of obstruction components (Urabe [4]) plays an essential role. We give a proof of following Main Theorem, which have been announced in Part I (Urabe [4]). Every algebraic variety is assumed to be defined over the complex number field **C**. As for the exact definition of Dynkin graphs, see Part I.

We study a set of Dynkin graphs $PC = PC(W_{1,0})$ in this article. Recall that a Dynkin graph G with components of type A, D, or E only belongs to PC if and only if there exists a fiber Y in the semi-universal deformation family of a singularity of type $W_{1,0}$ satisfying the following two conditions depending on G.

(1) The fiber Y has only rational double points as singularities.

(2) The combination of rational double points on Y just corresponds to the graph G.

Main Theorem. *A Dynkin graph G belongs to $PC(W_{1,0})$ if and only if G can be made from one of the following essential basic Dynkin graphs with distinguished obstruction components by elementary or tie transformations applied 2 times (We can apply 2 different kinds of transformations once for each, or can apply 2 transformations of the same kind.), and G contains no vertex corresponding to a short root and no obstruction component.*

The essential basic Dynkin graphs:
$E_8 + B_1 + G_2, \quad E_7 + B_3 + G_1, \quad B_9 + G_2, \quad A_{11}$
(The component A_{11} is the obstruction component.)

Recall that by results in Part I that the "if" part under the condition (2) is true. Thus in this Part III we show the "only if" part.

Let Λ_3 denote the even unimodular lattice of signature (19, 3), and P be the lattice associated with $W_{1,0}$. (See Part I.) P has rank 7. Let $Q(G)$ be the root lattice associated with a Dynkin graph G with components of type A, D or E only. By the results in Part I, the proof of the "only if" part can be reduced to showing only the following.

Proposition 0.1. *Assume that $G \in PC(W_{1,0})$. Then, with respect to some full embedding $Q(G) \hookrightarrow \Lambda_3/P$ without an obstruction component A_{11}, there exists a primitive isotropic element u in Λ_3/P in a nice position, i.e., such that either u is orthogonal to $Q(G)$, or there is a root basis $\Delta \subset Q(G)$ and a long root $\alpha \in \Delta$ such that $\beta \cdot u = 0$ for every $\beta \in \Delta$ with $\beta \neq \alpha$ and $\alpha \cdot u = 1$.*

To show this proposition we use the theories developed in Part II (Urabe [5]).

Now, if $G \in PC$, then we have an embedding $S = P \oplus Q(G) \hookrightarrow \Lambda_3$ come from an actual deformation fiber Y. The embedding satisfies Looijenga's conditions $\langle a \rangle$ and $\langle b \rangle$ and the induced embedding $Q(G) \hookrightarrow \Lambda_3/P$ is full.

Also we have an elliptic K3 surface $\Phi : Z \to C$ corresponding to the embedding. By $\Sigma = \{c_1, \dots, c_t\}$ we denote the set of critical values of Φ. By F_i with $1 \le i \le t$ we denote the singular fiber $\Phi^{-1}(c_i)$. The elliptic surface $Z \to C$ has a singular fiber F_1 of type I_0^* in our situation by definition. Moreover it has two sections $s_0, s_1 : C \to Z$ whose images $C_5 = s_0(C)$ and $C_6 = s_1(C)$ are disjoint. The union $IF = F_1 \cup C_5 \cup C_6$ is called the curve at infinity. The lattice P has signature $(6, 1)$ and it is defined associated with the dual graph of components of IF. The union $\mathcal{E}$ of smooth rational curves on Z not intersecting IF coincides with the union of components not intersecting IF of singular fibers of $Z \to C$. The dual graph of $\mathcal{E}$ is G by definition.

We use the same division of the case into three subcases as in Part II.

((1)) The surface $Z \to C$ has another singular fiber of type I^* apart from F_1.

((2)) $Z \to C$ has a singular fiber of type II^*, III^* or IV^*.

((3)) $Z \to C$ has no singular fiber of type I^*, II^*, III^* or IV^* apart from F_1.

For each case we apply the theory in Part II. However, in the case $W_{1,0}$ IF contains 2 of images of sections, and the theories in Part II are not sufficient to treat $W_{1,0}$. Thus in the first step of the proof we write down the list of possible Dynkin graphs, and then we check each item G in the list case by case. We show either G can be made from one of the basic graphs by two transformations, or $G \notin PC$. To show $G \notin PC$ we apply the theory of symmetric bilinear forms, the theory of elliptic surfaces, and the theory of K3 surfaces, etc.

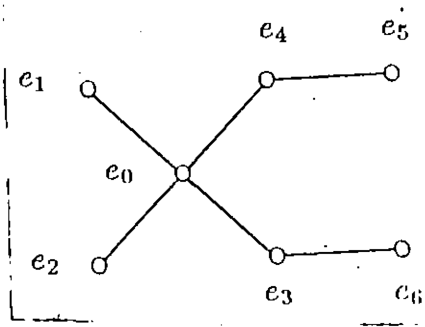
The case ((i)) is discussed in section i.

Assume that a Dynkin graph G with components of type A, D or E only can be made from one of the essential basic graphs by elementary or tie transformations applied twice and G has no obstruction component. Then we can construct a full embedding $Q(G) \hookrightarrow \Lambda_3/P$ without an obstruction component of type A_{11} which has a primitive isotropic element in a nice position. This is a consequence of the theories in [2], [3], [4]. (See Theorem 1.1 in [3], Theorem 4.4 etc. in [4].) Of course, the constructed embedding may not be equivalent to the given embedding. However, we can use this to show Proposition 0.1 without any problem. Note moreover that under the assumption we have $G \in PC(X)$ by the "if" part of Main Theorem.

§1. Two singular fibers of type I^*

By G we denote a Dynkin graph belonging to $PC(W_{1,0})$ with the number of vertices r. By $\Phi : Z \to C$ we denote the corresponding elliptic K3 surface. We assume that apart from the singular fiber F_1 of type I_0^*, F_2 is of type I^* in this section.

We have an embedding $S = P \oplus Q(G) \hookrightarrow \Lambda_3$ satisfying $\langle a \rangle$ and $\langle b \rangle$. Recall that the lattice P has a basis $e_0, e_1, \dots, e_6$ whose mutual intersection numbers are described by the dual graph in the figure associated with the curve at infinity IF.



The bilinear form is denoted by a dot $\cdot$. We set

$$u_0 = 2e_0 + e_1 + e_2 + e_3 + e_4, \quad v_0 = -u_0 - e_5, \quad f = e_6 - e_5 - 2u_0.$$

$u_0^2 = v_0^2 = 0$, $u_0 \cdot v_0 = 1$, $P = P' \oplus (\mathbf{Z}u_0 + \mathbf{Z}v_0)$, and P' has a basis e_0, e_1, e_2, e_3, f.

By Proposition 1.2 in Part II there exists an element $u \in \Lambda_3$ satisfying the following conditions.

(1) u is isotropic.

(2) u is orthogonal to $Q(G)$.

(3) u is orthogonal to e_0, e_1, e_2, e_3, e_4, and e_5.

(4) $u \cdot e_6 \geq 0$.

Set $m = u \cdot e_6 = u \cdot f$. If $m = 0$, then the lattice S has an isotropic element in the orthogonal complement in Λ_3, and it is in a nice position. Thus in the following we consider the case $m \neq 0$. The element u is orthogonal to $\mathbf{Z}u_0 + \mathbf{Z}v_0$. Set $M = P' + \mathbf{Z}u$. M has signature $(5,1)$.

Lemma 1.1. *Assume that M is not primitive in Λ_3 and $u \cdot e_6 \neq 0$. Then, the primitive hull $\widetilde{M}$ of M in Λ_3 contains an element u' also satisfying the above conditions (1)-(4) and such that $u \cdot e_6 > u' \cdot e_6 > 0$.*

Proof. By $w_0, \ldots, w_3, z$ we denote the dual basis of $e_0, \ldots, e_3, f$. In particular $z = (2e_0 + e_1 + e_2 + 2e_3 + 2f)/6$. Set $\xi = m(2e_0 + e_1 + e_2 + 2e_3 + 2f) - 6u$. One knows that $\xi \in M$, $\xi \cdot e_i = 0$ $(0 \leq i \leq 3)$, $\xi \cdot f = 0$, $\xi \cdot u = 2m^2$ and $\xi^2 = -12m^2$. The element $y_0 = \xi/12m^2$ satisfies $y_0 \cdot \xi = -1$ and $u = mz - 2m^2 y_0 = m(z - 2my_0)$. Set $N = P' \oplus \mathbf{Z}\xi$. We have $N \subset M \subset \widetilde{M} \subset \widetilde{M}^* \subset M^* \subset N^*$. (Recall that by * we denote the dual module.) Consider the discriminant group $N^*/N = P'^*/P' \oplus \mathbf{Z}y_0/\mathbf{Z}\xi$. On this group we can define the discriminant bilinear form b and the discriminant quadratic form q. For $x \in N^*$ we denote $\overline{x} = x \bmod N^* \in N^*/N$. Set $I = M/N$. $I^\perp = M^*/N$ is the orthogonal complement of I with respect to b. I is generated by a unique element $\overline{u} = m\overline{z} - 2m^2\overline{y}_0$. On the other hand P'^*/P' is a cyclic group of order 12 generated by $\overline{w}_1$, and $\overline{z} = 2\overline{w}_1$. (Recall $w_1^2 = 13/12$ and thus $\overline{w}_1^2 \equiv 13/12 \bmod 2\mathbf{Z}$). Since

$$b(2m\overline{w}_1 - 2m^2\overline{y}_0, a\overline{w}_1 + b\overline{y}_0) \equiv \frac{1}{6}ma + \frac{1}{6}b \bmod \mathbf{Z},$$

$a\overline{w}_1 + b\overline{y}_0 \in I^\perp \Leftrightarrow ma + b \equiv 0 \pmod 6$. Choose an element $\overline{x}_0 \in \widetilde{M}/N$ with $\overline{x}_0 \notin I = M/N$. Since it is contained in $I^\perp$, we can write $\overline{x}_0 = a(\overline{w}_1 - m\overline{y}_0) + 6c\overline{y}_0$. Moreover,

$$0 \equiv q(\overline{x}_0) \equiv a^2 + \frac{acm - 3c^2}{m^2} \bmod 2\mathbf{Z}.$$

In particular $c^2 \equiv am(am + c) \pmod 2$, and c is even. We set $c = 2d$.

First we would like to show that there is $\overline{x}_1 \in \widetilde{M}/N$ with $\overline{x}_1 \notin I$ in the form either $\overline{x}_1 = 2A(\overline{w}_1 - m\overline{y}_0)$ (In this case no restriction on m.), or $\overline{x}_1 = 2A(\overline{w}_1 + m\overline{y}_0)$ and $m \equiv 0 \pmod 3$.

◇ Case 1. $am - 12d \not\equiv 0 \pmod{2m}$.

Set $\overline{x}_1 = m\overline{x}_0$. Since $12m\overline{y}_0 = -12(\overline{w}_1 - m\overline{y}_0)$, we have $\overline{x}_1 = am(\overline{w}_1 - m\overline{y}_0) + d(12m\overline{y}_0) = a'(\overline{w}_1 - m\overline{y}_0)$ for $a' = am - 12d$. The assumption $a' \not\equiv 0 \pmod{2m}$

implies $\overline{x}_1 \notin I$. Since $0 \equiv q(\overline{x}_1) \equiv a'^2 \bmod 2\mathbf{Z}$, we can write $a' = 2A$. This $\overline{x}_1$ satisfies the condition.

◇ Case 2. $am - 12d \equiv 0 \pmod{2m}$.

Set $12d = am + 2me$. $\overline{x}_0 = a\overline{w}_1 + 2me\overline{y}_0$. Then $q(\overline{x}_0) \equiv 0 \bmod 2\mathbf{Z} \Leftrightarrow 13a^2 - 4e^2 \equiv 0 \pmod{24}$. We can write $a = 2A'$ and we have $(A'+e)(A'-e) \equiv 0 \pmod 6$.

If $A' + e \equiv 0 \pmod 3$, set $3C = A' + e$. We have $C^2 \equiv C(2A' - 3C) \equiv 0 \pmod 2$. C is even. Thus $\overline{x}_1 = \overline{x}_0 = 2A'(\overline{w}_1 - m\overline{y}_0) + 6mC\overline{y}_0 = 2A(\overline{w}_1 - m\overline{y}_0)$ for $A = A' - 3C$.

If $A' - e \equiv 0 \pmod 3$, set $3C = -A' + e$. We have $C^2 \equiv 0 \pmod 2$. C is even. Thus $\overline{x}_1 = \overline{x}_0 = 2A'(\overline{w}_1 + m\overline{y}_0) + 6mC\overline{y}_0 = 2A(\overline{w}_1 + m\overline{y}_0)$ for $A = A' + 3C$. If $m \equiv 0 \pmod 3$ we are done. Thus we can assume $m \not\equiv 0 \pmod 3$. Since $\overline{x}_1 \in I^\perp$, $Am \equiv 0 \pmod 3$, and thus $A \equiv 0 \pmod 3$. Then $\overline{x}_1 = 2A(-\overline{w}_1 + m\overline{y}_0) = 2(-A)(\overline{w}_1 - m\overline{y}_0)$, since $6\overline{w}_1 = -6\overline{w}_1$.

Next, we consider the case $\overline{x}_1 = 2A(\overline{w}_1 - m\overline{y}_0)$. If we write $A = Cm + B$ $(0 \le B < m)$, then $B \neq 0$, since $\overline{x}_1 \notin I$. The element $\overline{x}_2 = \overline{x}_1 - 2Cm(\overline{w}_1 - m\overline{y}_0) = 2B(\overline{w}_1 - m\overline{y}_0)$ satisfies $\overline{x}_2 \in \widetilde{M/N}$ and $\overline{x}_2 \notin I$.

On the other hand we can check that $u' = Bu/m$ is an element in N^*, $u'^2 = 0, u' \neq 0$, $u' \cdot e_i = 0$ for $0 \le i \le 5$ and $m = u \cdot e_6 > B = u' \cdot e_6$. Moreover $u' \in \widetilde{M}$ since $\overline{u}' = \overline{x}_2$. We have the desired element.

The case $\overline{x}_1 = 2A(\overline{w}_1 + m\overline{y}_0)$, $m \equiv 0 \pmod 3$ is remaining. In this case $2m\overline{w}_1 = -2m\overline{w}_1$ since $\overline{w}_1$ has order 12. If A is a multiple of m, then $\overline{x}_1 = 2A(\overline{w}_1 + m\overline{y}_0) = (-A)\cdot 2(\overline{w}_1 - m\overline{y}_0) \in I$, which is a contradiction. Thus we can write $A = Cm + B$ $(0 < B < m)$. Setting $\overline{x}_2 = 2B(\overline{w}_1 + m\overline{y}_0) = \overline{x}_1 - 2Cm(\overline{w}_1 + m\overline{y}_0) = \overline{x}_1 + 2Cm(\overline{w}_1 - m\overline{y}_0)$, we have $\overline{x}_2 \in \widetilde{M/N}$, $\overline{x}_2 \notin I$.

On the other hand setting $u'' = B(z + 2my_0) = B(2z - (u/m))$, we can check $u''^2 = 0$, $u'' \neq 0$, $u'' \cdot e_i = 0$ $(0 \le i \le 5)$, $u'' \cdot u_0 = u'' \cdot v_0 = 0$, and $u'' \cdot f = u'' \cdot e_6 = B$. Since $u'' \in N^*$ and since $\overline{u}'' = \overline{x}_1$, one knows $u'' \in \widetilde{M}$. This u'' is the desired element.

Q.E.D.

By induction on $u \cdot e_6$ we can assume that u satisfies the following (5) in addition to (1)–(4).

(5) $M = P' + \mathbf{Z}u$ is primitive in Λ_3.

Then, of course, $M \oplus (\mathbf{Z}u_0 + \mathbf{Z}v_0) = P + \mathbf{Z}u$ is also primitive in Λ_3.

Proposition 1.2. *Assume that we have an element $u \in \Lambda_3$ satisfying above (1)–(4) and (5).*

1. *If $u \cdot e_6 = 0$, then the orthogonal complement of $P \oplus Q(G)$ contains an isotropic element.*
2. *If $u \cdot e_6 = 1$, then G is a subgraph of the Coxeter-Vinberg graph Γ of the lattice $Q(D_{12}) \oplus H$ (H denotes a hyperbolic plane.).*
3. *If $u \cdot e_6 \ge 2$, then G can be obtained from a subgraph of the above Γ by one elementary transformation.*

Proof. 1 is obvious.

2. Setting $v = f - 2u$, one has

$$M = (\mathbf{Z}e_0 + \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}(e_3 - u)) \oplus (\mathbf{Z}u + \mathbf{Z}v),$$

and $u^2 = v^2 = 0$, $u \cdot v = 1$. Thus $P + \mathbf{Z}u \cong Q(D_4) \oplus H \oplus H$. For their discriminant quadratic forms we know $q_{P+\mathbf{Z}u} = q_{Q(D_4)}$.

Let L be the orthogonal complement of $P + \mathbf{Z}u$ in Λ_3. L has signature (13, 1). The discriminant quadratic form of L is $-q_{P+\mathbf{Z}u} = -q_{Q(D_4)} = q_{Q(D_4)}$. Since $Q(D_{12}) \oplus H$ and L have the same signature and the same discriminant quadratic form, they are isomorphic by Nikulin [1] Theorem 1.14.2 and Corollary 1.9.4. In particular the Coxeter-Vinberg graph of L coincides with Γ. Since $Q(G)$ is full in L, G is a subgraph of Γ.

3. Assume that $m = u \cdot e_6 \geq 2$.

By L we denote the orthogonal complement of $R = P + \mathbf{Z}u$ in Λ_3. We have a natural isomorphism $R^*/R \cong L^*/L$ which preserves discriminant quadratic forms up to sign.

Set $u_1 = u/m$ and $R_1 = R + \mathbf{Z}u_1$. R_1 is an even overlattice of R with index m, and is isomorphic to $P + \mathbf{Z}u$ in the case of $m = u \cdot e_6 = 1$. In particular the discriminant quadratic form of R_1 is the same as that of $Q(D_4)$.

By the above isomorphism one knows that L has an overlattice L_1 with index m whose discriminant quadratic form is $q_{Q(D_4)} = -q_{Q(D_4)}$. By reasoning in **2** $L_1 \cong Q(D_{12}) \oplus H$.

Let $\widetilde{Q}_1$ (resp. $\widetilde{Q}$) be the primitive hull of $Q(G)$ in L_1 (resp. L). The Dynkin graph of $\widetilde{Q}_1$ is a subgraph of the Coxeter-Vinberg graph Γ of L_1. Since $\widetilde{Q}_1/\widetilde{Q} \subset L_1/L \cong \mathbf{Z}/m$ is cyclic, the Dynkin graph of $\widetilde{Q}$ is obtained from that of $\widetilde{Q}_1$ by one elementry transformation. Besides, by the fullness the Dynkin graph of $\widetilde{Q}$ is G. Q.E.D.

Now, we would like to draw the Coxeter-Vinberg graph Γ of $Q(D_{12}) \oplus H$.

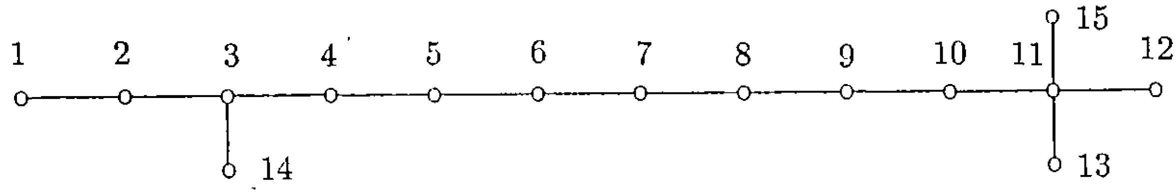
Set $K = \sum_{i=0}^{13} \mathbf{Z}v_i$ where $v_0, \dots, v_{13}$ is a free basis with $v_0^2 = -1, v_i^2 = 1$ $(1 \leq i \leq 13), v_i \cdot v_j = 0$ $(i \neq j)$. The sublattice $L = \left\{\sum_{i=0}^{13} x_i v_i \in K \mid \sum_{i=0}^{13} x_i \in 2\mathbf{Z}\right\}$ is isomorphic to $Q(D_{12}) \oplus H$. We use v_0 as the controlling vector. We can take

$$\begin{aligned} \gamma_i &= -v_i + v_{i+1} \ (1 \leq i \leq 12) \\ \gamma_{13} &= -(v_{12} + v_{13}) \end{aligned}$$

as a root basis for the orthogonal complement of v_0 in L. By Vinberg's algorithm we get succeedingly;

$$\begin{aligned} \gamma_{14} &= v_0 + v_1 + v_2 + v_3 \\ \gamma_{15} &= 3v_0 + v_1 + v_2 + \cdots + v_{11}. \end{aligned}$$

Drawing the graph for these 15 vectors, we get the following graph. This has no dotted edges, no Lannér subgraph and any extended Dynkin subgraph is a component of an extended Dynkin subgraph of rank 12. Thus this is Γ.



The Coxeter-Vinberg graph Γ for $Q(D_{12}) \oplus H$

Lemma 1.3. *There are 26 kinds of maximal Dynkin subgraphs of* Γ *with 13 vertices. The following is the list.*

(1) D_{13}	(2) $D_{12} + A_1$	(3) $D_{10} + A_2 + A_1$
(4) $D_9 + A_4$	(5) $D_8 + D_5$	(6) $D_7 + E_6$
(7) $E_7 + D_6$	(8) $E_8 + D_5$	(9) $A_{10} + 3A_1$
(10) $D_{10} + 3A_1$	(11) $A_9 + 4A_1$	(12) $A_7 + A_2 + 4A_1$
(13) $A_6 + A_4 + 3A_1$	(14) $D_5 + A_5 + 3A_1$	(15) $E_6 + A_4 + 3A_1$
(16) $E_7 + A_3 + 3A_1$	(17) $E_8 + A_2 + 3A_1$	(18) $A_9 + D_4$
(19) $D_9 + D_4$	(20) $A_8 + D_4 + A_1$	(21) $A_6 + D_4 + A_2 + A_1$
(22) $A_5 + A_4 + D_4$	(23) $D_5 + D_4 + A_4$	(24) $E_6 + D_4 + A_3$
(25) $E_7 + D_4 + A_2$	(26) $E_8 + D_4 + A_1$	

We consider the following two conditions here.

(1) The Picard number $\rho = 7 + r$.

(2) The orthogonal complement of $\mathrm{Pic}(Z)$ in $H^2(Z, \mathbf{Z})$ does not contain an isotropic element.

Note that $7 + r = \mathrm{rank}\,(P \oplus Q(G))$ and it is also equal to the rank of the subgroup S of $\mathrm{Pic}(Z)$ generated by classes of components of IF and all components of singular fibers not intersecting IF. In particular, if $r = 13$, then the above (1) and (2) are automatically satisfied. Moreover, by Theorem 1.2 in Part I (Urabe [4]) we can always assume the condition (1). It implies that $\mathrm{Pic}(Z)$ is the primitive hull of S. On the other hand, under the condition (1), the condition (2) is not satisfied if and only if the arithmetic condition in Part I Theorem 0.5 [II](A) holds. By Part I Theorem 0.5 we have nothing to verify, if (2) is not satisfied.

Thus we assume the above (1) and (2) in the following.

Lemma 1.4. *Under the above assumptions,* G *has a component of type* D*, has 11, 12, or 13 vertices, and can be obtained from one of the 26 Dynkin graphs in Lemma 1.3 by one elementary transformation. Besides, the group* E *of sections of* Φ *has rank 1 and only one component of the singular fiber* F_i *intersects* IF *for every* $2 \le i \le t$.

Proof. For $2 \le i \le t$ by $n(F_i)$ we denote the number of components of F_i not intersecting the curve IF at infinity. The equality $\sum_{i=2}^{t} n(F_i) = r$ holds. Thus by the equality (2) in Part II section 1 and by assumption we have

$$1 + \sum_{i=2}^{t} n(F_i) = a + \sum_{i=2}^{t} (m(F_i) - 1),$$

since $m(F_1) = 5$.

Assume $a = 0$. Then all elements in E have finite order. In particular the element u in Part II Proposition 1.2 is orthogonal also to the class of $C_6 = s_1(C)$. Thus u is orthogonal to the subgroup S generated by the classes in the union of the set of components of IF and the set of components not intersecting IF of singular fibers. On the other hand by (1) rank S = rank Pic, and Pic is the primitive hull of S. Consequently u is orthogonal to Pic. This contradicts the assumption (2). Thus $a > 0$.

Since $n(F_i) \leq m(F_i)-1$, by the above equality we have $a = 1$ and $n(F_i) = m(F_i)-1$ for $2 \leq i \leq t$. In particular only one component of F_2 intersects IF. Note that the intersecting component has multiplicity 1. The dual draph of components of F_2 without intersection with IF is of type D and it is a component of G. By Proposition 1.2 and Lemma 1.3 we have the characterization of G in terms of elementary transformations.

Since $\rho \leq 20$, $r \leq 13$. If $r \leq 10$, then the orthogonal complement of Pic in H^2 is an indefinite lattice with rank≥ 5. In this case by Meyer's theorem it contains an isotripic element. Q.E.D.

For every singular fiber F_i with $2 \leq i \leq t$, let G_i be the Dynkin graph defined as the dual graph of components of F_i not intersecting C_5. Note that by Lemma 1.4 $G = \sum G_i$.

Lemma 1.5. *(1)* $\nu(A) + 2\nu(D) + 2\nu(E) \leq 18 - r$, *where* $\nu(T)$ *denotes the number of components of* G *of type* T.
(2) If $r = 13$, *then* G *has no component of type* D_4.

Proof. (1) In our situation we can substitute $\rho = 7 + r$, and $a = 1$ into the equality (3) in the beginning of section 1 in Part II. Moreover, by the note just above one has $t - t_1 = 1 + \nu(D) + \nu(E) + \nu(II) + \nu(III) + \nu(IV)$, $t_1 = \nu(A) - \nu(III) - \nu(IV) + \nu(I_1)$. The inequality (1) follows from these ones.
(2) First assume that the functional invariant J is constant. Then $t_1 = 0$ in the equality (3) section 1 Part II, since J has never poles. Since $\rho = 20$, and $a = 1$ under our assumptions, we have $2t = 7$, which is a contradiction. Thus J is not constant. We can apply the inequality (4) in the beginning of Part II section 1. We have the claim, since F_1 is of type I_0^* and $\nu(I_0^*) \geq 1$. Q.E.D.

Lemma 1.6. *(1)* $Q(G)$ *is primitive in* Λ_3.
(2) Let $\widetilde{S}$ *be the primitive hull of* $S = P \oplus Q(G)$ *in* Λ_3. *Then, the restriction to* $\widetilde{S}/S$ *of the projection* $S^*/S = P^*/P \oplus Q(G)^*/Q(G) \to P^*/P$ *is injective.*
(3) $(\widetilde{S}/S)_p = 0$ *for any prime* $p \geq 5$, *where* M_p *denotes the p-Sylow subgroup of a finite abelian group* M.
(4) $l((\widetilde{S}/S)_p) \leq 1$ *for* $p = 2, 3$, *where* $l(M)$ *denotes the minimum number of generators of an abelian group* M.

Proof. (1) Let $\widetilde{Q}(G)$ be the primitive hull of $Q(G)$ in Λ_3. We will deduce a contradiction, assuming that $I_Q = \widetilde{Q}(G)/Q(G) \neq 0$.

Let $\overline{P}$ be the sublattice of rank 6 in P generated by $e_0, e_1, \ldots, e_5$. Set $\overline{S} = \overline{P} \oplus Q(G)$ and $\widetilde{\overline{S}}$ be the primitive hull of $\overline{S}$ in Λ_3. Note that $J = \widetilde{\overline{S}}/\overline{S}$ can be identified with the group of sections of finite order. Since $I_Q \subset J$, we have an image C' of a section corresponding to a non-zero element $\overline{\alpha}$ in I_Q. In $J = \overline{P}^*/\overline{P} \oplus Q(G)^*/Q(G)$, $\overline{\alpha}$ is

contained in the direct summand $Q(G)^*/Q(G)$. It implies that C' and C_5 intersect F_1 on the same component. Thus $C' = C_5$ since the homomorphism from $\widetilde{\overline{S}}/\overline{S}$ to the group $F_1^{\#}$ of the singular fiber F_1 is injective, and every component of F_1 contains at most one point of finite order. We have $\overline{\alpha} = 0$, a contradiction.
(2) If it is not injective, then $(\widetilde{S}/S) \cap (Q(G)^*/Q(G)) = \widetilde{Q}(G)/Q(G)$ is not zero, which contradicts (1).
(3), (4) By (2) $I = \widetilde{S}/S$ is isomorphic to a subgroup of $P^*/P \cong \mathbf{Z}/12$. Since $l(I_p) \leq l((P^*/P)_p)$, $l(I_p) \leq 1$ if $p = 2$ or 3, and it is zero if $p \geq 5$. Q.E.D.

From here for a while we consider the case where G has 13 vertices. We can easily list up all the Dynkin graphs which can be made from each one of the above 26 graphs by one elementary transformation.

(1) D_{13}

(1.1) D_{13} (1.2) $D_{11} + 2A_1$ (1.3) $D_{10} + A_3$ (1.4) $D_9 + D_4$
(1.5) $D_8 + D_5$ (1.6) $D_7 + D_6$

Among these 6 graphs (1.1) and (1.5) can be made from an essential basic Dynkin graph A_{11} by tie transformations repeated twice. (By the first transformation we can make D_{12}.) Thus these graphs belong to $PC = PC(W_{1,0})$.

On the other hand (1.4) can never be realized by Lemma 1.5. We can show that the remaining three graphs (1.2), (1.3), and (1.6) do *not* belong to PC by considering discriminant quadratic forms.

Consider the case (1.2). Let $G = D_{11} + 2A_1$. Assume that we have an embedding $S = P \oplus Q(G) \hookrightarrow \Lambda_3$ satisfying Looijenga's condition $\langle a \rangle$ and $\langle b \rangle$. By $\widetilde{S}$ we denote the primitive hull of S in Λ_3, and by T the orthogonal complement of S in Λ_3. The discriminant group of S can be written $S^*/S \cong \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/3$, since $P^*/P \cong \mathbf{Z}/4 \oplus \mathbf{Z}/3$, $Q(D_{11})^*/Q(D_{11}) \cong \mathbf{Z}/4$, and $Q(A_1)^*/Q(A_1) \cong \mathbf{Z}/2$. The discriminant quadratic form can be written

$$q(a, b, c_1, c_2, x) \equiv -\frac{1}{4}a^2 + \frac{3}{4}b^2 + \frac{1}{2}(c_1^2 + c_2^2) - \frac{2}{3}x^2 \bmod 2\mathbf{Z}$$

for an element $(a, b, c_1, c_2, x) \in (\mathbf{Z}/4)^2 \oplus (\mathbf{Z}/2)^2 \oplus (\mathbf{Z}/3)$ of the discriminant group.

$l((S^*/S)_2) = 4$. Since $\widetilde{S}^*/\widetilde{S} \cong T^*/T$, $2 = \operatorname{rank} T \geq l(T^*/T) = l(\widetilde{S}^*/\widetilde{S}) \geq l((\widetilde{S}^*/\widetilde{S})_2)$. Thus we have an element $\overline{\alpha} = (a, b, c_1, c_2, 0) \in (\widetilde{S}/S)_2$ with $\overline{\alpha} \neq 0$. It satisfies $q(\overline{\alpha}) \equiv 0 \bmod 2\mathbf{Z}$.

By solving the congruence equation, one knows that $\overline{\alpha} = (2,2,0,0,0), (2,0,1,1,0)$ or $(0,2,1,1,0)$.

If $\overline{\alpha} = (2,2,0,0,0)$, $\widetilde{S}$ contains an element in the form $\alpha = \xi + \omega$ with $\xi \in P^*$, $\omega \in Q(D_{11})^*$ and $\omega^2 = 1$. The image of α by the quotient morphism $\Lambda_3 \to \Lambda_3/P$ has self-intersection number 1 ($= \omega^2$), and it is a short root. This short root is contained in the primitive hull of $Q(G)$ in Λ_3/P. It contradicts the fullness of $Q(G)$.

In the case $\overline{\alpha} = (2,0,1,1,0)$, we can also conclude that the primitive hull of $Q(G)$ contains a short root, which is a contradiction.

When $\overline{\alpha} = (0,2,1,1,0)$, we have an element $\alpha \in \widetilde{S}$ with $\alpha^2 = 2$ such that $\alpha \notin S$, which also contradicts the fullness.

Thus we can conclude $D_{11} + 2A_1 \notin PC$.

For the cases (1.3) and (1.6) we can deduce a contradiction by the same argument as in (1.2).

(2) $D_{12} + A_1$

(2.1) $D_{12} + A_1$ (2.2) $D_{10} + 3A_1$ (2.3) $D_9 + A_3 + A_1$ (2.4) $D_8 + D_4 + A_1$
(2.5) $D_7 + D_5 + A_1$ (2.6) $2D_6 + A_1$

For the case (2.1) $G = D_{12} + A_1$ we can make it from a basic graph A_{11}. Thus it belongs to $PC = PC(W_{1,0})$.

For the other 5 cases the Dynkin graph G is not an element of PC. For the case (2.4) it follows from Lemma 1.5. For the cases (2.3) and (2.5) by essentially same arguments as in the case (1.2) we can show it. (In these cases that we can choose an element of order 2 as the corresponging element to $\overline{\alpha}$ in the above. Under this note the argument becomes simpler.)

We can apply Lemma 1.6 to the cases (2.2) and (2.6).

We consider case (2.2). Set $G = D_{10} + 3A_1$. By S, $\widetilde{S}$, T we denote the same lattice as above. Now, we have $(S^*/S)_2 \cong \mathbf{Z}/4 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2$. Note that $2 \geq l((\widetilde{S}^*/\widetilde{S})_2) \geq l((S^*/S)_2) - 2l((\widetilde{S}/S)_2) = 6 - 2l((\widetilde{S}/S)_2)$. We have $l((\widetilde{S}/S)_2) \geq 2$, which contradicts Lemma 1.6.

The case (2.6) is similar.

(3) $D_{10} + A_2 + A_1$

(3.1) $D_{10} + A_2 + A_1$ (3.2) $D_8 + A_2 + 3A_1$ (3.3) $D_7 + A_3 + A_2 + A_1$
(3.4) $D_6 + D_4 + A_2 + A_1$ (3.5) $2D_5 + A_2 + A_1$

Among these graphs we can make (3.1) $D_{10} + A_2 + A_1$ from A_{11}. Thus it belongs to PC. The graphs (3.2), (3.4), (3.5) cannot be elements in PC because of Lemma 1.5. The graph (3.3) does not belong to PC, either. If it is in PC, we can show an extra root contradicting the fullness as in case (1.2).

(4) $D_9 + A_4$

(4.1) $D_9 + A_4$ (4.2) $D_7 + A_4 + 2A_1$ (4.3) $D_6 + A_4 + A_3$ (4.4) $D_5 + D_4 + A_4$

The graph (4.1) can be made from A_{11}. Thus it is in PC. The graph (4.4) cannot be in PC because of Lemma 1.5. For the remaining two cases (4.2) and (4.3), we can show an extra root contradicting the fullness by calculation on the discriminant group as in case (1.2), if they are in PC.

(5) $D_8 + D_5$

(5.1) $D_8 + D_5$ (5.2) $D_6 + D_5 + 2A_1$ (5.3) $2D_5 + A_3$
(5.4) $2D_4 + D_5$ (5.5) $D_8 + A_3 + 2A_1$ (5.6) $D_6 + A_3 + 4A_1$
(5.7) $D_5 + 2A_3 + 2A_1$ (5.8) $2D_4 + A_3 + 2A_1$

The graph (5.1) is equal to (1.5), and we can make the first graph (5.1) from A_{11}. It is in PC.

We can apply Lemma 1.5 to (5.2), (5.4), (5.6), (5.7), (5.8), and they are not in PC.

For the case (5.5) we can apply Lemma 1.6 as in (2.2). (5.5) is not in PC, either.

Here we explain the case (5.3). Set $S = P \oplus Q(2D_5 + A_3)$. We can consider only the 2-Sylow subgroup of the discriminant group $S^*/S \cong \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/3$. For an element $(a, b_1, b_2, c) \in (\mathbf{Z}/4)^4 = (S^*/S)_2$ the discriminant quadratic form can be written

$$q(a, b_1, b_2, c) \equiv -\frac{1}{4}a^2 - \frac{3}{4}(b_1^2 + b_2^2) + \frac{3}{4}c^2 \bmod 2\mathbf{Z}.$$

A solution of $q \equiv 0$ is one of the following; $(0,0,0,0)$, $\overline{\alpha}_1 = (2,0,0,2)$, $\overline{\alpha}_2 = (0,2,2,0)$, $\overline{\alpha}_3 = (2,2,2,2)$, $\overline{\alpha}_4 = (2,\{2,0\},0)$, $\overline{\alpha}_5 = (0,\{2,0\},2)$, $\overline{\alpha}_6 = (\pm 1,\{2,\pm 1\},0)$, $\overline{\alpha}_7 = (2,\{2,\pm 1\},\pm 1)$, $\overline{\alpha}_8 = (0,\{\pm 1,0\},\pm 1)$, $\overline{\alpha}_9 = (\pm 1,\{\pm 1,0\},2)$. Here $\{x_1,x_2,\ldots,x_k\}$ stands for $x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(k)}$ for some permutation σ. Among these, $\overline{\alpha}_6$, $\overline{\alpha}_7$, $\overline{\alpha}_8$, and $\overline{\alpha}_9$ have order four. The group $I = \widetilde{S}/S$ contains one of $\overline{\alpha}_i, i = 1,2,3,4,5$. We can see that if the contained element is $\overline{\alpha}_i, i = 1,2,4,5$, the induced embedding $Q(G) \hookrightarrow \Lambda_3/P$ is not full. Thus I contains $\overline{\alpha}_3 = (2,2,2,2)$. On the other hand since $2\overline{\alpha}_i \neq \overline{\alpha}_3$ for $i = 6,7,8,9$, one knows that I is generated by $(2,2,2,2)$. Let $I^\perp$ be the orthogonal complement of I with respect to the discriminant bilinear form b on S^*/S. By easy calculation one knows $l(I^\perp) \geq 4$. Thus $3 \leq l(I^\perp/I) = l(\widetilde{S}^*/\widetilde{S}) = l(T^*/T)$, where T is the orthogonal complement of S in Λ_3. However, $l(T^*/T) \leq \operatorname{rank} T = 2$, which is a contradiction. Thus $2D_5 + A_3 \notin PC$.

(6) $D_7 + E_6$

By Lemma 1.5 we can omit graphs with a D_4-component.

(6.1) $D_7 + E_6$ (6.2) $D_7 + A_5 + A_1$ (6.3) $D_7 + 3A_2$

(6.4) $D_5 + E_6 + 2A_1$ (6.5) $D_5 + A_5 + 3A_1$ (6.6) $D_5 + 3A_2 + 2A_1$

We can make (6.1) from A_{11}. Thus the graph (6.1) is in PC.

The others are not in PC. For (6.4), (6.5), and (6.6) it follows from Lemma 1.5. For (6.3) we can show an extra root in the primitive hull. In this case an extra short root with length $\sqrt{2/3}$ appears.

To treat (6.2) we have to use a p-adic method. (Nikulin [1] Theorem 1.12.2 etc.) Set $S = P \oplus Q(D_7 + A_5 + A_1)$. Assume that we have an embedding $S \hookrightarrow \Lambda_3$ satisfying Looijenga's conditions $\langle a \rangle$ and $\langle b \rangle$. We will deduce a contradiction. By $\widetilde{S}$ we denote the primitive hull of S in Λ_3, and by T we denote the orthogonal complement of S. Consider the discriminant group $S^*/S \cong \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/3 \oplus \mathbf{Z}/3$. The first component $\mathbf{Z}/4$ and the fifth component $\mathbf{Z}/3$ are associated with the lattice P. The second $\mathbf{Z}/4$ is associated with the component D_7. The third $\mathbf{Z}/2$ and the last $\mathbf{Z}/3$ are associated with A_5, and the fourth $\mathbf{Z}/2$ with A_1. We have a non-zero element $\overline{\alpha} = (a,b,c,d,x,y)$ in $I = \widetilde{S}/S$. For the discriminant quadratic form q,

$$q(\overline{\alpha}) \equiv -\frac{1}{4}(a^2+b^2) - \frac{1}{2}c^2 + \frac{1}{2}d^2 - \frac{2}{3}(x^2+y^2) \equiv 0 \bmod 2\mathbf{Z}.$$

$\overline{\alpha}$ is one of the following; $\overline{\alpha}_1 = (0,0,1,1,0,0)$, $\overline{\alpha}_2 = (2,2,0,0,0,0)$, $\overline{\alpha}_3 = (2,2,1,1,0,0)$, $\overline{\alpha}_4 = (\pm 1,\pm 1,0,1,0,0)$. If $\overline{\alpha} = \overline{\alpha}_1$, then $\widetilde{S}$ contains a long root orthogonal to P such that it is not in S. It contradicts the assumption. If $\overline{\alpha} = \overline{\alpha}_2$, then $\widetilde{S}/P$ contains a short root with length 1, which is a contradiction. If $\overline{\alpha} = \overline{\alpha}_4$, then I contains $2\overline{\alpha}_4 = \overline{\alpha}_2$ and we can reduce the problem to the second case. Thus we can assume that I is a cyclic group of order 2 generated by $\overline{\alpha}_3$. Set $\overline{\beta}_1 = (2,1,0,1,0,0)$, $\overline{\beta}_2 = (1,0,0,1,0,0)$, $\overline{\gamma}_1 = (0,0,0,0,1,0)$, and $\overline{\gamma}_2 = (0,0,0,0,0,1) \in S^*/S$. We can check that the orthogonal complement $I^\perp$ of I with respect to b is a direct sum of I and 4 cyclic groups generated by these 4 elements. $\overline{\beta}_i$ $(i = 1,2)$ generates a cyclic group of order 4, and $\overline{\gamma}_i$ $(i = 1,2)$ generates a cyclic group of order 3. We have $\widetilde{S}^*/\widetilde{S} \cong I^\perp/I \cong \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/3 \oplus \mathbf{Z}/3$. Here note that $\overline{\beta}_1$, $\overline{\beta}_2$, $\overline{\gamma}_1$, and γ_2 are mutually orthogonal with respect to b, and $q(\overline{\beta}_1) \equiv -3/4$, $q(\overline{\beta}_2) \equiv 1/4$, $q(\overline{\gamma}_i) \equiv -2/3 \bmod 2\mathbf{Z}$ $(i = 1,2)$. Thus we can compute the discriminant form of $\widetilde{S}$. Reversing the sign of the discriminant quadratic form on $\widetilde{S}^*/\widetilde{S}$ we get the discriminant form q_T on T^*/T. We have

$$q_T(a,b,x,y) \equiv \frac{3}{4}a^2 - \frac{1}{4}b^2 + \frac{2}{3}(x^2+y^2) \bmod 2\mathbf{Z}$$

for $(a,b,x,y) \in \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/3 \oplus \mathbf{Z}/3$.

Now we consider the lattice $T_2 = T \otimes \mathbf{Z}_2$ over 2-adic integers $\mathbf{Z}_2$. Note that $T_2^*/T_2 = (T^*/T)_2$. Thus the discriminant quadratic form of T_2 has the form $3a^2/4 - b^2/4$. This implies that T_2 is equivalent over $\mathbf{Z}_2$ to the lattice whose intersection form is defined by the matrix $\begin{pmatrix} 3\cdot 2^2 & 0 \\ 0 & -2^2 \end{pmatrix}$. Therefore we can conclude that the discriminant D of T satisfies $D \equiv -3\cdot 2^4 \bmod \mathbf{Z}_2^{*2}$. However, on the other hand, $|D|$ = the order of T^*/T = the order of $\widetilde{S}^*/\widetilde{S} = 3^2 \cdot 2^4$, and moreover $D = |D|$ since T has signature $(0,2)$. One knows that there exists an element $\xi \in \mathbf{Z}_2^* = \mathbf{Z}_2 - 2\mathbf{Z}_2$ with $3 = -\xi^2$. It implies $3 \equiv -1 \pmod 8$, which is a contradiction.

(7) $E_7 + D_6$

We can omit a graph with a D_4-component and a graph without a component of type D.

(7.1) $E_7 + D_6$ (7.2) $2D_6 + A_1$ (7.3) $D_6 + 2A_3 + A_1$

(7.4) $D_6 + A_5 + A_2$ (7.5) $D_6 + A_7$

We can make the graph (7.1) from A_{11}. Thus (7.1) belongs to PC.

The others are not in PC. The graph (7.2) is equal to (2.6).

For (7.2), (7.3), we can apply Lemma 1.6 as in (2.2). For (7.4) the method in the case (1.2) can be applied and we can show extra roots by considering the 3-Sylow subgroup. To (7.5) we can apply a 2-adic method and we can deduce a contradiction similarly as in the case (6.2).

(8) $E_8 + D_5$

We can omit graphs without a component of type D.

(8.1) $E_8 + D_5$ (8.2) $E_7 + D_5 + A_1$ (8.3) $E_6 + D_5 + A_2$

(8.4) $2D_5 + A_3$ (8.5) $D_5 + 2A_4$ (8.6) $D_5 + A_5 + A_2 + A_1$

(8.7) $A_7 + D_5 + A_1$ (8.8) $D_8 + D_5$ (8.9) $D_5 + A_8$

(8.10) $D_5 + 2A_3 + 2A_1$ (8.11) $D_8 + A_3 + 2A_1$

Note that (8.4) is equal to (5.3), (8.8) is equal to (1.5)=(5.1), and (8.11) is equal to (5.5), which have been previously discussed above.

Among them we can make the graph (8.1) from the graph A_{11}, (8.2) from $E_8 + B_1 + G_2$, and (8.8)=(1.5)=(5.1) from A_{11}. (8.1), (8.2) and (8.8) belong to PC.

On the other hand, the others do not belong to PC. For (8.4)=(5.3) and (8.11)= (5.5), we have shown it in the above. For (8.10) it follows from Lemma 1.5. For (8.3) and (8.6) we can apply similar arguments to those in (1.2). By a similar argument as in (5.3) we can conclude it for the case (8.7).

For the remaining (8.5) and (8.9) we apply a p-adic method.

Here we discuss (8.9) $D_5 + A_8$. Assume that there is an embedding $S = P \oplus Q(G) \hookrightarrow \Lambda_3$ satisfying Looijenga's $\langle a \rangle$ and $\langle b \rangle$ for $G = D_5 + A_8$. We will dedice a contradiction. The induced embedding $Q(G) \hookrightarrow \Lambda_3/P$ is full. By $\widetilde{S}$ we denote the primitive hull of S in Λ_3, and by T we denote the orthogonal complement of S. $\operatorname{rank} T = 2$.

First assume that $m = [\widetilde{S} : S]$ is prime to 3. Then $(T^*/T)_3 \cong (\widetilde{S}^*/\widetilde{S})_3 \cong (S^*/S)_3$. Here note that $(S^*/S)_3 \cong \mathbf{Z}/3 \oplus \mathbf{Z}/9$. The first $\mathbf{Z}/3$-component of $(S^*/S)_3$ corresponds to the lattice P and the second component $\mathbf{Z}/9$ corresponds to the A_8-component of G respectively. The discriminant quadratic form on $(S^*/S)_3$ can be written

$$q_3 \equiv -\frac{2}{3}x^2 + \frac{8}{9}y^2 \bmod 2\mathbf{Z}.$$

The discriminant quadratic form on $(T^*/T)_3$ is $-q_3$.

We consider the lattice $T_3 = T \otimes \mathbf{Z}_3$ over 3-adic integers $\mathbf{Z}_3$. Since $T_3^*/T_3 \cong (T^*/T)_3$, the discriminant quadratic form of T_3 coincides with $-q_3$. This implies that T_3 is equivalent over $\mathbf{Z}_3$ to the lattice defined by the diagonal matrix whose diagonal entries are 6, -72. Thus the discriminant D of T satisfies $D \equiv -2^4 \cdot 3^3 \bmod \mathbf{Z}_3^{*2}$. On the other hand, $|D| = \#(T^*/T) = \#(\widetilde{S}^*/\widetilde{S}) = \#(S^*/S)/m^2 = 2^4 \cdot 3^3/m^2$ (By $\#M$ we denote the order of an abelian group M.), and $D > 0$ since T has signature $(0, 2)$. In conclusion we have $-2^4 \cdot 3^3 \equiv 2^4 \cdot 3^3/m^2 \bmod \mathbf{Z}_3^{*2}$. It implies that $x^2 \equiv -1 \pmod 3$ has an integral solution, which is a contradiction.

Now, we can assume that there is a non-zero element $\overline{\alpha} \in (\widetilde{S}/S)_3 \subset (S^*/S)_3$. Since $q(\overline{\alpha}) \equiv 0 \bmod 2\mathbf{Z}$, one knows $\overline{\alpha} = (0, \pm 3)$. Then, $\widetilde{S}$ contains an element α with $\alpha^2 = 2$ which is not in S. This contradicts Looijenga's condition $\langle a \rangle$ and the fullness.

Thus we can conclude $D_5 + A_8 \notin PC$.

The case (8.5) is similar. We consider $p = 5$ in this case.

All the graphs in the remaining cases (9)–(26) turn out that they do not belong to PC.

(9) $A_{10} + 3A_1$

From this graph we can make no graph with a D-component by elementary transformations.

(10) $D_{10} + 3A_1$

(10.1) $D_{10} + 3A_1$ (10.2) $D_8 + 5A_1$ (10.3) $D_7 + A_3 + 3A_1$

(10.4) $D_6 + D_4 + 3A_1$ (10.5) $2D_5 + 3A_1$

We have already treated the case (10.1)=(2.2). It is not in PC. For (10.2)–(10.5) by Lemma 1.5 one knows that they are not in PC.

(11) $A_9 + 4A_1$ (12) $A_7 + A_2 + 4A_1$ (13) $A_6 + A_4 + 3A_1$

Obviously we cannot make a graph with a component of type D from any one of these three graphs (11)–(13).

(14) $D_5 + A_5 + 3A_1$ (15) $E_6 + A_4 + 3A_1$ (16) $E_7 + A_3 + 3A_1$

(17) $E_8 + A_2 + 3A_1$

Let G be a graph with 13 vertices and with a component of type D made by an elementary transformation from one of the above 4 graphs (14)–(17). G has the form $G = G' + A_k + 3A_1$ with $k \geq 2$. By Lemma 1.5 one knows $G \notin PC$.

(18) $A_9 + D_4$ (19) $D_9 + D_4$ (20) $A_8 + D_4 + A_1$

(21) $A_6 + D_4 + A_2 + A_1$ (22) $A_5 + A_4 + D_4$ (23) $D_5 + D_4 + A_4$

(24) $E_6 + D_4 + A_3$ (25) $E_7 + D_4 + A_2$ (26) $E_8 + D_4 + A_1$

Omitting graphs without a D-component or with a D_4-component, only the following items remain for these cases (18)–(26).

(19.1) $D_9 + 4A_1$ (19.2) $D_7 + 6A_1$ (19.3) $D_6 + A_3 + 4A_1$

(23.1) $D_5 + A_4 + 4A_1$

(25.1) $D_6 + A_2 + 5A_1$

(26.1) $D_5 + A_3 + 5A_1$ (26.2) $D_8 + 5A_1$

By Lemma 1.5 one knows that all of them cannot belong to PC.

By the above we complete the case of the number of vertices 13.

We would like to proceed to the case of 12 vertices.

Let G be a Dynkin graph with components of type A, D, or E only. For simplicity by $\epsilon_p(G)$ and by $d(G)$ we denote the Hasse invariant $\epsilon_p(Q(G))$ and the discriminant $d(Q(G))$ of the root lattice $Q(G)$ of type G respectively.

In the following we assume further that G has 12 vertices and has a component of type D. Here recall our assumption (1) and (2). By assumption (1) $\rho = 19$. By assumption (2) $\epsilon_p(G) \neq (3, d(G))_p$ for some prime number p.

Let G' be the sum of components of G of type A or E. The number of vertices of G' is less than or equal to 8. Then we have

$$\epsilon_p(G') \neq (3, d(G'))_p \text{ for } p = 3, 5, \text{or } 7.$$

In what follows we explain this assertion.

Here recall that $d(G_1 + G_2) = d(G_1)d(G_2)$ and $\epsilon_p(G_1 + G_2) = \epsilon_p(G_1)\epsilon_p(G_2) \cdot (d(G_1), d(G_2))_p$ for Dynkin graphs G_1, G_2, and that for $p \neq 2,\infty$ $(a,b)_p = 1$ if integers a, b satisfy $p \nmid a$ and $p \nmid b$. Besides, $(a, b^2)_p = 1$ for every a, b, p.

Let $G'' = G - G'$ be the sum of components of type D. Note that $d(G'') = 4^m$ for some m, and $\epsilon_p(G'') = 1$ for every prime p. We have $\epsilon_p(G) = \epsilon_p(G')\epsilon_p(G'') \cdot (d(G'), 4^m)_p = \epsilon_p(G')$, and $(3, d(G))_p = (3, 4^m)_p(3, d(G'))_p = (3, d(G'))_p$.

Thus $\epsilon_p(G') \neq (3, d(G'))_p$ for some prime p.

If $p \geq 3$ and $p \nmid d(G')$, then $\epsilon_p(G') = 1$, since $\epsilon_p(A_k) = (-1, d(A_k))_p$ and $\epsilon_p(D_l) = \epsilon_p(E_m) = 1$. Moreover, if $p \geq 5$ and $p \nmid d(G')$, then we have $(3, d(G'))_p = 1$. Therefore if $p \geq 5$ and $p \nmid d(G')$, then $\epsilon_p(G') = (3, d(G'))_p$. Thus we can consider only the case where $p = 2, 3$ or $p \mid d(G')$. Here note that $d(A_k) = k + 1$, $d(E_m) = 9 - m$ and G' has at most 8 vertices. Thus $p = 2, 3, 5$ or 7, if $p \mid d(G')$. Consequently we can assume $p = 2, 3, 5$ or 7.

Finally we can omit $p = 2$ further, because of the product formula:

$$\prod_{p,incl.\infty} \epsilon_p(G') = 1, \qquad \prod_{p,incl.\infty} (3, d(G'))_p = 1.$$

$\big(\epsilon_\infty(G') = 1$ since $Q(G')$ is positive definite. $(3, d(G'))_\infty = 1$ since $d(G') > 0.\big)$

Assume that $\epsilon_7(G') \neq (3, d(G'))_7$. (We omit the lower index $p = 7$ in the following.) We can write $G' = A_6 + G_1$ since $7 \mid d(G')$. $G_1 = A_2, 2A_1, A_1$ or $\emptyset$. In any case $7 \nmid d(G_1)$, and thus $(3, d(G')) = (3, d(A_6))(3, d(G_1)) = (3, 7) = -1$. Calculating $\epsilon(G')$ for the four possible cases, one has $G' = A_6 + A_2$.

Assume that $\epsilon_5(G') \neq (3, d(G'))_5$. (We omit the lower index $p = 5$ in the following.) We can write $G' = A_4 + G_1$. G_1 has at most 4 vertices.

Case 1. $5 \mid d(G_1)$

$G' = 2A_4$. However in this case $\epsilon(G') = 1 = (3, d(G'))$.

Case 2. $5 \not| \, d(G_1)$

$(3, d(G')) = -1$. We have only to check whether $\epsilon(A_4 + G_1) = (5, d(G_1))$ is equal to 1. This is equivalent to $d(G_1) \equiv \pm 1 \pmod 5$. Among the 11 possibilities only the following 6 graphs satisfy $\epsilon(G') = 1$.

$$A_4 + 2A_2, \quad A_4 + 4A_1, \quad A_4 + A_3, \quad A_4 + A_2 + A_1, \quad A_4 + 2A_1, \quad A_4.$$

Assume that $\epsilon_3(G') \neq (3, d(G'))_3$. (We omit the lower index $p = 3$ in the following.) Note that in this case we cannot conclude $3 \mid d(G')$.

Case 1. $3 \not| \, d(G')$

In this case $\epsilon(G') = 1$. Thus the assumption $\iff (3, d(G')) = -1 \iff d(G') \equiv -1 \pmod 3$. G' has at most 8 vertices and its component is either A_1, A_3, A_4, A_6, A_7, E_7, or E_8. We can pick up the following 15 graphs satisfying the assumptions from 34 possibilities.

E_7	A_7	$A_6 + A_1$	$A_4 + 4A_1$	$A_4 + A_3$	$A_4 + 2A_1$	A_4	$2A_3 + A_1$
$A_3 + 5A_1$	$A_3 + 3A_1$	$A_3 + A_1$	$7A_1$	$5A_1$	$3A_1$	A_1	

Case 2. G' contains A_8.

$G' = A_8$. $\epsilon(A_8) = (-1, 9) = 1$. $(3, d(A_8)) = (3, 9) = 1$. This does not satisfy the assumption.

Case 3. G' contains E_6.

There are only 4 possibilities for G'. Only the following three satisfy the assumption.

$$E_6 + 2A_1, \quad E_6 + A_1, \quad E_6.$$

Case 4. G' contains A_5.

There are only 7 possibilities for G'. Among them only the following six satisfy the assumption.

$$A_5 + A_3, \quad A_5 + A_2, \quad A_5 + 3A_1, \quad A_5 + 2A_1, \quad A_5 + A_1, \quad A_5.$$

Case 5. G' contains just 4 of A_2.

$G' = 4A_2$. In this case $\epsilon = 1 = (3, d)$. It does not satisfy the assumption.

Case 6. G contains just 3 of A_2.

We can write $G' = 3A_2 + G_1$ with $3 \not| \, d(G_1)$. Then we have $\epsilon(G') = (3, d(G_1))$, and $(3, d(G')) = -(3, d(G_1))$. Thus every possibility automatically satisfies the assumption. There are three possibilities.

$$3A_2 + 2A_1, \quad 3A_2 + A_1, \quad 3A_2.$$

Case 7. G contains just 2 of A_2.

Writing $G' = 2A_2 + G_1$, one knows $\epsilon \neq (3, d) \iff d(G_1) \equiv 1 \pmod 3$. Only the following four among possibilities satisfy the assumption.

$$2A_2 + 4A_1, \quad 2A_2 + A_3, \quad 2A_2 + 2A_1, \quad 2A_2.$$

Case 8. G contains just one of A_2 and it does not contain A_5 and E_6.

We can write $G' = A_2 + G_1$ with $3 \not| d(G_1)$. We have $\epsilon(G') = -(3, d(G_1))$ and $(3, d(G')) = -(3, d(G_1))$. Thus the assumption is never satisfied in this case.

We have the following proposition.

Proposition 1.7. *Let G be a Dynkin graph with components of type A, D, or E only. We assume that G has 12 vertices and it contains a component of type D. If $\epsilon_p(G) \neq (3, d(G))_p$ for some prime p, then G is one of the following 40.*

(1) $D_{11} + A_1$	(2) $D_8 + A_4$	(3) $D_8 + A_3 + A_1$
(4) $D_7 + A_5$	(5) $D_6 + D_5 + A_1$	(6) $D_6 + E_6$
(7) $D_6 + A_5 + A_1$	(8) $D_5 + A_7$	(9) $D_5 + E_7$
(10) $D_5 + A_6 + A_1$	(11) $D_5 + E_6 + A_1$	(12) $D_5 + A_5 + 2A_1$
(13) $D_5 + A_4 + A_3$	(14) $D_5 + A_4 + A_2 + A_1$	
(15) $D_9 + 3A_1$	(16) $D_7 + D_4 + A_1$	(17) $D_6 + A_4 + 2A_1$
(18) $D_6 + 3A_2$	(19) $D_6 + 2A_2 + 2A_1$	(20) $D_5 + 3A_2 + A_1$
(21) $D_4 + E_6 + 2A_1$	(22) $D_4 + A_5 + A_3$	
(23) $D_6 + A_3 + 3A_1$	(24) $D_4 + A_5 + 3A_1$	
(25) $D_5 + 2A_3 + A_1$		
(26) $D_8 + 2A_2$	(27) $D_5 + A_5 + A_2$	(28) $D_5 + A_3 + 2A_2$
(29) $D_4 + A_6 + A_2$	(30) $D_4 + A_4 + 2A_2$	
(31) $D_7 + 5A_1$	(32) $D_5 + D_4 + 3A_1$	(33) $D_5 + 7A_1$
(34) $2D_4 + A_4$	(35) $2D_4 + A_3 + A_1$	(36) $2D_4 + 2A_2$
(37) $D_4 + A_4 + 4A_1$	(38) $D_4 + A_3 + 5A_1$	(39) $D_4 + 3A_2 + 2A_1$
(40) $D_4 + 2A_2 + 4A_1$		

Among the above, 14 items (1)–(14) can be made from one of the essential basic Dynkin graphs by elementary or tie transformations repeated twice. (For example we can make D_{12} from a basic graph A_{11} by one tie transformation. From D_{12} we can make the graphs (1)–(14) by the same transformation.)

Lemma 1.8. *In addition to our assumptions on the elliptic K3 surfaces we assume that the number r of vertices of $G \in PC(W_{1,0})$ is 12. Then G has at most only one component of type D_4.*

Proof. If G has 2 or more components of type D_4, then the functional invariant must be constant. (See Lemma 1.5.) One has $\rho = 19$, $a = 1$, $t_1 = 0$, and $t = 4$ by assumption and by the equality (3) in Part II section 1. Thus the combination of singular fibers must be $4I_0^*$ and $G = 3D_4$. However, then, $\epsilon_p(3D_4) = (3, d(3D_4))_p$ for every p, which contradicts the assumption. Q.E.D.

The ten items (31)–(40) do not satisfy the condition in Lemma 1.8 or Lemma 1.5 (1). Thus they do not belong to PC.

Items (15)–(30) are remaining. For (15)–(22) we can apply a similar argument to that in the case (1.2), $r = 13$ in the above. Only by solving the congruence equation $q \equiv 0$, we can conclude that they are not in PC.

For (23) and (24) we can use the method applying Lemma 1.6 explained in the case (2.2) in the above. To (25) we can apply the method in the case (5.3) above.

To (26)–(30) we can apply the same method as in (8.9). Namely, first by the p-adic method, we show that $S = P \oplus Q(G)$ has no primitive emdedding into Λ_3. Next assuming $\widetilde{S}/S \neq 0$ by solving $q \equiv 0$ we can deduce a contradiction from the fullness.

Anyway we can show that any one of the items (15)–(40) does not belong to PC.

We complete the case of 12 vertices.

Now, the last remaining case is the case of 11 vertices. Also in this case we use the abbriviations $d(G)$ and $\epsilon_p(G)$.

Proposition 1.9. *Let G be a Dynkin graph with 11 vertices with components of type A, D, or E only. Assume that G has a component of type D and satisfies the following condition $\heartsuit$.*

$$\text{For some prime number } p,\ 3d(G) \in \mathbf{Q}_p^{*2} \text{ and } \epsilon_p(G) \neq (-1,3)_p. \qquad \heartsuit$$

Then, $G = D_5 + E_6$, $D_5 + A_5 + A_1$ or $D_5 + 3A_2$.

Proof. Let G' be the sum of components of G of type A or E. G' has at most 7 vertices. One knows easily that $\heartsuit$ is equivalent to the following $\heartsuit'$.

$$\text{For } p = 2, 3, 5 \text{ or } 7, 3d(G') \in \mathbf{Q}_p^{*2} \text{ and } \epsilon_p(G') \neq (-1,3)_p. \qquad \heartsuit'$$

In the first step we determine all the pairs (G', p) such that $\epsilon_p(G') \neq (-1,3)_p$, where G' is a Dynkin graph with at most 7 vertices with components of type A or E only, and $p = 2, 3, 5$ or 7. We can omit the calculation in the case $p = 2$ thanks to the product formula.

We do not present the result here. However, the list of such pairs contains 33 kinds of Dynkin graphs, and each graph corresponds to just 2 prime numbers. The two graphs $A_6 + A_1$ and A_6 correspond $p = 3$ and 7. The three graphs $A_4 + 3A_1$, $A_4 + A_2$ and $A_4 + A_1$ correspond to $p = 3$ and 5. The others correspond to $p = 2$ and 3.

Next, we check whether each item satisfies $3d(G') \in \mathbf{Q}_p^{*2}$ or not. (Note that an integer $p^m a$ with $p \not| a$ belongs to $\mathbf{Q}_p^{*2}$ if and only if m is even and $\left(\frac{a}{p}\right) = +1$ (when p is an odd prime number. $\left(\frac{-}{p}\right)$ is Legendre's quadratic residue symbol.), m is even and $a \equiv 1 \pmod 8$ (when $p = 2$).) Then, one knows that only three graphs satisfy the condition, and we have the above three graphs. Each of the three satisfies the condition for $p = 2$ and 3. Q.E.D.

If $\heartsuit$ is not satisfied, then there exists an isotropic element in a nice position by Theorem 0.5 [II] in Part I (Urabe [4]). Thus we can assume $\heartsuit$. Consequently we can consider only the above three graphs.

For $D_6 + E_6$ and $D_6 + A_5 + A_1$ we can make them from the basic graph A_{11} by tie transformations repeated twice. Thus they belong to $PC = PC(W_{1,0})$. Indeed we can make D_{12} from A_{11} easily. From D_{12} we can make them.

On the contrary, the third graph $D_5 + 3A_2$ does not belong to PC. To see this we can apply the same method as in (8.9) above.

We can complete this section. We have shown Proposition 0.1 under the assumption ((1)) in the introduction.

§2. A singular fiber of type II^*, III^*, IV^*

Let G be a Dynkin graph belonging to $PC(W_{1,0})$ with r vertices. Throughout this section we assume that the corresponding elliptic K3 surface $\Phi : Z \to C$ has a singular fiber of type II^*, III^*, or IV^*, which is denoted by F_2.

Proposition 2.1. *(1) G is a subgraph of the Coxeter-Vinberg graph of the unimodular lattice of signature* $(14, 1)$.
(2) G has a component of type E.

Proof. (1) We have the associated embedding $P \oplus Q(G) \hookrightarrow \Lambda_3$ satisfying Looijenga's conditions $\langle a \rangle$, $\langle b \rangle$. Let $\overline{P}$ denote the sublattice in P of rank 6 which has a basis $e_0, \ldots, e_5$ corresponding to C_5 and 5 components of F_1. This $\overline{P}$ is isomorphic to P defined in the case $J_{3,0}$. It is easy to check that the induced embedding $\overline{P} \oplus Q(G) \hookrightarrow \Lambda_3$ also satisfies the conditions $\langle a \rangle$, $\langle b \rangle$. Thus the embedding $Q(G) \hookrightarrow \Lambda_3/\overline{P}$ is full. By Proposition 2.1 in Part II the orthogonal complement of $Q(G)$ in $\Lambda_3/\overline{P}$ contains an element ξ with $\xi^2 = -4$. (Though we can write $\xi = 2\eta$ in $\Lambda_3/\overline{P}$, we do not use this fact.) The orthogonal complement L of $\mathbf{Z}\xi$ in $\Lambda_3/\overline{P}$ is a unimodular lattice of signature $(14, 1)$, and $Q(G)$ is full in L. The claim follows from these facts.
(2) Consider the dual graph G_1 associated with the set of components of F_2 not intersecting IF. The dual graph of all components of F_2 minus 1 or 2 vertices corresponding to components with multiplicity 1 is G_1.

Assume that G_1 is not of type E. We will deduce a contradiction. One knows immediately that F_2 is of type IV^* and G_1 is of type D_5.

For $2 \leq i \leq t$, let $n(F_i)$ denote the number of components not intersecting IF of the singular fiber F_i. $n(F_2) = 5$ and $r = \sum_{i=2}^{t} n(F_i)$. Recall that we can assume $\rho = 7 + r$ without loss of generality. Then, by the equality (2) in the beginning of section 1 in Part II, we have $\sum_{i=3}^{t}(m(F_i) - n(F_i) - 1) + a = 0$, since $m(F_1) = 5$ and $m(F_2) = 7$. We have $a = 0$ and $m(F_i) - 1 = n(F_i)$ for $3 \leq i \leq t$. In particular the group E of sections is finite.

Here we recall that we denoted by $\operatorname{Tor} M$ the subgroup of an abelian group M consisting of all elements of finite order.

In our case the section s_1 corresponding to C_6 belongs to $\operatorname{Tor} E = E$. We consider $F_i^{\#} = F_i \cap Z^{\#}$ for $i = 1, 2$. Recall that they carry the group structure. Since F_1 is of type I_0^*, $\operatorname{Tor} F_1^{\#} \cong \mathbf{Z}/2 + \mathbf{Z}/2$. One knows that s_1 has order 2 in E since $E \to \operatorname{Tor} F_1^{\#}$ is injective. Thus $\operatorname{Tor} F_2^{\#}$ has an element of order 2, since $E \to \operatorname{Tor} F_2^{\#}$ is injective. However, $\operatorname{Tor} F_2^{\#} \cong \mathbf{Z}/3$, since F_2 is of type IV^*. It is a contradiction. Q.E.D.

Here recall Proposition 3.5 in Part II, which claims that if a Dynkin graph G can be obtained from a basic Dynkin graph G_0 by elementary or tie transformations applied twice, then any subgraph G' of G can be obtained from G_0 by elementary or tie transformations applied twice.

By the concrete form of the Coxeter-Vinberg graph in Part II section 2, one knows that a Dynkin graph G satisfying the conditions (1) and (2) in Proposition 2.1 is a subgraph of $E_8 + E_6$ or $2E_7$.

By Theorem 0.5 in Part I [4] we can treat only the case $r = 13, 12$ or 11.

The first case is $r = 13$. Then G is one of the following 18 graphs.

(1) $E_8 + A_4 + A_1$ (2) $E_8 + D_5$ (3) $E_7 + A_6$
(4) $E_7 + D_6$ (5) $E_7 + D_5 + A_1$ (6) $2E_6 + A_1$
(7) $E_6 + D_7$
(8) $E_8 + 2A_2 + A_1$ (9) $E_6 + A_4 + A_2 + A_1$ (10) $E_6 + D_5 + A_2$
(11) $E_8 + A_5$ (12) $E_7 + A_5 + A_1$ (13) $E_7 + A_3 + A_2 + A_1$
(14) $E_7 + A_4 + A_2$ (15) $E_7 + E_6$ (16) $E_6 + A_7$
(17) $E_6 + A_6 + A_1$ (18) $E_6 + A_4 + A_3$

The 7 graphs (1)–(7) can be made from one of the essential basic Dynkin graphs by tie transformations repeated twice. The following shows an example of the initial basic graph.

(1) $\leftarrow E_8 + B_1 + G_2$ (2) $\leftarrow B_9 + G_2$ (3) $\leftarrow E_7 + B_3 + G_1$
(4) $\leftarrow E_7 + B_3 + G_1$ (5) $\leftarrow E_8 + B_1 + G_2$ (6) $\leftarrow E_8 + B_1 + G_2$
(7) $\leftarrow B_9 + G_2$

The other 11 graphs do not belong to $PC = PC(W_{1,0})$.

For (8), (9), and (10) we can use the method explained in (1.2) in section 1 to show it. We can construct an extra root for each non-zero solution of $q \equiv 0$.

For 7 graphs (11)–(18) we apply the p-adic method for $p = 3$, and argue as in the case (8.9) $D_5 + A_8$.

Let us proceed to the case of $r = 12$. Thanks to Proposition 3.5 in Part II, we can assume that G is not isomorphic to a subgraph of the above (1)–(7) of case $r = 13$ in addition to the conditions in Proposition 2.1. It is not difficult to see that a graph satisfying the conditions is one in the following list.

$\diamond 1 \diamond$ $E_8 + 2A_2$ $\diamond 2 \diamond$ $E_7 + 2A_2 + A_1$
$\diamond 3 \diamond$ $E_6 + A_3 + A_2 + A_1$ $\diamond 4 \diamond$ $E_6 + A_4 + A_2$

It turns out that all of these 4 graphs do not belong to PC.

For these graphs we can apply the p-adic method for $p = 3$. By arguments similar to the case (8.9) $D_5 + A_8$, we can show that any one of them is not in PC.

Lastly we consider the case $r = 11$. However, in this case every graph satisfying the conditions in Proposition 2.1 is isomorphic to a subgraph of the above (1)–(7) in the case $r = 13$. By Proposition 3.5 in Part II we can complete the proof.

We have shown Proposition 0.1 under the assumption ((2)) in the introduction in this section.

§3. Combinations of graphs of type A

In this section we consider under the assumption ((3)) in the introduction. As in the previous sections G denotes a Dynkin graph in $PC(W_{1,0})$ with r vertices. The corresponding elliptic K3 surface $\Phi : Z \to C$ has t singular fibers $F_1, \ldots, F_t$ and one of them, say F_1, is of type I_0^* and the others are of type I, II, III or IV. The union $IF = F_1 \cup C_5 \cup C_6$ is the curve at infinity. By $C_0, \ldots, C_4$ we denote the components of IF. We assume that C_0 has multiplicity 2, C_5 intersects C_4, and C_6 intersects C_3. Let A_{n_i} be the dual graph of the set of components not intersecting C_5 of a singular fiber F_i for $2 \leq i \leq t$. (A_0 stands for an empty graph $\emptyset$.)

Lemma 3.1._(1) If $\rho = r+7$, then $a = 0$ or 1. If $\rho = r+7$ and $a = 1$, then every singular fiber F_i with $2 \le i \le t$ has only one component intersecting IF. If $\rho = r+7$ and $a = 0$, then $G = 2A_5 + G_1$ for some Dynkin graph G_1._
(2) $r \le 13$. If $r = 13$, then $a = 1$.
_(3) $\nu(G) \le 18 - r$._

Proof. (1) For $2 \le i \le t$ by $n(F_i)$ we denote the number of components of F_i not intersecting IF. By definition $r = \sum_{i=2}^{t} n(F_i)$. By the equaliy (2) in Part II section 1, we have $1 - a = \sum_{i=2}^{t}(m(F_i) - n(F_i) - 1)$. Obviously $m(F_i) \ge n(F_i) + 1$ by definition, and $a \le 1$.

If $a = 1$, then $m(F_i) = n(F_i) + 1$ for all i with $2 \le i \le t$.

Assume $a = 0$. There is a unique singular fiber, say F_2, different from F_1 such that C_5 and C_6 hit different components of F_2. Let S_i be the subgroup of $\mathrm{Pic}(Z)$ generated by the classes of components of F_i not intersecting C_5. We can write $[C_6] = [F] + m[F + C_5] + \omega_3 + \chi$, where $\omega_3 \in S_1^*$, $\chi \in S_2^*$ and $m \in \mathbf{Z}$. We have $m = 1$, since $[C_6] \cdot [C_5] = 0$.

Under the isomorphism $S_1^* \cong Q(D_4)^*$ ω_3 corresponds to the fundamental weight associated with the vertex of the Dynkin graph D_4 with one edge corresponding to the component C_3 of F_1. In particular $\omega_3^2 = -1$.

On the other hand, under $S_2^* \cong Q(A_{n_2})^*$ χ corresponds to the fundamental weight associated with the vertex of A_{n_2} corresponding to the component of F_2 hit by C_6.

However, by injectivity of $E = \mathrm{Tor}\, E \to F_2^{\#}$, the image of χ in the quotient $Q(A_{n_2})^*/Q(A_{n_2})$ has order 2. Thus n_2 is odd, and χ corresponds to the central vertex of A_{n_2}. In particular $\chi^2 = -(n_2+1)/4$. We have $-2 = [C_6]^2 = 2 - 1 - (n_2+1)/4$. Thus $n_2 = 11$. G contains the graph A_{11} minus the central vertex, i.e., $2A_5$.
(2) Since $20 \ge \rho \ge 7 + r$, the first claim is obvious.

Assume $r = 13$. Then $\rho = 20 = 7 + r$. We have $a \ge \nu(I_0^*) \ge 1$ by the inequality (4) in Part II section 1 when J is not constant.

Thus we can assume moreover that J is constant. Then $t_1 = 0$ in the equality (3) in Part II section 1, and we have $2(t-1) = 4 + a \le 5$.

On the other hand, every component of G is of type A_1 or A_2 under our assumption, since every singular fiber except F_1 is of type II, III or IV. Therefore we have $7 \le \nu(G) \le t - 1 \le 2$, which is a contradiction.
(3) We can assume without loss of generality that $\rho = r+7$. If $a = 0$, then the inequality obviously holds by (1). Thus we assume moreover $a = 1$. We apply the equality (3) in Part II section 1. First obviously $t - t_1 - 1 = \nu(II) + \nu(III) + \nu(IV)$ under our assumption. Secondly $t_1 = \nu(G) + \nu(I_1)$ by (1) above. The claim follows from the equality (3) in Part II. Q.E.D.

Note that by Lemma 3.1 (1) $G = \sum A_{n_i}$ if $\rho = r + 7$ and $a = 1$.

Let P be the lattice associated with the singularity $W_{1,0}$. $\mathrm{rank}\, P = 7$. Recall that $e_0, \ldots, e_6$ denote the basis of P which has a one-to-one correspondence with the components C_i's of IF. The surface Z defines an embedding $P \oplus Q(G) \hookrightarrow \Lambda_3$ satisfying Looijenga's conditions $\langle a \rangle$ and $\langle b \rangle$. The induced embedding $Q(G) \hookrightarrow \Lambda_3/P$ is full and has no obstruction component of type A_{11}, i.e., if G has a component G_1 of type A_{11}, then $[P(Q(G_1), \Lambda_3/P) : Q(G_1)] < 12$. Here recall that we have denoted the primitive hull of a submodule M in L by $P(M, L) = \{x \in L \mid \text{For some non-zero integer } m,\ mx \in M\}$.

We regard $P \oplus Q(G)$ as a submodule of Λ_3 via the induced embedding.

Proposition 3.2. $PC(W_{1,0}) \subset PC(J_{3,0})$.

Proof. Let $\overline{P}$ be the sublattice of rank 6 in P generated by $e_0, e_1, \ldots, e_5$. This $\overline{P}$ is isomorphic to P defined in the case of $J_{3,0}$.

If $G \in PC(W_{1,0})$, then we have an embedding $P \oplus Q(G) \hookrightarrow \Lambda_3$ satisfying the conditions $\langle a \rangle$, $\langle b \rangle$. It is easy to check that the induced embedding $\overline{P} \oplus Q(G) \hookrightarrow \Lambda_3$ also satisfies $\langle a \rangle$ and $\langle b \rangle$. It implies $G \in PC(J_{3,0})$. Q.E.D.

Proposition 3.3. *Assume that* $\rho = r + 7$ *and* $a = 1$.
(1) $Q(G)$ *is primitive in* Λ_3.
(2) For every element $\overline{\alpha} \in P(P \oplus Q(G), \Lambda_3)/(P \oplus Q(G))$ *with order 2, there is a subset* $T \subset N(2) = \{i \mid 2 \leq i \leq t,\ n_i + 1 \equiv 0 \pmod 2\}$ *satisfying* $\sum_{i \in T}(n_i + 1) = 12$ *and* $\overline{\alpha}$ *can be written* $\overline{\alpha} = \overline{\omega} + \sum_{i \in T} \overline{\chi}_i$, *where* $\overline{\omega} \in P^*/P$, *and* $\overline{\chi}_i \in Q(A_{n_i})^*/Q(A_{n_i})$ $(i \in T)$ *have order 2.*

Proof. (1) The proof is same as in Lemma 1.6 (1).
(2) We can write $\overline{\alpha} = \overline{\omega} + \sum_{i=2}^{t} \overline{\chi}_i$, where $\overline{\omega} \in P^*/P \cong \mathbf{Z}/12$, and $\overline{\chi}_i \in Q(A_{n_i})^*/Q(A_{n_i}) \cong \mathbf{Z}/(n_i + 1)$ for $2 \leq i \leq t$.

Set $T = \{i \mid 2 \leq i \leq t, \overline{\chi}_i \neq 0\}$. If $\overline{\chi}_i \neq 0$ it has order 2 by assumption, and n_i is odd.

If $\overline{\omega} = 0$, we have a contradiction by above (1). Thus $\overline{\omega} \neq 0$. By assumption $\overline{\omega}$ has order 2. It can be checked that the element of order 2 in P^*/P is ω_0 mod P where $\omega_0 = (e_1 + e_2)/2 + e_0 + e_3 \in P^*$ and thus it is contained in $(P^* \cap \overline{P}^*) + P/P \cong (P^* \cap \overline{P}^*)/\overline{P}$. Namely, we can regard $\overline{\alpha}$ as an element in $(\overline{P} \oplus Q(G))^*/(\overline{P} \oplus Q(G))$. Thus we have a section $s' : C \to Z$ whose image $C' = s'(C)$ represents the class $\overline{\alpha}$. Since for every point $a \in C$ the homomorphism from $\widetilde{S}/\overline{S}$ to the group $F_a^{\#}$ of the fiber over a is injective, $C' \cdot C_5 = 0$. Let S_i be the same group as in the proof of Lemma 3.1 (1). We can write

$$[C'] = [F] + [F + C_5] + \omega + \sum_{i \in T} \chi_i,$$

where F denotes a general fiber, $\chi_i \in S_i^*$ is the fundamental weight associated with the central vertex of the Dynkin graph A_{n_i}. In particular $\chi_i^2 = -(n_i + 1)/4$. The element $\omega \in S_1^*$ corresponds to $\omega_0 \in P_0'^* \cong Q(D_4)$ under $S_1^* \xrightarrow{\sim} P_0'^*$. We have $-2 = [C']^2 = ([F] + [F + C_5])^2 + \omega^2 + \sum_{i \in T}(n_i + 1)/4$. Therefore $\sum_{i \in T}(n_i + 1) = 12$, since $([F] + [F + C_5])^2 = 2$ and $\omega^2 = -1$. Q.E.D.

Corollary 3.4. *We consider a Dynkin graph* $G = \sum_{i \in I} A_{k_i} \in PC = PC(W_{1,0})$ *under the condition ((3)). Assume moreover that* $\rho = r + 7$ *and* $a = 1$. *Set* $S = P \oplus Q(G)$ *and* $\widetilde{S} = P(S, \Lambda_3)$.
(1) For any prime p *with* $p \geq 5$ $(\widetilde{S}/S)_p = 0$. $\big(M_p$ *denotes the* p*-Sylow subgroup of* $M\big)$.
(2) For $p = 2, 3$, $l((\widetilde{S}/S)_p) \leq 1$. $\big(l(M)$ *denotes the minimum number of generators of* $M\big)$.

Proof. The proof is same as that of Lemma 1.6 (3), (4).

First we consider the case $r = 13$. Then, we have $\rho = r + 7 = 20$ and by Lemma 3.1 (2) we have $a = 1$.

By Lemma 3.1 (3) items to be checked are graphs corresponding to the division of 13 into 5 pieces. By Proposition 3.2 we can assume moreover that the graph belongs to $PC(J_{3,0})$. Recall that there are 25 kinds of such graphs. 24 graphs of 25 was explained in paragraph **A** in Part II section 3 and we can make them from $E_8 + F_4$ by two tie transformations. The last one is (57) 3+3+3+2+2, $3A_3 + 2A_2$. The following is the list of 25 graphs. We use the numbering in Part II section 3.

(1) A_{13}	(2) $A_{12} + A_1$	(3) $A_{11} + A_2$
(5) $A_9 + A_4$	(6) $A_8 + A_5$	(8) $A_{11} + 2A_1$
(9) $A_{10} + A_2 + A_1$	(10) $A_9 + A_3 + A_1$	(11) $A_9 + 2A_2$
(12) $A_8 + A_4 + A_1$	(17) $2A_6 + A_1$	(18) $A_6 + A_5 + A_2$
(20) $2A_5 + A_3$	(21) $A_5 + 2A_4$	(23) $A_9 + A_2 + 2A_1$
(26) $A_7 + A_4 + 2A_1$	(27) $A_7 + A_3 + A_2 + A_1$	(30) $A_6 + A_4 + A_2 + A_1$
(32) $A_6 + A_3 + 2A_2$	(34) $A_5 + A_4 + A_3 + A_1$	(43) $A_7 + 2A_2 + 2A_1$
(47) $2A_5 + 3A_1$	(49) $A_5 + 2A_3 + 2A_1$	(53) $2A_4 + 2A_2 + A_1$
(57) $3A_3 + 2A_2$		

(a). For (1) and (2) we can make the corresponding graph G from the essential basic Dynkin graph A_{11} by two tie transformations. Thus $G \in PC$.

In the other 23 cases the corresponding graph G is not a member of PC.

(b). Consider the case (3) $G = A_{11} + A_2$. Note the obstruction component A_{11}.

Set $S = P \oplus Q(G)$. We assume that there is an embedding $S \hookrightarrow \Lambda_3$ satisfying Looijenga's $\langle a \rangle$ and $\langle b \rangle$. By $\widetilde{S}$ we denote the primitive hull of S in Λ_3. We set $I = \widetilde{S}/S$. T denotes the orthogonal complement of S in Λ_3. D is the discriminant of T.

We have $S^*/S \cong (\mathbf{Z}/4)^2 \oplus (\mathbf{Z}/3)^3$. The first $\mathbf{Z}/4$-component and the third component isomorphic to $\mathbf{Z}/3$ correspond to P, the second and the fourth correspond to A_{11}, and the last fifth $\mathbf{Z}/3$-component corresponds to A_2. We have $D = 2^4 \cdot 3^3/m^2$ for $m = [\widetilde{S} : S]$.

Assume that $m = [\widetilde{S} : S]$ is odd. $(T^*/T)_2 \cong (S^*/S)_2$. Since the discriminant quadratic form on $(S^*/S)_2 \cong (\mathbf{Z}/4)^2$ can be written $-a^2/4 + b^2/4$, $D \equiv -2^4 \bmod \mathbf{Z}_2^{*2}$. Thus we have $-3^3 \equiv m^2 \pmod 8$, which is a contradiction. One knows that m is even.

Assume that m is not a multiple of 4. Then, by Proposition 3.3 (2) I_2 is generated by $(2,2) \in (S^*/S)_2$. One can check that the finite quadratic form on $(\widetilde{S}/\widetilde{S})_2 \cong I_2^{\perp}/I_2 \cong \mathbf{Z}/2 + \mathbf{Z}/2$ can be written ab for $(a,b) \in \mathbf{Z}/2 + \mathbf{Z}/2$. Thus one has $D \equiv -2^2 \bmod \mathbf{Z}_2^{*2}$ and $-3^3 \equiv m'^2 \pmod 8$ for $m' = m/2$, which is a contradiction. Thus m is a multiple of 4.

Next, we consider $p = 3$. $3 - 2l(I_3) = l((S^*/S)_3) - 2l(I_3) \leq l((\widetilde{S}^*/\widetilde{S})_3) \leq \operatorname{rank} T = 2$ and we have $l(I_3) \geq 1$. Let $\overline{\alpha} \in I_3$ be non-zero element. The discriminant quadratic form on $(S^*/S)_3$ can be written $q_3 \equiv -2x^2/3 + 2(y_1^2 + y_2^2)/3$. Thus $\overline{\alpha}$ is equal to either $\overline{\alpha}_1 = (\pm 1, \pm 1, 0)$, or $\overline{\alpha}_2 = (\pm 1, 0, \pm 1)$. If $\overline{\alpha} = \overline{\alpha}_2$, $\widetilde{S}$ contains an element $\alpha = \chi + \omega$ where $\chi \in P^*$, $\omega \in Q(A_2)^*$ and $\omega^2 = 2/3$. The image of α under $\Lambda_3 \to \Lambda_3/P$ defines a short root in the primitive hull of $Q(G)$. It contradicts fullness. Thus $\overline{\alpha} = \overline{\alpha}_1$. Note that the third component of $\overline{\alpha}_1 = (\pm 1, \pm 1, 0)$ is 0. It implies that $((P(P \oplus Q(A_{11}), \Lambda_3)/(P \oplus Q(A_{11})))_3 \neq 0$.

In conclusion one has $[P(P \oplus Q(A_{11}), \Lambda_3) : P \oplus Q(A_{11})] \geq 12$. It implies that A_{11} is an obsrtuction component with respect to $Q(G) \hookrightarrow \Lambda_3/P$. Thus $A_{11} + A_2 \notin PC$.

(c). Consider (8) $G = A_{11} + 2A_1$. We define $S = P \oplus Q(G)$, $\widetilde{S}$, T, D, and I as above.

In this case we have $S^*/S \cong (\mathbf{Z}/4)^2 \oplus (\mathbf{Z}/2)^2 \oplus (\mathbf{Z}/3)^2$ The first $\mathbf{Z}/4$-component and the fifth $\mathbf{Z}/3$-component correspond to P, the second and the sixth to A_{11}, and the middle third, fourth $\mathbf{Z}/2$-component correspond to 2 of A_1.

Since $l((S^*/S)_2) = 4 > 2$, $I_2 \neq 0$. Assume that I_2 has order 2. Then $(2,2,0,0) \in (S^*/S)_2$ is the generator of I_2 and $l(I_2^{\perp}) \geq 4$. One has $3 \leq l(I_2^{\perp}/I_2) = l((\widetilde{S}^*/\widetilde{S})_2) = l((T^*/T)_2) \leq \operatorname{rank} T = 2$, a contradiction. Thus I_2 is cyclic of order 4. Let $\overline{\alpha} = (a_1, b, c_1, c_2) \in (S^*/S)_2$ be the generator. Note that $2\overline{\alpha} = (2,2,0,0)$. Since the discriminant quadratic form q_2 on $(S^*/S)_2$ can be written $-a^2/4 + b^2/4 + (c_1^2 + c_2^2)/2$, solving $q_2(\overline{\alpha}) \equiv 0$, one has $\overline{\alpha} = (\pm 1, \pm 1, 0, 0)$.

On the other hand, by considering $T \otimes \mathbf{Z}_3$ one knows $I_3 \neq 0$.

In conclusion $[P(P \oplus Q(A_{11}), \Lambda_3) : P \oplus Q(A_{11})] \geq 12$, and A_{11} is an obstruction component. Thus $A_{11} + 2A_1 \notin PC$.

(d). For (47) and (49) the division of 13 consists of 5 odd numbers. Thus the corresponding graph is not a member of PC. Indeed, $l((S^*/S)_2) = 6$. By Corollary 3.4(2) $l(\widetilde{S}/S) \leq 1$. Thus $4 \leq l((\widetilde{S}/\widetilde{S})_2) = l((T^*/T)_2) \leq \operatorname{rank} T = 2$, which is a contradiction.

(e). For the following 7 cases we have $l((S^*/S)_3) \geq 3$. Thus $(\widetilde{S}/S)_3 \neq 0$. For every element in the 3-Sylow subgroup $(S^*/S)_3$ at which the discriminant quadratic form takes 0, one can construct an extra root with length $\sqrt{2/3}$ or $\sqrt{2}$ in the primitive hull of $Q(G)$ in Λ_3/P. Thus the corresponding graph $G \notin PC$.

(6), (11), (18), (32), (43), (53), (57).

(f). In the following 6 cases, by computing D by the 3-adic method, one can show $(\widetilde{S}/S)_3 \neq 0$. However, for each non-zero solution of $q \equiv 0$ in the 3-Sylow subgroup, we can construct an extra root in the primitive hull.

(9), (12), (23), (27), (30), (34).

(g). If we can conclude $(\widetilde{S}/S)_p \neq 0$ for $p = 5$ or 7 by calculating the discriminant of $T \otimes \mathbf{Z}_p$, then it contradicts Corollary 3.4 (1). Thus the corresponding graph $G \notin PC$. We can apply this method to the following 3 cases.

(5), (17), (21).

(h). The remaining cases are the following 3. For each one of them $l((S^*/S)_2) \geq 3$. Thus $I_2 = (\widetilde{S}/S)_2 \neq 0$. By Proposition 3.3 (2) we can easily find an element $\overline{\alpha} \in I_2$ of order 2. However, for every element $\overline{\beta} \in (S^*/S)_2$ $2\overline{\beta} \neq \overline{\alpha}$, and thus I_2 is generated by $\overline{\alpha}$. Computing the discriminant quadratic form of $\widetilde{S}$ and computing the discriminant D in two different ways, one can deduce a contradiction. Thus the corresponding graph $G \notin PC$.

(10), (20), (26).

Let us proceed to the case $r = 12$. We can assume $\rho = r + 7 = 19$. By Lemma 3.1 (1) $a = 0$ or 1.

If $a = 0$, then G is either $2A_5 + A_2$ or $2A_5 + 2A_1$ by Lemma 3.1 (1). Now, we can make $2A_5 + A_1$ from a basic graph A_{11} by a tie transformation. From $2A_5 + A_1$ we can make both of them by a tie transformation. In particular, $2A_5 + A_2$, $2A_5 + 2A_1 \in PC$.

In the following we assume $a = 1$ and $\rho = r + 7 = 19$. By Lemma 3.1 (3) it is enough to examine the divisions of 12 into 6 pieces. Recall that there are 57 kinds of them. We use the numbering [1]–[57] in Part II section 3.

[1] 12	[2] 11+1	[3] 10+2	[4] 9+3
[5] 8+4	[6] 7+5	[7] 6+6	[8] 10+1+1
[9] 9+2+1	[10] 8+3+1	[11] 8+2+2	[12] 7+4+1
[13] 7+3+2	[14] 6+5+1	[15] 6+4+2	[16] 6+3+3
[17] 5+5+2	[18] 5+4+3	[19] 4+4+4	[20] 9+1+1+1
[21] 8+2+1+1	[22] 7+3+1+1	[23] 7+2+2+1	[24] 6+4+1+1
[25] 6+3+2+1	[26] 6+2+2+2	[27] 5+5+1+1	[28] 5+4+2+1
[29] 5+3+3+1	[30] 5+3+2+2	[31] 4+4+3+1	[32] 4+3+3+2
[33] 3+3+3+3	[34] 8+1+1+1+1	[35] 7+2+1+1+1	[36] 6+3+1+1+1
[37] 6+2+2+1+1	[38] 5+4+1+1+1	[39] 5+3+2+1+1	[40] 5+2+2+2+1
[41] 4+4+2+1+1	[42] 4+3+3+1+1	[43] 4+3+2+2+1	[44] 4+2+2+2+2
[45] 3+3+3+2+1	[46] 3+3+2+2+2	[47] 7+1+1+1+1+1	[48] 6+2+1+1+1+1
[49] 5+3+1+1+1+1	[50] 5+2+2+1+1+1	[51] 4+4+1+1+1+1	[52] 4+3+2+1+1+1
[53] 4+2+2+2+1+1	[54] 3+3+3+1+1+1	[55] 3+3+2+2+1+1	[56] 3+2+2+2+2+1
[57] 2+2+2+2+2+2			

[a]. For some of 57 the corresponding graph is a subgraph of a graph which can be made from one of the essential basic Dynkin graph by tie transformations repeated twice. Thus we can apply Proposition 3.5 in Part II.

▷ A subgraph of A_{13}	[1], [3]–[7]
▷ A subgraph of $A_{12} + A_1$	[8], [9], [10], [12], [14]
▷ A subgraph of $E_7 + A_6$	[15], [25]
▷ A subgraph of $D_9 + A_4$	[18]
▷ A subgraph of $D_{12} + A_1$	[20]
▷ A subgraph of $E_8 + A_4 + A_1$	[24], [31], [41]
▷ A subgraph of $D_{10} + A_2 + A_1$	[35]

[b]. Some of them can be constructed from a basic graph by two transformations. Below every arrow except the left one at the bottom three lines indicates a tie transformation.

[2] $A_{11} + A_1$	$\longleftarrow$	A_{12}	$\longleftarrow$	A_{11}
[11] $A_8 + 2A_2$	$\longleftarrow$	$A_8 + A_2 + A_1$	$\longleftarrow$	$E_8 + B_1 + G_2$
[13] $A_7 + A_3 + A_2$	$\longleftarrow$	$A_7 + A_3 + A_1$	$\longleftarrow$	$E_7 + B_3 + G_1$
[17] $2A_5 + A_2$	$\longleftarrow$	$2A_5 + A_1$	$\longleftarrow$	A_{11}
[21] $A_8 + A_2 + 2A_1$	$\longleftarrow$	$A_8 + A_2 + A_1$	$\longleftarrow$	$E_8 + B_1 + G_2$
[22] $A_7 + A_3 + 2A_1$	$\longleftarrow$	$A_7 + A_3 + A_1$	$\longleftarrow$	$E_7 + B_3 + G_1$
[27] $2A_5 + 2A_1$	$\longleftarrow$	$2A_5 + A_1$	$\longleftarrow$	A_{11}
[30] $A_5 + A_3 + 2A_2$	$\overset{elementary}{\longleftarrow}$	$E_7 + B_3 + G_2$	$\longleftarrow$	$E_7 + B_3 + G_1$
[45] $3A_3 + A_2 + A_1$	$\overset{elementary}{\longleftarrow}$	$E_7 + B_3 + G_2$	$\longleftarrow$	$E_7 + B_3 + G_1$
[50] $A_5 + 2A_2 + 3A_1$	$\overset{elementary}{\longleftarrow}$	$E_8 + B_2 + G_2$	$\longleftarrow$	$E_8 + B_1 + G_2$

Note that in the case [2] $A_{11} + A_1$, the component A_{11} is not an obstruction.

[c]. The following 13 items do not belong to $PC(J_{3,0})$. (The 13 items treated in paragraph [A] in Part II section 3.) By Proposition 3.2 we can exclude them from our consideration.

[16], [32], [33], [34], [36], [40], [44], [47], [48], [51], [52], [56], [57].

In what follows we define $S = P \oplus Q(G)$, $\widetilde{S}$, $I = \widetilde{S}/S$, T and D corresponding to the graph G under consideration similarly as in the above paragraph **(b)**.

[d]. For the following two items the division of 12 consists of 6 odd numbers, and thus the corresponding graph $G \notin PC$.

[49], [54].

Indeed, by Corollary 3.4 (2) $l(I_2) \leq 1$. Thus we have $5 = 7 - 2 \leq l((S^*/S)_2) - 2l(I_2) \leq l((\widetilde{S}^*/\widetilde{S})_2) = l((T^*/T)_2) \leq \operatorname{rank} T = 3$, a contradiction.

[e]. For the following 3 cases we have $l((S^*/S)_3) \geq 4$. Thus we can apply reasoning like the one in the above paragraph **(e)**, we can conclude $G \notin PC$.

[26], [46], [53].

[f]. In the following 6 cases, we can apply the 3-adic method as in the above paragraph **(f)** and we can conclude $G \notin PC$.

[23], [28], [37], [39], [43], [55].

[g]. We can conclude $(\widetilde{S}/S)_p \neq 0$ for $p = 5$ in the case [19] by calculating D in 2 ways. Thus by Corollary 3.4 (1) $G \notin PC$.

[h]. In the cases [29], [38], [42] we have $l((S^*/S)_2) \geq 4$, and by the same argument as in the above paragraph **(h)** we can infer $G \notin PC$.

In the following we proceed to the last case $r = 11$.

Proposition 3.5. *For every division $(k_1, k_2, \ldots, k_7)$ of $11 = \sum_{i=1}^{7} k_i$ into a sum of 7 non-negative integers $k_1 \geq k_2 \geq \cdots \geq k_7 \geq 0$ consider the Dynkin graph $G = \sum A_{k_i}$, the root lattice $Q = Q(G)$ of type G, the discriminant $d(G) = d(Q)$ of Q, and the Hasse invariant $\epsilon_p(G) = \epsilon_p(Q)$ of Q, where p is a prime number. The arithmetic condition in Part I Theorem 0.5 [II] (A) (2)*

$$3d(G) \notin \mathbf{Q}_p^{*2} \text{ or } \epsilon_p(G) = (-1,3)_p$$

is not satisfied if and only if G and p are one in the following list:

$p = 3$, $A_5 + A_4 + 2A_1$, $A_5 + A_3 + 3A_1$, $A_4 + 3A_2 + A_1$, $A_3 + 3A_2 + 2A_1$. *(4 items.)*
$p = 2$, $A_5 + A_3 + 3A_1$, $A_3 + 3A_2 + 2A_1$. *(2 items.)*

Proof. It is not difficult to make the list of p and the division $(k_1, \ldots, k_7)$ such that $\epsilon_p(G) \neq (-1,3)_p$. We can omit the calculation for $p = 2$ by the product formula. We do not present the list here but it contains 36 kinds of graphs, and each graph in it corresponds just 2 prime numbers. $A_6 + A_3 + 2A_1$ and $A_6 + 5A_1$ correspond to $p = 7$ and 3. $A_6 + 2A_2 + A_1$ corresponds to $p = 7$ and 2. $A_7 + A_4$, $A_5 + A_4 + A_2$, $A_4 + 2A_3 + A_1$ and $A_4 + A_3 + A_2 + 2A_1$ correspond to $p = 5$ and 3. $A_9 + A_2$ and $A_4 + 2A_2 + 3A_1$ correspond to $p = 5$ and 2. The other 27 graphs correspond to $p = 3$ and 2.

Checking whether $3d(G) \in \mathbf{Q}_p^{*2}$ for each item, we get the proposition. Q.E.D.

By Theorem 0.5 [II] in Part I, we can consider only the 4 graphs in the above proposition.

Obviously $A_5 + A_4$ and $A_5 + A_3 + A_1$ are subgraphs of A_{11}. Thus we can make $A_5 + A_4 + 2A_1$ and $A_5 + A_3 + 3A_1$ from A_{11} by tie transformations repeated twice. In particular they are members in PC.

Next, let us consider $G = A_4 + 3A_2 + A_1$. By using 3-adic integers $\mathbf{Z}_3$, we can show $(\widetilde{S}/S)_3 \neq 0$ for every embedding $S = P \oplus Q(G) \hookrightarrow \Lambda_3$. Any element $\overline{\alpha} \in (\widetilde{S}/S)_3$ satisfies $q(\overline{\alpha}) \equiv 0 \bmod 2\mathbf{Z}$, where q is the discriminant quadratic form of S. However, for every non-zero element $\overline{\alpha} \in (S^*/S)_3$ with $q(\overline{\alpha}) \equiv 0$ we can construct an extra root not in $Q(G)$ but in the primitive hull of $Q(G)$ in Λ_3/P. It contradicts the fullness. Thus $A_4 + 3A_2 + A_1 \notin PC$.

By the same method we can show $A_3 + 3A_2 + 2A_1 \notin PC$.

In this section we have shown Proposition 0.1 under the assumption ((3)) in the introduction. We can complete this article.

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